Massive Gravity Potentials

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- Counting of # DoF for general Potentials $V[g^{00},\ g^{0\,i},\ g^{ij}]$ SO(3) invariants

$$\int d^4x\ \sqrt{g}\left[\ M_{\rm PL}^2\ (\ \underbrace{R}_{Diff.Inv.}\ -m^2\ \underbrace{V}_{Diff.non\,Inv.}\ )+\mathcal{L}_{\rm matter}\right]$$

  - Dirac's Constrained Hamiltonian Dynamics → Non Perturbative # DoF
  - Constraints → Partial Diff Eqs in field space
- Phenomenology around Minkowski → Perturbative # DoF
- FRW Cosmology

DoF inside $g_{\mu\nu}$

- 10 DoF in $g_{\mu\nu}$:

$+4$ scalars $\quad$ $+4$ vectors $\quad$ $+2$ tensors

- 4 DoF in $g_{0\mu}$ are non dynamical (for non derivative modifications of GR):

-2 scalars $\quad$ -2 vectors $\quad$ 0 tensors

- In GR, general coordinate invariance $x^\mu \to x^\mu + \xi^\mu$:

-2 scalars $\quad$ -2 vectors $\quad$ 0 tensors

- In GR DoF=2 (tensors)
- Without Diff. DoF$\leq$ 6 (2 scalars + 2 vectors + 2 tensors)
- A Massive spin 2 in d=4 has 5 DoF (1 scalar + 2 vectors + 2 tensor)

Non perturbative # DoF and structure of the Potentials

ADM variables: $g^{\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2}N^i \\ N^{-2}N^i & \gamma^{ij} - N^{-2}N^iN^j \end{pmatrix}, \quad (N^A \equiv (N, N^i), \ \gamma^{ij})$

$\mathcal{V} \equiv \sqrt{g_{\alpha\beta}}\, V = \sqrt{\gamma}\, N\, V$

- 6 DoF: $\mathcal{V} = \mathcal{V}[N^A, \gamma]$
- 5 DoF: $\mathcal{V} = \sqrt{\gamma}\,\left(N\,\mathcal{U}[\mathcal{K}] + \mathcal{E}[\xi,\gamma] + \mathcal{U}_{\xi^i}\ \mathcal{Q}^i\right)$

  where $N^i = N\,\xi^i + \mathcal{Q}^i$, $\mathcal{K}^{ij} = \gamma^{ij} - \xi^i\,\xi^j$ and $\mathcal{Q}^i = (\partial^2_{\xi^i\xi^j}\mathcal{U})^{-1}\partial_{\xi^j}\mathcal{E}$
- 3 DoF: $\mathcal{V} = \mathcal{V}[N, \gamma]$ with $\partial^2_N \mathcal{V} \neq 0$
- 3 DoF: $\mathcal{V} = \sqrt{\gamma}\,\left(N\,\mathbf{U}[\gamma] + \mathbf{E}[\gamma]\right)$
- 2 DoF: $\mathcal{V} = \sqrt{\gamma}\,\left(\Lambda\ N + \mathbf{E}[\gamma]\right)$

$$\text{Configuration space}: \phi_i, \qquad i = 1, ..., n, \qquad \textit{Lagrangian} \ \ \mathcal{L}(\phi_i, \ \dot{\phi}_i)$$

$$\text{Phase space}: (\phi_i, \ \Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}) \qquad r = Rank|\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \, \partial \dot{q}_j}| \leq n$$

$$\begin{aligned} \textit{Primary} \quad &: \quad \mathcal{P}_a(q, \ \dot{q}, \Pi) = 0, \quad a = n - r \\ \textit{Total Hamiltonian} &: \quad H_T = \Pi \, \dot{q} - \mathcal{L} + \lambda \, \mathcal{P} \\ \text{Conservation of the Constraints} &: \quad \dot{\mathcal{P}} = \{\mathcal{P}, \ H_T\} = 0 \\ \{\textit{Primary}, \ \textit{Secondary}, \ \textit{Tertiary}, \ ...\} &\rightarrow \{\{\mathcal{F}.\mathcal{C}.\}, \ \{\mathcal{S}.\mathcal{C}.\}\} \\ \{\mathcal{F}.\mathcal{C}., \ \mathcal{F}.\mathcal{C}.\} = 0, &\quad \{\mathcal{S}.\mathcal{C}., \ \mathcal{S}.\mathcal{C}.\} \neq 0 \end{aligned}$$

$$\boxed{DoF = \frac{2 \, n - \#(\mathcal{S}.\mathcal{C}.) - 2 \, \#(\mathcal{F}.\mathcal{C}.)}{2}}$$

Canonical Analysis for GR

Phase Space 2×10 dim: $(\mathrm{N},\, \mathrm{N}^{\mathrm{i}},\, \gamma_{\mathrm{ij}}) + (\Pi_0,\, \Pi_i,\, \pi^{ij})$, $\quad \mathrm{N}^{\mathrm{A}} \equiv (\mathrm{N},\, \mathrm{N}^{\mathrm{i}})$

ADM decomposition

$$\mathrm{g}^{\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2}\, N^i \\ N^{-2}\, N^i & \gamma^{ij} - N^{-2} N^i N^j \end{pmatrix}, \qquad H = \int d^3x\ \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}}[\gamma,\, \pi]\, \mathrm{N}^{\mathrm{A}}$$

