A quantization of the conformally covariant symmetric traceless field over conformally flat Einstein spacetimes at $d = 4$ and related results

(Part I)

Julien Queva

Équipe Physique Théorique – Projet COMPA
Laboratoire SPE – UMR 6134
Université de Corse

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Cargèse

Outline of the talk

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- A brief reminder on conformal invariance
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- The gauge fixing equation and related results
- The Field Strength F
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Introduction

Why, Oh Why!?

Once one has quantized the Maxwell field and still wants to play with conformally covariant fields some roads are available:

- spinors,
- p-forms,
- (bosonic) symmetric-traceless fields.

For the latter Erdmenger and Osborn solved the problem while leaving a gap to be filled: $d = 4$ (for reasons to be seen later).

J. Erdmenger, H. Osborn, *Class.Quant.Grav.* 14 (1997) 2061-2084, *Class.Quant.Grav.* 15 (1998) 273-280.

A brief reminder on conformal invariance

Weyl transformations (GR)

On the one hand consider the Weyl transformations (rescalings):

$$(M, g) \mapsto (M, \overline{g}) \text{ where } \overline{g}_{\mu\nu}(x) = \omega^2(x) g_{\mu\nu}(x).$$

Def. $E_g(\varphi) = 0$ is *Weyl invariant*, if $\exists h \in \mathbb{R}$ st:

$$E_{\overline{g}}(\overline{\varphi}) = \omega^t E_g(\varphi) = 0,$$

where $\overline{\varphi} = \omega^h \varphi$ and $t \in \mathbb{R}$.

(h is the conformal weight of the field)

A brief reminder on conformal invariance

Conformal group (PΦ)

On the other hand, for (M, g) lorentzian the group leaving the causality invariant (light cone) will be called the *conformal group.*

Rmk. The isometries of (M, g) obviously belong to the conformal group.

The conformal group of Minkowski is locally isomorphic to $SO_0(2, 4)$ which is then the conformal group of all conformally flat spacetimes.

Prop. (Characterization). The generators X of the conformal group fulfill the conformal Killing equation:

$$\mathcal{L}_X g = 2 f_X(x) g,$$

$\mathcal{L}_X$: Lie derivative along X, $f_X(x)$: function on the spacetime.

A brief reminder on conformal invariance

Conformal group (PΦ)

Def. An equation $E_g(\varphi) = 0$ will be *conformally invariant* if one can realize the Lie algebra $\mathfrak{g}$ of the conformal group $SO_0(2,4)$ such that

$$[E_g, \mathfrak{g}](\varphi) = \zeta E_g(\varphi),$$

where ζ is a function.

Then the space of solutions of $E_g(\varphi) = 0$ is closed under $SO_0(2,4)$.

From 0 to 2

0 – The scalar field

The conformal equation on the scalar φ reads as:

$$\left(\Box - \frac{1}{4}\frac{d-2}{d-1}R\right)\varphi = 0$$

It could be referred to as the conformal Laplacian or Yamabe operator.

Provided its conformal weight is given by $h(\varphi) = 1 - d/2$ it's conformally invariant under both Weyl rescalings and $SO_0(2, d)$.

From 0 to 2

1 – The vector field general case

Considering the vector field one can find that either:

$$\Box A^\mu - \frac{4}{d}\nabla^\mu\nabla\cdot A - \frac{2}{d-2}R^{\mu\nu}A_\nu - \frac{1}{4}\frac{d(d-4)}{(d-1)(d-2)}RA^\mu = 0,$$

or

$$\Box A^\mu - \frac{4}{d}\nabla_\nu\nabla^\mu A^\nu + \frac{2}{d}\frac{d-4}{d-2}R^{\mu\nu}A_\nu - \frac{1}{4}\frac{d(d-4)}{(d-1)(d-2)}RA^\mu = 0,$$

depending on your personal preferences, are conformally invariant.

For $d = 4$ these equations are that of Maxwell for the 4-potential.

