Dynamical features of scalar-torsion theories

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We investigate the cosmological dynamics in teleparallel gravity with nonminimal coupling. We analytically extract several asymptotic solutions and we numerically study the exact phase-space behavior. Comparing the obtained results with the corresponding behavior of nonminimal scalar-curvature theory, we find significant differences, such is the rare stability and the frequent presence of oscillatory behavior.

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I. INTRODUCTION

Gravitational modification is one of the main directions one can follow in order to describe the late-time universe acceleration and/or the early-time inflationary stage (see for instance [1, 2]), which has the potential advantage of avoiding introducing exotic fields and the concept of dark energy (see [3, 4] and references therein). Definitely one should note that, apart from the different physical interpretation, one can transform from one approach to the other, partially or completely, keeping track only of the number of extra degrees of freedom [5]. Thus, one can have various combined scenarios, with nonminimal couplings between gravity and scalar fields being the most used class.

Speaking of modified gravity, a natural question arises, namely what formulation of gravity to use as a basis of modification. The usual approach on the literature is to start from the standard curvature gravitational formulation, that is from the Einstein-Hilbert action of General Relativity, and extend it in various ways [1, 2]. However, a different but still very interesting class of modified gravity could arise starting from torsional formulations of General Relativity (GR). In particular, it is well known that Einstein also constructed the "Teleparallel Equivalent of General Relativity" (TEGR), where gravity is described not by the curvature tensor but by the torsion one [6–10]. The Lagrangian of this theory is given by contractions of this torsion tensor, namely the torsion scalar T, in a similar way that the Lagrangian of General Relativity is given by the Ricci scalar R, that is from contractions of the curvature tensor. Hence, instead of modifying GR one could try to modify TEGR. The most interesting feature is that although GR coincides with TEGR at the level of equations, their modifications do not, and thus they correspond to different classes of gravitational modification.

The simplest modification of TEGR is to replace T with $f(T)$ in the action, resulting to $f(T)$ gravity [11, 12], in a similar way with the $f(R)$ modification of GR. Since $f(T)$ gravity has no known curvature equivalent and is a novel modified class, its cosmological [12–14] and black hole [15] applications have attracted significant interest. A next extension of TEGR arises if one construct and use higher-order torsion invariants, in a similar way to the use of higher-order curvature invariants in GR modifications. Thus, constructing the teleparallel equivalent T_G of the Gauss-Bonnet term G, one can build the $f(T,T_G)$ paradigm [16], which is not spanned by the $f(R,G)$ class and thus is a novel gravitational modification. Furthermore, one could extend TEGR to $f\left(T,\mathcal{L}_m\right)$ scenario [17], with $\mathcal{L}_m$ the matter Lagrangian, or to $f(T,\mathcal{T})$ theory [18], with $\mathcal{T}$ the trace of the energy-momentum tensor, inspired respectively by the $f\left(R,\mathcal{L}_m\right)$ [19, 20] and $f(R,\mathcal{T})$ [21, 22] extensions of curvature-based gravity, where again both these theories are different from their curvature counterparts.

One could proceed further, and introduce nonminimal couplings in the framework of TEGR, in a similar way that one introduces these couplings in GR [23]. Thus, in [24] we formulated the scenario of "teleparallel dark energy", in which T is coupled to an extra scalar, and as expected this scalar-torsion theory is different from scalar-(curvature)tensor model, that is from nonminimal quintessence, with interesting phenomenology [25–29]. Since scalar-torsion theories are different than scalar-(curvature)tensor ones, in the present work we are interested in exploring various features of the cosmological dynamics of both theories, and investigate their differences and similarities. Indeed, we do find that the behavior of the phase space is different in the two constructions, and amongst others we find that the scalar-torsion can be free from run-away solutions, which are typical for the standard curvature-based theory of nonminimally coupled scalar fields.

The plan of the manuscript is outlined as follows: In Section II we briefly present teleparallel equivalent of general relativity and its scalar-torsion extension. In Sec-

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tion III we perform a detailed dynamical analysis, while in Section IV we compare the obtained results with the corresponding behavior of nonminimal scalar-curvature theory. Finally, Section V is devoted to the conclusions.

II. TELEPARALLEL AND SCALAR-TORSION THEORY

In this section we briefly review the teleparallel formulation of General Relativity and its modifications. In this construction the dynamical variables are the vierbein fields $\mathbf{e}_A(x^\mu)$, which can be expressed in a coordinate basis as $\mathbf{e}_A = e^\mu_A \partial_\mu$ [1]. The vierbeins at each space-time point form an orthonormal basis for the tangent space, and they are related to the metric tensor through

$$g_{\mu\nu} = \eta_{AB}\, e^A_\mu\, e^B_\nu, \tag{1}$$

where $\eta_{AB} = \mathrm{diag}(1,-1,-1,-1)$. Concerning the independent object that defines the parallel transportation, i.e. the connection, we use the Weitzenböck one $\overset{\mathbf{w}}{\Gamma}{}^\lambda_{\nu\mu} \equiv e^\lambda_A\, \partial_\mu e^A_\nu$ [30], which leads to zero curvature. Thus, the gravitational field is described by the torsion tensor, which reads

$$T^\lambda_{\mu\nu} = \overset{\mathbf{w}}{\Gamma}{}^\lambda_{\nu\mu} - \overset{\mathbf{w}}{\Gamma}{}^\lambda_{\mu\nu} = e^\lambda_A\, (\partial_\mu e^A_\nu - \partial_\nu e^A_\mu). \tag{2}$$

In particular, constructing the torsion scalar T as

$$T \equiv \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^{\rho} T^{\nu\mu}{}_{\nu}, \tag{3}$$

and using it as a Lagrangian, variation with respect to the vierbeins leads to exactly the same equations with General Relativity [6–10]. That is why Einstein named this theory "Teleparallel Equivalent of General Relativity" (TEGR).

