Energy of massive gravitons and the related issues (cosmology, wormholes. etc.)

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Motivations for massive gravity

- Dark energy = modification of gravity:

$$\text{Newton} \quad \frac{1}{r} \quad \rightarrow \quad \text{Yukawa} \quad \frac{1}{r}\, e^{-mr}$$

hence the gravity is weaker at large distances $\Rightarrow$ the cosmic acceleration, $m \sim 1/(\text{cosm. horizon size})$.

- Theoretical problems, but it seems there is a consistent theory

Massive gravity = potential for the metric

Physical metric $g_{\mu\nu}$ and flat reference metric $f_{\mu\nu}$

$$S = \frac{1}{\kappa^2} \int \sqrt{-g} \left(\frac{1}{2} R - m^2 \mathcal{U} \right) d^4x \equiv \frac{1}{\kappa^2} \int \mathcal{L}\, d^4x$$

in the generic case ($H^\alpha_\beta = g^{\alpha\sigma} f_{\sigma\nu}$) /Fierz-Pauli 1939/

$$\mathcal{U} = \frac{1}{8} \left((H^\alpha_\alpha)^2 - H^\alpha_\beta H^\beta_\alpha \right)^2 + \ldots$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \Rightarrow \quad \boxed{\Box h_{\mu\nu} + \ldots = m^2 (h_{\mu\nu} - \; h\, \eta_{\mu\nu}) + 16\pi G T_{\mu\nu}}$$

In the dRGT case the ... are uniquely fixed

$$\mathcal{U} = b_0 + b_1 \sum_a \lambda_a + b_2 \sum_{a<b} \lambda_a \lambda_b + b_3 \sum_{a<b<c} \lambda_a \lambda_b \lambda_c + b_4\, \lambda_0 \lambda_1 \lambda_2 \lambda_3$$

where λ_a are eigenvalues of $\sqrt{H^\mu_\nu} = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$

How to compute the energy ?

/Hassan and Rosen 2012/
/Comelli, Nesti, Pilo 2012/

ADM decomposition

With

$$
\begin{aligned}
ds_g^2 &= -N^2dt^2 + \gamma_{ik}(dx^i + N^i dt)(dx^k + N^k dt) \\
ds_f^2 &= -dt^2 + \delta_{ik}dx^i dx^k
\end{aligned}
$$

the Lagrangian becomes

$$
\mathcal{L} = \sqrt{\gamma}N\left(\frac{1}{2}\{K_{ik}K^{ik} - K^2 + R^{(3)}\} - m^2\mathcal{U}(N^\nu, \gamma_{ik})\right)
$$

where the second fundamental form

$$
K_{ik} = \frac{1}{2N}(\partial_t\gamma_{ik} - \nabla_i^{(3)}N_k - \nabla_k^{(3)}N_i).
$$

Variables are γ_{ik} and $N^\mu = (N, N^k)$.

Hamiltonian

The conjugate momenta

$$\pi^{ik} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ik}} = \frac{1}{2}\sqrt{\gamma}\,(K^{ik} - K\gamma^{ik}), \qquad p_{N_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{N}^\mu} = 0.$$

$\Rightarrow$ N^μ are non-dynamical. Hamiltonian

$$\boxed{\mathcal{H} = \pi^{ik}\dot{\gamma}_{ik} - \mathcal{L} = N^\mu \mathcal{H}_\mu + m^2 \mathcal{V}(N^\alpha, \gamma_{ik})}$$

with $\mathcal{V} = \sqrt{\gamma} N \mathcal{U}$ and

$$\mathcal{H}_0 = \frac{1}{\sqrt{\gamma}}\,(2\pi^{ik}\pi_{ik} - (\pi^k_k)^2) - \frac{1}{2}\sqrt{\gamma} R^{(3)}, \quad \mathcal{H}_k = -2\nabla^{(3)}_i \pi^i_k$$

phase space is spanned by 12 (π^{ik}, h_{ik})

$$\boxed{0 = -\dot{p}_{N_\mu} = \frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu},}$$

this condition determines the number of Dof.

GR, $m = 0$

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) = 0. \qquad \text{4 constraints}$$

They are fist class,

$$\{\mathcal{H}_\mu, \mathcal{H}_\nu\} \sim \mathcal{H}_\alpha$$

they generate gauge symmetries $\Rightarrow$ one can impose 4 gauge conditions. There remain

$$12 - 4 - 4 = 4 = 2 \times (2 \text{ DoF})$$

independent variables $\Rightarrow$ 2 graviton polarizations.

