Energy in Massive Gravity Theories

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Theories with massive gravitons

Motivations for massive gravity

- Modification of gravity:

$$\text{Newton} \quad \frac{1}{r} \quad \rightarrow \quad \text{Yukawa} \quad \frac{1}{r}\, e^{-mr}$$

  hence the gravity is weaker at large distances $\Rightarrow$ the cosmic acceleration, $m \sim 1/(\text{cosm. horizon size})$.

- Purely theoretical, interesting history.

Linear massive gravity of Fierz and Pauli /1939/

massless : $\Box\phi = 0 \quad \Rightarrow \quad$ massive : $\Box\phi = m^2\phi$

linearized GR : $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \Rightarrow \quad \Box h_{\mu\nu} + \ldots = 16\pi G T_{\mu\nu} \quad \Rightarrow$

$$\boxed{\Box h_{\mu\nu} + \ldots = m^2(h_{\mu\nu} - \alpha\, h\, \eta_{\mu\nu}) + 16\pi G T_{\mu\nu}}$$

One should have $2s+1=5$ Dof. Taking the divergence gives 4 constraints

$$m^2(\partial^\mu h_{\mu\nu} - \alpha\partial_\nu h) = 0.$$

Taking the trace,

$$(1-\alpha)\Box h + m^2(4\alpha - 1)h = 16\pi T.$$

If $\alpha = 1$ one gets the fifth constraint

$$h = \frac{16\pi T}{3m^2}$$

$\Rightarrow$ 5 Dof= polarizations of massive graviton. For $\alpha \neq 1$ there is a sixth Dof with a negative kinetic energy.

VdVZ discontinuity /1970/

$$\begin{aligned}(\Box - m^2)h_{\mu\nu} &= 16\pi G T_{\mu\nu}\\ \partial^\mu h_{\mu\nu} &= \partial_\nu h\\ h &= \frac{16\pi G T}{3m^2}\end{aligned}$$

The limit $m \to 0$ is not smooth. Let us make $5 = 2 + 2 + 1$ split

$$h_{\mu\nu} = \gamma_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{m^2}\partial_\mu\partial_\nu\phi$$

where $\partial^\mu\gamma_{\mu\nu} = \gamma^\mu_\mu = \partial^\mu A_\mu = 0$. Then for $m \to 0$ one gets

$$\begin{aligned}\Box\gamma_{\mu\nu} &= 16\pi G T_{\mu\nu} && \text{tensor modes}\\ \Box A_\mu &= 0 && \text{vector modes}\\ \Box\phi &= \frac{16\pi G}{3} T && \text{scalar graviton}\end{aligned}$$

$\Rightarrow$ extra attraction, different Newton's laws for massive bodies.

Explicit VdVZ

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}R'^2(r)dr^2 - r^2 e^{\mu(r)}d\Omega^2$$

where $R = re^{\mu}$ and $\nu, \lambda, \mu \ll 1$. Linearizing,

$$\nu = -\frac{r_g}{r}e^{-mr}, \qquad \lambda = \frac{r_g}{2r}(1+mr)e^{-mr}$$

$$\mu = r_g \frac{1+mr+(mr)^2}{2m^2r^3} e^{-mr}$$

For $mr \ll 1$ one has

$$\nu = -\frac{r_g}{r}, \qquad \lambda = \frac{r_g}{2r}, \qquad \mu \sim \frac{1}{r^3}$$

One has $\nu = -2\lambda$ instead of $\nu = -\lambda \Rightarrow$ either the Newton law is wrong or the bending of light is wrong, depending on choice of r_g.

Is massive gravity ruled out by the Solar System observations?