One can be based on TEGR and start constructing various extensions. As we discussed in the Introduction, the simplest extension is to replace T by $T + f(T)$ in the Lagrangian, i.e. formulating $f(T)$ gravity. Introducing also the matter sector, the total action will be

$$S = \frac{1}{2\kappa^2} \int d^4x e\, [T + f(T)] + S_m, \tag{4}$$

where S_m is the matter action, $e = \det(e^A_\mu) = \sqrt{-g}$ and κ^2 is the gravitational constant (we set the light speed to one). Variation in terms of the vierbeins leads to

$$\begin{aligned} & e^{-1}\partial_\mu (e e^\rho_A S_\rho{}^{\mu\nu})[1 + f_T] + e^\rho_A S_\rho{}^{\mu\nu} \partial_\mu(T) f_{TT} \\ & \quad -[1 + f_T] e^\lambda_A T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} + \frac{1}{4} e^\nu_A [T + f(T)] \\ & = \frac{\kappa^2}{2}\, e^\rho_A \overset{em}{T}{}_\rho{}^\nu, \end{aligned} \tag{5}$$

where $f_T = \partial f/\partial T$, $f_{TT} = \partial^2 f/\partial T^2$, $\overset{em}{T}{}_\rho{}^\nu$ denotes the matter energy-momentum tensor, and we have defined the convenient tensor $S_\rho{}^{\mu\nu} \equiv \frac{1}{2}\Big(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha\Big)$, constructed with the help of the contorsion tensor $K^{\mu\nu}{}_\rho \equiv -\frac{1}{2}\Big(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\Big)$. We stress that although for $f(T) = $ const. one recovers TEGR with a cosmological constant, and thus GR with a cosmological constant, for $f(T) \neq$ const. the theory is different from $f(R)$ gravity. Thus, $f(T)$ gravity is a new gravitational modification and that is why it has attracted a significant interest in the literature [12–15].

An alternative extension of TEGR is to introduce a scalar field nonminimally coupled with T, in a similar way that in curvature-based gravity one introduces a scalar field nonminimally coupled to R. In particular, the total action will be [24]

$$S = \int d^4x\, e\, \left[\frac{T}{2\,\kappa^2} + \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - V(\phi) + \frac{\xi}{2}\,B(\phi)\,T + \mathcal{L}_m\right], \tag{6}$$

where ϕ is a canonical scalar field, $V(\phi)$ its potential, and $B(\phi)$ its arbitrary nonminimal coupling with the torsion scalar T. Variation with respect to the vierbein yields the coupled field equation

$$\begin{aligned} & \left[\frac{2}{\kappa^2} + 2\xi\, B(\phi)\right] \left[e^{-1}\partial_\mu (e e^\rho_A S_\rho{}^{\mu\nu}) - e^\lambda_A T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} + \frac{1}{4} e^\nu_A T\right] \\ & \qquad - e^\nu_A \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] + e^\mu_A \partial^\nu\phi\partial_\mu\phi, \\ & \qquad + 2\xi e^\rho_A S_\rho{}^{\mu\nu} B'(\phi)\,(\partial_\mu\phi) = e^\rho_A \overset{\mathbf{em}}{T}{}_\rho{}^\nu, \end{aligned} \tag{7}$$

where primes denote the derivative of a function with respect to its argument.

In order to focus on the cosmological application of this theory, we impose the vierbein ansatz

$$e^A_\mu = \mathrm{diag}(1, a(t), a(t), a(t)), \tag{8}$$

which leads to the flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a^2(t)\,\delta_{ij} dx^i dx^j\,, \tag{9}$$

where $a(t)$ is the scale factor. Thus, with this vierbein ansantz, the equations of motion (7) give rise to the modified Friedmann equations

$$3H^2 = \kappa^2 \left[\frac{\dot{\phi}^2}{2} + V(\phi) - 3\xi H^2 B(\phi) + \rho_m\right], \tag{10}$$

$$2\dot{H} = -\kappa^2 \left[\dot{\phi}^2 + 2\xi H B'(\phi)\dot{\phi} + 2\xi \dot{H} B(\phi) + \rho_m(1 + \omega_m)\right], \tag{11}$$

where $H = \dot{a}/a$ is the Hubble function, and dots denote differentiation with respect to t. Note that in the above relations we have used the relation $T = -6H^2$,

[1] Greek indices run over coordinate space-time, while capital Latin indices span the tangent space-time.

which holds for the vierbein choice (8). Additionally, we have considered the matter Lagrangian to correspond to a perfect fluid with energy density and pressure ρ_m and p_m respectively, and we have defined its equation-of-state parameter to be $w_m \equiv p_m/\rho_m$. Finally, varying the action (6) with respect to ϕ, and imposing the FRW ansatz, we acquire the scalar field equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + 3\xi H^2 B'(\phi) + V'(\phi) = 0. \tag{12}$$

Lastly, note that, as expected, for $\xi = 0$ the scenario at hand coincides completely with standard quintessence, since in this case the scalar field is minimally coupled to TEGR, which in turn coincides with General Relativity.

III. DYNAMICAL ANALYSIS

In the cosmological application of any gravitational theory one can find many analytical solutions, and an infinite number of numerical ones. However, the most important issue is to investigate the global features of the dynamics, that is to extract information about the given cosmological model that is independent of the initial conditions and the specific evolution. This is obtained using the powerful method of dynamical analysis, which allows to examine in a systematic way all the possible asymptotic cosmological behaviours, that is all the possibilities of the universe behaviour at late times. In particular, if stable late-time solutions are revealed, it is implied that the universe will result to them independently of the initial conditions and the model parameters.

The phase-space and stability analysis is performed by transforming the given cosmological model into its autonomous form, which in general will be of the form $d\mathbf{Y}/d\ln a = \mathbf{f(Y)}$, where $\mathbf{Y}$ is a vector constituted by suitable chosen variables and $\mathbf{f(Y)}$ the corresponding vector of the autonomous equations [31, 32]. Then the critical points $\mathbf{Y_c}$ of this dynamical system are extracted imposing the condition $d\mathbf{Y}/d\ln a = 0$. Thus, in order to examine the stability of these critical points one expands the system around $\mathbf{Y_c}$ as $\mathbf{Y} = \mathbf{Y_c} + \mathbf{U}$, with $\mathbf{U}$ the column vector of the variable perturbations, and for each critical point he expands the equations for the perturbations up to first order as $\mathbf{U}' = \mathbf{Q}\cdot\mathbf{U}$, where the matrix $\mathbf{Q}$ contains all the coefficients of the perturbation equations. Hence, the stability properties and the type of a specific critical point are determined by the eigenvalues of $\mathbf{Q}$, namely if all eigenvalues have positive real parts then this point will be unstable, if they all have negative real parts then it will be stable, while if they change sign it will be a saddle point.

Let us apply the above method to the scalar-torsion cosmology that was presented in the previous section. As an example we will focus on power-law potentials and power-law coupling functions, since the exponential cases have been investigated elsewhere [25, 26]. In particular, in the following we consider the case where

$$V(\phi) = V_0\phi^n \tag{13}$$

and

$$B(\phi) = \phi^N, \tag{14}$$

where we focus on even n in order for the potential to be non-negative. For convenience we choose $N > 0$, although the incorporation of the $N < 0$ case is straightforward. Additionally, since for $\xi = 0$ the scenario at hand coincides completely with standard quintessence, in the following we focus on the case of interest of this work, that is on $\xi \neq 0$. The dynamical analysis proves to be different for the cases $N \neq 2$ and $N = 2$, and thus in the following subsections we examine these cases separately.