- <u>Primary</u> $\quad \Pi_{\mathrm{A}} \approx 0$ and then $H \to H_{\mathrm{T}} = H + \int d^3x\, \lambda^A\, \Pi_A$
- <u>Secondary</u> $\mathcal{S}_{\mathrm{A}} = \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}} \approx 0$
- Tertiary $\quad \mathcal{T}_{\mathrm{A}} = \{\mathcal{S}_{\mathrm{A}},\, H\} \propto \underbrace{\{\mathcal{H}^{\mathrm{GR}}_{\mathrm{A}},\, \mathcal{H}^{\mathrm{GR}}_{\mathrm{B}}\} \propto \mathcal{H}^{\mathrm{GR}}_{\mathrm{C}}}_{GR\ Algebra} \approx 0$

$\mathcal{F}.\mathcal{C}. = \{\Pi_A,\, \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}}\}$

$$\#DoF = \frac{20 - 2 \times 4\, (\Pi_{\mathrm{A}}) - 2 \times 4\, (\mathcal{S}_{\mathrm{A}})}{2} = 2$$

Canonical Analysis for modified gravity

$$H = \int d^3x \; [\, \mathrm{N^A} \;\; \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}}[\gamma,\; \pi] + \mathcal{V}[\mathrm{N},\; \mathrm{N^i},\; \gamma^{ij}] \,]$$

$$\mathrm{N^A} = (\mathrm{N},\; \mathrm{N^i}), \qquad \boxed{\mathcal{V} \equiv \mathrm{N}\; \sqrt{\gamma}\; \mathrm{V}}, \qquad \mathcal{V}_{\mathrm{A}} \equiv \frac{\partial \mathcal{V}}{\partial \mathrm{N^A}}, \qquad \mathcal{V}_{\mathrm{AB}} \equiv \frac{\partial^2 \mathcal{V}}{\partial \mathrm{N^A} \partial \mathrm{N^B}}$$

- Primary $\Pi_{\mathrm{A}} \approx 0$ and then $H \to H_{\mathrm{T}} = H + \int d^3x\; \lambda^A\; \Pi_A$
- Secondary $\mathcal{S}_{\mathrm{A}} = \mathcal{H}^{\mathrm{GR}}_{\mathrm{A}} + \mathcal{V}_{\mathrm{A}} \approx 0$
- Tertiary $\mathcal{T}_{\mathrm{A}} = \{\mathcal{S}_{\mathrm{A}},\; H\} + \lambda^{\mathrm{B}}\; \mathcal{V}_{\mathrm{AB}} \approx 0$

two cases

$$(a)\;\; Det|\mathcal{V}_{\mathrm{AB}}| \neq 0$$

$$(b)\;\; Det|\mathcal{V}_{\mathrm{AB}}| = 0$$

6 DoF Potentials

$$\mathcal{T}_{\rm A} = \{\mathcal{S}_{\rm A},\ H\} + \lambda^{\rm B}\, \mathcal{V}_{\rm AB} \approx 0$$

$$\text{If} \quad Det|\mathcal{V}_{\rm AB}| \neq 0 \to \text{all } \lambda^{\rm A} \text{ are determined}$$

end of the counting of the constraints:

$$\boxed{\#DoF = \frac{20 - 4\,(\Pi_{\rm A}) - 4\,(\mathcal{S}_{\rm A})}{2} = 6}$$

Around Minkowski: 5 (massive spin 2) +1 (Scalar→ B.D. Ghost)

Potentials with $\#DoF \leq 5$

When $\quad Det|\mathcal{V}_{\rm AB}| = 0 \quad$ compute $\quad \boxed{\rm r \equiv Rank||\mathcal{V}_{AB}||}$

- For SO(3) invariant Potentials, $\rm r$: 3, ~~2~~, 1, 0
- $(4-\rm r)$ zero eigenstate $\chi^{\rm A}_{\alpha=1,\dots,4-\rm r}$: $\quad \mathcal{V}_{\rm AB}\ \chi^{\rm B}_{\alpha} = 0$
- Decomposition of the Lagrange Multipliers:
  $\lambda^{\rm A} = z_\alpha\ \chi^{\rm A}_\alpha + \sum_{\rm n} d_{\rm n}\ {\rm E}^{\rm A}_{\rm n}$
- $(4-\rm r)$ Tertiary Constraints $\mathcal{T}_\alpha$:
  $\mathcal{T}_{\alpha=1,..,4-r} \equiv \chi^{\rm A}_\alpha\ \mathcal{T}_{\rm A} = \chi^{\rm A}_\alpha\ \{\mathcal{S}_{\rm A},\ H\} \approx 0$
- Quaternary: $\mathcal{Q}_\alpha \propto\ z_\beta\ \chi^{\rm A}_\beta\ \{\mathcal{S}_{\rm A},\ \mathcal{S}_{\rm B}\}\ \chi^{\rm B}_\alpha + \bar{\lambda}^{\rm A} \dots$
- Independent quaternaries $(4-r-s)$:

$$\boxed{\theta_{\alpha\beta} \equiv \chi^{\rm A}_\alpha\ \{\mathcal{S}_{\rm A}, \mathcal{S}_{\rm B}\}\ \chi^{\rm B}_\beta} \qquad \boxed{s = {\rm Rank}||\theta_{\alpha\beta}||}$$

General Counting Rules

We recap the steps that are required to compute the number of propagating DoF for a given deforming potential $\mathcal{V}$:

(1) Compute the rank $\boxed{r}$ of the hessian matrix $\|\mathcal{V}_{\mathrm{AB}}\|$ ($4 \otimes 4$ matrix).

(2) Compute the $(4 - r)$ null eigenvectors $\chi^{\mathrm{A}}_{\alpha}$ of the matrix $\mathcal{V}_{\mathrm{AB}}$.