Vainshtein /1972/

Non-linear corrections to the VdVZ solution are proportional to

$$\frac{r_g}{m^4 r^5}$$

This should be small as compared to unity, but it is $\sim 10^{32}$ at the edge of solar system. Becomes small only for

$$r \gg r_V = \left(\frac{r_g}{m^4}\right)^{1/5} \sim 100\, Kps$$

For $r \ll r_V$ one cannot use the linear theory $\Rightarrow$ linearized equations for ν, λ but non-linear for μ

$$\nu = -\frac{r_g}{r} + \ldots, \quad \lambda = \frac{r_g}{r} + \ldots, \quad \mu = \sqrt{\frac{a r_g}{r}} + \ldots$$

The GR is recovered, the scalar graviton is bound. The VdVZ discontinuity is visible only for $r \gg r_V$, for $r \ll r_V$ it is cured by the non-linear effects. check = /Babichev, Deffayet, Ziour 2009/

Hamiltonian formulation
Boulware-Deser ghost

Non-linear Fierz-Pauli

$$S = \frac{1}{\kappa^2} \int \sqrt{-g} \left(\frac{1}{2} R - m^2 \mathcal{U} \right) d^4x \equiv \frac{1}{\kappa^2} \int \mathcal{L} \, d^4x \, .$$

The potential $\mathcal{U}$ is a scalar function of $H^\mu_\nu = \delta^\mu_\nu - g^{\mu\alpha} f_{\alpha\nu}$ where $f_{\mu\nu}$ is a flat reference metric. The potential should reproduce the Fierz-Pauli in the weak field, therefore

$$\mathcal{U} = \frac{1}{8} \left(H^\mu{}_\nu H^\nu{}_\mu - (H^\mu{}_\mu)^2 \right) + \dots$$

Higher order term can be arbitrary.

ADM decomposition

With

$$ds^2 = -N^2 dt^2 + \gamma_{ik}(dx^i + N^i dt)(dx^k + N^k dt)$$

the Lagrangian becomes

$$\mathcal{L} = \sqrt{\gamma} N \left(\frac{1}{2} \{ K_{ik} K^{ik} - K^2 + R^{(3)} \} - m^2 \mathcal{U}(N^\nu, \gamma_{ik}) \right) + \text{total derivative} \, ,$$

where the second fundamental form

$$K_{ik} = \frac{1}{2N} (\dot{\gamma}_{ik} - \nabla_i^{(3)} N_k - \nabla_k^{(3)} N_i).$$

Hamiltonian

Canonical momenta for γ_{ik} and $N^{\mu} = (N, N^k)$

$$\pi^{ik} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ik}} = \frac{1}{2}\sqrt{\gamma}(K^{ik} - K\gamma^{ik}), \qquad p_{N_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{N}^{\mu}} = 0.$$

Hamiltonian

$$\mathcal{H} = \pi^{ik}\dot{\gamma}_{ik} - \mathcal{L} = N^{\mu}\mathcal{H}_{\mu} + m^2\mathcal{V}$$

with $\mathcal{V} = \sqrt{\gamma} N \mathcal{U}$ and

$$\mathcal{H}_0 = \frac{1}{\sqrt{\gamma}}(2\pi^{ik}\pi_{ik} - (\pi^k_k)^2) - \frac{1}{2}\sqrt{\gamma}R^{(3)}, \quad \mathcal{H}_k = -2\nabla^{(3)}_i \pi^i_k$$

N^{μ} are non-dynamical, phase space is spanned by 12 (π^{ik}, h_{ik}).

$$0 = -\dot{p}_{N_\mu} = \frac{\partial \mathcal{H}}{\partial N^{\mu}} = \mathcal{H}_{\mu}(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^{\alpha}, \gamma_{ik})}{\partial N^{\mu}},$$

This condition determines the number of Dof.

GR, $m = 0$

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) = 0. \qquad \text{4 constraints}$$

Since

$$\{\mathcal{H}_\mu, \mathcal{H}_\nu\} \sim \mathcal{H}_\alpha$$

they are first class and generate gauge symmetries $\Rightarrow$ one can impose 4 gauge conditions. There remain

$$12 - 4 - 4 = 4 = 2 \times (2 \text{ DoF})$$

independent phase space variables describing 2 graviton polarizations.