A. $N \neq 2$

In order to proceed, and as we described above, we introduce the following dimensionless auxiliary variables:

$$x = \frac{\kappa^2\dot{\phi}^2}{6H^2[1+\kappa^2\xi B(\phi)]} \tag{15}$$

$$y = \frac{\kappa^2 V(\phi)}{3H^2[1+\kappa^2\xi B(\phi)]} \tag{16}$$

$$z = \frac{\kappa^2 \rho_m}{3H^2[1+\kappa^2\xi B(\phi)]} \tag{17}$$

$$m = \frac{\dot{\phi}}{H\phi} \tag{18}$$

$$A = \frac{B'(\phi)\phi}{1+\kappa^2\xi B(\phi)} = \frac{N}{1/\phi^N + \kappa^2\xi}. \tag{19}$$

Note the useful relation between A, x, m, namely

$$6Nx(N-\kappa^2\xi A)^{\frac{2-N}{N}} = \kappa^2 A^{\frac{2}{N}} m^2. \tag{20}$$

Furthermore, it proves convenient to additionally define the following functions, where in our case become just dimensionless parameters:

$$b = \frac{B''(\phi)\phi}{B'(\phi)} = N-1 \tag{21}$$

$$c = \frac{V'(\phi)\phi}{V(\phi)} = n. \tag{22}$$

In terms of the auxiliary variables the first Friedmann equation (10) gives rise to the following constraint

$$1 = x + y + z, \tag{23}$$

while using additionally the above parameters and the scalar-field evolution equation (12), we can obtain the following expressions for the physically interesting quantities:

$$\frac{\ddot{\phi}}{H\dot{\phi}} = -3 - \frac{m}{2x}(\kappa^2\xi A + ny) \tag{24}$$

$$\frac{\dot{H}}{H^2} = -3x - \kappa^2\xi A m. \tag{25}$$

Since we are interested in investigating the dynamical features of the pure modified gravitational sector, we focus on the vacuum case $\rho_m = 0$. Hence $z = 0$ and the dimensionality of the phase space is reduced by one. In this case, the cosmological equations of the scenario at hand are finally transformed to their autonomous form

$$\frac{dm}{d\ln a} = -\frac{3N(\kappa^2\xi A+n)(N-\kappa^2\xi A)^{\frac{2-N}{N}}}{\kappa^2 A^{\frac{2}{N}}} + \frac{\kappa^2 A^{\frac{2}{N}} m^3}{2N(N-\kappa^2\xi A)^{\frac{2-N}{N}}} + m^2\left(\frac{n}{2}+\kappa^2\xi A-1\right) - 3m \qquad (26)$$

$$\frac{dA}{d\ln a} = Am(N-\kappa^2\xi A), \qquad (27)$$

where we have replaced $b+1 = N$ as it arises from (21).

The system (26),(27) admits only one real critical point:

1. *Point* P_1: $m = 0$, $A = -\frac{n}{\kappa^2\xi}$

The critical point P_1 corresponds to $m = 0$, $A = -\frac{n}{\kappa^2\xi}$, which according to (20) leads to $x = 0$, and then using (23) to $y = 1$. Since $A =$ const. from (19) we deduce that the scalar field is also constant at P_1, namely

$$\phi^N = {\phi_0}^N = -\frac{n}{\kappa^2\xi(N+n)}, \qquad (28)$$

while according to (25) we find that $\dot{H}/H^2$ is zero. Thus, knowing additionally that y is non-zero, from (16) we deduce that the Hubble function obtains a constant non-zero value at P_1, namely

$$H = H_0 = \sqrt{-\frac{nV_0{\phi_0}^{n-N}}{3N\xi}}, \qquad (29)$$

and thus the universe is in a de Sitter phase where the scale factor expands exponentially as

$$a(t) = a_0 e^{H_0(t-t_0)}, \qquad (30)$$

where a_0,t_0 are constants. We mention that the de Sitter fixed point always exist for N odd, while for N even it exists only for $\xi < 0$.

Linearizing the perturbation equations around P_1 and extracting the eigenvalues of the corresponding perturbation matrix, as we discussed earlier, we find:

$$\lambda_{1,2} = -\frac{3}{2} \pm \frac{\sqrt{3}}{2\kappa^2}\left(-\frac{n}{\kappa^2\xi}\right)^{-\frac{N}{2}} \cdot \left\{\frac{n\kappa^2}{\xi}\left[4\xi N\left(-\frac{n}{\kappa^2\xi}\right)^{\frac{N}{2}}(N+n)^{\frac{2}{N}} - 3\left(-\frac{n}{\kappa^2\xi}\right)^{N-1}\right]\right\}^{\frac{1}{2}}. \qquad (31)$$

Due to the complexity of the above expressions it is not possible to derive analytical results for the signs of the real parts of these eigenvalues. Thus, we perform a numerical scanning in the parameter space.

In Figs. 1 and 2 we present the stability regions in the parameter subspace (n,ξ) for $N = 2$ and $N = 4$ respectively. Since for $\xi < 0$, $n < 0$ the de Sitter solution exists only for negative ϕ_0^{n-N} (see Eq. (29)), we restrict ourselves by odd integer $|n|$, in other quadrants n can be a continious variable (in white zones the de Sitter solution does not exist). As we observe, in the case where both N and n are positive we find that there is not any stability region, and thus point P_1 is saddle. However, in the case where $n < 0$, and for $N > 2$, we do find stability regions, that is in this case the de Sitter point P_1 is stable. We do not consider negative n further in the present paper.

