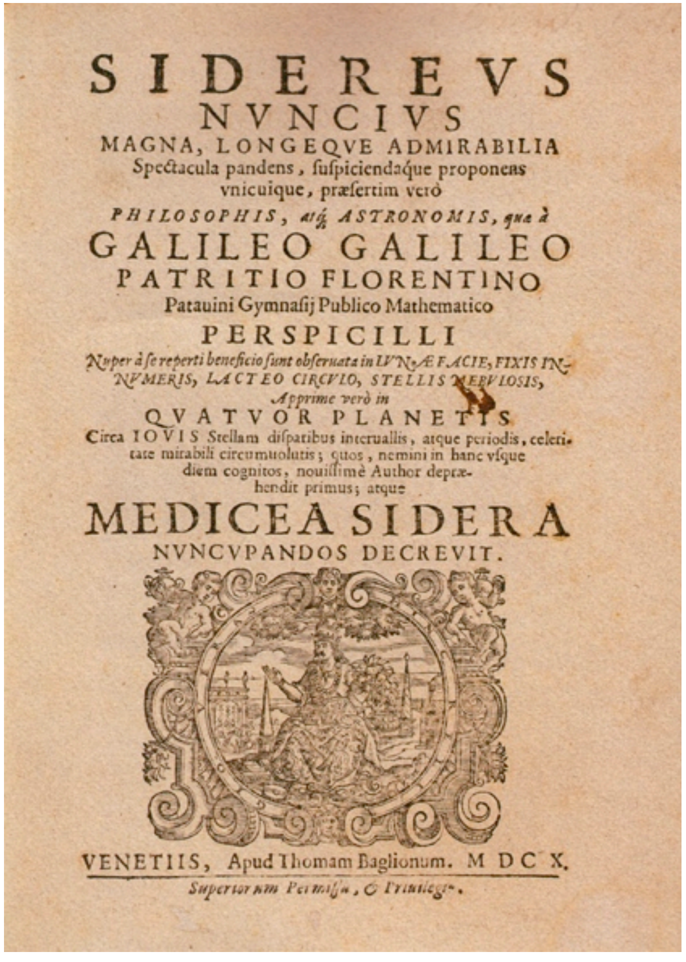
Recent results in (anti)-de Sitter QFT/Strings

Cargese, May 12, 2014

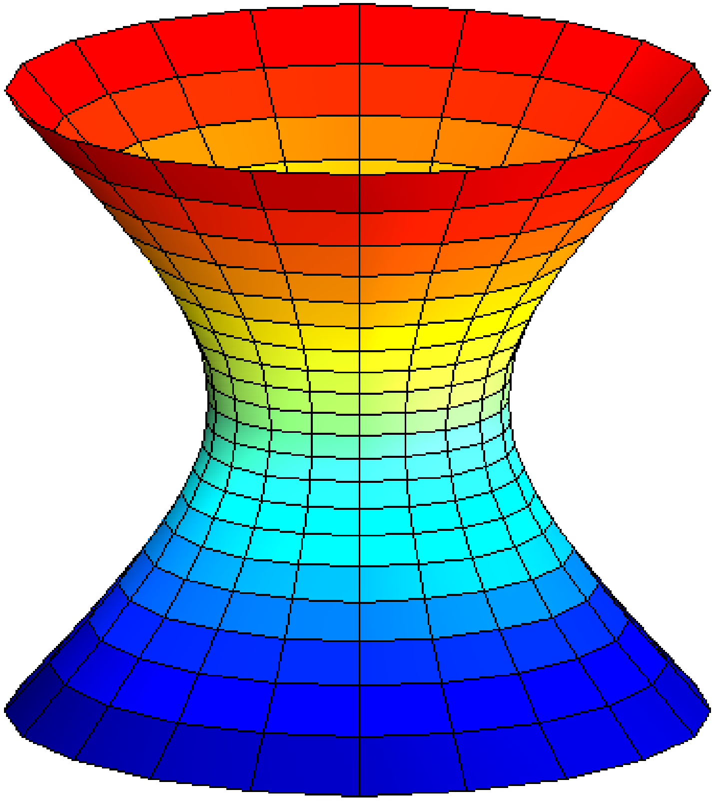
Ugo Moschella
Università dell'Insubria – Como
ugomoschella@gmail.com

The shape of our universe

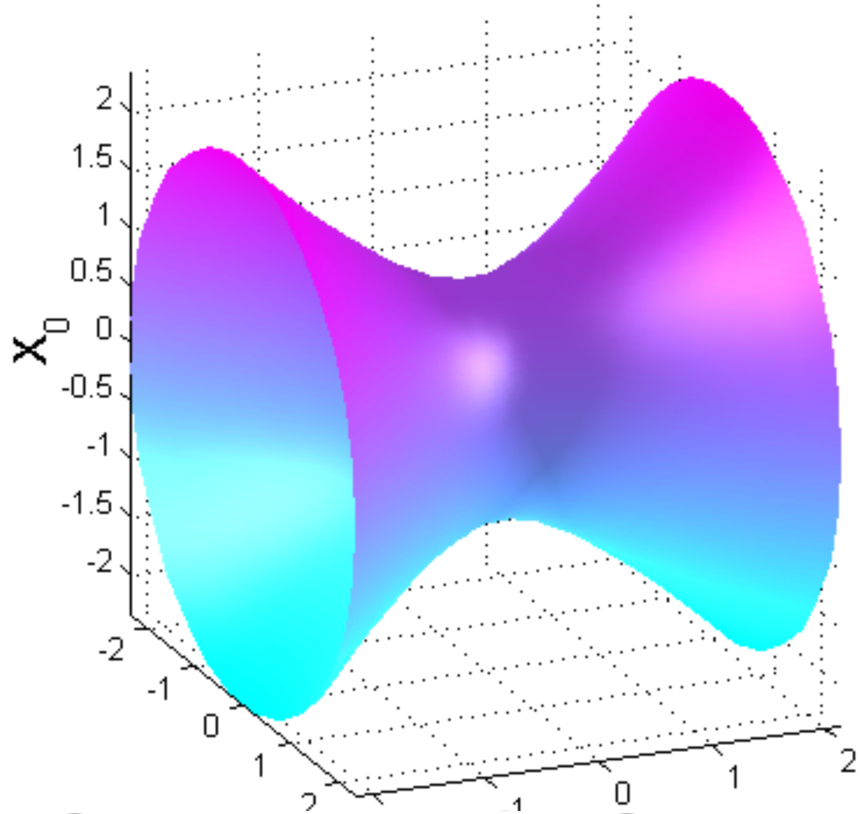
SN 1997 = Sidereus Nuncius 1997


SIDEREVS
NVNCIVS
MAGNA, LONGEQVE ADMIRABILIA
Spectacula pandens, suspiciendaque proponens
vnicuique, præsertim verò
PHILOSOPHIS, atq; ASTRONOMIS, quæ à
GALILEO GALILEO
PATRITIO FLORENTINO
Patauini Gymnasij Publico Mathematico
PERSPICILLI
Nuper à se reperti beneficio sunt obseruata in LVNÆ FACIE, FIXIS IN-NVMERIS, LACTEO CIRCVLO, STELLIS NEBVLOSIS,
Apprime verò in
QVATVOR PLANETIS
Circa IOVIS Stellam disparibus interuallis, atque periodis, celeritate mirabili circumuolutis; quos, nemini in hanc vsque diem cognitos, nouissimè Author depræhendit primus; atque
MEDICEA SIDERA
NVNCVPANDOS DECREVIT.