(3) Determine secondary constraints $\mathcal{S}_{\mathrm{A}} = \mathcal{H}^{GR}_{\mathrm{A}} + \mathcal{V}_{\mathrm{A}}$.

(4) Compute the ranks $\boxed{s}$ of the matrix $\|\chi^{\mathrm{A}}_{\alpha} \{\mathcal{S}_{\mathrm{A}}, \mathcal{S}_{\mathrm{B}}\} \chi^{\mathrm{B}}_{\beta}\|$ ($4 - r \otimes 4 - r$ matrix).

(5) Plug these numbers into the formula

$$\boxed{\#DoF \leq 2 + r + \frac{s}{2}}$$

$0 \leq \mathrm{r} \leq 4,\ 0 \leq \mathrm{s} \leq 4 - r \rightarrow \#DoF \leq 4 + \frac{\mathrm{r}}{2}$

General Classification of Potentials

r =Rank$\lvert\mathcal{V}_{\rm AB}\rvert$	s =Rank$\lvert\theta_{\alpha\beta}\rvert$	#DoF $\leq$	Rotations?	Realized?
4	**0**	**6**	**OK**	**Yes**
3	**0**	**5**	**OK**	**Yes**
3	**1**	**5+$\frac{1}{2}$**	**OK**	**Yes**
2	0	4	NO	No
2	1	4+$\frac{1}{2}$	NO	No
2	2	5	NO	No
1	**0**	**3**	**OK**	**Yes**
1	1	3+$\frac{1}{2}$	NO	No
1	2	4	NO	No
1	3	4+ $\frac{1}{2}$	OK	No
0	**0**	**2**	**OK**	**Yes**
0	1	2+ $\frac{1}{2}$	OK	No
0	**2**	**3**	**OK**	**Yes**
0	3	3+$\frac{1}{2}$	OK	No
0	4	4	OK	No

Table: Deforming potentials classified according to the rank r of the Hessian and the rank s of the matrix θ.

(1) $\mathcal{V}_{\rm AB}$ $4 \otimes 4$ matrix with $\mathrm{Det}|\mathcal{V}_{\rm AB}| = 0$ and $\mathrm{r} = 3$

(2) One zero eigenstate χ^A i.e. $\mathcal{V}_{\rm AB}\,\chi^{\rm B} = 0$

(3) $\mathcal{S}_{\rm A} = \mathcal{H}^{GR}_{\rm A} + \mathcal{V}_{\rm A}$

(4) $\theta = \|\chi^{\rm A}\,\{\mathcal{S}_{\rm A}, \mathcal{S}_{\rm B}\}\,\chi^{\rm B}\|$ $(1 \otimes 1$ matrix$) \to s = 0,\ 1$

(5) $\#DoF = 2 + (r = 3) + \frac{(s=0,1)}{2}$
so that

$\#DoF = 5$ (for $s = 0$) $\boxed{\text{or}}$ $\#DoF = 5 + \frac{1}{2}$ (for $s = 1$)

5 DoF Conditions

Necessary Conditions for a 5 DoF Potential $\mathcal{V}$

⊗ Monge-Ampere equation

$$\boxed{Det|\mathcal{V}_{AB}| = 0} \to \boxed{\partial^2_{N^0}\tilde{\mathcal{V}} - \partial^2_{N^0\,N^i}\tilde{\mathcal{V}}\ \left(\partial^2_{N^i\,N^j}\tilde{\mathcal{V}}\right)^{-1}\ \partial^2_{N^j\,N^0}\tilde{\mathcal{V}} = 0} \Leftrightarrow \mathrm{DoF} = 5.5$$

⊗ Extra Differential equation

$$\boxed{\chi^A\ \{\mathcal{S}_A,\ \mathcal{S}_B\}\ \chi^B = 0} \to \boxed{\frac{\partial\tilde{\mathcal{V}}}{\partial N^i} + 2\,\xi^A\xi^j\frac{\partial^2\tilde{\mathcal{V}}}{\partial N^A\ \partial\gamma^{ij}} = 0} \Leftrightarrow \mathrm{DoF} = 5$$

where $\tilde{\mathcal{V}} = \gamma^{-1/2}\mathcal{V}, \quad \chi^A\ \mathcal{V}_{AB} = 0, \quad \xi^A = \chi^A/\chi^0 = (1,\ \xi^i)$

D.B. Fairlie and A.N. Leznov, (1995)

Implicit change of variables $N^i \to \xi^i = \frac{\chi^i}{\chi^0}$ such that

$$\boxed{N^{\rm i} = N\,\xi^{\rm i} + \mathcal{Q}^{\rm i}[\xi,\gamma]} \to \xi^{\rm i} = \xi^{\rm i}(N,\, N^i,\, \gamma)$$

The solution is given by:

$$\boxed{\boxed{\mathcal{V} = \sqrt{g}\,\mathrm{V} = \sqrt{\gamma}\,\left(\mathrm{N}\,\mathcal{U} + \mathcal{E} - \mathcal{U}_\xi \cdot (\mathcal{U}_{\xi\,\xi})^{-1} \cdot \,\mathcal{E}_\xi\right)}}$$

Where the two free functions $\mathcal{U}[\xi^{\rm i},\,\gamma^{\rm ij}]$ and $\mathcal{E}[\xi^{\rm i},\,\gamma^{\rm ij}]$ fix also

$$\mathcal{Q}^{\rm i}[\xi,\gamma] \equiv -(\,\partial^2_{\xi^{\rm i}\,\xi^{\rm j}}\mathcal{U})^{-1}\,\partial_{\xi^{\rm j}}\mathcal{E}$$

$$\boxed{\mathcal{V} = \sqrt{\gamma}\,\left(\mathrm{N}\,\mathcal{U} + \mathcal{E} + \partial_{\xi^{\rm i}}\mathcal{U}\;\mathcal{Q}^{\rm i}\right)}$$