B. $N = 2$

In this case we also use the auxiliary variables (15)-(19), the constraint (23) remains the same, but now relation (20) simplifies to

$$x = \frac{\kappa^2 A m^2}{12}. \qquad (32)$$

Moreover, instead of (24),(25) we now have

$$\frac{\ddot{\phi}}{H\dot{\phi}} = -3 - 6\frac{\xi}{m} - 6\frac{n}{\kappa^2 m A} + \frac{nm}{2} \qquad (33)$$

$$\frac{\dot{H}}{H^2} = -\frac{\kappa^2 A m^2}{4} - \kappa^2\xi A m. \qquad (34)$$

Hence, the autonomous form of the cosmological system becomes

$$\frac{dm}{d\ln a} = -3m - 6\xi - \frac{6n}{\kappa^2 A} + \frac{\kappa^2 A m^3}{4} + m^2\left(\frac{n}{2}+\kappa^2\xi A-1\right) \qquad (35)$$

$$\frac{dA}{d\ln a} = Am(2-\kappa^2\xi A). \qquad (36)$$

The system (35),(36) admits four real critical points:

1. *Point* Q_1: $m = 0$, $A = -\frac{n}{\kappa^2\xi}$

The first critical point Q_1 corresponds to $m = 0$, $A = -\frac{n}{\kappa^2\xi}$, which according to (32) leads to $x = 0$, and then using (23) gives $y = 1$ (again we focus on the case $\rho_m = 0$ i.e. $z = 0$). Hence, this point can be obtained from point P_1 of the previous subsection, setting $N = 2$. In particular, from the definition of A in (19) we find that the scalar field is constant too, namely

$$\phi^2 = {\phi_0}^2 = -\frac{n}{\kappa^2\xi(2+n)}, \qquad (37)$$

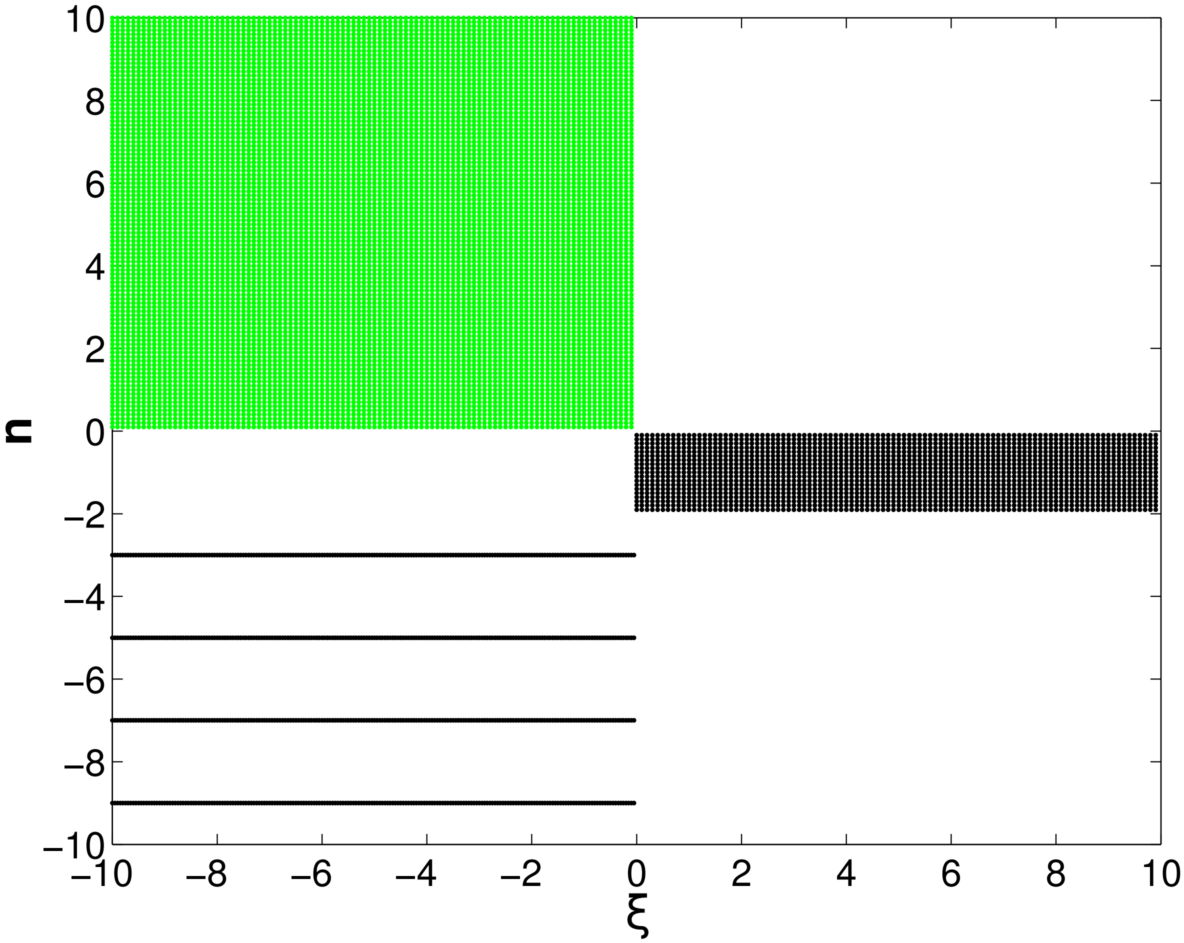

FIG. 1: *Stability regions of de Sitter point P_1 in the parameter subspace (n, ξ) for $N = 2$. Green regions correspond to saddle behavior, while black regions denote stable behavior.*

while from (34) we deduce that $\dot{H}/H^2$ is zero. Thus, knowing additionally that y is non-zero, from (16) we deduce that the Hubble function obtains a constant non-zero value at Q_1, namely

$$H = H_0 = \sqrt{-\frac{nV_0{\phi_0}^{n-2}}{6\xi}}. \tag{38}$$

Therefore, the universe is in a de Sitter phase where the scale factor expands exponentially as

$$a(t) = a_0 e^{H_0(t-t_0)}. \tag{39}$$

Note that this de Sitter fixed point exists only for $\xi < 0$.

The eigenvalues of the perturbation matrix can be obtained from (31) setting $N = 2$, and thus we have

$$\lambda_{1,2} = -\frac{3}{2} \pm \frac{1}{2}\sqrt{9 - 24\xi(n+2)}. \tag{40}$$

Therefore, we can easily see that this point is always a saddle one.