VENETIIS, Apud Thomam Baglionum. M DC X.
Superiorum Permissu, & Priuilegio.



1997. The anti de Sitter universe



$$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 + X_d^2 = R^2$$

$E^{(2,d-1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1,1**) $SO(2, d-1)$

Classical Strings

$$t, s \to Y(t,s) \in dS(AdS)$$

String equations

$$\partial_t^2 Y_i - \partial_s^2 Y_i + [(\partial_t Y)^2 - (\partial_s Y)^2] Y_i = 0$$

Conformal gauge constraints:

$$(\partial_t Y \pm \partial_s Y)^2 = 0.$$

GKP's rotating folded string (2002)

$$AdS_3 = \{Y \in \mathbf{R}^4 :\ Y^2 = Y\cdot Y = Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 = 1\}.$$

$$\begin{cases} Y_0 = \cosh\rho(s)\cos(\omega_1 t) \\ Y_1 = \cosh\rho(s)\sin(\omega_1 t) \\ Y_2 = \sinh\rho(s)\cos(\omega_2 t) \\ Y_3 = \sinh\rho(s)\sin(\omega_2 t) \end{cases} \qquad \rho(s) = \rho(s + 2L)$$

Conformal gauge constraints:

$$(\partial_t Y \pm \partial_s Y)^2 = 0.$$

$$\left(\frac{d\rho}{ds}\right)^2 = \omega_1^2 \cosh^2\rho(s) - \omega_2^2 \sinh^2\rho(s)$$

GKP's rotating string (2002)

$$\left(\frac{d\rho}{ds}\right)^2 = \omega_1^2 \cosh^2 \rho(s) - \omega_2^2 \sinh^2 \rho(s)$$

$$2L = \int_0^{2L} ds = 4 \int_0^{\rho_0} \frac{d\rho}{\sqrt{\omega_1^2 \cosh^2 \rho - \omega_2^2 \sinh^2 \rho}} = \frac{4K(k)}{\omega_2}$$

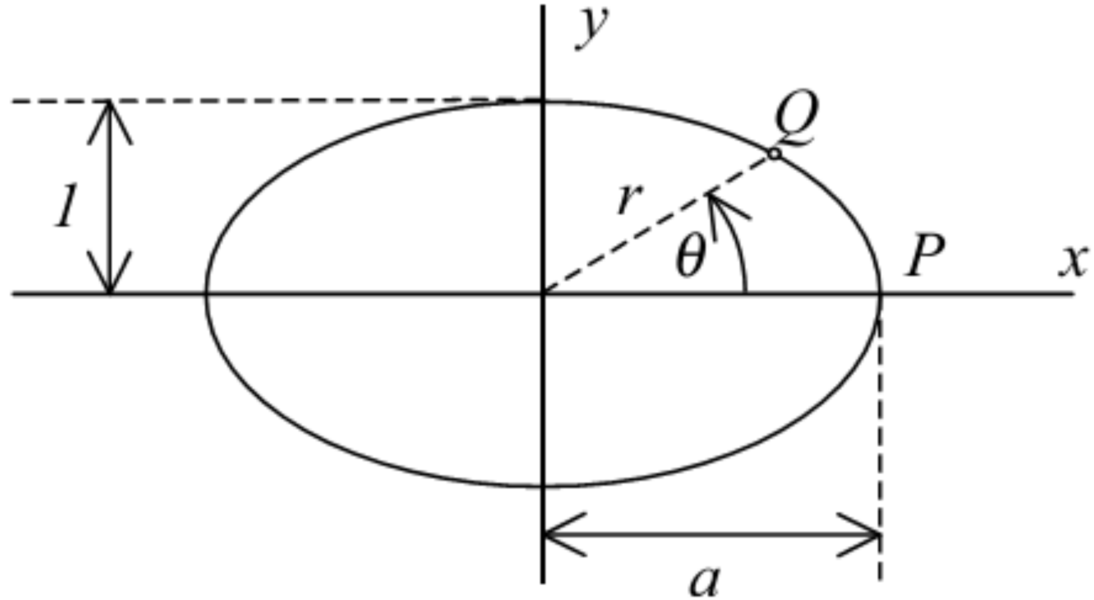
$$\tanh \rho_0 = \pm \frac{\omega_1}{\omega_2} = \pm k \qquad \omega_2 > \omega_1$$

Jacobi elliptic functions: a reminder

$$\left(\frac{x}{a}\right)^2 + y^2 = 1.$$

$$x^2 + y^2 = r^2.$$

$$\epsilon \equiv k = \sqrt{1 - \frac{1}{a^2}},$$



$$u = \int_P^Q r\,d\theta = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$\mathrm{sn}(u,k) = y = \sin\phi \qquad \mathrm{dn}(u,k) = \frac{r}{a}$$

$$\mathrm{cn}(u,k) = \frac{x}{a} = \cos\phi$$

Jacobi elliptic functions: a reminder

$$\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u = \sqrt{1-\operatorname{sn}^2 u}\sqrt{1-k^2\operatorname{sn}^2 u},$$

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2 y^2).$$

GKP's rotating string (2002)

$y = \cosh\rho$ transforms the constraint into a nonlinear Jacobian differential equation:

$$(y')^2 = \omega_2^2 \left[-1 + \left(2 - k^2\right) y^2 - \left(1 - k^2\right) y^4 \right]$$

Initial condition $\rho(0) = 0$; solution

$$\cosh\rho = \mathrm{nd}\,(\omega s; k)\,, \qquad \sinh\rho = k \;\; \mathrm{sd}(\omega s; k),$$

$$\begin{cases} Y_0 = \mathrm{nd}\,(\omega s; k) \cos(k\omega t), & Y_1 = \mathrm{nd}\,(\omega s; k) \sin(k\omega t), \\ Y_2 = k \;\mathrm{sd}(\omega s; k) \cos(\omega t), & Y_3 = k \;\mathrm{sd}(\omega s; k) \sin(\omega t). \end{cases}$$

GKP's rotating string (2002)

$$\mathcal{E} = \int_0^{2L} (\dot{Y}_0 Y_1 - Y_0 \dot{Y}_1) d\sigma \ = \frac{4kE(k)}{1-k^2}$$

$$\mathcal{S} = \int_0^{2L} (\dot{Y}_2 Y_3 - Y_3 \dot{Y}_2) d\sigma \ = \frac{4E(k)}{1-k^2} - 4K(k)$$

$$\mathcal{E} = \mathcal{E}(\mathcal{S})$$

3D -> 2D

$$AdS_3 = \{Y \in \mathbf{R}^4 : \ Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 = 1\}.$$

$$\begin{cases} Y_0 = \cosh\rho(s)\cos(\omega_1 t) \\ Y_1 = \cosh\rho(s)\sin(\omega_1 t) \\ Y_2 = \sinh\rho(s)\cos(\omega_2 t) \\ Y_3 = \sinh\rho(s)\sin(\omega_2 t) \end{cases} \qquad \rho(s) = \rho(s + 2L)$$