Energy vanishes on the constraint surface (up to a surface term)

$$H = N^\mu \mathcal{H}_\mu = 0$$

$m \neq 0$ generic case, the BD problem

4 conditions

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu} = 0$$

determine the laps and shifts $N^\mu = N^\mu(\pi^{ik}, \gamma_{ik})$. Inserting N^μ to $\mathcal{H}$ gives

$$\boxed{\mathcal{H} = \mathcal{E}(\pi^{ik}, \gamma_{ik})}$$

$\Rightarrow$ no constraints

$\Rightarrow$ there are $12 = 2 \times (5 + 1$ DoF). The kinetic part of the energy $\mathcal{H}$ is has the wrong sign – the energy is unbounded from below, which is associated to the extra Dof=BD ghost /1972/

$m \neq 0$, dRGT theory

4 conditions

$$\frac{\partial \mathcal{H}}{\partial N^\mu} = \mathcal{H}_\mu(\pi^{ik}, \gamma_{ik}) + m^2 \frac{\partial \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu} = 0$$

determine three shifts $N^k = N^k(N, \pi^{ik}, \gamma_{ik})$. The lapse N remains undetermined because the Hessian matrix

$$\frac{\partial^2 \mathcal{V}(N^\alpha, \gamma_{ik})}{\partial N^\mu \partial N^\nu}$$

has rank 3. Inserting N^k to $\mathcal{H}$ gives

$$\boxed{\mathcal{H} = \mathcal{E}(\pi^{ik}, \gamma_{ik}) + N\mathcal{C}(\pi^{ik}, \gamma_{ik})}$$

$$\Rightarrow \text{ constraints } \quad \mathcal{C} = 0, \quad \mathcal{S} = \{\mathcal{C}, H\} = 0$$

$\Rightarrow$ there are $12 - 2 = 10 = 2 \times (5 \text{ DoF})$. The energy density is $\mathcal{E}(\pi^{ik}, \gamma_{ik})$ computed on the constraint surface.
The energy can be negative as well. However, there are ways to overcome this.

Restricting to the s-sector

Spherical symmetry

$$
\begin{aligned}
ds_g^2 &= -N^2 dt^2 + \frac{1}{\Delta^2}(dr + \beta\, dt)^2 + R^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \\
ds_f^2 &= -dt^2 + dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)
\end{aligned}
$$

where N, β, Δ, R depend on t, r. Lapse N and shift β are non-dynamical. Dynamical variables are Δ and R and their momenta

$$
p_\Delta = \frac{\partial \mathcal{L}}{\partial \dot{\Delta}}, \quad p_R = \frac{\partial \mathcal{L}}{\partial \dot{R}}
$$

Phase space is spanned by 4 variables $\Delta, R, p_\Delta, p_R \equiv (q^i, p_k)$.

Hamiltonian

$$\mathcal{H} = p\dot{q} - \mathcal{L} = \dot{\Delta}\pi_\Delta + \dot{R}\pi_R - \mathcal{L} = N\mathcal{H}_0 + \beta\mathcal{H}_r + m^2\mathcal{V}$$

where

$$\begin{aligned} \mathcal{H}_0 &= \frac{\Delta^3}{4R^2}\, p_\Delta^2 + \frac{\Delta^2}{2R}\, p_\Delta p_R + \Delta R R'^2 + 2R(\Delta R')' - \frac{1}{\Delta}\,, \\ \mathcal{H}_r &= \Delta p_\Delta' + 2\Delta' p_\Delta + R' p_R\,. \end{aligned}$$

and the potential

$$\mathcal{V} = \frac{NR^2 P_0}{\Delta} + \frac{R^2 P_1}{\Delta}\sqrt{(\Delta N + 1)^2 - \beta^2} + R^2 P_2$$

with

$$P_n = b_n + 2b_{n+1}\,\frac{r}{R} + b_{n+2}\,\frac{r^2}{R^2}$$

Dof's

One has

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial N} &= \mathcal{H}_0 + m^2 \frac{\partial \mathcal{V}}{\partial N} = 0 \\ \frac{\partial \mathcal{H}}{\partial \beta} &= \mathcal{H}_r + m^2 \frac{\partial \mathcal{V}}{\partial \beta} = 0 \end{aligned}$$

If $m = 0 \Rightarrow$ 2 first class constraints $\mathcal{H}_0 = 0$, $\mathcal{H}_r = 0 \Rightarrow$ $4 - 2 - 2 = 0$ Dof's $\Rightarrow$ no dynamics = Birkhoff theorem.