Energy is zero on the constraint surface (up to surface terms)

$$H = N^\mu \mathcal{H}_\mu = 0$$

Boulware-Deser ghost /1972/

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu} = 0,$$

These are not constraints but equations for N^μ whose solution is $N^\mu(\pi^{ik}, h_{ik})$. No constraints $\Rightarrow$ all 12 phase space variables are independent $\Rightarrow$ $6 = 5 + 1$ DoF $=$ 5 graviton polarizations $+$ ghost

Inserting $N^\mu(\pi^{ik}, h_{ik})$ to $\mathcal{H}$ gives a non-positive-definite in π^{ik} expression. No constraints $\Rightarrow$ energy is unbounded from below.

In the non-linear theory one can overcome the VdVZ problem, but on the other hand there is the BD ghost

Pauli-Fierz theory – weak fields

The argument goes differently in this case.

$$N = 1 + \nu, \qquad N^k = \nu^k, \qquad \gamma_{ik} = f_{ik} + h_{ik}$$

then

$$-\frac{1}{2}\sqrt{\gamma}\, R^{(3)} = \sqrt{f}\,(V_1 + V_2) + \ldots$$

with

$$\begin{aligned}
V_1 &= \frac{1}{2}(\nabla^k\nabla_k h - \nabla^i\nabla^k h_{ik}) \\
V_2 &= \frac{1}{4} h^{ik}\left(-\frac{1}{2}\nabla^s\nabla_s h_{ik} + \frac{1}{2} f_{ik}\nabla^s\nabla_s h - \nabla_i\nabla_k h + \nabla_i\nabla^s h_{sk}\right), \\
\mathcal{U} &= \frac{1}{8}\left(h^i_k h^k_i - h^2 - 2\,\nu_k\,\nu^k - 4\nu h\right) + \ldots
\end{aligned}$$

Shifts ν_k appear quadratically, there are equations for them, but the lapse ν arises linearly $\Rightarrow$ constraint.

Pauli-Fierz Hamiltonian

$$\mathcal{H}_{\rm FP} = \mathcal{H}_{\rm FP}(\pi) + \mathcal{H}_{\rm FP}(h) + \nu\, \mathcal{C}_{\rm FP}\,,$$

where

$$\begin{aligned}
\mathcal{H}_{\rm FP}(\pi) &= \frac{1}{\sqrt{f}}\left(2\,\pi^i_k\pi^k_i - (\pi^k_k)^2 + \frac{4}{m^2}\,\nabla_i\pi^i_k\nabla^j\pi^k_j\right),\\
\mathcal{H}_{\rm FP}(h) &= \sqrt{f}\,\Bigg(\frac{1}{8}\,\nabla^j h^i_k\nabla_j h^k_i - \frac{1}{8}\,\nabla_k h\nabla^k h + \frac{1}{4}\,\nabla_j h^j_k\,\nabla^k h\\
&- \frac{1}{4}\,\nabla_j h^j_k\nabla^i h^k_i + \frac{m^2}{8}\left(h^i_k h^k_i - h^2\right)\Bigg),
\end{aligned}$$

$$\mathcal{C}_{\rm FP} = \frac{1}{2}\sqrt{f}\left(\nabla^k\nabla_k h - \nabla^i\nabla^k h_{ik} - m^2 h\right) = 0\,.$$

Poisson bracket of $\mathcal{C}_{\rm FP}$ with $H_{\rm FP} = \int \mathcal{H}_{\rm FP}\, d^3x$ gives the secondary constraint

$$\mathcal{S}_{\rm FP} = \{\mathcal{C}_{\rm FP}, H_{\rm FP}\} = m^2\pi^k_k + 2\,\nabla^i\nabla^k\pi_{ik} = 0\,.$$

