Status of Hořava Gravity

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based on
DV & T. P. Sotiriou, PRD **85***, 064003 (2012) [arXiv:1112.3385 [hep-th]]*
DV & T. P. Sotiriou, JPCS **453***, 012022 (2013) [arXiv:1212.4402 [hep-th]]*
DV, arXiv:1502.06607 [hep-th] (2015)

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Lovelock's Theorem

- *In 4 dimensions the most general 2-covariant divergence-free tensor, which is constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, is the Einstein tensor $G_{\mu\nu}$ plus a cosmological constant (CC) term $\Lambda g_{\mu\nu}$.*

- *This result suggests a natural route to Einstein's equations in vacuum:*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu}.$$

The Action

With the additional requirement that the eqs. for the gravitational field and the matter fields be derived by a diff.-invariant action, Lovelock's theorem singles out in 4 dimensions the action of GR with a CC term:

$$S_{GR} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_M[g_{\mu\nu}, \psi_M]\,.$$

The variation with respect to the metric gives rise to the field equations of GR in presence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N\, T_{\mu\nu}\,,$$

where

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}\,.$$

The Problems

- **GR is not a Renormalizable Theory**
  Renormalization at one-loop demands that GR should be supplemented by higher-order curvature terms, such as R^2 and $R_{\alpha\beta\sigma\gamma}R^{\alpha\beta\sigma\gamma}$ (Utiyama and De Witt '62). However such theories are not viable as they contain ghost degrees of freedom (Stelle '77).

- **The Cosmological Constant**
  The observed cosmological value for the CC is smaller than the value derived from particle physics at best by 60 orders of magnitude.

- **The Dark Side of the Universe**
  The most recent data tell us that about the 95% of the current Universe is made by unknown components, Dark Energy and Dark Matter.

Beyond General Relativity

- **Higher-Dimensional Spacetimes**
  One can expect that for any higher-dimensional theory, a 4-dimensional effective field theory can be derived in the IR, that is what we are interested in.

- **Adding Extra Fields (or Higher-Order Derivatives)**
  One can take into account the possibility to modify the gravitational action by considering more degrees of freedom. This can be achieved by adding extra dynamical fields or equivalently considering theories with higher-order derivatives.

- **Giving Up Diffeomorphism Invariance**
  Lorentz symmetry breaking can lead to a modification of the graviton propagator in the UV, thus rendering the theory power-counting renormalizable.

The Lifshitz Scalar

We take a scalar field theory whose action has the following form:

$$S_\phi = \int dt dx^d \left[\dot{\phi}^2 - \sum_{m=1}^{z} a_m \phi (-\Delta)^m \phi + \sum_{n=1}^{N} b_n \phi^n \right].$$

Space and time coordinates have the following dimensions in the units of spatial momentum p:

$$[dt] = [p]^{-z}\,, \qquad [dx] = [p]^{-1}\,.$$

A theory is said to be "power-counting renormalizable" if all of its interaction terms scale like momentum to some non-positive power, as in this case Feynman diagrams are expected to be convergent or have at most a logarithmic divergence.

The Lifshitz Scalar

The dimensions for the scalar field are then immediately derived to be

$$[\phi] = [p]^{(d-z)/2} .$$

Since the action has to be dimensionless, the interaction terms scale like momentum to some non-positive power when the couplings of these interaction terms scale like momentum to some non-negative power.