Solution Monge Ampere + Extra eq

$$\boxed{\frac{\partial\tilde{\mathcal{V}}}{\partial N^{\rm i}} + 2\,\xi^{\rm A}\xi^{\rm j}\frac{\partial^2\tilde{\mathcal{V}}}{\partial N^{\rm A}\partial\gamma^{\rm ij}} = 0} \rightarrow \frac{\partial\mathcal{U}}{\partial\xi^{\rm i}} + 2\,\xi^{\rm j}\frac{\partial\mathcal{U}}{\partial\gamma^{\rm ij}} = 0$$

$$\mathcal{U}[\xi^{\rm i},\gamma^{\rm ij}] = \boxed{\mathcal{U}[\gamma^{\rm ij} - \xi^{\rm i}\,\xi^{\rm j}]} \equiv \mathcal{U}[\mathcal{K}^{\rm ij}]$$

$$\mathcal{V}[{\rm N},\ {\rm N}^{\rm i},\ \gamma^{\rm ij}] = \sqrt{\gamma}\,\left({\rm N}\,\mathcal{U}[\mathcal{K}^{\rm ij}] + \mathcal{E}[\xi^{\rm i},\gamma^{\rm ij}] + \partial_{\xi^{\rm a}}\mathcal{U}[\mathcal{K}^{\rm ij}]\ \mathcal{Q}^{\rm a}[\xi^{\rm i},\gamma^{\rm ij}]\right)$$

$$\boxed{\mathcal{V}[{\rm N},\ {\rm N}^{\rm i},\ \gamma^{\rm ij}] = \sqrt{\gamma}\,\left({\rm N}\,\mathcal{U} + \mathcal{E} + \partial_{\xi^{\rm i}}\mathcal{U}\ \mathcal{Q}^{\rm i}\right)}$$

$$\boxed{{\rm N}^{\rm i} = {\rm N}\,\xi^{\rm i} + \mathcal{Q}^{\rm i},\quad \mathcal{Q}^{\rm i} = -\,\mathcal{U}^{-1}_{\xi^{\rm i}\xi^{\rm j}}\ \mathcal{E}_{\xi^{\rm j}}}$$

with $\mathcal{K}^{\rm ij} = \gamma^{\rm ij} - \xi^{\rm i}\,\xi^{\rm j}$

The Energy for the 5 DoF Potential

$$H = H_{\rm Back.} + H_{\rm Surf.} = \int d^3x \, \left(\mathcal{H}\ N + \mathcal{H}_{\rm i}\ N^{\rm i} + \mathcal{V}\right) + H_{ADM}$$

$$H|_{\rm on\ shell} = \int d^3x \, \sqrt{\gamma} \;\; \underbrace{\mathcal{E}[\xi,\gamma]}_{Bulk\ Energy\ Density} \;\; + H_{ADM}$$

Example of 5 DoF Potential

$$\boxed{\mathcal{V} = \sqrt{\gamma}\,(\mathrm{N}\,\mathcal{U} + \mathcal{E} + \mathcal{U}_{\xi^{\mathrm{i}}}\,\mathcal{Q}^{\mathrm{i}})} \quad \boxed{\mathrm{N}^{\mathrm{i}} = \mathrm{N}\,\xi^{\mathrm{i}} + \mathcal{Q}^{\mathrm{i}}, \quad \mathcal{Q}^{\mathrm{i}} = -\,\mathcal{U}^{-1}_{\xi^{\mathrm{i}}\xi^{\mathrm{j}}}\;\mathcal{E}_{\xi^{\mathrm{j}}}}$$

If $\boxed{\boxed{\mathcal{E}[\xi,\ \gamma] = \ \mathcal{E}[\gamma]}}$

$$\xi^{\mathrm{i}} = \mathrm{N}^{\mathrm{i}}/\mathrm{N} \;\rightarrow\; \mathcal{K}^{\mathrm{ij}} = \gamma^{\mathrm{ij}} - \frac{\mathrm{N}^{\mathrm{i}}\,\mathrm{N}^{\mathrm{j}}}{\mathrm{N}^2} = \mathrm{g}^{\mathrm{ij}} \;\text{ and }\; \mathcal{Q}^{\mathrm{i}} = 0$$

$$\boxed{\mathcal{V} = \sqrt{\gamma}\,\left(\mathrm{N}\,\mathcal{U}[\mathrm{g}^{\mathrm{ij}}] + \mathcal{E}[\gamma^{\mathrm{ij}}]\right)}$$

$$H|_{\text{on shell}} = \int d^3x\,\sqrt{\gamma}\,\mathcal{E}[\gamma] + Boundaries$$