Pauli-Fierz – degrees of freedom

$\{\mathcal{C}_{\rm FP}, \mathcal{S}_{\rm FP}\} \neq 0 \Rightarrow$ the two constraints are second class $\Rightarrow$

$$12 - 2 = 10 = 2 \times (5 \text{ DoF})$$

$\Rightarrow$ 5 graviton polarizations. Preservation of $\mathcal{S}_{\rm FP}$ gives

$$\{\mathcal{S}_{\rm FP}, H_{\rm FP}\} = \frac{3}{4}\, m^4 (h - \nu) + \frac{3}{2}\, m^2 \partial^2_{kk} h + (\partial^2_{kk})^2 h = 0,$$

$\Rightarrow$ equation with the solution

$$\nu = h + \frac{2}{m^2}\, \partial^2_{kk} h + \frac{4}{3m^4}\, (\partial^2_{kk})^2 h\,.$$

$\Rightarrow$ all ν, ν_k are determined.

Pauli-Fierz energy

$$E_{\mathrm{FP}} = \int \mathcal{H}_{\mathrm{FP}}(\pi) d^3x + \int \mathcal{H}_{\mathrm{FP}}(h)\, d^3x$$

where π, h satisfy the constraints

$$\mathcal{C}_{\mathrm{FP}}(h) = 0, \qquad \mathcal{S}_{\mathrm{FP}}(\pi) = 0.$$

The energy is quadratic in fields $\Rightarrow$ one can Fourier-decompose

$$\pi_{ik}(\vec{x}) = \int \pi_{ik}(\vec{k}) e^{i\vec{x}\vec{k}} d^3k, \qquad h_{ik}(\vec{x}) = \int h_{ik}(\vec{k}) e^{i\vec{x}\vec{k}} d^3k$$

The constraints can be resolved in the momentum space, they relate the trace to the diagonal traceless part of the tensors. There remains two tensor modes, two vector modes, and a scalar, for which

$$E_{\mathrm{FP}} \geq 0.$$

Ghost-free massive gravity
Energy

dRGT theory

$$S = \frac{1}{\kappa^2}\int \sqrt{-g}\left(\frac{1}{2}R - m^2\,\mathcal{U}\right) d^4x \equiv \frac{1}{\kappa^2}\int \mathcal{L}\, d^4x\,,$$

with the potential made of $H^\mu_{\ \nu} = \delta^\mu_\nu - g^{\mu\alpha} f_{\alpha\nu}$

$$\mathcal{U} = \frac{1}{8}\,(H^\mu_{\ \nu} H^\nu_{\ \mu} - (H^\mu_{\ \mu})^2) + \ldots$$

For the dRGT theory /de Rham, Gabadadze, Tolley 2010/ the higher order terms are chosen such that

$$\mathcal{U} = b_0 + b_1 \sum_a \lambda_a + b_2 \sum_{a<b} \lambda_a \lambda_b + b_3 \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + b_4\, \lambda_0 \lambda_1 \lambda_2 \lambda_3$$

where λ_a are eigenvalues of $\gamma^\mu_{\ \nu} = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$ and

$$\begin{aligned} b_0 &= 4c_3 + c_4 - 6, \quad b_1 = 3 - 3c_3 - c_4, \quad b_2 = 2c_3 + c_4 - 1, \\ b_3 &= -(c_3 + c_4), \quad b_4 = c_4 \end{aligned}$$

Degrees of freedom

The Hessian matrix

$$\frac{\partial^2 \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu \partial N^\nu}$$

has rank 3 $\Rightarrow$ equations

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu} = 0$$

determine $\boxed{N^k = N^k(N, \pi^{ik}, \gamma_{ik})}$ but N remains undetermined.
Inserting N^k to $\mathcal{H}$ gives

$$\mathcal{H} = \mathcal{E}(\pi^{ik}, \gamma_{ik}) + N\mathcal{C}(\pi^{ik}, \gamma_{ik})$$

Varying with respect to N gives the primary constraint $\mathcal{C} = 0 \Rightarrow$ the secondary constraint $\mathcal{S} = \{\mathcal{C}, H\} = 0 \Rightarrow$ only 5 DoF. The energy density is $\mathcal{E}(\pi^{ik}, \gamma_{ik})$ restricted to the constraint surface.
No explicit expressions for $\mathcal{E}, \mathcal{C}$.