It can be easily verified that

$$[a_m] = [p]^{2(z-m)} , \qquad [b_n] = [p]^{d+z-n(d-z)/2} .$$

It follows that a_m *has non-negative momentum dimension for all values of* m*, while* b_n *for* $z \geq d$ *has non-negative momentum dimensions for all values of* n*.*

Dispersion Relations and Propagators

In general the dispersion relation one gets for such a Lorentz-violating field theory is of the following form

$$\omega^2 = m^2 + a_1 p^2 + \sum_{n=2}^{z} a_n \frac{p^{2n}}{P^{2n-2}} \,,$$

where P *is the momentum-scale suppressing the higher-order operators. The resulting Quantum Field Theory (QFT) propagator is then*

$$G(\omega, p) = \frac{1}{\omega^2 - \left[m^2 + a_1 p^2 + \sum_{n=2}^{z} a_n p^{2n} / P^{2n-2}\right]} \,.$$

The very rapid fall-off as $p \to \infty$ *improves the behaviour of the integrals one encounters in the QFT Feynman diagram calculations.*

Hořava's Proposal

- *In 2009, Hořava proposed an UV completion to GR modifying the graviton propagator by adding to the gravitational action higher-order spatial derivatives without adding higher-order time derivatives.*
- *This prescription requires a splitting of spacetime into space and time and leads to Lorentz violations.*
- *Lorentz violations in the IR are requested to stay below current experimental constraints.*

P. Hořava, JHEP **0903**, *020 (2009)*

P. Hořava, PRD **79**, *084008 (2009)*

Foundations of the Theory

The theory is constructed using the full ADM metric

$$ds^2 = N^2 dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

and it is invariant under foliation-preserving diffeomorphysms, i.e.,

$$t \to \tilde{t}(t), \qquad\qquad x^i \to \tilde{x}^i(t, x^i).$$

The most general action is

$$S_H = S_K - S_V.$$

The Kinetic Term

$$S_K = \frac{M_{\rm pl}^2}{2} \int dt d^3x \sqrt{h} N \left(K_{ij}K^{ij} - \lambda K^2\right).$$

Foundations of the Theory

The Potential Term

$$S_V = \frac{M_{\rm pl}^2}{2} \int dt d^3x \sqrt{h} N \left[L_2 + \frac{1}{M_4^2} L_4 + \frac{1}{M_6^4} L_6 \right].$$

- *Power-counting renormalizability requires as a minimal prescription at least 6th-order spatial derivatives in V.*
- *The most general potential V with operators up to 6th-order in derivatives, contains tens of terms $\sim \mathcal{O}(10^2)$.*
- *The theory propagates both a spin-2 and a spin-0 graviton.*

Foundations of the Theory

In the most general theory some of the terms that one can consider in the potential are:

$$L_2 = R,\ a_i a^i,$$

$$L_4 = R^2,\ R_{ij}R^{ij},\ R\nabla_i a^i,\ a_i \Delta a^i,\ (a_i a^i)^2,\ a_i a_j R^{ij},\ \ldots,$$

$$L_6 = (\nabla_i R_{jk})^2,\ (\nabla_i R)^2,\ \Delta R \nabla_i a^i,\ a_i \Delta^2 a^i,\ (a_i a^i)^3,\ \ldots,$$

where $a_i = \partial_i lnN$.

D. Blas, O. Pujolas & S. Sibiryakov, PRL **104**, *181302 (2010)*

Projectability & Detailed Balance

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$$N = N(t).$$

2. *We can impose as an additional symmetry to the theory the so called "***Detailed Balance***": it requires that V should be derivable from a superpotential W as follows:*

$$V = E^{ij} G_{ijkl} E^{kl} \,,$$

where E^{ij} is given in term of a superpotential W as

$$E^{ij} = \frac{1}{\sqrt{h}} \frac{\delta W \left[h_{kl} \right]}{\delta h_{ij}} \,.$$