If $m \neq 0 \Rightarrow$ the second condition determines β,

$$\beta = (N\Delta + 1)\frac{\Delta \mathcal{H}_r}{Y}$$

while the first condition gives a constraint $\mathcal{C}(\Delta, R, p_\Delta, p_R) = 0$

Constraints

$$\mathcal{C} \equiv \mathcal{H}_0 + Y + m^2 \frac{R^2 P_0}{\Delta} = 0$$

The Hamiltonian is $\mathcal{H} = \mathcal{E} + N\mathcal{C}$ where

$$\mathcal{E} = \frac{Y}{\Delta} + m^2 R^2 P_2 \quad \text{with} \quad Y \equiv \sqrt{(\Delta \mathcal{H}_r)^2 + (m^2 R^2 P_1)^2}$$

Secondary constraint

$$\begin{aligned}
\mathcal{S} &= \{\mathcal{C}, H\} = \frac{m^4 R^2 P_1^2}{2Y} (\Delta p_\Delta + R p_R) - Y \left(\frac{\Delta \mathcal{H}_r}{Y} \right)' \\
&- \frac{\Delta^2 p_\Delta}{2R} \left\{ \frac{m^4}{2\Delta Y} \partial_R (R^4 P_1^2) + m^2 \partial_R (R^2 P_2) \right\} \\
&- \frac{m^2 \mathcal{H}_r}{Y} \left\{ \Delta \left(R^2 P_2 \right)' + R^2 \partial_r (P_0 - \Delta^2 P_2) \right\} = 0
\end{aligned}$$

$\Rightarrow 4 - 2 = 2 \times 1$ Dof. Energy $E = \int_0^\infty \mathcal{E} dr$ *assuming* $\mathcal{C} = \mathcal{S} = 0$.

Weak fields=Fierz-Pauli limit

If $N = \Delta = 1$, $R = r$ and $p_\Delta = p_R = 0 \Rightarrow \mathcal{C} = \mathcal{S} = \mathcal{E} = 0$.

If $N = 1 + \nu$, $\Delta = 1 + \delta$, $R = r + \rho$ then

$$\mathcal{C} = \mathcal{C}_{\rm FP} + \ldots, \qquad \mathcal{S} = \mathcal{S}_{\rm FP} + \ldots, \quad \mathcal{H} = \mathcal{E}_{\rm FP} + \nu\,\mathcal{C}_{\rm FP} + \ldots$$

where $\mathcal{E}_{\rm FP}$ is the quadratic part,

$$\mathcal{E}_{\rm FP} \;=\; (\qquad)^2 + (\qquad)'$$

$\Rightarrow E = \int \mathcal{E}_{\rm FP} dr \geq 0$.

Strong fields – kinetic energy sector

With $\quad \Delta = 1, \quad R = r, \quad p_\Delta = \frac{\sqrt{xz}}{m}, \qquad p_R = -\frac{(xz + 4x^4 f)}{(2x\sqrt{xz})}$

and $x = mr$ the constraints reduce to

$$\begin{aligned} \frac{dz}{dx} &= 4x^2 f + 2x\sqrt{xz}\sqrt{f(f+2)}\,, \\ \frac{df}{dx} &= \frac{\{4\,zf - 4x^3 f - 3z\}\sqrt{f(f+2)}}{4x\sqrt{xz}} - \frac{2}{x} f(f+2)\,, \end{aligned}$$

while $\boxed{\mathcal{E} = x^2 f}$. One should have $f(f+2) \geq 0 \Rightarrow$ two solution branches: either $f \geq 0$ and $\mathcal{E} > 0$ or $f \leq -2$ and $\mathcal{E} < 0$.

Solutions with $\mathcal{E} < 0$ are not globally defined and are singular.

Strong fields – potential energy sector

$$\text{Let} \quad p_\Delta = p_R = 0, \qquad \Delta = \frac{g}{h}, \quad R = rh, \qquad \Rightarrow \qquad \mathcal{S} = 0,$$

$$\begin{aligned} \mathcal{C} \;&=\; -h'' - \frac{2}{x}\,h' + \frac{h'^2}{2h} - \frac{(xh)'g'}{xg} + \frac{h(1-g^2)}{2x^2g^2} \\ &+\; \frac{h(2-3h)}{2g} + \frac{h(1-6h+6h^2)}{2g^2} = 0, \end{aligned}$$

asymptotic flatness: $h_0 \leftarrow h \rightarrow 1, \;\; 1 \leftarrow g \rightarrow 1$ for $\; 0 \leftarrow x \rightarrow \infty$