From 0 to 2

1 – Maxwell field and its gauge invariance

These equations, for $d = 4$ admit a differential gauge invariance of the form:

$$A_\mu \mapsto {}^\varphi A_\mu = A_\mu + \nabla_\mu \varphi$$

which can be tamed thanks to the conformally invariant gauge fixing equation due to Eastwood and Singer:

$$\Box \nabla \cdot A + 2\nabla^\mu R_{\mu\nu} A^\nu - \frac{2}{3} \nabla_\mu R A^\mu = 0$$

which indeed restricts the scalar φ while leaving a residual gauge freedom to be talked about soon.

M.G. Eastwood, M. Singer, *Phys.Lett. A107* (1985) 73-74

From 0 to 2

2 – generic case

Considering $A_{\mu\nu}$ s.t. $A_{\mu\nu} = A_{\nu\mu}$ and $g_{\mu\nu}A^{\mu\nu}$ and generalizing one can find that

$$\begin{aligned}\Box A^{\mu\nu} &- \frac{4}{d+2}(\nabla^\mu\nabla_\rho A^{\nu\rho} + \nabla^\nu\nabla_\rho A^{\mu\rho}) + \frac{8}{d(d+2)}g^{\mu\nu}\nabla_\rho\nabla_\sigma A^{\rho\sigma} \\ &- \frac{(d^2-2d+8)}{4d(d-1)}RA^{\mu\nu} + \frac{2}{d}(R^\mu{}_\rho A^{\nu\rho} + R^\nu{}_\rho A^{\mu\rho}) \\ &- \frac{4(d-1)}{d}R^\mu{}_{\rho\sigma}{}^\nu A^{\rho\sigma} - \frac{4}{d}g^{\mu\nu}R_{\rho\sigma}A^{\rho\sigma} = 0\end{aligned}$$

is conformally invariant, but is no longer unique as a *natural conformally invariant differential operator* as one can add a term

$$+\lambda C^\mu{}_{\rho\sigma}{}^\nu A^{\rho\sigma}$$

with C the Weyl tensor, conformally covariant on its own, and keep the conformal invariance of the system $\forall\lambda \in \mathbb{R}$.

Even if one restricts one self to conformally flat spacetimes ($C = 0$) it remains too messy to tackle one such equation head-on[1] and one is led to consider Einstein spaces for which:

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} &= \frac{R}{d(d-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \\
R_{\mu\nu} &= \frac{R}{d}g_{\mu\nu}, \\
R &= \text{Const.}
\end{aligned}
$$

Examples of such spacetimes are: Minkowski, (A)dS and static Einstein spacetime (obviously).

[1] Not true actually.

From 0 to 2

2 – Content of the eq.

Then on a CFES spacetime our former equation reads as:

$$\Box A^{\mu\nu} - \frac{4}{d+2}(\nabla^\mu \nabla_\rho A^{\nu\rho} + \nabla^\nu \nabla_\rho A^{\mu\rho}) + \frac{8}{d(d+2)} g^{\mu\nu} \nabla_\rho \nabla_\sigma A^{\rho\sigma} - \frac{(d^2 - 2d + 8)}{4d(d-1)} R A^{\mu\nu} = 0$$

it admits, for $d = 4$, a differential gauge invariance of the form

$$A_{\mu\nu} \mapsto {}^\varphi A_{\mu\nu} = A_{\mu\nu} + \left(\nabla_\mu \nabla_\nu - \frac{1}{4} g_{\mu\nu} \Box\right)\varphi$$

which can be restricted by the gauge fixing equation

$$\left(\Box - \frac{1}{6} R\right) \nabla_\mu \nabla_\nu A^{\mu\nu} = 0$$

The equation and its feature

Now on CFES we consider the following equation:

$$\Box A^{\mu_1\cdots\mu_s} - \frac{4s}{d+2s-2}\nabla^{(\mu_1}\nabla_\rho A^{\mu_2\cdots\mu_s)\rho} + \frac{4s(s-1)}{(d+2s-4)(d+2s-2)}g^{(\mu_1\mu_2}\nabla_\rho\nabla_\sigma A^{\mu_3\cdots\mu_s)\rho\sigma} - \frac{d^2-2d+4s}{4d(d-1)}RA^{\mu_1\cdots\mu_s} = 0$$

with $A^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_s} = A^{\mu_1\cdots\mu_j\cdots\mu_i\cdots\mu_s}$ and $g_{\mu_i\mu_j}A^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_s} = 0$.