2. *Point Q_2: $m = \sqrt{6}\xi$, $A = \frac{2}{\kappa^2\xi}$*

The second critical point Q_2 exists only for $\xi > 0$ and corresponds to $m = \sqrt{6}\xi$, $A = \frac{2}{\kappa^2\xi}$, and thus according to (32) we get $x = 1$, and then using (23) we obtain $y = 0$. Therefore, from (34) we find that

$$\frac{\dot{H}}{H^2} = -3 - 2\sqrt{6}\xi, \tag{41}$$

which leads to

$$a(t) = a_0|t - t_0|^{\frac{1}{3+2\sqrt{6}\xi}}, \tag{42}$$

where a_0,t_0 are constants. Note that at $t \to \infty$ both H and $\dot{H}$ tend to zero ($H \sim t^{-1}$ and $\dot{H} \sim t^{-2}$), but with the ratio $H/\dot{H}^2$ being the constant given in (41). Inserting the scale-factor expression into the m-definition in (18), we extract the scalar-field solution as

$$\phi(t) = \phi_0|t - t_0|^{\frac{\sqrt{6}\xi}{3+2\sqrt{6}\xi}}. \tag{43}$$

Finally, substituting the coordinates of Q_2 into (19) we deduce that $\phi \to \infty$, and thus the only case is when

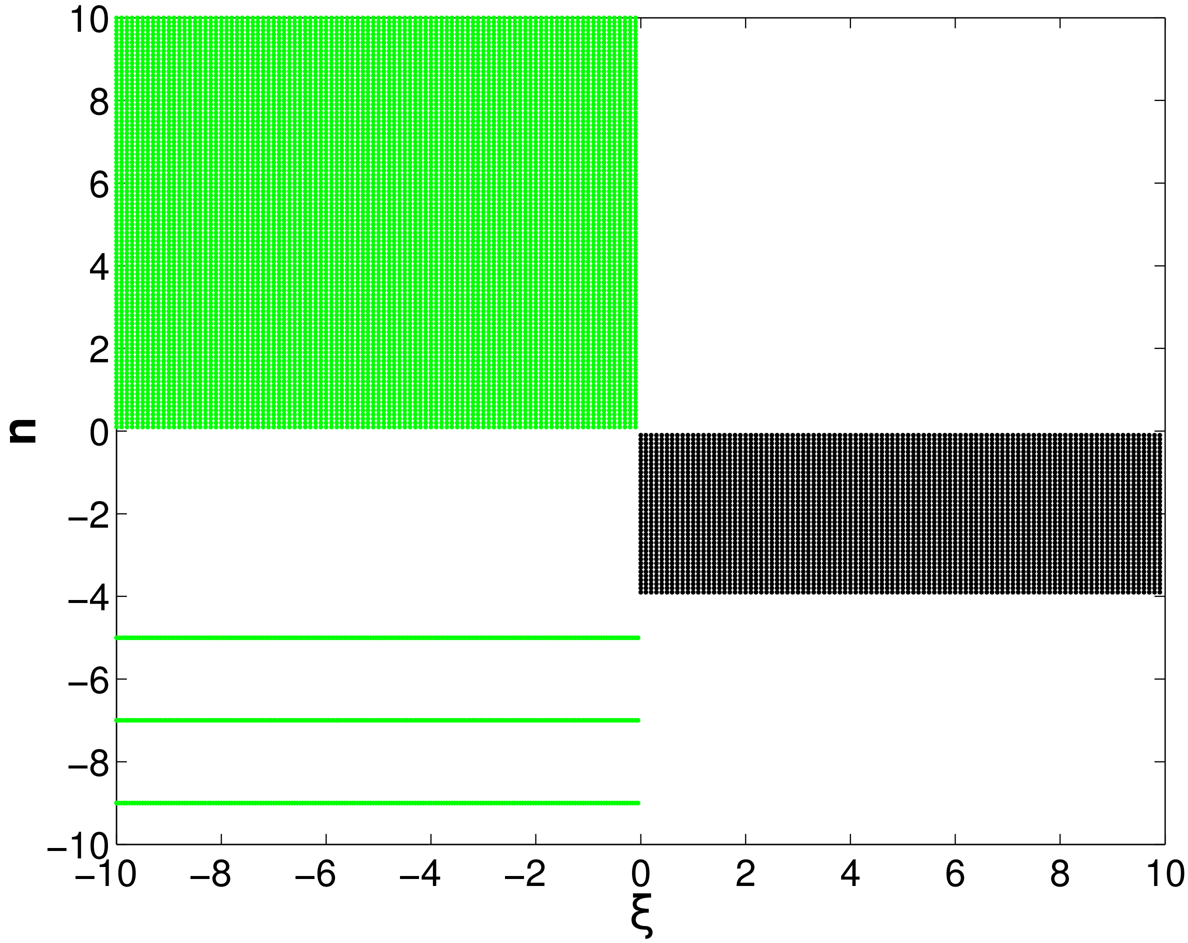

FIG. 2: *Stability regions of de Sitter point P_1 in the parameter subspace (n, ξ) for $N = 4$. Green regions correspond to saddle behavior, while black regions denote stable behavior.*

$t_0 < t \to \infty$. Hence point Q_2 cannot give rise to a Big Rip [33].

Lastly, substituting these expressions into the y-definition in (16), we deduce that in order to have self-consistency the potential must have $n < -2$, which is not of interest for the scope of the present work.

3. Point Q_3: $m = -\sqrt{6\xi}$, $A = \frac{2}{\kappa^2\xi}$

The third critical point Q_3 exists only for $\xi > 0$ and corresponds to $m = -\sqrt{6\xi}$, $A = \frac{2}{\kappa^2\xi}$. Thus, from (32) we acquire $x = 1$, and then from (23) we get $y = 0$. From (34) we find that

$$\frac{\dot{H}}{H^2} = -3 + 2\sqrt{6\xi}, \tag{44}$$

which, for $\xi \neq 3/8$, leads to

$$a(t) = a_0|t - t_0|^{\frac{1}{3-2\sqrt{6\xi}}}, \tag{45}$$

where a_0,t_0 are constants. Note that at $t \to \infty$ both H and $\dot{H}$ tend to zero ($H \sim t^{-1}$ and $\dot{H} \sim t^{-2}$), but with the ratio $H/\dot{H}^2$ being the constant given in (44). Inserting the scale-factor expression into the m-definition in (18), we obtain the scalar-field solution as

$$\phi(t) = \phi_0|t - t_0|^{\frac{\sqrt{6\xi}}{2\sqrt{6\xi}-3}}. \tag{46}$$

Substituting these expressions into (16), and contrary to point Q_2, we deduce that self-consistency is obtained for both positive and negative values of n. Restricting ourselves to the positive-n cases, we extract the following requirements for the existence of this critical point:

$$\begin{aligned} &\text{For } n < 2 \text{ it exists for} \quad && \xi \in \left(0; \frac{3}{8}\right) \cup \left(\frac{3}{8}; \frac{6}{(n+2)^2}\right) \\ &\text{For } 2 \leq n \text{ it exists for} \quad && 0 < \xi < \frac{6}{(n+2)^2}. \end{aligned}$$

Concerning its stability, the eigenvalues of the perturbation matrix are found to be

$$\begin{aligned} \lambda_1 &= 2\sqrt{6\xi} \\ \lambda_2 &= 6 - \sqrt{6\xi}(n+2), \end{aligned} \tag{47}$$

and thus Q_3 is always an unstable node, since $\sqrt{\xi}(n+2) < \sqrt{6}$.