Restricting to the s-sector

Spherical symmetry

$$
\begin{aligned}
ds_g^2 &= -N^2 dt^2 + \frac{1}{\Delta^2}(dr + \beta\, dt)^2 + R^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \\
ds_f^2 &= -dt^2 + dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)
\end{aligned}
$$

where N, β, Δ, R depend on t, r. With the canonical momenta $p_\Delta = \partial\mathcal{L}/\partial\dot{\Delta}$ and $p_R = \partial\mathcal{L}/\partial\dot{R}$ the Hamiltonian

$$
\mathcal{H} = \dot{\Delta}\pi_\Delta + \dot{R}\pi_R - \mathcal{L} = N\mathcal{H}_0 + \beta\mathcal{H}_r + m^2\mathcal{V}
$$

where

$$
\begin{aligned}
\mathcal{H}_0 &= \frac{\Delta^3}{4R^2}\, p_\Delta^2 + \frac{\Delta^2}{2R}\, p_\Delta p_R + \Delta R R'^2 + 2R(\Delta R')' - \frac{1}{\Delta}\,, \\
\mathcal{H}_r &= \Delta p_\Delta' + 2\Delta' p_\Delta + R' p_R
\end{aligned}
$$

Phase space is spanned by 4 variables $\Delta, R, p_\Delta, p_R \equiv (q^i, p_k)$.

Generic case

- $m = 0 \Rightarrow$ varying $\mathcal{H}$ with respect to N, β yields 2 first class constraints $\mathcal{H}_0 = 0$, $\mathcal{H}_r = 0 \Rightarrow$ there are $4 - 2 - 2 = 0$ independent variables $\Rightarrow$ no dynamics $=$ Birkghoff theorem.
- $m \neq 0$ and generic $\mathcal{V} \Rightarrow$

$$\mathcal{H}_0 + m^2 \frac{\partial \mathcal{V}}{\partial N} = 0, \qquad \mathcal{H}_r + m^2 \frac{\partial \mathcal{V}}{\partial \beta} = 0.$$

$\Rightarrow N = N(q^i, p_k)$, $\beta = \beta(q^i, p_k)$, no constraints $\Rightarrow$ all 4 phase space variables are independent $\Rightarrow$

2 DoF=scalar graviton+ghost

Inserting $N(q^i, p_k)$, $\beta(q^i, p_k)$ to $\mathcal{H}$, the result is unbounded from below

dRGT theory

$$\mathcal{V} = \frac{NR^2P_0}{\Delta} + \frac{R^2P_1}{\Delta}\sqrt{(\Delta N+1)^2-\beta^2} + R^2P_2\,,$$

with

$$P_n = b_n + 2b_{n+1}\frac{r}{R} + b_{n+2}\frac{r^2}{R^2}.$$

in which case

$$\begin{aligned}
\frac{\partial\mathcal{H}}{\partial N} &= \mathcal{H}_0 + m^2\frac{R^2P_0}{\Delta} + m^2R^2P_1\frac{N\Delta+1}{\sqrt{(N\Delta+1)^2-\beta^2}} = 0 \\
\frac{\partial\mathcal{H}}{\partial\beta} &= \mathcal{H}_r - m^2\frac{R^2P_1}{\Delta}\frac{\beta}{\sqrt{(N\Delta+1)^2-\beta^2}} = 0.
\end{aligned}$$

The second of these conditions determines β,

$$\beta = (N\Delta+1)\frac{\Delta\mathcal{H}_r}{Y}$$

while the first condition gives

Constraints

$$\mathcal{C} \equiv \mathcal{H}_0 + Y + m^2 \frac{R^2 P_0}{\Delta} = 0$$

The Hamiltonian becomes $\mathcal{H} = \mathcal{E} + N\mathcal{C}$ where

$$\mathcal{E} = \frac{Y}{\Delta} + m^2 R^2 P_2 \quad \text{with} \quad Y \equiv \sqrt{(\Delta \mathcal{H}_r)^2 + (m^2 R^2 P_1)^2}$$

