Static spherically symmetric solutions on geometric scalar theory of gravity

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INTRODUCTION

The Geometrical Scalar Gravity [GSG]

Novello, Bittencourt, Moschella, Salim, Toniato and Goulart, 2012

It is a metric theory of gravity with the following properties:

- The gravitational interaction is described by a scalar field Φ;
- The field Φ satisfies a nonlinear dynamics;
- The theory satisfies the principle of general covariance.

  In other words, this is not a theory restricted to the realm of special relativity;
- All kind of matter and energy interact with Φ only through the pseudo-Riemannian metric $q_{\mu\nu} = \alpha\, \eta_{\mu\nu} + \beta\, \partial_\mu \Phi\, \partial_\nu \Phi$ **(fundamental hypothesis)**;
- Test particles follow geodesics relative to the gravitational metric $q_{\mu\nu}$;
- Φ is related in a nontrivial way with the Newtonian potential Φ_N;
- Electromagnetic waves propagate along null geodesics relative to the metric $q^{\mu\nu}$.

The theory was constructed following the main idea of General Relativity [GR] and assuming as an *a priori* that gravity is described by a Riemannian geometry.

The auxiliary (Minkowski) metric $\eta_{\mu\nu}$ is unobservable because the gravitational field couples to matter *only* through $q_{\mu\nu}$.

v e r s u s

Old days of Einstein-Grosmann proposal

- The theory is described in a conformally flat geometry and the background Minkowski metric is observable;
- The source of the gravitational field is the trace of the energy-momentum tensor;
- The scalar field is the (special) relativistic generalization of the Newtonian potential.

In GSG *all* these assumptions fail.

The motivation to GSG could be understood as proceeding from the lemma:

Given the Lagrangian $L = V(\Phi)\, \eta^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi$ with an arbitrary potential $V(\Phi)$, the field theory satisfying the equation of motion in Minkowski spacetime is equivalent to a massless Klein-Gordon field $\Box\Phi = 0$ in the metric $q^{\mu\nu}$ provided that the functions $\alpha(\Phi)$ and $\beta(\Phi)$ satisfy the condition

$$\alpha + \beta = \alpha^3 V.$$

Remarkable, this equivalence is valid for any dynamics described in the Minkowski background by the Lagrangian L. This fact can be extended to other kinds of nonlinear Lagrangian .

Novello and Goulart, 2011

Summary of GSG's equations

Metric:

$$q^{\mu\nu} = \alpha\, \eta^{\mu\nu} + \frac{\beta}{w}\, \eta^{\mu\rho}\, \eta^{\nu\sigma}\, \partial_\rho \Phi\, \partial_\sigma \Phi \;;$$

$$q_{\mu\nu} = \frac{1}{\alpha}\, \eta_{\mu\nu} - \frac{\beta}{\alpha(\alpha+\beta)w}\, \partial_\mu \Phi\, \partial_\nu \Phi \;;$$

$$w = \eta^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi \;.$$

If the Newtonian limit is extrapolated:

$$q_{00} = \frac{1}{\alpha} \approx 1 + 2\,\Phi_N \quad \Rightarrow \quad \alpha = e^{-2\,\Phi} \;;$$

From the analysing of the planetary orbits, GSG gives

$$V(\Phi) = \frac{1}{4}\frac{(\alpha-3)^2}{\alpha^3} \;.$$

Energy-momenum tensor for a perfect fluid:

$$T_{\alpha\beta} = (\rho + p) V_\alpha V_\beta - p\, g_{\alpha\beta} \;.$$

Projection of conservation energy equation on the hypersurface ortonormal to the observers class V^α:

$$\boxed{p(r)' = -(\rho + p)\, \Phi'} \;. \qquad \text{(II)}$$

Equation of motion:

$$\frac{1}{\sqrt{-\eta}}\partial_\mu(\sqrt{-\eta}\, \eta^{\mu\nu}\, \partial_\nu \Phi) + \frac{w}{2}\frac{d}{d\Phi}(\ln V)) = 0 \,.$$

Dynamic equation:

$$\text{(I)} \qquad \boxed{\sqrt{V}\, \Box\Phi = k\,\chi} \;;$$

$$\chi = \tfrac{1}{2}\left[\tfrac{T-E}{2}\, \tfrac{d}{d\Phi} \ln\alpha + \tfrac{E}{2}\, \tfrac{d}{d\Phi} \ln(\alpha+\beta) - \nabla_\lambda C^\lambda\right];$$

$$C^\lambda = \tfrac{\beta}{\alpha\Omega}\, (E\, q^{\lambda\nu} - T^{\lambda\nu})\partial_\nu \Phi \;;$$

$$T = T^{\mu\nu}\, q_{\mu\nu} \;; \qquad E = \tfrac{T^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi}{\Omega} \;;$$

$$\Omega = q^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi \;;$$

$$k \equiv 8\pi G/c^4 \,.$$

$$X' \equiv \frac{dX}{dr} \,.$$

Static spherically symmetric metric of GSG

Using a auxiliary Minkowski background metric in spherical coordinates, we are able to do a change of coordinates upon the original metric of GSG and with this result we calculate the other quantities associated with the dynamic equation (II).

Metric: $ds^2 = (1/\alpha)\, dt^2 - B(r)\, dr^2 - r^2\, d\Omega^2$;

where $B(r) \equiv \frac{\alpha}{\alpha+\beta}\left(\frac{r}{2\,\alpha}\alpha' + 1\right)^2$.