Lorentz Invariant ghost free DeRGT potential

$$\boxed{\mathcal{V} = \sqrt{\gamma}\,(\mathrm{N}\,\mathcal{U} + \mathcal{E} + \mathcal{U}_{\xi^{\mathrm{i}}}\,\mathcal{Q}^{\mathrm{i}})} \qquad \boxed{\mathrm{N}^{\mathrm{i}} = \mathrm{N}\,\xi^{\mathrm{i}} + \mathcal{Q}^{\mathrm{i}},\quad \mathcal{Q}^{\mathrm{i}} = -\,\mathcal{U}^{-1}_{\xi^{\mathrm{i}}\xi^{\mathrm{j}}}\ \mathcal{E}_{\xi^{\mathrm{j}}}} \qquad \mathcal{K}^{\mathrm{ij}} = \gamma^{\mathrm{ij}} - \xi^{\mathrm{i}}\xi^{\mathrm{j}}$$

$$\mathcal{U} = \mathrm{Tr}(\mathcal{K}^{1/2}) - 3, \qquad \mathcal{E} = \frac{1}{\sqrt{1-\xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}}}, \qquad \mathcal{U}_{\xi^{\mathrm{i}}}\,\mathcal{Q}^{\mathrm{i}} = -\frac{\xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}}{\sqrt{1-\xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}}}$$

$$\mathrm{V}_{\mathrm{LI}} = (\mathrm{Tr}(\mathcal{K}^{1/2}) - 3) + \mathrm{N}^{-1}\,\sqrt{1-\xi^{\mathrm{i}}\gamma_{\mathrm{ij}}\xi^{\mathrm{j}}} = \ \mathrm{Tr}\left[(\mathrm{g}^{\mu\alpha}\ \eta_{\alpha\nu})^{1/2}\right] - 3$$

$$H|_{\text{on shell}} = \int d^3x\,\sqrt{\gamma}\,\mathcal{E}[\xi,\gamma] + \mathit{Boundaries}$$

There are others Lorentz Invariant 5 DoF Potentials?

$$\mathrm{X} \equiv \mathrm{g}^{-1} \cdot \eta \qquad \text{with } d \, (= \textit{spacetime dim.}) \text{ eigenvalues} \quad \lambda_{i=1,\ldots,d}$$

$$\mathrm{X} = \mathrm{D}^t \, \mathrm{X_D} \, \mathrm{D} \qquad \text{with } \mathrm{X_D} = \mathrm{Diag}\{\lambda_1, \ldots, \lambda_d\}$$

$$\mathrm{X}^{1/2} = \mathrm{D}^t \, \mathrm{X_D}^{1/2} \, \mathrm{D} \qquad \text{with } \mathrm{X_D}^{1/2} = \mathrm{Diag}\{\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_d}\}$$

there are 2^d branches
Only DGRT potentials have Minkowski as a background.

$r = 1$ Potentials

- $r = Rank\|\mathcal{V}_{\rm AB}\| = 1 \to \boxed{\boxed{\mathcal{V}[{\rm N},\, \gamma]}}$ with $\mathcal{V}_{\rm NN} \neq 0$
- $\chi_i^{\rm A} = \delta_i^{\rm A}$ with $i = 1, 2, 3$
- $\mathcal{S}_0 = \mathcal{H} + \mathcal{V}_{\rm N}, \quad \mathcal{S}_{\rm i} = \mathcal{H}_{\rm i}$
- $\theta_{\alpha\beta} = \theta_{ij} \sim \mathcal{H}_j \sim 0 \to s = 0$

$$\boxed{\#DoF = 2 + (r = 1) + \frac{s = 0}{2} = 3}$$

$r = 0$ Potentials

- $rank\|\mathcal{V}_{\rm AB}\| = 0 \to \boxed{\boxed{\mathcal{V}[{\rm N},\,\gamma] = \sqrt{\gamma}({\rm N}\ \mathbf{U}[\gamma] + \mathbf{E}[\gamma])}}$
- $\chi^{\rm A}_{\alpha} = \delta^{\rm A}_{\alpha}$ with $\alpha = 0, 1, 2, 3$
- $\mathcal{S}_0 = \mathcal{H} + \mathcal{V}_{\rm N} = \mathcal{H} + \mathbf{U}, \quad \mathcal{S}_{\rm i} = \mathcal{H}_{\rm i}$
- $\theta_{0\,0} = \theta_{i\,j} \sim 0, \qquad \boxed{\theta_{0\,i} \propto \partial_{\gamma}\mathbf{U}}$

$$s = 2 \to \boxed{\#DoF = 2 + (r = 0) + \frac{s = 2}{2} = 3}$$

- For $\mathbf{U} = 0$ or $\mathbf{U} = \sqrt{\gamma}$ we have $\theta_{\alpha\beta} = 0$

$$s = 0 \to \boxed{\#DoF = 2 + (r = 0) + \frac{s = 0}{2} = 2}$$

$$\boxed{\boxed{\mathcal{V} = \sqrt{\gamma}\,(\Lambda\,{\rm N} + \mathbf{E}[\gamma])}}$$

Quadratic Perturb. in Minkowsky

Quadratic fluctuation's Lagrangian (the most general SO(3) invariant case) for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\mathcal{L}_{(2)} = \mathcal{L}^{GR}_{Kin} + \frac{M_{Pl}^2}{2}\left(m_0^2\, h_{00}^2 + 2\, m_1^2\, h_{0i}^2 - m_2^2\, h_{ij}^2 + m_3^2\, h_{ii}^2 - 2\, m_4^2\, h_{00}\, h_{ii}\right)$$

- $\boxed{m_0 = 0}$ & $m_1 \neq 0$ $\quad 2_{Tensor} + 2_{Vector} + 1_{Scalar} = 5$ DoF
- $\boxed{m_1 = 0}$ $\quad 2_{Tensor} = 2$ DoF

Minkowski perturbations for General Potentials

- $\boxed{m_0^2} = -\left.\frac{\partial^2 \mathcal{V}}{4\,\partial \mathrm{N}^2}\right|_\eta$
- $\boxed{m_1^2} = -\left.\frac{\partial^2 \mathcal{V}}{2\,\partial \mathrm{N}^i \partial \mathrm{N}^j}\right|_\eta$