$\omega_1 \to 0$ The string becomes pointlike

$\omega_2 \to 0$ Also trivial but less

2D "GKP's" string

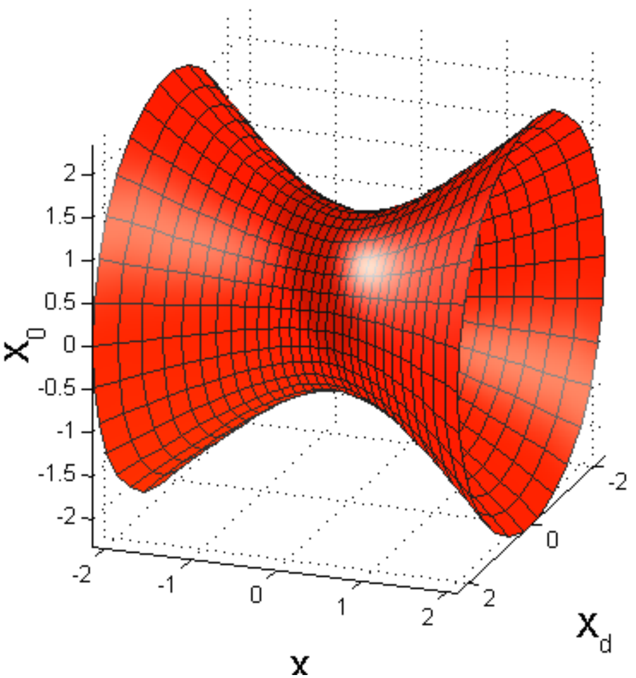
$$AdS_3 = \{Y \in \mathbf{R}^4 : \; Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 = 1\}.$$

$$\begin{cases} Y_0 = \cosh\rho(s)\cos(\omega_1 t) \\ Y_1 = \cosh\rho(s)\sin(\omega_1 t) \\ Y_2 = \sinh\rho(s) \end{cases}$$

$$(\partial_t Y \pm \partial_s Y)^2 = 0.$$

$$\cosh\rho(s) \to \frac{1}{\cos(\omega_1 s)}$$

$$\sinh\rho(s) \to \tan(\omega_1 s)$$



Elliptic function and theta functions

$$\vartheta_3(z,\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi n^2\tau + 2\pi i z)$$

$$\mathrm{sn}(z,k) = \frac{\vartheta_3\vartheta_1(z/\vartheta_3^2)}{\vartheta_2\vartheta_4(z/\vartheta_3^2)}, \quad \mathrm{cn}(z,k) = \frac{\vartheta_4\vartheta_2(z/\vartheta_3^2)}{\vartheta_2\vartheta_4(z/\vartheta_3^2)}, \quad \mathrm{dn}(z,k) = \frac{\vartheta_4\vartheta_3(z/\vartheta_3^2)}{\vartheta_3\vartheta_4(z/\vartheta_3^2)}.$$

$$k = \frac{\vartheta_2^2}{\vartheta_3^2} = \frac{\vartheta_2^2(0\,|\,\tau)}{\vartheta_3^2(0\,|\,\tau)}, \qquad k' = \frac{\vartheta_4^2}{\vartheta_3^2} = \frac{\vartheta_4^2(0\,|\,\tau)}{\vartheta_3^2(0\,|\,\tau)}$$

$$\begin{cases} Y_0 = \mathrm{nd}\,(\omega s; k)\cos(k\omega t), & Y_1 = \mathrm{nd}\,(\omega s; k)\sin(k\omega t), \\ Y_2 = k\ \mathrm{sd}(\omega s; k)\cos(\omega t), & Y_3 = k\ \mathrm{sd}(\omega s; k)\sin(\omega t). \end{cases}$$

$$\begin{cases} Y_0 = \dfrac{\vartheta_3\vartheta_4(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\cos(kt), & Y_1 = \dfrac{\vartheta_3\vartheta_4(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\sin(kt), \\ Y_2 = \dfrac{\vartheta_2\vartheta_1(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\cos(t), & Y_3 = \dfrac{\vartheta_2\vartheta_1(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\sin(t), \end{cases}$$

$$\omega = 1, \quad \widehat{s} = s/\vartheta_3^2.$$

The trick of the tale: GKP's string in hom. coordinates

$$AdS_3 = \{Y \in \mathbf{R}^4 :\ Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 = 1\}.$$

$$\begin{cases} Y_0 = \frac{\vartheta_3\vartheta_4(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\cos(kt), & Y_1 = \frac{\vartheta_3\vartheta_4(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\sin(kt), \\ Y_2 = \frac{\vartheta_2\vartheta_1(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\cos(t), & Y_3 = \frac{\vartheta_2\vartheta_1(\widehat{s})}{\vartheta_4\vartheta_3(\widehat{s})}\sin(t), \end{cases}$$

$$C_{2,3} = \{\xi \in \mathbf{R}^{d+2} :\ \xi^2 = \xi \cdot \xi = \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 = 0\}$$

String on the cone

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_3\vartheta_4(\widehat{s})\cos(kt), \\ \xi_1 = \vartheta_3\vartheta_4(\widehat{s})\sin(kt), \\ \xi_2 = \vartheta_2\vartheta_1(\widehat{s})\cos(t), \\ \xi_3 = \vartheta_2\vartheta_1(\widehat{s})\sin(t), \\ \xi_4 = \vartheta_4\vartheta_3(\widehat{s}); \end{cases}$$

GKP's string on the cone

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_3\vartheta_4(\hat{s}) \cos(kt), \\ \xi_1 = \vartheta_3\vartheta_4(\hat{s}) \sin(kt), \\ \xi_2 = \vartheta_2\vartheta_1(\hat{s}) \cos(t), \\ \xi_3 = \vartheta_2\vartheta_1(\hat{s}) \sin(t), \\ \xi_4 = \vartheta_4\vartheta_3(\hat{s}); \end{cases}$$

$\xi \in C_{2,3}$ is a well-known quadratic identity between theta functions (Whittaker, p. 466):

$$\xi^2 = \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 =$$

$$= {\vartheta_3}^2\vartheta_4(\hat{s})^2 - {\vartheta_2}^2\vartheta_1(\hat{s})^2 - {\vartheta_4}^2\vartheta_3(\hat{s})^2 = 0.$$

Constraints on the cone

AdS string in homogeneous coordinates

$$t, s \to Y_i(t,s) = \frac{\xi_i(t,s)}{\xi_{d+1}(t,s)}, \quad i = 0, 1, \ldots, d; \tag{1}$$

$t, s \to \xi_\mu(t,s)$, $\mu = 0, 1, \ldots, d+1$, is a two-surface in $C_{2,d}$.

$$\xi^2 = 0 \quad \longrightarrow \quad \partial_z Y^i \partial_w Y_i = \frac{1}{\xi_{d+1}^2} \partial_z \xi^\mu \partial_w \xi_\mu, \tag{2}$$

z, w can be either t or s.

If Y_i satisfy the constraints in AdS_d, the functions ξ_μ also do in $C(2,d)$ and viceversa. .