Dynamic equation: $\sqrt{V}\, \Box\Phi = \kappa\, \left[\frac{3-2\alpha}{3-\alpha}\, p - \frac{\rho}{2}\right]$;

where $T = \rho - 3\,p \,,\quad E = -\,p \,,\quad C^\lambda = 0$.

It is possible to write a system to GSG analogous to the Tolman-Oppenheimer-Volkoff [TOV] set of equations. This system is defined by the following equations:

TOV-*like* equations for GSG:

$$\tfrac{1}{H(r)\, r^2}(1-\sigma)^{1/2}\left(1 - \tfrac{3}{2}\sigma\right)^2 \left[\tfrac{r^2\sigma'}{H(r)}\left(1 - \tfrac{3}{2}\sigma\right)\right]' = -\,k\,\left[p(1-3\sigma)\left(1-\tfrac{3}{2}\sigma\right)^{-1} - \rho\right] ,$$

$$p(r)' = \tfrac{1}{2}(\rho + p)(1-\sigma)^{-1}\, \sigma' ,$$

$p = p\,(\rho)$, (equation of state)

where $H(r) \equiv 1 - \sigma + r\sigma'/2$, $\sigma(r) \equiv 1 - \frac{1}{\alpha} = 1 - e^{2\Phi}$.

associated with q^{00}

TOV equations for GR:

$$\Sigma' = r\rho - \tfrac{\Sigma}{r} \,,$$

$$p(r)' = -\,(\rho + p)\,(1-\Sigma)^{-1}\left(r\,p + \tfrac{\Sigma}{2\,r}\right) ,$$

$p = p\,(\rho)$, (equation of state)

where $\Sigma(r) \equiv 1 - e^{-2\,\xi(r)}$.

associated with g^{11}

Work in progress:

The aim of this work is to find exact spherically symmetric solutions for the static case of GSG. To do this we rewrite the analogous TOV system in such a manner that we can obtain exact solutions once we realize the integrability conditions of a specific function. We already have some solutions, however we must study them a little longer before present the results.

Main references:

M. Novello et al, *Geometric scalar theory of gravity*, JCAP06 (2013) 014.

R. C. Tolman, *Static Solutions of Einstein's Field Equations for Spheres of Fluid*, Phys. Rev **55**, 364 (1939).

Newtonian stars

The Newtonian limit of the equations (I) and (II) is given by:

$$\nabla^2\Phi = -\tfrac{k}{2}\,\rho,$$

$$f(r)' = -\tfrac{k}{2}\rho\, r^2,$$

where the function f(r) is define as $f(r) \equiv r^2\, \Phi'$.

Hydrostatic equation for GSG:

$$p(r)' = -f(r)\, \tfrac{\rho(r)}{r^2} \;.$$

Hydrostatic equation for GR:

$$p(r)' = -m(r)\, \tfrac{\rho(r)}{r^2} \;.$$

For the Newtonian limit the functions m(r) and f(r) match with the mass of the fluid (or for instance star in hydrostatic equilibrium). Once more is important to remember that m(r) is associated with the g^{11} component of the metric, while f(r) is with the q^{00} component.

Theorems

Birkhoff's theorem is valid for GSG

The solution for the empty space, meaning p = 0 and ρ = 0, is given by solving the dynamic equation:

$$\Box\Phi = 0 \,,$$

given the following cubic equation

$$a\,\sqrt{\alpha} + 1/\alpha + b/r = 0 \,,$$

where a and b are constants.

This equation underwent a very careful examination of its roots and if we assume the Newtonian limit we find that the **only possible solution** is

$$\alpha(r) = (1 - 2M/r)^{-1},$$

which implies in the Schwarzschild metric

$$ds^2 = (1-2\,M/r)\, dt^2 - (1-2\,M/r)^{-1}\, dr^2 - r^2 d\Omega^2 \,.$$

Moreover: Once one imposes the solution to be asymptotically flat, GSG *always* give a unique vaccuum solution, for *any* potentials!

Deruelle, 2013 (on informal comments)

Gauss's theorem on GSG

We are still investigating the physical interpretation of differents forms of applying the Gauss's theorem on GSG. Nevertheless even an unpretentious approach can exemplify the kind of new features which GSG brings to gravitation and to the understanding of gravitaional energy.

For example, the dynamic equation (II) can be written in its integral form as:

$$\oint \sqrt{-q}\, G^\alpha\, dS_\alpha - \int \sqrt{-q}\, G^\alpha (\ln\, \sqrt{V})_{,\alpha}\; d^4x = k \int \sqrt{-q}\, \chi\, d^4x \;,$$

where $G^\alpha \equiv \sqrt{V}\, q^{\alpha\beta}\, \Phi_{,\beta}$.

If we take χ(r) from the static spherically symmetric case:

$$\underbrace{\oint \sqrt{-q}\, G^\alpha\, dS_\alpha}_{\text{on the hypersurface}} - \underbrace{\int \sqrt{-q}\, G^\alpha (\ln\, \sqrt{V})_{,\alpha}\; d^4x}_{\text{on all space}} = \underbrace{k \int \sqrt{-q}\; \tfrac{3-2\,\alpha}{3-\alpha}\; p(r)\; d^4x \;-\; \overbrace{k \int \sqrt{-q}\;\; \rho(r)/2\; d^4x}^{\text{mass term in Newtonian gravity}}}_{\text{interior of the energy-matter distribution}} \,.$$