The Superpotential W and the Action

The most general superpotential containing all of the possible terms up to 3rd-order in spatial derivatives is

$$W = \frac{M_{\rm pl}^2}{2M_6^2}\int \omega_3(\Gamma) + \frac{M_{\rm pl}^2}{M_4}\int d^3x\sqrt{h}\left[R - 2\xi(1-3\lambda)M_4^2\right],$$

where $\omega_3(\Gamma)$ is the gravitational Chern-Simons term. Then the action is

$$\begin{aligned} S_H \;=\; & \frac{M_{\rm pl}^2}{2}\int dt d^3x\sqrt{h}N\Bigg[K_{ij}K^{ij} - \lambda K^2 + \xi R - 2\Lambda - \frac{1}{M_4^2}R_{ij}R^{ij} \\ & + \frac{1-4\lambda}{4(1-3\lambda)}\frac{1}{M_4^2}R^2 + \frac{2}{M_6^2 M_4}\epsilon^{ijk}R_{il}\nabla_j R^l_k - \frac{1}{M_6^4}C_{ij}C^{ij}\Bigg]. \end{aligned}$$

The Sign of the Cosmological Constant

The bare CC is

$$\Lambda = \frac{3}{2}\xi^2(1-3\lambda)M_4^2\,.$$

If we want the theory to be close to General Relativity in the IR, then

$$\lambda, \xi \sim 1,$$

to high accuracy.

Therefore it is obvious that Λ has to be negative in this case!

The Size of the Cosmological Constant

The size of the cosmological constant is related to the size of the energy scale M_4 (suppressing the fourth order operators), at which Lorentz-violating effects will become manifest.
For Lorentz violations to have remained undetected in sub-mm precision tests, as an optimistic estimate one would need roughly

$$M_4 \geq 1 \div 10\mathrm{meV}.$$

Considering this mildest constraint coming from purely gravitational experiments, the value of the bare CC would be (roughly)

$$\Lambda \sim 10^{-60} M_{\mathrm{pl}}^4.$$

There is at best a 60 orders of magnitude discrepancy between the value required by detailed balance and the observed value!

Problems

Obvious:

1. *There is a parity violating term (the term which is 5th-order in derivatives).*
2. *The scalar mode does not satisfy a 6th-order dispersion relation and is not power-counting renormalizable.*
3. *The bare CC has the opposite sign and it has to be much larger than the observed value.*

T. P. Sotiriou, M. Visser & S. Weinfurtner, PRL **102**, *251601 (2009)*

C. Appignani, R. Casadio & S. Shankaranarayanan, JCAP **1004**, *006 (2010)*

Less Obvious:

4. *The IR behaviour of the scalar mode is plagued by instabilities and strong coupling at unacceptably low-energies.*

C. Charmousis, G. Niz, A. Padilla & P. M. Saffin, JHEP **0908**, *070 (2009)*

D. Blas, O. Pujolas & S. Sibiryakov, JHEP **0910**, *029 (2009)*

Projectable Version of the Theory without DB

$$S_P = \frac{M_{\rm pl}^2}{2}\int d^3x dt N\sqrt{h}\Big\{K^{ij}K_{ij} - \lambda K^2 - d_0\, M_{\rm pl}^2 - d_1 R - g_2\, M_{\rm pl}^{-2}\, R^2$$
$$-d_3\, M_{\rm pl}^{-2}\, R_{ij}R^{ij} - d_4\, M_{\rm pl}^{-4}\, R^3 - d_5\, M_{\rm pl}^{-4}\, R(R_{ij}R^{ij})$$
$$-d_6\, M_{\rm pl}^{-4}\, R^i{}_j R^j{}_k R^k{}_i - d_7\, M_{\rm pl}^{-4}\, R\nabla^2 R - d_8\, M_{\rm pl}^{-4}\, \nabla_i R_{jk}\, \nabla^i R^{jk}\Big\}.$$

- *Parity violating terms have been suppressed.*
- *Power-counting renormalizability is achieved.*
- *The CC is controlled by d_0 and it is not restricted.*
- *Strong coupling and instabilities plague the scalar mode in the IR.*