Since $\{\mathcal{C}(r_1), \mathcal{C}(r_2)\} = 0 \Rightarrow$ the secondary constraint

$$\begin{aligned} \mathcal{S} &= \{\mathcal{C}, H\} = \frac{m^4 R^2 P_1^2}{2Y} (\Delta p_\Delta + R p_R) - Y \left(\frac{\Delta \mathcal{H}_r}{Y} \right)' \\ &- \frac{\Delta^2 p_\Delta}{2R} \left\{ \frac{m^4}{2\Delta Y} \partial_R (R^4 P_1^2) + m^2 \partial_R (R^2 P_2) \right\} \\ &- \frac{m^2 \mathcal{H}_r}{Y} \left\{ \Delta \left(R^2 P_2 \right)' + R^2 \partial_r (P_0 - \Delta^2 P_2) \right\} = 0 \end{aligned}$$

No ghost. $E = \int_0^\infty \mathcal{E} dr$ *assuming that* $\mathcal{C} = \mathcal{S} = 0$.

Weak fields

If $N = \Delta = 1$, $R = r$ and $\beta = p_\Delta = p_R = 0 \Rightarrow \mathcal{C} = \mathcal{S} = \mathcal{E} = 0$.

If $N = 1 + \nu$, $\Delta = 1 + \delta$, $R = r + \rho$ then

$$\mathcal{C} = \mathcal{C}_{\rm FP} + \ldots, \qquad \mathcal{S} = \mathcal{S}_{\rm FP} + \ldots, \qquad \mathcal{H} = \mathcal{E}_{\rm FP} + \nu\, \mathcal{C}_{\rm FP} + \ldots$$

$$\mathcal{C}_{\rm FP} = (2r(\delta + \rho'))' + m^2(r^2\delta - 2r\rho),$$
$$\mathcal{S}_{\rm FP} = \frac{m^2}{2}(rp_R - p_\Delta) - (p'_\Delta + p_R)'.$$

where $\mathcal{E}_{\rm FP}$ is the quadratic part of $\mathcal{E} + \mathcal{C}$,

$$\begin{aligned} \mathcal{E}_{\rm FP} \quad &= \quad \frac{p_\Delta^2}{4r^2} + \frac{p_\Delta p_R}{2r} + \frac{(p'_\Delta + p_R)^2}{m^2 r^2} \\ &+ \quad 2\rho\,\delta' - \rho'^2 - \delta^2 + m^2(2r\delta\rho - \rho^2). \end{aligned}$$

Resolving the constraints

$$\delta = -\rho' + \frac{Q'}{r^2}, \qquad \rho = \frac{Q}{r^2} + \frac{2Q'}{m^2 r^3},$$

$$p_R = -p'_\Delta + \frac{F'}{r}, \qquad p_\Delta = \frac{F}{r} - \frac{2F'}{m^2 r^2},$$

where Q, F are arbitrary,

$$\mathcal{E}_{\mathrm{FP}} = 3\left(Q'^2 + m^2 Q^2 + \frac{F^2}{4}\right)\frac{1}{r^4} + \text{derivative}$$

therefore

$$\int_0^\infty \mathcal{E}_{\mathrm{FP}}\, dr \geq 0$$

Strong fields – kinetic energy sector

$$\text{Let} \quad \Delta = 1, \quad R = r, \quad \pi_\Delta = \pi_\Delta(r), \quad \pi_R = \pi_R(r)$$

$$\text{Setting} \quad x = mr, \quad p_\Delta = \frac{\sqrt{xz}}{m}, \quad p_R = -\frac{(xz + 4x^4 f)}{(2x\sqrt{xz})}$$