Finally, in the case $\xi = 3/8$ with $n < 2$ equation (44) leads to the trivial solution $H = H_0 = const$, and thus for the scale factor we acquire

$$a(t) = a_0 e^{H_0(t-t_0)}, \tag{48}$$

while relation (18) leads to

$$\phi(t) = \phi_0 e^{-\sqrt{6\xi}H_0(t-t_0)}. \tag{49}$$

In this case the eigenvalues of the perturbation matrix are still given by (47) but for $\xi = 3/8$, and thus we deduce that Q_3 is an unstable node.

4. ***Point*** Q_4***:*** $m = -\xi(n+2)$, $A = \frac{2}{\kappa^2\xi}$

The fourth critical point corresponds to $m = -\xi(n+2)$, $A = \frac{2}{\kappa^2\xi}$, and thus according to (32) we get $x = \frac{\xi(n+2)^2}{6}$, and then using (23) we acquire $y = 1 - \frac{\xi(n+2)^2}{6}$. Hence, from (34) we find that

$$\frac{\dot{H}}{H^2} = \frac{-\xi(n+2)(n-2)}{2}. \tag{50}$$

For $n \neq 2$ the above equation leads to

$$a(t) = a_0|t - t_0|^{\frac{2}{\xi(n^2-4)}}, \tag{51}$$

where t_0 is a constant. Then, inserting this expression into the m-definition in (18), we extract the scalar-field solution as

$$\phi(t) = \phi_0|t - t_0|^{\frac{2}{2-n}}. \tag{52}$$

As $A = \frac{2}{\kappa^2\xi}$ then $\phi \to \infty$ at this stationary point.

Substituting found solution (51), (52) to the initial system (10), (11), (12) we get corresponding conditions of existence ($V_0 < 0$ is not considered in this paper):
for $N = 2$ and
1). $n \neq 2$ — even, $V_0 > 0$, $0 < \xi < \frac{6}{(n+2)^2}$, $\forall\phi_0$
2). $n \neq 2$ — odd, $V_0 > 0$, $0 < \xi < \frac{6}{(n+2)^2}$, $\phi_0 > 0$ or $V_0 > 0$, $\xi \in (-\infty; 0) \cup (\frac{6}{(n+2)^2}; +\infty)$, $\phi_0 < 0$.

The eigenvalues of the perturbation matrix for Q_4 read

$$\begin{aligned}\lambda_1 &= 2\xi(n+2)\\ \lambda_2 &= -3 + \frac{\xi(n+2)^2}{2},\end{aligned} \tag{53}$$

This point for $\xi > \frac{6}{(n+2)^2}$ — an unstable node, for $0 < \xi < \frac{6}{(n+2)^2}$ — a saddle, for $\xi < 0$ – a stable node.

For $n = 2$, equation (50) does not accept the solution (51), but it has the trivial solution $H = H_0 = const$, which inserted into y-definition relation (16) gives $H_0^2 = \frac{V_0}{\xi(3-8\xi)}$. Since we focus on positive potential, i.e. with $V_0 > 0$, we deduce that this solution exist for $0 < \xi < 3/8$. Hence, for the scale factor we obtain

$$a(t) = a_0 e^{\sqrt{\frac{V_0}{\xi(3-8\xi)}}(t-t_0)}, \tag{54}$$

while relation (18) gives

$$\phi(t) = \phi_0 e^{\sqrt{\frac{16V_0\xi}{(3-8\xi)}}(t-t_0)}. \tag{55}$$

In this case the eigenvalues of the perturbation matrix are still given by (53) but for $n = 2$, and since $0 < \xi < 3/8$ we deduce that Q_4 for $n = 2$ is always a saddle point.

IV. COMPARISON BETWEEN SCALAR-TORSION AND SCALAR-CURVATURE BEHAVIOR

In the previous section we performed the dynamical analysis of some general scalar-torsion models. In this section we present specific figures in the phase space, and we discuss the physical features of the obtained cosmology. Then, we compare these results with the known behavior of the corresponding scalar-curvature scenarios [34, 35]. Since in scalar-curvature models only the case of positive ξ has been studied in detail, in the following we focus on this case too. Similar to the scalar-curvature models the important regime of dumped scalar field oscillations can not be extracted from fixed points analysis for the set of variables we used.

As it is usual in the majority of cosmological scenarios, the scalar field and the scale factor diverge at the critical points, and therefore for presentation reasons it proves more convenient to define suitable compact variables, projecting the dynamics in the unit circle. In particular, we define

$$\alpha = \frac{\varphi}{\sqrt{1+\varphi^2+\dot{\varphi}^2}}, \qquad \beta = \frac{\dot{\varphi}}{\sqrt{1+\varphi^2+\dot{\varphi}^2}}, \tag{56}$$

and thus the inverse transformation reads

$$\varphi = \frac{\alpha}{\sqrt{1-\alpha^2-\beta^2}}, \qquad \dot{\varphi} = \frac{\beta}{\sqrt{1-\alpha^2-\beta^2}}. \tag{57}$$

Obviously we have

$$\begin{aligned}\dot{\alpha} &= \beta\left(1-\alpha^2-\alpha\ddot{\varphi}\sqrt{1-\alpha^2-\beta^2}\right),\\ \dot{\beta} &= \ddot{\varphi}\sqrt{1-\alpha^2-\beta^2}(1-\beta^2)-\alpha\beta^2.\end{aligned} \tag{58}$$

A. $N \neq 2$

We start by examining the phase portraits for the $N \neq 2$ case. As we discussed in the previous section, the only critical point in this case is the de Sitter point P_1, but since it exists only for $\xi < 0$ we are not going to discuss it in detail. Hence, remaining in the case $\xi > 0$ we conclude that the only possible evolution is that of scalar-field oscillations, and the global picture of such dynamics does not depend on the particular value of ξ. In order to present this behavior more transparently, in Fig. 3 we depict the phase-space evolution in the (α, β) plane, in the case where $N = 4$, $n = 2$, $\xi = 1$, with $V_0 = 1$, $\kappa^2 = 1$. Note that this behavior is similar to that in scalar-curvature case [34, 35].