$$\mathcal{E} \;=\; x^2 h^2 (3h - g - 2)/g\,.$$

Special solutions, also fulfill the Hamilton equations:

flat space: $h = 1$, $g = 1$, $ds_g^2 = ds_f^2$, $\qquad \mathcal{E} = 0$

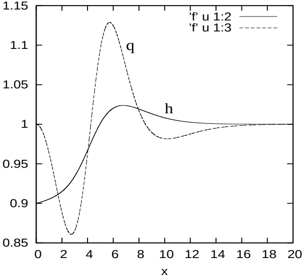
tachyon space: $h = \frac{1}{2}$, $g = 1$, $ds_g^2 = \frac{1}{4}\,ds_f^2$, $\mathcal{E} = -\frac{3x^2}{8}$, $m_{FP}^2 = -\frac{m^2}{2}$

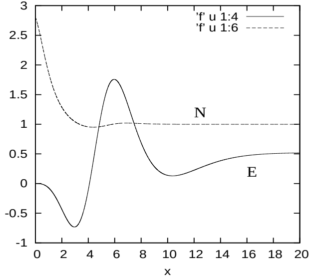
Deformations of flat space

Setting $g = qh/(xh)'$, the constraint is solved with

$$\begin{aligned} Q &= xh(1-q^2) + x^3 h(2h-1)(h-1), \\ Q' &= r^2 h(3h-2)(q-1) \end{aligned}$$

for any $Q(x)$. Let $Q = A\Theta(x-x_0)(x-x_0)^p e^{-x}$





Energy is positive for smooth, asymptotically flat fields.

Deformations of tachyon space

Solutions with

$$h_0 \leftarrow h \rightarrow \frac{1}{2}$$

the energy is negative and infinite.

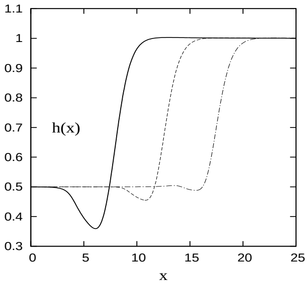
Negative energies cannot affect the positive energy solutions, because the asymptotic conditions at infinity are different:

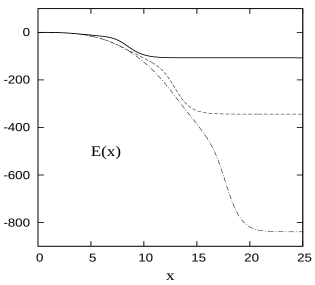
$$h_0 \leftarrow h \rightarrow 1 \text{ positive energy branch}$$
$$h_0 \leftarrow h \rightarrow \tfrac{1}{2} \text{ negative energy branch}$$

When deforming positive branch to the negative one the energy shows a pole $\Rightarrow$ the two branches are completely disjoint

Tachyon bubbles

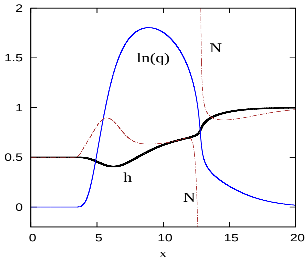
Asymptotically flat solutions that start from the tachyon branch at the origin and approach the flat space at infinity. The energy is finite and negative.





Does this affect the stability of flat space ?

Tachyon bubbles



The lapse N shows poles. Can be proven for $c_3 = c_4 = 0$, in other cases – numerical evidence.

One does not find globally regular and asymptotically flat negative energy solutions which would describe initial data for a decay of flat space $\Rightarrow$ *negative energy decouple and are harmless.*

Summary of part I

- The energy is positive for globally regular and asymptotically flat fields constituting the "physical sector"
- The energy can be unbounded from below, but only for fields that are either singular or not asymptotically flat. Such fields can show superluminal features.
- It seems that negative energies are completely disjoint from the positive sector. Negative energy solutions cannot describe initial data for a decay of flat space $\Rightarrow$ negative energy decouple and are harmless $\Rightarrow$ the physical sector is protected from negative energies and superluminal phenomena by a potential barrier.

II. Stability of cosmological solutions

arXiv:1503.03042

Problems of dRGT cosmology

- dRGT theory does not admit spatially flat FLRW solutions
- There is a spatially open FLRW solution, but it is unstable
- Therefore, one should rather study extensions of the dRGT – bigravity, quasidilaton etc.

dRGT theory admit solutions whose $g_{\mu\nu}$ is de Sitter while $f_{\mu\nu}$ depends on a Stuckelberg scalar which should satisfy a complicated nonlinear PDE. Only some simple solutions of the PDE are known.