Doubly elliptic strings on the cone

A fundamental quadratic identity between theta functions:

$$\vartheta_1(\hat{t})^2\,\vartheta_1(\hat{s})^2-\vartheta_2(\hat{t})^2\,\vartheta_2(\hat{s})^2+\vartheta_3(\hat{t})^2\,\vartheta_3(\hat{s})^2-\vartheta_4(\hat{t})^2\,\vartheta_4(\hat{s})^2=0$$

$$(t,s)\to\xi(t,s)=\begin{cases}\xi_0=\vartheta_1(\hat{t})\,\vartheta_1(\hat{s}), & \xi_1=\vartheta_3(\hat{t})\,\vartheta_3(\hat{s}),\\ \xi_2=\vartheta_2(\hat{t})\,\vartheta_2(\hat{s}), & \xi_3=\vartheta_4(\hat{t})\,\vartheta_4(\hat{s}).\end{cases}$$

$$\xi^2 \;=\xi_0^2+\xi_1^2-\xi_2^2-\xi_3^2=0$$

$$(t,s)\to\xi(t,s)\in C_{2,2}$$

Constraints

Amounts to another (possibly unknown) identity among theta functions and their derivatives

$$\partial_t\xi \cdot \partial_t\xi + \partial_s\xi \cdot \partial_s\xi = -\theta_3^{-2} \sum_{i=1}^{4} (-1)^\alpha (\vartheta'_\alpha(\hat{t})^2\, \vartheta_\alpha(\hat{s})^2 + \vartheta_\alpha(\hat{t})^2\, \vartheta'_\alpha(\hat{s})^2) = 0.$$

Proof: apply the Laplace operator to the defining identity

$$0 = \frac{1}{2}(\partial_x^2 + \partial_y^2) \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta_\alpha(x)^2 \vartheta_\alpha(y)^2) =$$

$$= \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2 + \vartheta_\alpha(x)\vartheta''_\alpha(x)\vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2\vartheta_\alpha(y)\vartheta''_\alpha(y)) =$$

$$= \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2) + \frac{2i}{\pi}\frac{\partial}{\partial\tau} \sum (-1)^\alpha (\vartheta_\alpha(x)^2 \vartheta_\alpha(y)^2) =$$

$$= \sum_{\alpha=1}^{4} (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2) = 0.$$

Finite open strings

Project back to AdS/dS

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_1(\hat{t})\,\vartheta_1(\hat{s}), & \xi_1 = \vartheta_3(\hat{t})\,\vartheta_3(\hat{s}), \\ \xi_2 = \vartheta_2(\hat{t})\,\vartheta_2(\hat{s}), & \xi_3 = \vartheta_4(\hat{t})\,\vartheta_4(\hat{s}). \end{cases}$$

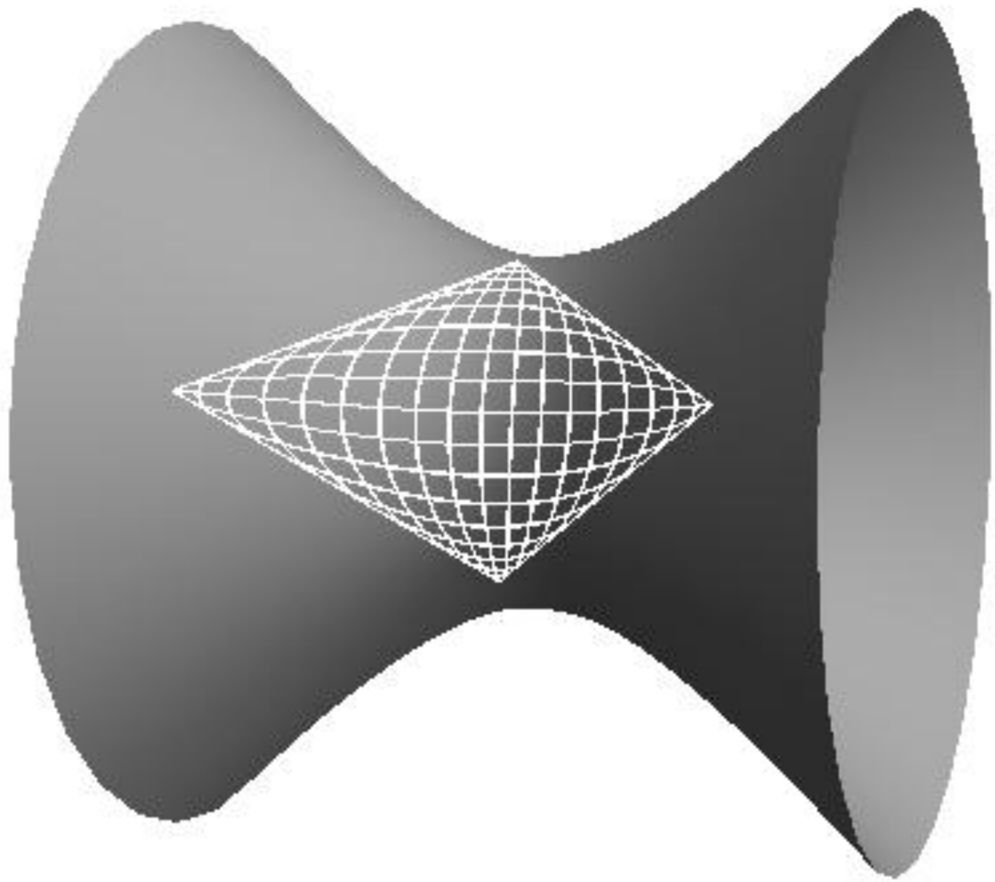
$$(t,s) \to Y^{(1)} = \begin{cases} Y_0(t,s) = \frac{\xi_0}{\xi_3} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = k\ \mathrm{sn}(t,k)\,\mathrm{sn}(s,k), \\ Y_1(t,s) = \frac{\xi_1}{\xi_3} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = \frac{1}{k'}\,\mathrm{dn}(t,k)\,\mathrm{dn}(s,k), \\ Y_2(t,s) = \frac{\xi_2}{\xi_3} = \frac{\vartheta_2(\hat{t})\vartheta_2(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = \frac{k}{k'}\,\mathrm{cn}(t,k)\,\mathrm{cn}(s,k), \end{cases}$$

Constraints are satisfied.

String equations also (I leave this as an exercise!)