Potential	**Nonpert. #DoF**	**LB Masses**	**Pert. #DoF**
$\mathcal{V}[\mathrm{N}^A, \gamma]$	6	$m^2_{0,\dots,4} \neq 0$	6=5+ghost
$\sqrt{\gamma}\left(\mathrm{N}\,\mathcal{U}[\mathcal{K}] + \mathcal{E}[\xi, \gamma] + \mathcal{U}_i \mathcal{Q}^i\right)$	5	$m_0^2 = 0$	5
As above + Lorentz Invariance	5	$m_0^2 = 0$	5
$\mathcal{V}[\mathrm{N}, \gamma]$	3	$m_1^2 = 0$	2
$\sqrt{\gamma}\left(\mathrm{N}\,\mathbf{U}[\gamma] + \mathbf{E}[\gamma]\right)$	3	$m^2_{0,1} = 0$	2
$\sqrt{\gamma}\left(\Lambda\,\mathrm{N} + \mathbf{E}[\gamma]\right)$	2	$m^2_{0,1,4} = 0$	2

Strong Coupling

$$\mathcal{L} = M_{Pl}^2\,(\partial h)^2 + m^2\,M_{Pl}^2(\partial\phi)^2 + (h + m^2\ \phi)\ T_m$$

canonical fields: $h_c = M_{Pl}\ h, \quad \phi_c = m\,M_{Pl}\ \phi$

$$\mathcal{L}_c = \ (\partial h_c)^2 + (\partial\phi_c)^2 + \frac{(h_c + m\ \phi_c)}{M_{Pl}}\ T_m$$

Schw. solutions: $\nabla^2 h_c = \frac{T_m}{M_{Pl}}, \quad \nabla^2\phi_c = m\frac{T_m}{M_{Pl}} \to \bar{h}_c \sim \frac{\bar{\phi}_c}{m} \sim \frac{M_\odot}{M_{Pl}\ r}$

Perturbativity: $\nabla\bar{h}_c \ll M_{Pl}^2$ and $\nabla\bar{\phi}_c \ll m\,M_{Pl} \equiv \Lambda_2^2 \Rightarrow r \gg r_S = \frac{M_\odot}{M_{Pl}^2}$

- No VDVZ discontinuity

$$\Phi \sim 1 - \frac{r_S}{r} + m^2 r^2 (c_1 \frac{r_S}{r} + c_2 \frac{r_S^2}{r^2} \log mr), \qquad mr \ll 1$$

$$\Phi \sim \frac{r_S}{r}(A_1 e^{-M_1 r} + A_2 e^{-M_2 r}), \qquad mr \gg 1$$

- Strong coupling scale $\Lambda_2 \sim \sqrt{m\ M_{Pl}}$:

$$\phi \sim M_{Pl}\ \frac{m_4}{m_1}\sqrt{m_4^2 - m_1^2}\,\nabla\,\phi_c \sim \Lambda_2\,\nabla\,\phi_c$$

$$A_i \sim M_{pl}\ m_1\ A_{i\,c}, \quad h \sim M_{Pl}\ h_c$$

$$g_{\mu\nu} = \mathrm{Diag}||\mathrm{N}^2(t), -\alpha^2(t), -\alpha^2(t), -\alpha^2(t)||$$

$\mathrm{N}^i = \xi^i = 0$, $\partial_{\gamma_{ij}}\mathcal{E} \equiv \gamma_{ij}\mathcal{E}_\gamma$, $\dot{\mathcal{E}} = -6\,\frac{\dot\alpha}{\alpha}\,\mathcal{E}_\gamma$, $\mathcal{E}_\gamma = -\frac{\partial\mathcal{E}}{6\,\partial\log\alpha}$

$$\rho_{eff} = \frac{\mathcal{V}_\mathrm{N}}{\sqrt{\gamma}}, \qquad p_{eff} = 2\,\frac{\mathcal{V}_\gamma}{\mathrm{N}\,\sqrt{\gamma}}$$

Bianchi

$$\dot{\mathrm{N}}\,\mathcal{V}_{\mathrm{NN}} - 6\,\frac{\dot\alpha}{\alpha}\left(\mathcal{V}_{\mathrm{N}\,\gamma} - \frac{\mathcal{V}_\gamma}{\mathrm{N}}\right) = 0$$

If $\mathcal{V}_{\mathrm{NN}} \neq 0$ then N evolves.

If $\mathcal{V}_{\mathrm{NN}} = 0 \to \mathcal{V} = \boxed{\sqrt{\gamma}\,(\mathrm{N}\,\mathcal{B}[\gamma] + \mathcal{D}[\gamma])}$ with $\left.\boxed{\mathcal{D}_\gamma - \frac{1}{2}\mathcal{D}}\right|_{\mathrm{FWR}} = 0$

$$\boxed{w_{eff} = -1 + \frac{2\,\mathcal{B}_\gamma}{\mathcal{B}}}$$

Conditions to have a FRW background

If $\mathcal{V}_{\rm NN} \neq 0 \Rightarrow \mathcal{V} = \boxed{\sqrt{\gamma}\,(\mathrm{N}\,\mathcal{B}[\gamma] + \mathcal{D}[\gamma])}$ with $\boxed{\mathcal{D}_\gamma - \frac{1}{2}\mathcal{D} = 0}$ and $\boxed{w_{eff} = -1 + \frac{2\,\mathcal{B}_\gamma}{\mathcal{B}}}$