Rmk. It's a restriction of a conformally invariant equation due to Wünsch.

Rmk. On (A)dS these fields are pretty much, yet not quite, *maximal depth* partially massless fields.

V. Wünsch, *Math. Nachr.* 129 (1986) 269281.

S. Deser, A. Waldron, *Phys. Rev. Lett* 87 (2001) 031601; *Phys. Lett. B* 603 (2004) 30-34

The equation and its feature

The restricted Weyl rescalings

Since we consider CFES only the conformal factor $\omega(x)$ can no longer be arbitrary. Not going into the details one can find that a rescaling mapping a CFES onto a CFES needs to fulfill

$$\left(\nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \Box\right) \frac{1}{\omega} = 0$$

and also

$$\Box \nabla_\mu \omega = (\nabla_\mu \omega)\left(3\omega^{-1}(\Box\omega) - \frac{R}{d(d-1)}\right) - (d-4)\omega^3 \left(\nabla^\alpha \frac{1}{\omega}\right)\left(\nabla_\mu \nabla_\alpha \frac{1}{\omega}\right),$$

put in a, rather convenient, form for computations to be carried out later.

Rmk. The $SO_0(2,d)$ invariance remains implied, on conformally flat spacetimes, by those restricted Weyl rescalings.

H.W. Brinkmann, *Math. Ann.* 94 (1925), 119–145

A.R. Gover, *Math. Ann.* 336 (2006), 311–334

The equation and its features

Properties

Now one can find that the equation fulfills the following:

- It is conformally invariant (of course),
- For $d = 4$ it admits a differential gauge invariance of the form

$$A^{\mu_1 \dots \mu_s} \mapsto {}^{\varphi}A^{\mu_1 \dots \mu_s} = A^{\mu_1 \dots \mu_s} + \nabla^{(\mu_1} \dots \nabla^{\mu_s)} \varphi - \text{traces}$$

  (and that's the only GI).

Skipping on the gory details of the computation one can show that the gauge fixing equation:

$$\left(\Box - \frac{1}{6} R\right) \nabla_{\mu_1} \dots \nabla_{\mu_s} A^{\mu_1 \dots \mu_s} = 0$$

is conformally invariant *on the space of solutions* of the equation.

Rmk. It is actually <u>not</u> obvious due to the fact that $\nabla_{\mu_1} \dots \nabla_{\mu_s} A^{\mu_1 \dots \mu_s}$ does <u>not</u> behave like a scalar density *at all*.

The gauge fixing equation and related results

Residual gauge invariance

Using the gauge fixing equation

$$\left(\Box - \frac{1}{6}R\right)\nabla_{\mu_1}\dots\nabla_{\mu_s}A^{\mu_1\dots\mu_s} = 0$$

constrains the scalar φ while leaving a residual gauge invariance which can be found by plugging a pure gauge solution into the cgfe

$$\left(\Box - \frac{1}{6}R\right)\Box\left(\Box + \frac{1}{3}R\right) \times \cdots \times \left(\Box + \frac{(s-1)(s+2)}{12}R\right)\varphi = 0.$$

And that is interesting at least for two reasons.

The gauge fixing equation and related results

On de Sitter

We recall that the group of isometries of the de Sitter spacetime is $SO_0(1,4)$, and obviously $SO_0(1,4) \subset SO_0(2,4)$.

In its scalar representation the first Casimir $\mathcal{C}^1(SO(1,4))$ has a well-known spectrum, namely:

- The principal serie: $\langle \mathcal{C}^1 \rangle \geq \left(\frac{d-1}{2}\right)^2$,
- The complementary serie: $0 < \langle \mathcal{C}^1 \rangle < \left(\frac{d-1}{2}\right)^2$,
- The discrete serie: $\langle \mathcal{C}^1 \rangle = -j(j+d-1)$, with $j \in \mathbb{N}$.