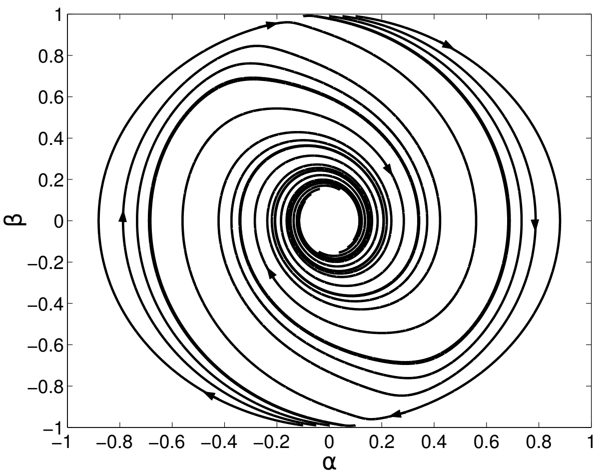

FIG. 3: *The projection of the phase-space evolution of the system (26),(27) on the (α, β) plane, in the case where $N = 4$, $n = 2$, $\xi = 1$, with $V_0 = 1$, $\kappa^2 = 1$, i.e. in the case of quartic nonminimal coupling function and quadratic potential. There are no stable fixed points, and the scalar field exhibits oscillations.*

B. $N = 2$

We now examine the more interesting $N = 2$ case. For this quadratic coupling function, the quadratic potential $n = 2$ is an exceptional one, since in this case the obtained solutions are not power laws but exponentials. Having in mind the existence and the stability conditions of points Q_1 to Q_4 of subsection III B, we deduce that when $n = 2$, two different types of cosmological dynamics are possible:

- In the case where the coupling parameter is small, namely $0 < \xi < 3/8$, only two critical points exist, namely Q_3 and Q_4. The evolution starts with a massless regime (point Q_3 which is unstable node) and ultimately results to $(0, 0)$ point, except for a measure-zero set of initial conditions which ends in the saddle point Q_4 representing the exponential solution (54),(55). In Fig. 4 we depict the phase-space evolution in the (α, β) plane, in the case where $N = 2$, $n = 2$, $\xi = 1/4$, with $V_0 = 1$, $\kappa^2 = 1$.

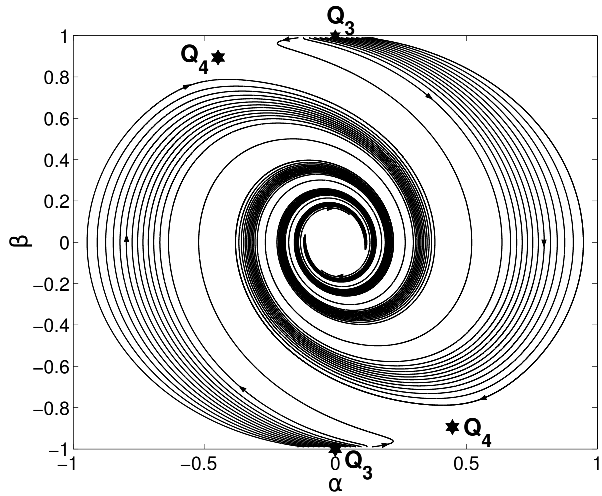

FIG. 4: *The projection of the phase-space evolution of the system (35),(36) on the (α, β) plane, in the case where $N = 2$, $n = 2$, $\xi = 1/4$, with $V_0 = 1$, $\kappa^2 = 1$, i.e. in the case of quadratic nonminimal coupling function and quadratic potential.*

- In the case where the coupling parameter is large, namely $\xi > 3/8$, no critical point exist, and thus the scalar field exhibits oscillations. This behavior is depicted in Fig. 5, for the case of $N = 2$, $n = 2$, $\xi = 4$, with $V_0 = 1$, $\kappa^2 = 1$.

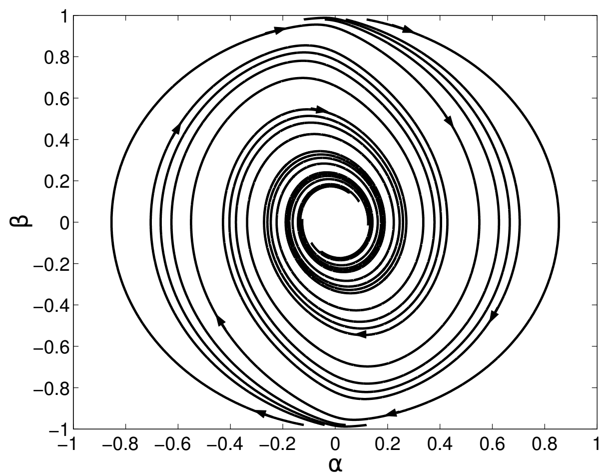

FIG. 5: *The projection of the phase-space evolution of the system (35),(36) on the (α, β) plane, in the case where $N = 2$, $n = 2$, $\xi = 4$, with $V_0 = 1$, $\kappa^2 = 1$, i.e. in the case of quadratic nonminimal coupling function and quadratic potential. There are no stable fixed points, and the scalar field exhibits oscillations.*

Let us now examine the $n > 2$ case. In this case, the solutions, when they exist, are power laws instead of exponentials. Having in mind the existence and the stability conditions of points Q_1 to Q_4 of subsection III B, we deduce that two different types of cosmological dynamics are possible:

- In the case of small coupling parameter, namely

$\xi < 6/(n+2)^2$, the phase-space trajectories begin from the unstable node Q_3 and tend to dumping oscillations near $(0,0)$ point, since Q_4 is saddle. This behavior can be observed in Fig. 6, in the case where $N=2$, $n=4$, $\xi=1/10$, with $V_0=1$, $\kappa^2=1$. Note that the points Q_3 and Q_4 coincide, making the resulting phase portrait less informative.

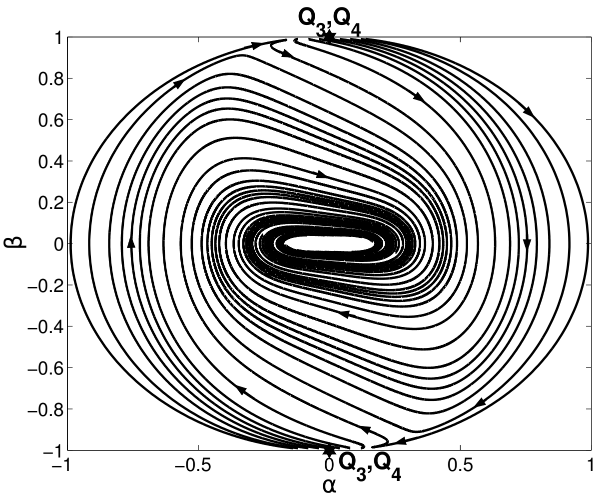

FIG. 6: *The projection of the phase-space evolution of the system (35),(36) on the (α,β) plane, in the case where $N=2$, $n=4$, $\xi=1/10$, with $V_0=1$, $\kappa^2=1$, i.e. in the case of quadratic nonminimal coupling function and quartic potential.*

- In the case of large coupling parameter, namely $\xi > 6/(n+2)^2$, the critical points do not exist, and thus the scalar field exhibits oscillations. This behavior can be seen in Fig. 7.