T. P. Sotiriou, M. Visser & S. Weinfurtner, PRL **102***, 251601 (2009)*

T. P. Sotiriou, M. Visser & S. Weinfurtner, JHEP **0910***, 033 (2009)*

K. Koyama & F. Arroja, JHEP **1003***, 061 (2010)*

Detailed Balance without Projectability

- *Abandoning Projectability: one can use not only the Riemann tensor of h_{ij} and its derivatives in order to construct invariants under foliation-preserving diffeomorphisms, but also the vector $a_i = \partial_i \ln N$.*
- *In the version without DB this leads to a proliferation of terms $\sim \mathcal{O}(10^2)$, while here there is only one 2nd-order operator one can add to the superpotential W in the version with DB: $a_i a^i$.*

New Superpotential and Action

The superpotential W then becomes

$$W=\frac{M_{\rm pl}^2}{2M_6^2}\int\omega_3(\Gamma)+\frac{M_{\rm pl}^2}{M_4}\int d^3x\sqrt{h}\left[R-2\xi(1-3\lambda)M_4^2\right]+\frac{M_{\rm pl}^2\eta}{M_4\xi}\int d^3x\sqrt{h}\,a_ia^i.$$

The total action now looks as

$$S_H=\frac{M_{\rm pl}^2}{2}\int dt d^3x\sqrt{g}N\Bigg\{K_{ij}K^{ij}-\lambda K^2+\xi R-2\Lambda+\eta\, a^ia_i$$

$$-\frac{1}{M_4^2}R_{ij}R^{ij}+\frac{1-4\lambda}{4(1-3\lambda)}\frac{1}{M_4^2}R^2+\frac{2\eta}{\xi M_4^2}\left[\frac{1-4\lambda}{4(1-3\lambda)}Ra^ia_i-R_{ij}a^ia^j\right]$$

$$-\frac{\eta^2}{4\xi^2M_4^2}\frac{3-8\lambda}{1-3\lambda}(a^ia_i)^2+\frac{2}{M_6^2M_4}\epsilon^{ijk}R_{il}\nabla_jR^l_k+\frac{2\eta}{\xi M_6^2M_4}C^{ij}a_ia_j-\frac{1}{M_6^4}C_{ij}C^{ij}\Bigg\}.$$

DV & T. P. Sotiriou, PRD **85***, 064003 (2012)*

DV & T. P. Sotiriou, JPCS **453***, 012022 (2013)*

Linearization at Quadratic Order in Perturbations

The quadratic action for scalar perturbations reads

$$S^{(2)}_{DB} = \frac{M_{\rm pl}^2}{2}\int dt d^3x \left\{ \frac{2(1-3\lambda)}{1-\lambda}\dot{\zeta}^2 + 2\xi\left(\frac{2\xi}{\eta}-1\right)\zeta\partial^2\zeta - \frac{2(1-\lambda)}{1-3\lambda}\frac{1}{M_4^2}(\partial^2\zeta)^2 \right\}.$$

Dispersion Relation for the Spin-0 Graviton

$$\omega^2 = \xi\left(\frac{2\xi}{\eta}-1\right)\frac{1-\lambda}{1-3\lambda}p^2 + \frac{1}{M_4^2}\left(\frac{1-\lambda}{1-3\lambda}\right)^2 p^4 .$$

Dispersion Relation and Stability

- *The scalar has positive energy as well as the spin-2 graviton for*

$$\lambda < \frac{1}{3} \qquad \text{or} \qquad \lambda > 1\,.$$

- *Classical stability of the scalar requires that*

$$c_\zeta^2 = \xi \left(\frac{2\xi}{\eta} - 1 \right) \frac{1-\lambda}{1-3\lambda} > 0\,.$$

- *The spin-2 graviton is stable for $\xi > 0$.*

Stability Window for Spin-0 and Spin-2 Gravitons

$$0 < \eta < 2\xi$$

Recovering Power-Counting Renormalizability

- *Adding 4th-order terms to W would lead to both 6th- and 8th-order terms for the scalar, rendering the theory power-counting renormalizable. The 4th-order terms one could add to W are:*

$$R^2\,, \quad R^{ij}R_{ij}\,, \quad R\nabla^i a_i\,, \quad R^{ij}a_i a_j\,,$$

$$R\, a_i a^i\,, \quad (a_i a^i)^2\,, \quad (\nabla^i a_i)^2\,, \quad a_i a_j \nabla^i a^j\,.$$

- *After adding these terms and imposing parity invariance in total there would be 12 free couplings in the theory.*
  This is roughly an order of magnitude less than the number of couplings in the theory without DB.