Koyama, Niz, Tasinato, 2011
Chamseddine and M.S.V., 2011
dAmico, de Rham, Dubovsky, Gabadadze, Pirtskhalava, 2011
Gumrukcuoglu, Lin, Mukohyama, 2011
Gratia, Hu, and Wyman, 2011
M.S.V., 2012
Kobayashi, Siino, Yamaguchi, Yoshida, 2012
Khosravi, Niz, Koyama, Tasinato, 2013

de Sitter space

Hyperboloid

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = \alpha^2$$

is 5D Minkowski space with metric

$$ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2$$

The induced geometry solves Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

provided that

$$\frac{1}{\alpha^2} = \frac{\Lambda}{3}$$

Changing coordinates gives expanding FLRW cosmologies.

Gordon ansatz

$$f_{\mu\nu} = \omega^2 \left(g_{\mu\nu} + (1-\zeta^2)V_\mu V_\nu\right),$$

with

$$g^{\mu\nu}V_\mu V_\nu = -1.$$

If

$$P_1(\omega) = 0$$

with $P_m(\omega) = b_m + 2b_{m+1}\,\omega + b_{m+2}\,\omega^2$ then

$$T^\mu_\nu = -\Lambda\delta^\mu_\nu$$

with $\Lambda = m^2 P_0(u) \Rightarrow$

$$G^\mu_\nu + \Lambda\delta^\mu_\nu = 0$$

/Baccetti, Martin-Moruno, Visser, 2012/

de Sitter in dRGT

physical metric, $1/\alpha^2 = m^2\, P_0(u)/3$

$$
\begin{aligned}
ds_g^2 &= \alpha^2 \left\{ -dt^2 + dr^2 + dx^2 + dy^2 + dz^2 \right\} \qquad (g) \\
1 &= -t^2 + r^2 + x^2 + y^2 + z^2
\end{aligned}
$$

reference metric, $P_1(u) = 0$,

$$
ds_f^2 = \alpha^2 u^2 \left\{ -dT^2(t,r) + dx^2 + dy^2 + dz^2 \right\} \qquad (f)
$$

$\Rightarrow$ compatible with Gordon if

$$
\boxed{(\partial_t T)^2 - (\partial_r T)^2 = 1}
$$

$\Rightarrow$ infinitely many solutions with the same (g) but different (f).

Simplest solution $T = t$

Transformation to the flat slicing

$$t = \sinh\tau + \frac{\rho^2}{2}e^\tau,\ \ r = \cosh\tau - \frac{\rho^2}{2}e^\tau,\ \ R \equiv \sqrt{x^2+y^2+z^2} = e^\tau\rho$$

gives spatially flat FLRW g-metric, with $a(\tau) = e^\tau$,

$$ds_g^2 = \alpha^2\{-d\tau^2 + a^2(\tau)(d\rho^2 + \rho^2 d\Omega^2)\},$$

and

$$ds_f^2 = \alpha^2 u^2\{-dT^2(\tau,\rho) + dR^2 + R^2 d\Omega^2)\},$$

with

$$T(\tau,\rho) = \frac{1}{2}\int\frac{d\tau}{\dot{a}(\tau)} + \frac{1}{2}\left(1+\rho^2\right)a(\tau)$$

f-metric is inhomogeneous, whole solution is not FLRW

/d'Amico, de Rham, Dubovsky, Gabadadze, Pirtskhalava, 2011/

Simplest solution $T = t$

Transformation to the open slicing

$$t = \sinh(\tau)\cosh(\rho), \qquad r = \cosh(\tau), \quad R = \sinh(\tau)\sinh(\rho)$$

gives

$$\begin{aligned} ds_g^2 &= \alpha^2\{-d\tau^2 + a^2(\tau)(d\rho^2 + \sinh^2(\rho)d\Omega^2)\}, \\ ds_f^2 &= \alpha^2 u^2\{-\cosh(\tau)^2 d\tau^2 + a^2(\tau)(d\rho^2 + \sinh^2(\rho)d\Omega^2)\}. \end{aligned}$$

with $a(\tau) = \cosh(\tau)$. Solution is manifestly FLRW.