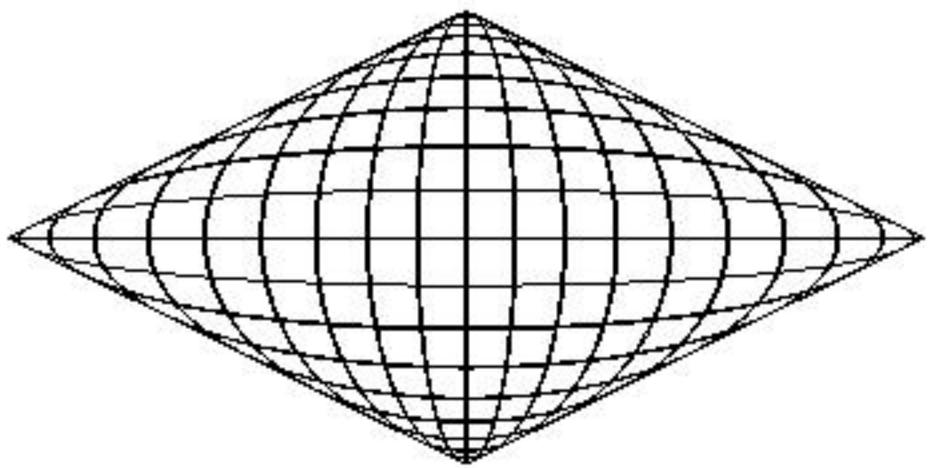
Finite open strings

$$(t,s) \to Y^{(1)} = \begin{cases} Y_0(t,s) &= k\ \mathsf{sn}(t,k)\,\mathsf{sn}(s,k), \\ Y_1(t,s) &= \frac{1}{k'}\,\mathsf{dn}(t,k)\,\mathsf{dn}(s,k), \\ Y_2(t,s) &= \frac{k}{k'}\,\mathsf{cn}(t,k)\,\mathsf{cn}(s,k), \end{cases}$$



Finite open strings

$$(t,s) \to Y^{(1)} = \begin{cases} Y_0(t,s) &=& k\ \mathsf{sn}(t,k)\,\mathsf{sn}(s,k), \\ Y_1(t,s) &=& \frac{1}{k'}\,\mathsf{dn}(t,k)\,\mathsf{dn}(s,k), \\ Y_2(t,s) &=& \frac{k}{k'}\,\mathsf{cn}(t,k)\,\mathsf{cn}(s,k), \end{cases}$$

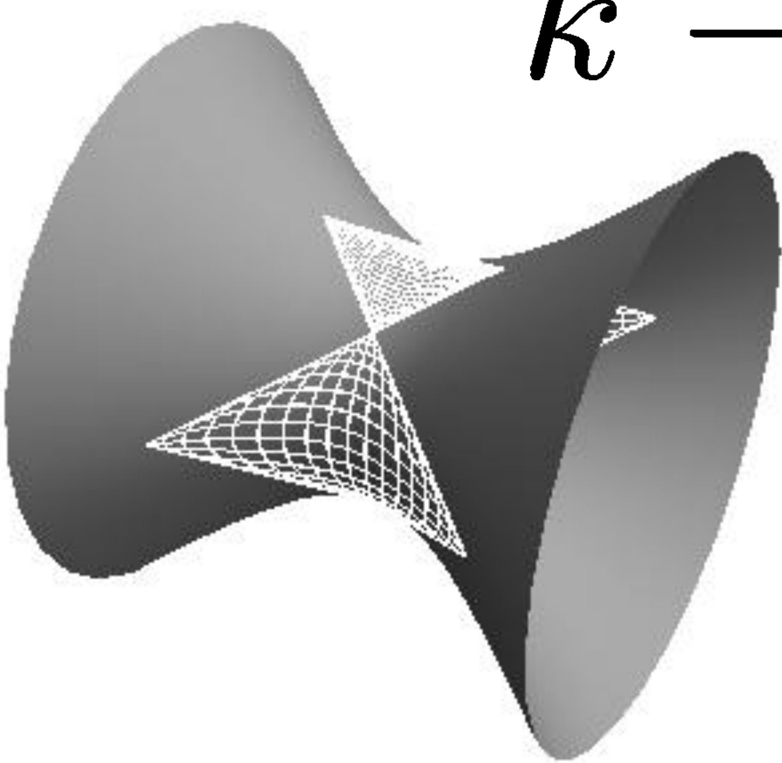


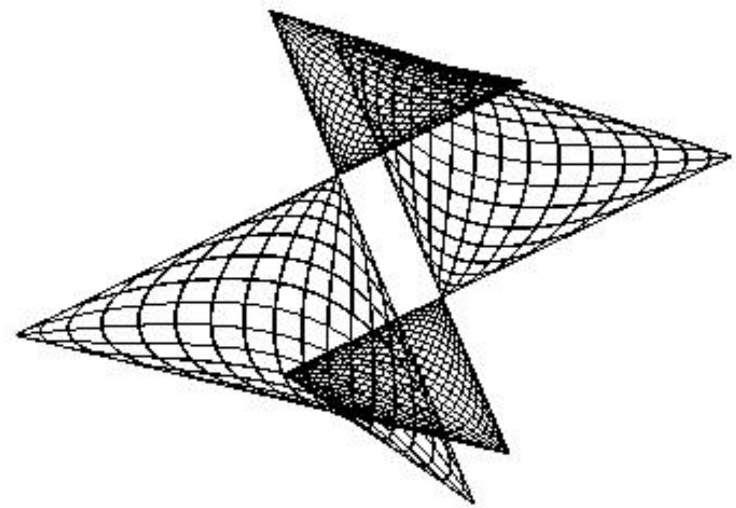
$$\mathcal{A} = \int \sqrt{h}\,dtds = k^2 \int \sqrt{\left(\mathsf{sn}\,(t,k)^2 - \mathsf{sn}\,(s,k)^2\right)^2}\,dtds =$$

$$= 8(K(k) - E(k))K(k).$$

Finite open strings

$$k \to k'$$





$$\mathcal{A} \; = \; \int \sqrt{h} dt ds = 8[K(k)^2 + K(k')^2 - E(k)K(k) - E(k')K(k')].$$

Semi infinite Strings

Project back to AdS/dS

$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_1(\hat{t})\,\vartheta_1(\hat{s}), & \xi_1 = \vartheta_3(\hat{t})\,\vartheta_3(\hat{s}), \\ \xi_2 = \vartheta_2(\hat{t})\,\vartheta_2(\hat{s}), & \xi_3 = \vartheta_4(\hat{t})\,\vartheta_4(\hat{s}). \end{cases}$$

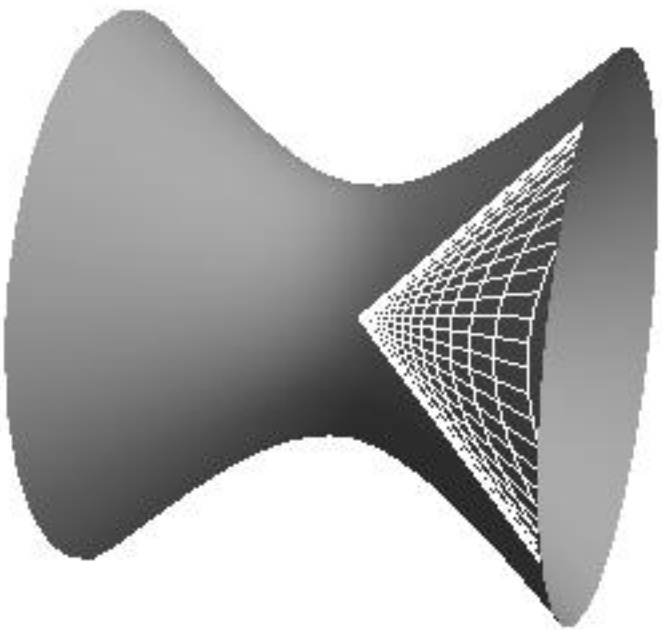
$$(t,s) \to Y^{(2)}(t,s;k) = \begin{cases} Y_0(t,s) = \frac{\xi_0}{\xi_2} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = k' \,\mathrm{sc}(t,k)\,\mathrm{sc}(s,k), \\ Y_1(t,s) = \frac{\xi_1}{\xi_2} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{1}{k}\,\mathrm{dc}(t,k)\,\mathrm{dc}(s,k), \\ Y_2(t,s) = \frac{\xi_3}{\xi_2} = \frac{\vartheta_4(\hat{t})\vartheta_4(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{k'}{k}\,\mathrm{nc}(t,k)\,\mathrm{nc}(s,k); \end{cases}$$