The Fourth-Order Superpotential

Let us consider the extra terms with fourth-order derivatives in the superpotential:

$$
\begin{aligned}
W_{extra} \;=\; & \int d^3x\sqrt{g}\,\Big[\gamma\, R^2 + \nu\, R^{ij}R_{ij} + \rho\, R\nabla^i a_i + \chi\, R^{ij}a_i a_j \\
& + \tau\, R\, a_i a^i + \varsigma\,\left(a_i a^i\right)^2 + \sigma\,\left(\nabla^i a_i\right)^2 + \theta\, a_i a_j \nabla^i a^j\Big]\,.
\end{aligned}
$$

Quite surprisingly it is found that perturbing the resulting action to quadratic order, sixth and eight-order operators do not give any contribution to the dispersion relation of the spin-0 graviton, which is still not power-counting renormalizable.

DV, arXiv:1502.06607 [hep-th] (2015)

Fine-Tuning Coefficients for the Spin-0 Graviton

Nevertheless, if and only if $\rho = 0$, we get a dispersion relation for the spin-0 graviton where sixth and eight-order contributions are instead present, leading to

UV Dispersion Relation for the Spin-0 Graviton

$$\omega^2 \sim \frac{1}{M_{pl}^4}\left(\frac{1-\lambda}{1-3\lambda}\right)^2\left[2\mu\,(3\nu+8\gamma)\,p^6+(3\nu+8\gamma)^2\,p^8\right].$$

Notice that the coefficient in front of p^8 is always positive and then it cannot lead to instabilities in the UV.

Spin-2 Graviton

Considering tensor perturbations we find that the operators $R_{ij}\nabla^2 R^{ij}$ *and* $\nabla^2 R_{kl}\nabla^2 R^{kl}$ *generated in the potential generically yield non-trivial contributions, respectively at sixth and eight-order, to the dispersion relation of the spin-2 graviton:*

UV Dispersion Relation for the Spin-2 Graviton

$$\omega_T^2 \sim \frac{\nu}{M_{pl}^4}\left[-2\mu p^6 + \nu p^8\right].$$

So, we generically end up with a power-counting renormalizable theory for the spin-2 graviton. The latter is also classically stable at very high-energies for any choice of the couplings since the coefficient in front of p^8 *is always positive.*

Conclusions

- *The necessity to go beyond General Relativity.*
- *Lorentz symmetry breaking as an UV regulator: Hořava gravity.*
- *The full theory has a very large number of operators allowed by the symmetry: Implementation of projectability and detailed balance.*
- *Only with a suitable fine-tuning of the couplings a power-counting renormalizable version with DB does exist.*

Future Perspectives

- *Need of further proposals in order to reduce the number of independent couplings in the full action.*

- *Hořava gravity with mixed derivative terms: power-counting renormalizability with lower-order dispersion relations.*

  M. Colombo, A. E. Gumrukcuoglu & T. P. Sotiriou, PRD **91**, *044021 (2015)*
  M. Colombo, A. E. Gumrukcuoglu & T. P. Sotiriou, arXiv:1503.07544 (2015)

- *The issue of renormalizability beyond the power-counting arguments is still open in Hořava gravity.*

  G. D'Odorico, F. Saueressig & M. Schutten, PRL **113**, *171101 (2014)*

- *The vacuum energy problem in Hořava gravity also deserves further investigation.*