/ Gumrukcuoglu, Lin, Mukohyama, 2011/

Lesson: the two metrics can have common non-manifest isometries

Simplest solution $T = t$

Transformation to the static slicing

$$t = \sqrt{1-\rho^2}\,\sinh(\tau), \quad r = \sqrt{1-\rho^2}\,\cosh(\tau), \quad R = \rho.$$

gives

$$\begin{aligned} ds_g^2 &= \alpha^2 \left\{ -(1-\rho^2)\,d\tau^2 + \frac{d\rho^2}{1-\rho^2} + \rho^2 d\Omega^2 \right\}, \\ ds_f^2 &= \alpha^2 u^2 \{ -dT^2(\tau,\rho) + d\rho^2 + \rho^2 d\Omega^2) \}, \end{aligned}$$

with

$$T(\tau,\rho) = \sqrt{1-\rho^2}\,\sinh(\tau),$$

Solution is not invariant under the action of the timelike de Sitter isometry $\partial/\partial\tau$. This probably explains why it is unstable

/de Felice, Gumrukcuoglu, Mukohyama, 2012/

What about other solutions of $(\partial_t T)^2 - (\partial_r T)^2 = 1$?

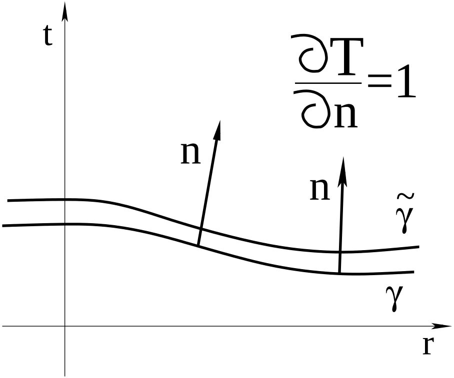
$(\partial_t T)^2 - (\partial_r T)^2 = 1$

- fairly general solution

$$\begin{aligned} T &= \cosh(\xi)\, t + \sinh(\xi)\, r + W(\xi)\,, \\ 0 &= \sinh(\xi)\, t + \cosh(\xi)\, r + \frac{dW(\xi)}{d\xi} \end{aligned}$$

$(\partial_t T)^2 - (\partial_r T)^2 = 1$

- method of characteristics



$(\partial_t T)^2 - (\partial_r T)^2 = 1$

- separation of variables. E.g. in static coordinates

$$\frac{1}{1-\rho^2}\left(\frac{\partial T}{\partial \tau}\right)^2 - \frac{1-\rho^2}{\rho^2}\left(\frac{\partial T}{\partial \rho}\right)^2 = 1.$$

gives genuinely static solutions

$$T = \sqrt{1+q^2}\,\tau + \int \frac{\rho\, d\rho}{1-\rho^2}\sqrt{q^2+\rho^2}$$

$\Rightarrow$ f-metric is static $\Rightarrow$ a one-parameter family labeled by $q \geq 0$. If $q = 0$ then

Static solutions

$$
\begin{aligned}
ds_g^2 &= \alpha^2\{-\Sigma dV^2 + 2dVd\rho + \rho^2 d\Omega^2\}, \\
ds_f^2 &= u^2\alpha^2\{-dV^2 + 2dVd\rho + \rho^2 d\Omega^2\}.
\end{aligned}
$$

with

$$
V = t + \int \frac{d\rho}{1-\rho^2}
$$

The canonical energy is time-independent

Canonical energy

$$E = \int \mathcal{E}\, d\rho,$$

where the radial energy density

$$\mathcal{E} = u^2 P_2(u) \rho^2 \partial_\tau T.$$

If $T = t$ then

$$\mathcal{E} = u^2 P_2(u) \rho^2 \sqrt{1-\rho^2}\cosh(\tau)$$

depends on time. For the static solutions

$$\mathcal{E} = u^2 P_2(u) \sqrt{1+q^2}\rho^2$$

which corresponds to the constant volume energy density

$$\epsilon = u^2 P_2(u) \sqrt{1+q^2}$$

Conjecture: *E is minimal and the solutions are stable.*

Summary of part II

- In dRGT theory there are infinitely many de Sitter solutions labeled by $T(t, r)$ subject to $(\partial_t T)^2 - (\partial_r T)^2 = 1$.

- Solutions can be FLRW (g and f have common rotational and translational isometries) in a *non-manifest way*.

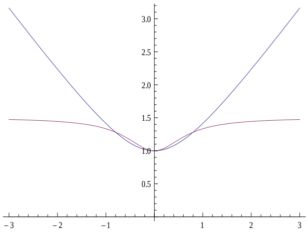
- For a one-parameter family of solutions both metrics are invariant under the timelike isometry and the energy is time-independent. If the energy is minimal then the solution is stable $\Rightarrow$ "de Sitter vacuum".