Comparing that spectrum with the residual gauge invariance left by the gauge fixing equation one views that it picks:

- one term of the complementary serie: *the massless conformally coupled*,
- the s-first terms of the discrete serie: *massless minimally coupled*, *tachyons* . . .

The gauge fixing equation and related results

Conf. Diff. Geom.

Let us rewrite the equation ruling the residual gauge invariance in the following manner:

$$P_{2n}\varphi = \Big[\prod_{\ell=1}^{n}\Big(\Box + \frac{(\ell+1)(\ell-2)}{12}R\Big)\Big]\varphi = 0, \quad n = s+1.$$

This differential operator is not unknown.

But before spelling out its name a little story ...

The gauge fixing equation and related results

On flat spacetime the differential operator on scalars:

$$\Box^n \varphi = 0, \quad n \in \mathbb{N}_*$$

is known to be $SO_0(2, d)$ invariant.

Then appears the natural question: is there a conformally invariant curved analog P_{2n} from densities of weight $-d/2 + n$ to densities of weight $-d/2 - n$ whose leading term is $\Box^n$?

H.P. Jakobsen, M. Vergne, *J.Funct.Anal.* 24:52-106(1977).

The gauge fixing equation and related results

GJMS Theorem

There is a definitive answer to that question namely

Theorem (GJMS): P_{2n} exists on generic conformal manifolds if

- d is odd,
- the bounds $1 \leq n \leq \frac{1}{2}d$ are fulfilled for d even.

Rmk. Then for $d = 4$ there exists only 2 such operators:

- The conformal laplacian: $\left(\Box - \frac{1}{6}R\right)\varphi = 0$,
- The Paneitz operator $\left(\Box^2 + 2\nabla^\mu R_{\mu\nu}\nabla^\nu - \frac{2}{3}\nabla_\mu R \nabla^\mu\right)\varphi = 0$.

C.R. Graham, R. Jenne, L.J. Mason, G.A.J Sparling, *J. London Math. Soc.* 46 (1992) 557–565;
C.R. Graham *J. London Math. Soc.* 46 (1992) 566–576
A.R. Gover, K. Hirachi, *J. Amer. Math. Soc.* 17 (2004) 389 – 405

S. Paneitz, *preprint* (1983); R.J. Riegert, *Phys. Lett.* B134 (1984), 56–60

The gauge fixing equation and related results

The factorization theorem

The obstruction tensor, on generic conformal manifolds, at $d = 4$ to the existence is the (conformally invariant at $d = 4$) Bach tensor:

$$B_{\mu\nu} = C_{\mu\alpha\nu\beta} P^{\alpha\beta} + \nabla^{\alpha}(\nabla_{\mu} P_{\alpha\nu} - \nabla_{\alpha} P_{\mu\nu}),$$

$P_{\mu\nu} \equiv$ Schouten tensor.

- Conformally Einstein ($\exists g$ Einstein in $[g]$) are Bach flat,
- There however are Bach flat manifolds which are not conformally Einstein.

On Einstein manifolds P_{2n} exists and fulfills Branson's factorization formula:

$$P_{2n}\varphi = \Big[\prod_{\ell=1}^{n}\Big(\Box + \frac{(2\ell - 2 + d)(2\ell - d)}{4d(d-1)} R\Big)\Big]\varphi.$$

setting $d = 4$ we recover the equation ruling the residual gauge invariance.

C.R. Graham, K. Hirachi, volume 8 of *IRMA Lect. Math. Theor. Phys.* (2005)
H-S Liu, H L, C N Pope, J F Vázquez-Poritz, *Class. Quant. Grav.*30 (2013) 165015.

T.P. Branson, *Trans. Amer. Math. Soc.* 347 (1995) 3671–3742

The gauge fixing equation and related results

So What?

We provide a (new) proof of the factorization formula linking two conformally invariant equations through the inpection of gauge invariance.

It actually has a physical significant meaning: the (conformally invariant) space of gauged solutions admits a conformally invariant subspace of solutions.