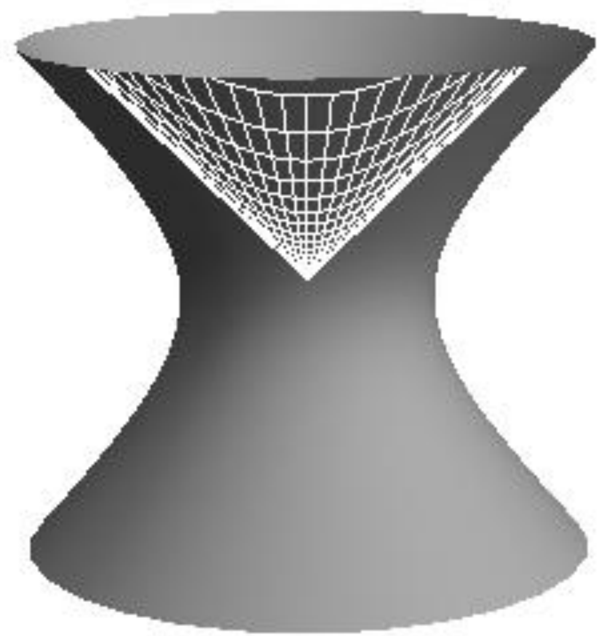
Constraints are satisfied.

String equations also (I leave this as another exercise!)

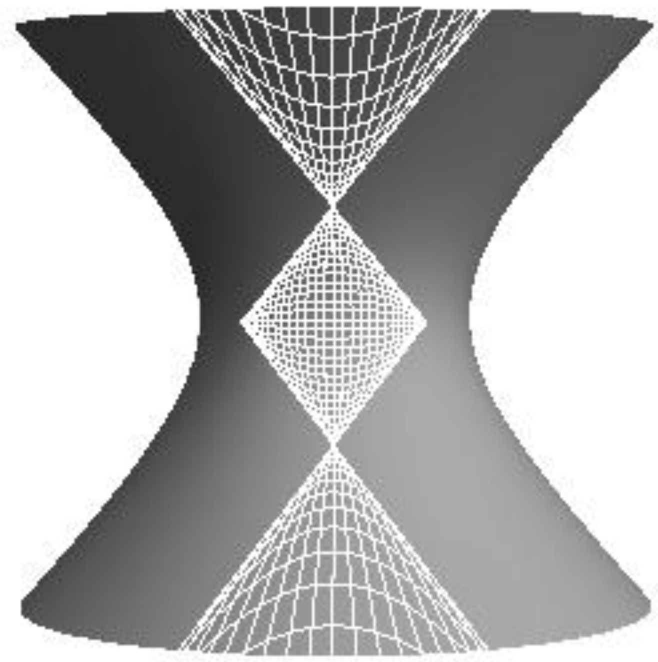
Finite open strings

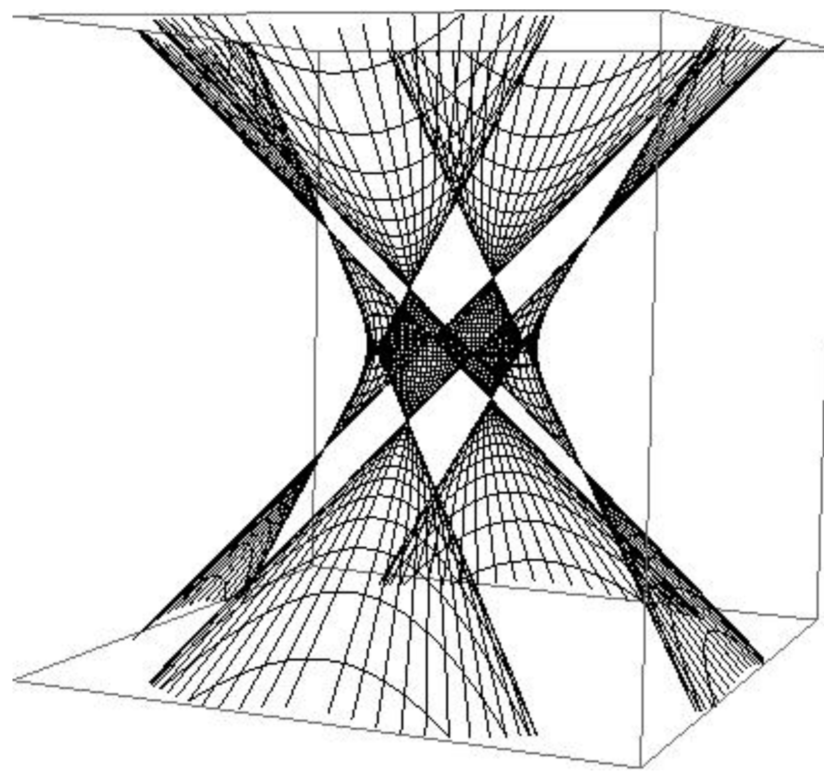
$$(t,s) \to Y^{(2)}(t,s;k) = \begin{cases} Y_0(t,s) &= \frac{\xi_0}{\xi_2} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = k' \,\mathsf{sc}(t,k)\,\mathsf{sc}(s,k), \\ Y_1(t,s) &= \frac{\xi_1}{\xi_2} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{1}{k}\,\mathsf{dc}(t,k)\,\mathsf{dc}(s,k), \\ Y_2(t,s) &= \frac{\xi_3}{\xi_2} = \frac{\vartheta_4(\hat{t})\vartheta_4(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{k'}{k}\,\mathsf{nc}(t,k)\,\mathsf{nc}(s,k); \end{cases}$$





All in all





Infinite open (AdS)/ closed (dS) strings

A second well-known relation between theta functions

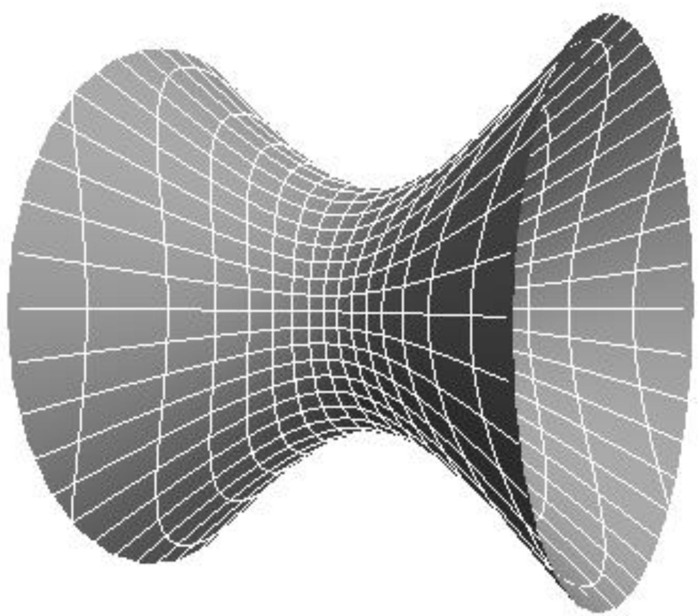
$$\xi^2 = \vartheta_1(\hat{t})^2\,\vartheta_3(\hat{s})^2 + \vartheta_2(\hat{t})^2\,\vartheta_4(\hat{s})^2 - \vartheta_3(\hat{t})^2\,\vartheta_1(\hat{s})^2 - \vartheta_4(\hat{t})^2\,\vartheta_2(\hat{s})^2 = 0.$$