III. Wormholes in ghost-free bigravity

S.V.Sushkov and M.S.V., arXiv.1502.03712

Wormholes – bridges between universes

$$ds^2 = -Q^2(r)dt^2 + dr^2 + R^2(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$



$G_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow \rho + p < 0,\ p < 0 \Rightarrow$ violation of the null energy condition: $T_{\mu\nu}v^\mu v^\nu \geq 0$ for any null v^μ.

$\Rightarrow$ one needs vacuum polarization, phantom fields, higher derivative gravity, Gauss-Bonnet, branworld gravity, non-minimal couplings, Horndeski, Galileon, massive gravitons.

Ghost-free bigravity – two dynamical metrics $g_{\mu\nu}$ and $f_{\mu\nu}$.

$$S = \frac{M_{\rm Pl}^2}{m^2} \int \left(\frac{1}{2\kappa_1} R(g)\sqrt{-g} + \frac{1}{2\kappa_2} R(f)\sqrt{-f} - \mathcal{U}\sqrt{-g} \right) d^4x$$

$$\begin{aligned} \mathcal{U} &= b_0 + b_1 \sum_A \lambda_A + b_2 \sum_{A<B} \lambda_A \lambda_B \\ &+ b_3 \sum_{A<B<C} \lambda_A \lambda_B \lambda_C + b_4 \lambda_0 \lambda_1 \lambda_2 \lambda_3 \end{aligned}$$

where λ_A are eigenvalues of $\gamma^\mu_{\ \nu} = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$.

$$\begin{aligned} G^\mu_\nu(g) &= \kappa_1 \, T^\mu_{\ \nu}(g, f), \\ G^\mu_\nu(f) &= \kappa_2 \, \mathcal{T}^\mu_{\ \nu}(g, f), \end{aligned}$$

A massive + a massless graviton = 7 DoF.

Reduction to the S-sector

$$
\begin{aligned}
ds_g^2 &= -Q^2 dt^2 + \frac{R'^2}{N^2}\, dr^2 + R^2 d\Omega^2 \\
ds_f^2 &= -q^2 dt^2 + \frac{U'^2}{Y^2}\, dr^2 + U^2 d\Omega^2
\end{aligned}
$$

Q, N, R, q, Y, U depend on r, one can impose 1 gauge condition.
5 independent equations

$$
\begin{aligned}
G_0^0(g) &= \kappa_1\, T_0^0, \\
G_r^r(g) &= \kappa_1\, T_r^r, \\
G_0^0(f) &= \kappa_2\, \mathcal{T}_0^0, \\
G_r^r(f) &= \kappa_2\, \mathcal{T}_r^r, \\
T_r^{r\prime} &+ \frac{Q'}{Q}(T_r^r - T_0^0) + \frac{2}{r}(T_\vartheta^\vartheta - T_r^r) = 0.
\end{aligned}
$$

Local solution at the throat

$$
\begin{aligned}
ds_g^2 &= -Q^2 dt^2 + dr^2 + R^2 d\Omega^2 \\
ds_f^2 &= -q^2 dt^2 + \frac{U'^2}{Y^2} dr^2 + U^2 d\Omega^2
\end{aligned}
$$

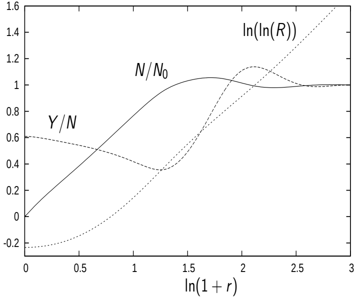
$$
\begin{aligned}
Y &= Y_1 r + Y_3 r^3 + \dots \quad Q = Q_0 + Q_2 r^2 + \dots \quad R = h + R_2 r^2 + \dots \\
q &= q_0 + q_2 r^2 + \dots \qquad U = \sigma h + U_2 r^2 + \dots
\end{aligned}
$$

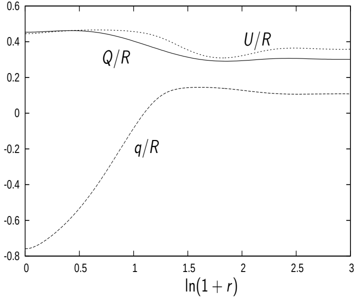
Expanding the field equations gives in the leading order

$$
\begin{aligned}
\left(\kappa_1 \mathrm{P}_0 - \frac{1}{h^2}\right) Q_0 + \kappa_1 \mathrm{P}_1\, q_0 &= 0, \\
\left(\kappa_2 \mathrm{P}_2 - \frac{1}{h^2}\right) q_0 + \kappa_2 \mathrm{P}_1\, Q_0 &= 0,
\end{aligned}
$$

with $\mathrm{P}_m = b_m + 2b_{m+1}\sigma + b_{m+2}\,\sigma^2$. To have non-zero Q_0, q_0, the determinant of this system must vanish. This gives