6DoF $\mathcal{V}[\mathrm{N}^{\mathcal{A}},\, \gamma] \to \mathrm{N}$ evolution

5DoF $\mathcal{V} = \sqrt{\gamma}\,\left(\mathrm{N}\,\mathcal{U}[\mathcal{K}] + \mathcal{E}[\xi,\, \gamma] + \mathcal{U}_{\rm i}\,\mathcal{Q}^{\rm i}\right)$:

$$\mathrm{w}_{\rm eff} = -1 + 2\,\frac{\mathcal{U}_\gamma}{\mathcal{U}},\quad \mathcal{E}_\gamma - \tfrac{1}{2}\mathcal{E} = 0$$

3DoF $\mathcal{V}[\mathrm{N},\, \gamma] \to \mathrm{N}$ evolution

3DoF $\mathcal{V} = \sqrt{\gamma}\left(\mathrm{N}\ \mathbf{U}[\gamma] + \mathbf{E}[\gamma]\right) \to \mathrm{w}_{\rm eff} = -1 + 2\,\frac{\mathbf{U}_\gamma}{\mathbf{U}},\quad \mathbf{E}_\gamma - \frac{1}{2}\mathbf{E} = 0$

2DoF $\mathcal{V} = \sqrt{\gamma}\left(\Lambda\,\mathrm{N} + \mathbf{E}[\gamma]\right) \to \rho_{\rm eff} = -\mathrm{p}_{\rm eff} = \Lambda,\quad \mathbf{E}_\gamma - \frac{1}{2}\mathbf{E} = 0$

Cosmology of the 5DoF Potentials

$$\mathcal{L}_{5DoF} = \sqrt{g} R - m^2 \sqrt{\gamma} \left(N\,\mathcal{U} + (\mathcal{E} + \mathcal{Q}^i \, \mathcal{U}_i) \right)$$

$\boxed{\text{Minkowski}}$: $\quad \mathcal{U}\big|_\eta = 0, \quad \mathcal{U}_\gamma + \mathcal{E}_\gamma - \frac{1}{2}\mathcal{E}\Big|_\eta = 0$

$\boxed{\text{FRW}}$: $\quad \mathcal{E}_\gamma - \frac{1}{2}\mathcal{E}\Big|_{FRW} = 0 \qquad w_{eff} = -1 + 2\,\frac{\mathcal{U}_\gamma}{\mathcal{U}}$

Masses

$\boxed{m_0^2 = 0}$, $\quad \boxed{m_{1,\,4}^2 \propto m^2\, \mathcal{U}_\gamma}$, $\quad m_{2,\,3}^2 \propto \; m^2\, (\mathcal{E}, \mathcal{E}_{\gamma\gamma}, \mathcal{U}_\gamma, \mathcal{U}_{\gamma\gamma})$.

$\boxed{Mink + FRW} \Rightarrow \mathcal{U}_\gamma = 0 \Rightarrow m_1^2 = 0 \Rightarrow Strong\;\; Coupling$

$\boxed{deSitter} \Rightarrow w_{eff} = -1 \Rightarrow \mathcal{U}_\gamma = 0 \Rightarrow m_1^2 = 0 \Rightarrow Strong\;\; Coupling$

Cosmology of the 5DoF Potentials

$$\mathrm{H}^2 \simeq \mathrm{m}^2\, \mathcal{U}, \quad \mathrm{w_{eff}} = -1 + \frac{2\,\mathcal{U}_\gamma}{\mathcal{U}} \;\Rightarrow\; \mathcal{U}_\gamma \simeq \frac{\mathrm{H}^2}{\mathrm{m}^2}\,(1 + \mathrm{w_{eff}})$$

$$\mathcal{L} \sim \Lambda_2^4\, \left[\mathcal{U}_\gamma\, \dot{\phi}^2 - \nabla^2\, \phi^2 + \ldots\right]$$

$$\Lambda_{eff} \sim \underbrace{\Lambda_2}_{\sqrt{m\, M_{Pl}}}\, (\mathcal{U}_\gamma)^{1/4} \sim \underbrace{\sqrt{\mathrm{H_0\ M_{Pl}}}}_{10^{-3}\,\mathrm{mm}}\,(1 + \mathrm{w_{eff}})^{1/4}$$

Intriguing connection between small (short distance Newtonian gravity test at ($1/\Lambda_{eff} \sim 10^{-3}$mm)) and large scales (deviation of w_{eff} from -1).

NB: $w_{eff} = w_0 + w_\alpha(1-\alpha[t]) \to w_0 = -1.04^{+0.7}_{-0.7}, \;\; w_\alpha < 1.3$ (From Planck+BAO+WP)

Conclusions: Classifications of SO(3) invariant Potentials

Potential	Nonpert. #DoF	LB Masses	Pert. #DoF	FRW Cosmo	Weak Coupling
$\mathcal{V}[\mathrm{N}^{\mathrm{A}}, \gamma]$	6	$m^2_{0,\dots,4} \neq 0$	6=5+ghost	no	$\surd$
$\sqrt{\gamma}(\mathrm{N}\,\mathcal{U}[\mathcal{K}] + \mathcal{E}[\xi, \gamma] + \mathcal{U}_{\mathrm{i}}\,\mathcal{Q}^{\mathrm{i}})$	5	$m^2_0 = 0$	5	$\surd^*$	$\surd$ but No dS
As above + LI	5	$m^2_0 = 0$	5	no	no
$\mathcal{V}[\mathrm{N}, \gamma]$	3	$m^2_1 = 0$	2	$\surd$	no
$\sqrt{\gamma}\,(\mathrm{N}\,\mathbf{U}[\gamma] + \mathbf{E}[\gamma])$	3	$m^2_{0,1} = 0$	2	$\surd^*$	no
$\sqrt{\gamma}\,(\Lambda\,\mathrm{N} + \mathbf{E}[\gamma])$	2	$m^2_{0,1,4} = 0$	2	$\surd^*$(CC)	$\surd$

Table: The allowed potentials supporting spatial rotations, and the number of perturbative and non perturbative DoF. For perturbative DoF the reference background is Minkowski space. Whether a realistic FRW cosmology is admitted is also shown, as well as whether the theory is weakly coupled up to Λ_2. The symbol * denotes that a tuning of the functional form of V is required.