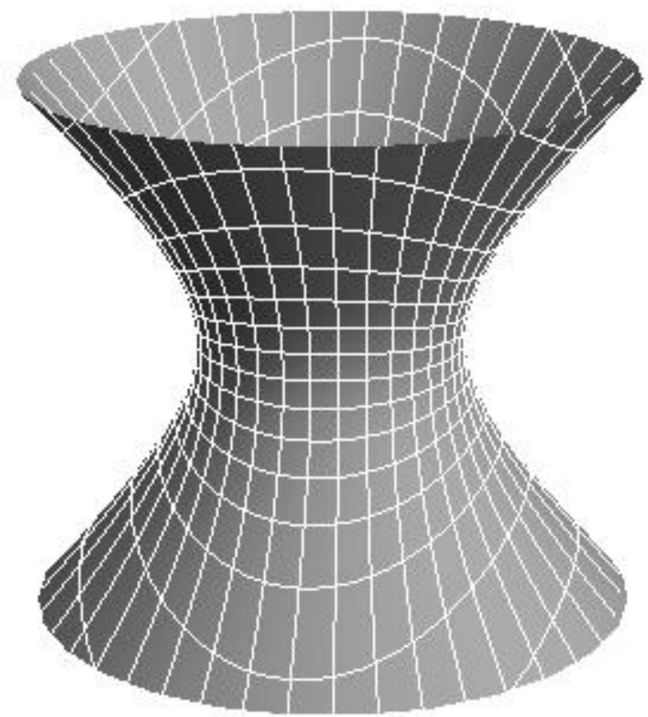
$$(t,s) \to \xi(t,s) = \begin{cases} \xi_0 = \vartheta_1(\hat{t})\,\vartheta_3(\hat{s}), & \xi_1 = \vartheta_2(\hat{t})\,\vartheta_4(\hat{s}), \\ \xi_2 = \vartheta_3(\hat{t})\,\vartheta_1(\hat{s}), & \xi_3 = \vartheta_4(\hat{t})\,\vartheta_2(\hat{s}). \end{cases}$$

$$(t,s) \to \begin{cases} Y_0(t,s) = \frac{\xi_0}{\xi_3} = \mathrm{sn}(t,k)\ \mathrm{dc}(s,k), \\ Y_1(t,s) = \frac{\xi_1}{\xi_3} = \mathrm{cn}(t,k)\ \mathrm{nc}(s,k), \\ Y_2(t,s) = \frac{\xi_2}{\xi_3} = \mathrm{dn}(t,k)\ \mathrm{sc}(s,k) \end{cases}$$

Infinite open (AdS)/ closed (dS) strings

$$(t,s) \to \begin{cases} Y_0(t,s) &= \frac{\xi_0}{\xi_3} &= \mathsf{sn}(t,k)\ \mathsf{dc}(s,k), \\ Y_1(t,s) &= \frac{\xi_1}{\xi_3} &= \mathsf{cn}(t,k)\ \mathsf{nc}(s,k), \\ Y_2(t,s) &= \frac{\xi_2}{\xi_3} &= \mathsf{dn}(t,k)\ \mathsf{sc}(s,k) \end{cases}$$





Integrability

Jacobi and Neumann's conoidal coordinates on the sphere $S_2 = \{X_0^2 + X_1^2 + X_2^2 = 1\}$

$$a_0 < \zeta_1 < a_1 < \zeta_2 < a_2$$

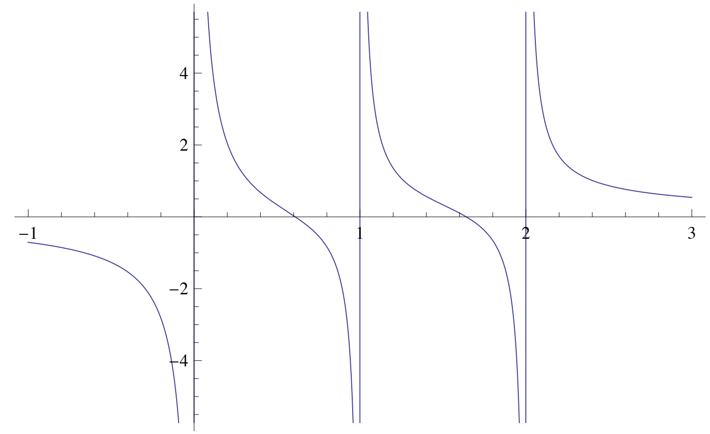
$$X_0(\zeta_1,\zeta_2) = \sqrt{\frac{(a_0-\zeta_1)(a_0-\zeta_2)}{(a_0-a_1)(a_0-a_2)}},$$

$$X_1(\zeta_1,\zeta_2) = \sqrt{\frac{(a_1-\zeta_1)(a_1-\zeta_2)}{(a_1-a_0)(a_1-a_2)}},$$

$$X_2(\zeta_1,\zeta_2) = \sqrt{\frac{(a_2-\zeta_1)(a_2-\zeta_2)}{(a_2-a_0)(a_2-a_1)}},$$

The conoidal coordinates are the zeros of the function

$$q_X(\zeta) = \frac{X_0^2}{\zeta - a_0} + \frac{X_1^2}{\zeta - a_1} + \frac{X_2^2}{\zeta - a_2}$$



$$q_X(\zeta) = \frac{(\zeta - \zeta_1)(\zeta - \zeta_2)}{(\zeta - a_0)(\zeta - a_1)(\zeta - a_2)}$$

$$X_0^2(\zeta_1, \zeta_2) = \frac{(a_0 - \zeta_1)(a_0 - \zeta_2)}{(a_0 - a_1)(a_0 - a_2)}$$

The off-diagonal components of the metric vanish and the conoidal coordinates are orthogonal

$$ds^2 = -\frac{1}{4}\, q'(\zeta_1)\, d\zeta_1^2 - \frac{1}{4}\, q'(\zeta_2)\, d\zeta_2^2$$