both constraints are fulfilled if z, f fulfill two equations

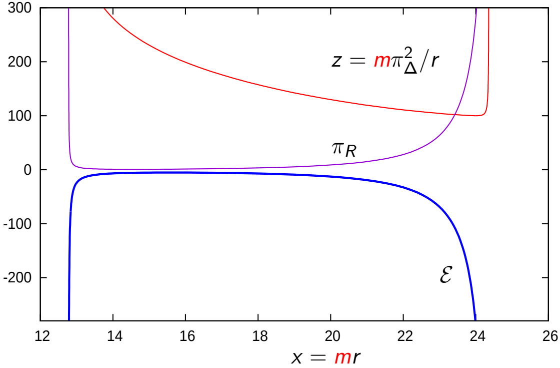
$$z' = 4x^2 f + 2x\sqrt{xz}\sqrt{f(f+2)},$$

$$f' = \frac{\{4\,zf - 4x^3 f - 3z\}\sqrt{f(f+2)}}{4x\sqrt{xz}} - \frac{2}{x} f(f+2),$$

while the energy density $\boxed{\mathcal{E} = x^2 f}$. One should have $f(f+2) \geq 0$ $\Rightarrow$ two solution branches: either $f \geq 0$, positive energy, or $f \leq -2$, negative energy.

$f = 0,\ z = z_0,\ E = 0.$
$f = -2,\ z = \frac{8}{3}\,(x_{\max}^3 - x^3),\ E = \int_0^{x_{\max}} \mathcal{E}\, dr = -\frac{2}{3m}\, x_{\max}^3.$

Negative energy solutions



Singular solutions on compact intervals, $\mathcal{E}$ is everywhere negative, $E = \int \mathcal{E} dr = -\infty$. Energy is unbounded from below. However, this *does not* imply that flat space is unstable, because solutions *do not* describe regular initial data.

Strong fields – potential energy sector

Let $p_\Delta = p_R = 0, \quad \Delta = \frac{g}{h}, \quad R = rh, \quad \Rightarrow \quad \mathcal{S} = 0,$

$$\begin{aligned}\mathcal{C} &= -h'' - \frac{2}{x}\,h' + \frac{h'^2}{2h} - \frac{(xh)'g'}{xg} + \frac{h(1-g^2)}{2x^2g^2} \\ &+ \frac{h(2-3h)}{2g} + \frac{h(1-6h+6h^2)}{2g^2} = 0,\end{aligned}$$

with $h_0 \leftarrow h \rightarrow 1, \;\; 1 \leftarrow g \rightarrow 1$ for $\; 0 \leftarrow x \rightarrow \infty;$ the energy

$$\mathcal{E} = \frac{x^2h^2(3h-g-2)}{g}\,.$$

Special solutions, also fulfill the Hamilton equations

- $h = 1,\; g = 1,\; ds_g^2 = ds_f^2, \quad \mathcal{E} = 0$ flat space
- $h = \frac{1}{2},\; g = 1,\; ds_g^2 = \frac{1}{4}\,ds_f^2, \quad \mathcal{E} = -3x^2/8$ tachyon universe

Deformations of flat space – normal branch

$$g = 1 + \frac{A\,x^2}{1+(x-x_0)^4}\,,$$

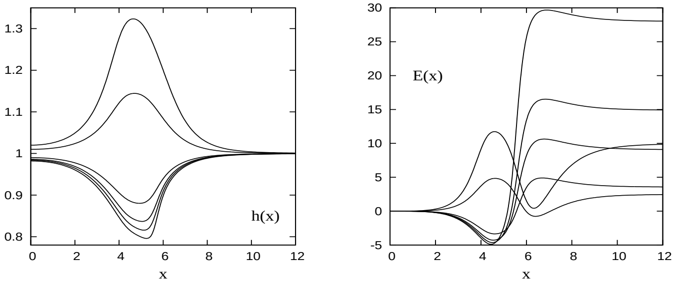


Figure : Profiles of h, E for several positive energy solutions.