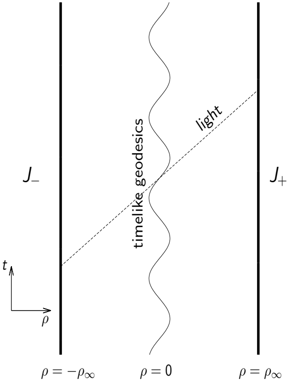
W1 wormholes – asymptotically AdS

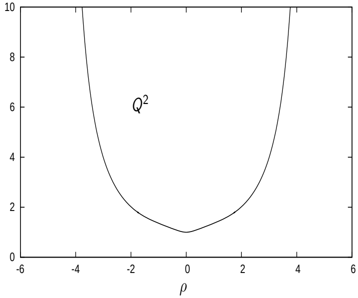




W1 wormhole solution – both metrics are asymptotically AdS.

Particle motion





Conformal structure of the g-geometry for the W1 solutions and the effective potential in the geodesic equation $\left(\frac{d\rho}{dt}\right)^2 + \frac{\mu^2}{\mathcal{E}^2}\, Q^2 = 1$.

Asymptotic behavior

For $R \to \infty$ approach the AdS solution, $ds_f^2 = \lambda^2 ds_g^2$ where

$$ds_g^2 = -N^2 dt^2 + \frac{dR^2}{N^2} + R^2 d\Omega^2$$

with $N^2 \to N_0^2 = 1 - \frac{\Lambda r^2}{3}$. One has

$$N^2/N_0^2 = 1 + \frac{C}{R^3} + \frac{A}{R\sqrt{R}} \cos\left(\omega \ln(R) + \varphi\right)$$

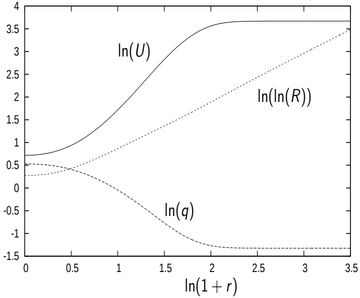
C-term is the Newtonian tail, the A-term is the effect of the massive mode – scalar polarization of the massive graviton. Oscillations: the massive graviton becomes a tachyon, with

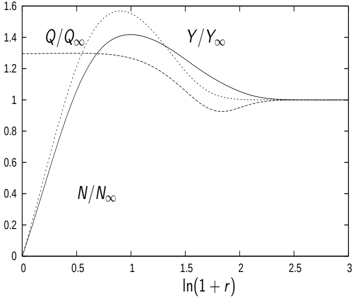
$$m_{\rm FP}^2 = \left(\frac{\kappa_2}{\lambda} + \kappa_1 \lambda\right)(b1 + 2b_2\lambda + b_3\lambda^2) < 0$$

May or may not exceed the BF bound

$$m_{\rm BF}^2 = \frac{3}{4}\Lambda$$

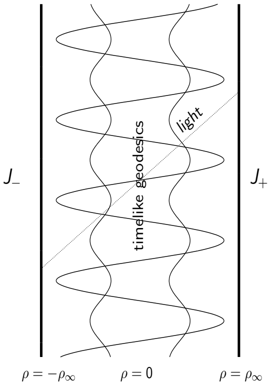
W2 wormholes

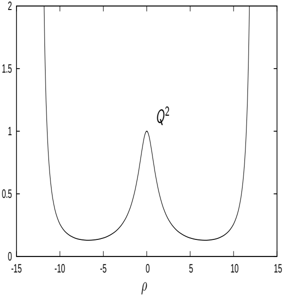




g-metric is asymptotically AdS, but f-metric is different and geodesically incomplete

Particles





Spacetime structure and geodesics

Summary of part III

- The ghost-free bigravity theory admits wormhole solutions for which the g-metric interpolates between AdS spaces
- The wormhole throat is cosmologically large (could we live inside it ?)
- Some wormholes show the tachyon instability in the AdS limit, but others could be stable.

14th Marcel Grossmann Meeting, Rome, July 12-18

Section Alternative theories – AT4: *Localized self-gravitating field systems in the Einstein and alternatives theories of gravity*

chairpersons: Dmitry Galtsov and Michael Volkov