AdS I

Jacobi and Neumann's conoidal coordinates on

$$AdS_2 = \{X_0^2 + X_1^2 - X_2^2 = 1\}$$

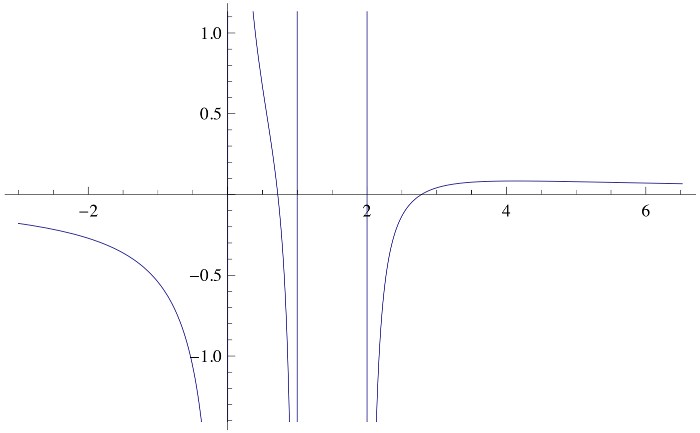
$$a_0 < a_1 < a_2$$

$$X_0(\zeta_1, \zeta_2) = \sqrt{\frac{(a_0 - \zeta_1)(a_0 - \zeta_2)}{(a_0 - a_1)(a_0 - a_2)}},$$

$$X_1(\zeta_1, \zeta_2) = \sqrt{\frac{(a_1 - \zeta_1)(a_1 - \zeta_2)}{(a_1 - a_0)(a_1 - a_2)}},$$

$$X_2(\zeta_1, \zeta_2) = \sqrt{-\frac{(a_2 - \zeta_1)(a_2 - \zeta_2)}{(a_2 - a_0)(a_2 - a_1)}},$$

The conoidal coordinates are the zeros of the function



$$q_X(\zeta) = \frac{X_0^2}{\zeta - a_0} + \frac{X_1^2}{\zeta - a_1} - \frac{X_2^2}{\zeta - a_2}$$

$$q_X(\zeta) = \frac{(\zeta-\zeta_1)(\zeta-\zeta_2)}{(\zeta-a_0)(\zeta-a_1)(\zeta-a_2)}$$

$$a_0 < \zeta_1 < a_1 < a_2 < \zeta_2$$

The conoidal coordinates are one spacelike and one timelike

$$ds^2 = -\frac{1}{4}\, q'(\zeta_1)\, d\zeta_1^2 - \frac{1}{4}\, q'(\zeta_2)\, d\zeta_2^2$$

Integrability AdS II

Jacobi and Neumann's conoidal coordinates on

$$AdS_2 = \{X_0^2 + X_1^2 - X_2^2 = 1\}$$

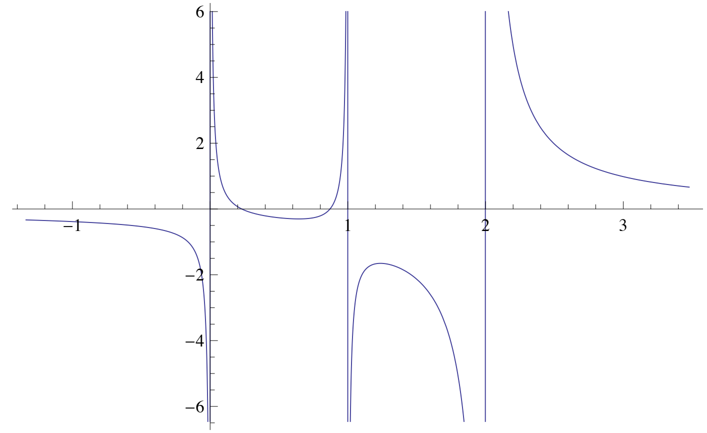
$$a_0 < a_2 < a_1$$

$$X_0(\zeta_1, \zeta_2) = \sqrt{\frac{(a_0 - \zeta_1)(a_0 - \zeta_2)}{(a_0 - a_1)(a_0 - a_2)}},$$

$$X_1(\zeta_1, \zeta_2) = \sqrt{\frac{(a_1 - \zeta_1)(a_1 - \zeta_2)}{(a_1 - a_0)(a_1 - a_2)}},$$

$$X_2(\zeta_1, \zeta_2) = \sqrt{-\frac{(a_2 - \zeta_1)(a_2 - \zeta_2)}{(a_2 - a_0)(a_2 - a_1)}},$$

The conoidal coordinates are the zeros of the function



$$q_X(\zeta) = \frac{X_0^2}{\zeta - a_0} + \frac{X_1^2}{\zeta - a_1} - \frac{X_2^2}{\zeta - a_2}$$

$$\zeta_1, \zeta_2 < a_0 < a_2 < a_1$$

$$a_0 < \zeta_1, \zeta_2 < a_2 < a_1$$

$$a_0 < a_2 < \zeta_1, \zeta_2 < a_1$$

$$a_0 < a_2 < a_1 < \zeta_1, \zeta_2$$

The conoidal coordinates are one spacelike and one timelike

$$ds^2 = -\frac{1}{4}\, q'(\zeta_1)\, d\zeta_1^2 - \frac{1}{4}\, q'(\zeta_2)\, d\zeta_2^2$$

Integrability: separation of variables

$$X_0(\zeta_1(t),\zeta_2(s)) = \sqrt{\tfrac{(a_0-\zeta_1(t))(a_0-\zeta_2(s))}{(a_0-a_1)(a_0-a_2)}}, \quad X_1(\zeta_1(t),\zeta_2(s)) = \sqrt{\tfrac{(a_1-\zeta_1(t))(a_1-\zeta_2(s))}{(a_1-a_0)(a_1-a_2)}},$$

$$X_2(\zeta_1(t),\zeta_2(s)) = \sqrt{-\frac{(a_2-\zeta_1(t))(a_2-\zeta_2(s))}{(a_2-a_0)(a_2-a_2)}},$$

$$\partial_t X\cdot\partial_t X = -\frac{1}{4}q'(\zeta_1)\ \dot{\zeta}_1^2 = -\frac{\zeta_1(t)-\zeta_2(s)}{4\,D(\zeta_1(t))}\dot{\zeta}_1(t)^2,$$

$$\partial_s X\cdot\partial_s X = -\frac{1}{4}q'(\zeta_2)\ {\zeta_2'}^2 = -\frac{\zeta_2(s)-\zeta_1(t)}{4\,D(\zeta_2(s))}\zeta_2'(s)^2.$$

$$D(\zeta) = (\zeta-a_0)(\zeta-a_1)(\zeta-a_2).$$

Constraints

$$\left(\frac{d\zeta_1}{dt}\right)^2 = (\zeta_1 - a_0)(\zeta_1 - a_1)(\zeta_1 - a_2),$$
$$\left(\frac{d\zeta_2}{ds}\right)^2 = (\zeta_2 - a_0)(\zeta_2 - a_1)(\zeta_2 - a_2)$$

$$\partial_t X \cdot \partial_t X = -\partial_s X \cdot \partial_s X = \frac{1}{4}(\zeta_2(s) - \zeta_1(t))$$

String equations

$$\partial_{tt}X - \partial_{ss}X + [(\partial_t X)^2 - (\partial_s X)^2]X = 0$$

$$\frac{\ddot{\zeta}_1}{a_i - \zeta_1} + \frac{\dot{\zeta}_1^2}{2(a_i - \zeta_1)^2} + \zeta_1 + C_i = 0$$

$$\frac{\zeta_2''}{a_i - \zeta_2} + \frac{{\zeta_2'}^2}{2(a_i - \zeta_2)^2} + \zeta_2 + C_i = 0$$

Automatically satisfied provided

$$2C_i = A - a_i, \quad A = a_1 + a_2 + a_3$$

Perspectives

- Higher genus case – nontrivial examples
- Physical interpretation
- Work in collaboration with Michel Gaudin (Saclay)