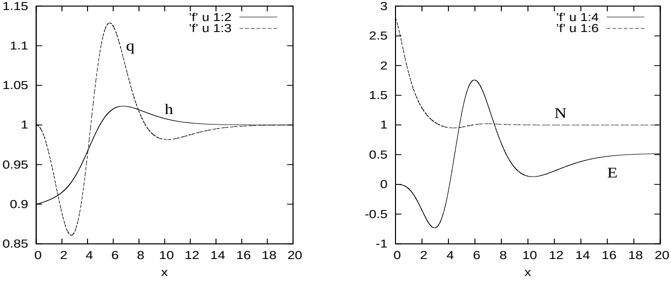
Energy is positive.

General solution of the constraint

Setting $g = qh/(xh)'$, the constraint is solved with

$$Q = xh(1-q^2) + x^3 h(2h-1)(h-1),$$
$$Q' = r^2 h(3h-2)(q-1)$$

for any $Q(x)$. Let $Q = A\Theta(x-x_0)(x-x_0)^p e^{-x}$



Energy is positive for smooth, asymptotically flat fields.

Tachyon branch

Solutions of the constraint with

$$h_0 \leftarrow h \rightarrow \frac{1}{2}$$

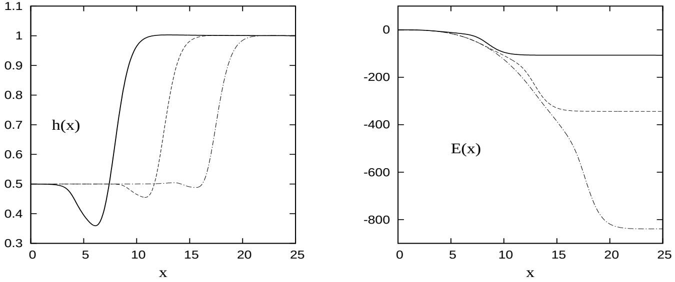
The energy is negative and infinite. However, this does not affect stability of flat space, because the asymptotic condition at infinity is different:

$h_0 \leftarrow h \rightarrow 1$ flat space branch
$h_0 \leftarrow h \rightarrow \frac{1}{2}$ tachyon branch

Negative energies comprise a disjoint branch and so they are harmless.

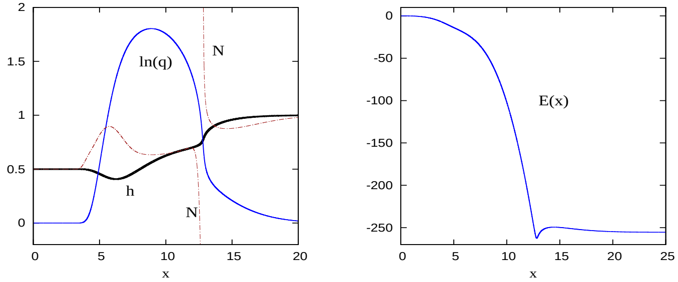
Tachyon branch vs. normal branch

There are solutions which start from the tachyon branch at the origin and approach the flat space at infinity.
The energy is finite and negative – tachyon bubbles.



Does this affect the stability of flat space ?

Tachyon bubbles



The lapse N is singular. Can be proven for $c_3 = c_4 = 0$. For other values of the parameters – numerical evidence. One does not find negative energy solutions which would describe initial data for a decay of flat space $\Rightarrow$ negative energy decouple and are harmless.

Summary

- Two constraints of the dRGT massive gravity remove one of the 2 DoF in s-sector $\Rightarrow$ only 1 DoF propagates.
- It is natural to think that the removed mode is the ghost. Then the energy should be positive, but in fact it is unbounded from below.
- However, for smooth deformations of flat space the energy is positive – the physical sector. It seems that negative energy states belong to disjoint sectors, so they are harmless.

$\Rightarrow$ The evidence that the theory is healthy in its physical sector, where the energy is positive and the ghost is suppressed.