Let us now compare the above behavior with the corresponding one in the scalar-curvature case, which will help to reveal the special features of the nonminimal coupling in teleparallel gravity. We can easily remark two very important qualitative differences concerning possible future and past asymptotic in these two theories of non-minimally coupled scalar field.

Firstly, in the case of standard, scalar-curvature models with power-law potentials and coupling functions, the future scalar-field behavior can be of two possible types: either solutions with $\phi \to \infty$ or damped scalar field oscillations. The former behavior is realized for $n < 2N$ (in the present paper notations), and moreover it represents a Big Rip singularity if $N < n < 2N$ [34, 35]. On the contrary, in teleparallel nonminimal coupling with positive even N and n the only stable future asymptotics are the oscillations (note that in this case one needs to be based on the numerical investigation in order to study the phase-space behavior since the scalar field oscillations cannot be obtained analytically through the dynamical analysis). In some sense, the theory under investigation resembles better the theory with minimally coupled scalar field, where stable future regimes apart from oscillations are impossible for simple power-law potentials.

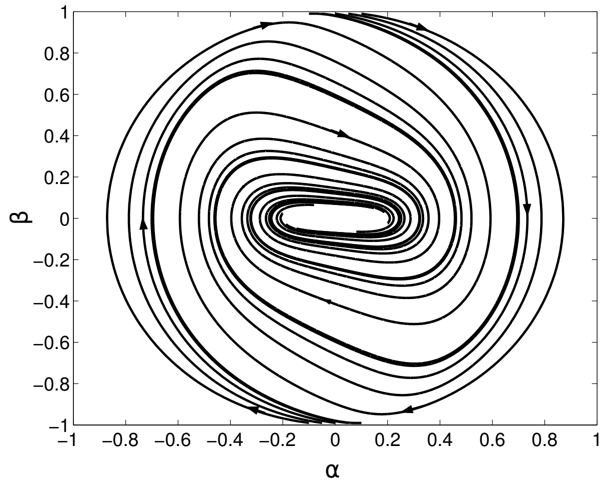

FIG. 7: *The projection of the phase-space evolution of the system (35),(36) on the (α,β) plane, in the case where $N=2$, $n=4$, $\xi=1$, with $V_0=1$, $\kappa^2=1$, i.e. in the case of quadratic nonminimal coupling function and quartic potential. There are no stable fixed points, and the scalar field exhibits oscillations.*

The reason behind this difference is quite clear. In the equation of motion for the scalar field the additional term originating from the non-minimal coupling, for even N has the same sign as the potential term for even n, and, as a result, can only enhance the driven force pushing the scalar field towards the minimum of its potential. On the contrary, the additional term in the case of curvature non-minimal coupling has a more complicated structure, with different signs, and under some conditions can push the scalar field into infinity.

Hence, it is evident that in order to make non-trivial future asymptotics in the present scenario, we need to balance the potential and correction terms and thus we need either to consider negative ξ or to match an increasing coupling function with a decreasing potential (or vice versa, however this case is more "exotic"). For instance, as we saw, for positive even N and n the fixed point P_1 corresponding to de Sitter solution exists only for negative ξ. The detailed study of such suitable constructed solutions, based on the interplay between the potential and the coupling function, lies beyond the scope of the present work and is left for a future investigation.

Secondly, concerning the past behavior, we also see that the past asymptotics in the scenario at hand are different from both minimal and nonminimal curvature theory. In both these scalar-curvature theories the source point on phase diagrams represents the massless field regime, existing if the potential is not too steep (exponentially steep for curvature minimal coupling [36] and $n < 5N$ for curvature non-minimal coupling [34, 35]). However, in teleparallel gravity with power-law potentials, this regime exists only if N does not exceed 2. For $N > 2$ (as well as for $N=2$, and $\xi > 6/(n+2)^2$) we have infinite scalar field oscillations near a singularity, similar to those in the theory of a minimally coupled field with exponential potentials. Finally, note that in the present scenario we find that the case of $N = n = 2$ is exceptional since only under this condition the unstable exponential solution exists. This feature has no analogies in the curvature theory.

V. CONCLUSIONS

In this work we investigated cosmological scenarios in the framework of teleparallel gravity with nonminimal coupling. Although in the case of absence of the scalar field, or in the case of minimal coupling, teleparallel gravity is completely equivalent with general relativity at the level of equations, when the nonminimal coupling is switched on the two theories become different, corresponding to distinct classes of gravitational modification. Hence, in order to reveal the differences between nonminimal scalar-curvature and nonminimal scalar-torsion theories, we use the powerful method of dynamical analysis, which allow us to study their global behavior without the need of extracting exact analytical solutions.

Focusing on the cases of power-law potentials and nonminimal coupling functions, we showed that contrary to the case of scalar-curvature gravity, teleparallel gravity has no stable future solutions for positive nonminimal coupling, and the scalar field results always in oscillations. Additionally, concerning the past behavior, while scalar-curvature exhibits a massless field regime for not too steep potentials, nonminimal scalar-torsion gravity exhibits this feature for not too steep coupling functions.

The reason behind these differences is the specific and simple form of the nonminimal correction in the scalar field equation in the case of nonminimal scalar-torsion theory, comparing to the more complicated corresponding term in nonminimal scalar-curvature theory. In particular, in the former case the correction term is relatively simple and thus it can lead to well-determined changes in the dynamics comparing to the minimal model, while in the latter case the correction term is complicated and thus it cannot lead to well-determined, one-way changes in the dynamics comparing to the minimal model. Hence, since minimal scalar-curvature is equivalent with minimal scalar-torsion gravity, nonminimal scalar-curvature gravity is relatively close to them, while nonminimal scalar-torsion gravity is radically different. The significant difference of the two theories was already known (for instance in the scalar-torsion case one can obtain the phantom regime, which is impossible in the scalar-curvature one [24, 25]), however in the present work we verified it analyzing in detail the dynamics of the pure scalar-gravity sector, in order to remove possible effects of the matter part.

Clearly, the torsional modification of gravity, and its coupling with scalar and matter sectors, brings novel and significant features, with no known curvature counterparts. Thus, it would be interesting to include these capabilities in the model-building of cosmological scenarios.

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