Those make the first two spaces of a Gupta-Bleuler triplet if someone wishes to canonicaly quantize those fields.

The field strength F

Introduction

The field A being gauge variant one needs to find a corresponding field strength F.

Introducing the derivative $\mathcal{D}$ as

$$\begin{aligned}(F(A))^{\alpha\ \mu_1\dots\mu_s} &= (\mathcal{D}A)^{\alpha\ \mu_1\dots\mu_s} \\ &= \nabla^\alpha A^{\mu_1\dots\mu_s} - \nabla^{(\mu_1} A^{\mu_2\dots\mu_s)\alpha} \\ &\quad - \frac{(s-1)}{(d+s-3)}\left[g^{\alpha(\mu_1}\nabla_\sigma A^{\mu_2\dots\mu_s)\sigma} - g^{(\mu_1\mu_2}\nabla_\sigma A^{\mu_3\dots\mu_s)\alpha\sigma}\right],\end{aligned}$$

one shows that F is indeed, on CFES, a field strength namely:

$$F(\nabla^{(\mu_1}\dots\nabla^{\mu_s)}\varphi + \dots) = 0, \quad \forall d.$$

The field strength F

Maxwell's equation

Then the equation on the potential is recovered as the divergence of F.

Generalizing further the differential operator $\mathcal{D}$ one can show that these equations can be written on F as

$$\left|\begin{array}{l} \mathcal{D}F = 0 \\ \mathrm{div} F = 0, \end{array}\right. \qquad g \in [\eta] \text{ or } s = 1$$

with

$$\mathcal{D}^2 A = 0, \quad g \in [\eta] \text{ or } s = 1$$

Which are free Maxwell-like equations (and for $s = 1$ those are Maxwell's equations).

One then shows that this system is conformally invariant iff $d = 4$.

The field strength F

E–B duality

Working out the mixed symmetry of the F one shows:

$$\mathrm{dof}(F) = \frac{(d+s-4)!}{(d-3)!(s+1)!} s(d+2s-2)(d+s-2)$$

and in particular

$$\mathrm{dof}(F)\Big|_{d=4} = 2s(s+2) = 2\sum_{j=1}^{s}(2j+1).$$

That is F has the right number of indep. components to be written as a sum over $2s$ symmetric traceless (wrt 3–"space"). Leaving aside the decomposition and the proof it establishes a bijection we indeed have:

$$F = \bigoplus_{j=1}^{s}(E_j \oplus B_j).$$

and the system exhibits a Electric-Magnetic duality:

$$(E_j, B_j) \to (-B_j, E_j)$$

The field strength F

The equations on the E–B fields, $SO_0(2,4)$ invariance

The system can be translated into first order eq. on $H_j = E_j + iB_j$

$$\partial_t H_j - i\Big(\frac{s+1}{j+1}\Big)\mathrm{curl} H_j \\ - \Big(\frac{s-j}{j+1}\Big)\mathrm{div} H_{j+1} + \frac{1}{j}\Big(1 + \frac{2(s-j)}{(j+1)(j+2)}\Big)\mathrm{grad} H_{j-1} = 0,$$

The action of $SO(2,4)$ on F carries to the E–B fields and behaves *"nicely"* apart from boosts and SCT, which was to be expected.

The conformal invariance remains.

The field strength F

Finally

Thanks to that construction we can build a conformally invariant ($d = 4$, $\forall g$) action

$$S(g, A) = \frac{s}{2(s+1)} \int (F(A))^2 \, \mathrm{dVol}_g.$$

out of which you can derive the symplectic product to endow the space of solutions A with:

$$\sigma(A, A') = \int [A\mathcal{D}A' - A'\mathcal{D}A] \mathrm{d}\Sigma$$

The product is conformally invariant, **vanishes on pure gauges solutions** and is positive otherwise.

What's next

Part II, of course!

Namely, we paved the way for the quantization of such fields when $d = 4$ which, we remind, was the missing case.

The next step is to quantize A (and F) with the help of the gauge fixing equation.