Note about the last Constraints

Structure of the last constraint: $z\,\chi^{\rm A}\,\{\mathcal{S}_{\rm A},\ \mathcal{S}_{\rm B}\}\,\chi^{\rm B}$

$$\int dy\ {\color{red}z}(y)\ \underbrace{\chi^{\rm A}(x)\,\chi^{\rm B}(y)}_{\text{even } (A,\,x)\leftrightarrow(B,\,y)}\ \underbrace{\{\mathcal{S}_{\rm A}(x),\ \mathcal{S}_{\rm B}(y)\}}_{\text{odd } (A,\,x)\leftrightarrow(B,\,y)} \rightarrow \text{always zero?}$$

$$\{\mathcal{S}_{\rm A}(x),\ \mathcal{S}_{\rm B}(y)\} = \underbrace{\underbrace{{\rm P}_{\rm A\ B}(x)}_{\text{odd } A\leftrightarrow B}\ \underbrace{\delta(x-y)}_{\text{even } x\leftrightarrow y}}_{\text{odd } (A,\,x)\leftrightarrow(B,\,y)} + \underbrace{\underbrace{{\rm R}_{\rm A\ B}(x)}_{\text{even } A\leftrightarrow B}\ \underbrace{{\color{red}\partial_x\,\delta(x-y)}}_{\text{odd } x\leftrightarrow y}}_{\text{odd } (A,\,x)\leftrightarrow(B,\,y)}$$

$$\int dy\ {\color{red}z}(y)\ \underbrace{\chi^{\rm A}(x)\,\chi^{\rm B}(y){\rm R}_{\rm A\ B}(x,\ y)}_{F(x,y)\ \text{ even } x\leftarrow y}\ \partial_x\,\delta(x-y) = -\frac{1}{2\,z(x)}\partial_x(z(x)^2\ F(x,\ x))$$

Structure of the Potential

Any scalar diff invariant function of g is trivial so to build a gravitational potential $V(g_{\mu\nu})$ we need to break diff. invariance.

- *Lorentz invariant Theories*: Add an external tensor $\eta_{\mu\nu}$:

$$V(X^{\mu}_{\nu} = g^{\mu\alpha}\ \eta_{\alpha\nu}) = V(Tr[X^a]), \qquad a = 1, .., 4$$

  We can preserve some symmetry in some specific background: Global Lorentz symmetry of the frozen metric $\Lambda^t_\eta \cdot \eta \cdot \Lambda_\eta = \eta$

$$V(Tr[(\eta + h)^{-1} \cdot \eta]) \propto Tr[(h \cdot \eta))], \qquad g = \eta + h$$

  symmetry of the perturbations: $h \to \Lambda^t_\eta \cdot h \cdot \Lambda_\eta$. But for $g = \bar{g} + h$ with $\bar{g} \neq \eta$ no Λ_η Symmetry!!
  No Local Lorentz symmetry: given g and η in the *Local Lorentz frame*

$$g \to \eta \qquad and \qquad \eta \to Diag||\eta_0,\ \eta_1,\ \eta_2,\ \eta_3||$$

- *Rotational SO(3) Invariance*: Add an external spatial metric δ_{ij}

$$V(N\ scalar,\ N^i\ vector,\ \gamma_{ij}\ tensor) \propto Rotational\ scalars$$

  Unbroken rotations only around $\gamma_{ij} \propto \delta_{ij}$ as for homogeneous representation of FRW.

Both model can become diff. invariants:

Add Fields (Stuckelberg trick) $\Leftrightarrow$ More symmetries

Present Cosmology

For $\alpha \leq 1$, $\quad \mathcal{U} = \sum_{\rm n} \bar{\mathcal{U}}_{\rm n}\,(\alpha - 1)^{\rm n}$

$$w_{eff} = -1 - \frac{\bar{\mathcal{U}}_1}{3\,\bar{\mathcal{U}}_0} - (1-\alpha)\frac{\bar{\mathcal{U}}_1^2 - \bar{\mathcal{U}}_0(2\,\bar{\mathcal{U}}_2 + \bar{\mathcal{U}}_1)}{3\,\bar{\mathcal{U}}_0^2} + \ldots$$

From Planck+BAO+WP

$$w = w_0 + w_\alpha(1-\alpha) \to w_0 = -1.04^{+0.7}_{-0.7}, \quad w_\alpha < 1.3$$

$$\left|\frac{\bar{\mathcal{U}}_1}{\bar{\mathcal{U}}_0}\right| = 0.12^{+2.1}_{-2.1} \qquad \left|\frac{\bar{\mathcal{U}}_2}{\bar{\mathcal{U}}_0}\right| \leq 3.3$$

$$\frac{m_1^2}{m_4^2} = \frac{3\,\bar{\mathcal{U}}_1}{\bar{\mathcal{U}}_1 + 3\,\bar{\mathcal{U}}_0} + \ldots \sim 0.1^{+1.2}_{-5.9} \qquad \textit{for}\ \ \xi^i_{N^i} = 3/2\ (\textit{as in L.I.})$$