Propagation peculiarities of mean-field massive gravity

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Motivation and goals

- Consistency requirements for higher spin theories
  (i) Degree of freedom count
  (ii) Stability viz. no ghosts
  (iii) Predictivity i.e. hyperbolicity
  (iv) (Sub)luminal propagation
- dRGT massive gravity: no Boulware-Deser ghost
  C. de Rham, G. Gabadadze, A. Tolley [2010,2011]
- Presence of superluminalities (cf Galileons in the decoupling limit)
- Interpretation of non-linear propagation analysis results
  S. Deser, M. Sandora, A. Waldron, GZ [arXiv:1408.0561]
  S. Deser, A. Waldron, GZ [arXiv:1504.02919]

Methodology

- Linearize perturbations over an mGR solution
- Compute covariant constraints
- Compute characteristic matrix for perturbations
- Make analogy with charged spin 3/2 model
  W. Rarita, J. Schwinger [1941]
- Infer propagation properties
  G. Velo, D. Zwanziger [1969,1970]
  1. Hyperbolicity requirements
  2. Superluminality and causal structure(s)

Non-linear theory

Perturbative analysis

Spin 3/2 analogy

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General setting

- dRGT massive gravity: first order Cartan formalism
  - 4 vierbein 1-forms $e^m := e_\mu{}^m dx^\mu$ (16 fields)
  - 6 connection 1-forms $\omega^{mn} := \omega_\mu{}^{mn} dx^\mu$ (24 fields)
- dRGT action (4 fiducial vierbein 1-forms $f^m := f_\mu{}^m dx^\mu$)

$$S = -\frac{1}{4} \int \epsilon_{mnrs}\, e^m e^n \, \left[d\omega^{rs} + \omega^r{}_t \omega^{ts} \right] \\ + m^2 \int \epsilon_{mnrs}\, e^m \left[\frac{\beta_0}{4} e^n e^r e^s + \frac{\beta_1}{3} e^n e^r f^s + \frac{\beta_2}{2} e^n f^r f^s + \beta_3 f^n f^r f^s \right]$$

Equations of motion

- Zero torsion condition

$$\mathcal{T}^m := \nabla e^m := de^m + \omega^m{}_n e^n \approx 0$$

- Einstein equations

$$\mathcal{G}_m := G_m - m^2 t_m \approx 0$$

  - Einstein 3-forms
    $G_m := \frac{1}{2}\epsilon_{mnrs} e^n \left[d\omega^{rs} + \omega^r{}_t \omega^{ts}\right]$
  - Mass term 3-forms
    $t_m := \epsilon_{mnrs} \left[\beta_0 e^n e^r e^s + \beta_1 e^n e^r f^s + \beta_2 e^n f^r f^s + \beta_3 f^n f^r f^s\right]$
- 40 eoms for 40 dynamical fields

Primary constraints

- Space-time decomposition of a p-form ($p < 4$)

$$\theta := \mathring{\theta} + \boldsymbol{\theta}$$

where $\mathring{\theta} \wedge dt = 0$

- Primary constraints: purely spatial part of the eoms

$$\boldsymbol{\mathcal{T}}^m = \boldsymbol{d}\boldsymbol{e}^m + \boldsymbol{\omega}^m{}_n \boldsymbol{e}^n \approx 0$$

$$\boldsymbol{G}_m \approx m^2 \boldsymbol{t}_m$$

- 12+4 =16 constraints

Secondary constraints

- Symmetry constraint

$$G_{[m}e_{n]} = \frac{1}{2}\epsilon_{mnrs}e^r\nabla\mathcal{T}^s \approx 0$$

So $t_{[m}e_{n]} \approx 0$ and generically

$$\mathcal{F} := e_m f^m \approx 0$$

- Vector constraint

$$\nabla G_m = \frac{1}{2}\epsilon_{mnrs}\mathcal{T}^n\left[d\omega^{rs} + \omega^r{}_t\omega^{ts}\right] \approx 0$$

So $\nabla t_m \approx 0$ which reduces to

$$\mathcal{V} := \epsilon_{mnrs}M^{mn}K^{rs} \approx 0$$

where $M^{mn}(e^r, f^s)$ and $K^{mn} := \omega^{mn} - \bar{\omega}^{mn}(f^r)$

- 6+4=10 constraints

Tertiary constraints

- Curl of symmetry constraint

$$\nabla\mathcal{F} := K_{mn}e^m f^n \approx 0$$

where the purely spatial part is not new!

- Curl of vector constraint

$$\nabla\mathcal{V} := \epsilon_{mnrs}M^{mn}\nabla K^{rs} + \cdots \approx 0$$

  - ∇K^{rs}: no time derivatives on-shell
  - Scalar constraint:

$$\mathcal{S} := \nabla\mathcal{V} \approx 0$$

- 3+1=4 constraints
- 40-16-10-4=10 first order dofs OR 5 physical dofs

Superluminal propagation analysis

- Are wavefronts allowed to propagate along spacelike hypersurfaces?
- Superluminal shock solutions

$$\begin{pmatrix} e \\ \omega \end{pmatrix} = \begin{pmatrix} \mathfrak{e} \\ \mathfrak{w} \end{pmatrix} e^{i\xi_\mu x^\mu}$$

with $|\xi_\mu| \to \infty$ and ξ_μ timelike

→ Solutions with discontinuous normal derivatives along some spacelike hypersurface?

- Effective linearization of the theory about some background...

! Which metric defines spacelike/timelike?

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Perturbations about a mean-field solution

- Replace

$$e^m \to e^m + \varepsilon^m$$

$$\omega^{mn} \to \omega^{mn} + \lambda^{mn}$$

- Expand to quadratic order in (ε, λ)

$$S^{(2)} := -\frac{1}{2}\int \epsilon_{mnrs}\Big[e^m\varepsilon^n\nabla\lambda^{rs} + \frac{1}{2}\left(e^m e^n \lambda^r{}_t\lambda^{ts} + R^{mn}\varepsilon^r\varepsilon^s\right) - m^2\Big(3\beta_0 e^m e^n\varepsilon^r\varepsilon^s + 2\beta_1 e^m f^n\varepsilon^r\varepsilon^s + \beta_2 f^m f^n\varepsilon^r\varepsilon^s\Big)\Big]$$

- (e, ω) solution of non-linear system of eoms and constraints

Degrees of freedom and constraints

- Linearized equations of motion

$$\nabla\varepsilon^m + \lambda^{mn} e_n \approx 0$$

$$\frac{1}{2}\epsilon_{mnrs}\Big[e^n \nabla\lambda^{rs} + \varepsilon^n R^{rs}\Big] \approx m^2\,\tau_m$$

where $\tau_m := \epsilon_{mnrs}\Big[3\beta_0\, e^n e^r \varepsilon^s + 2\beta_1\, e^n f^r \varepsilon^s + \beta_2\, f^n f^r \varepsilon^s\Big]$

- Linearized constraints

$$\boldsymbol{\nabla\varepsilon}^m + \boldsymbol{\lambda}^{mn}\boldsymbol{e}_n \approx 0 \approx \frac{1}{2}\epsilon_{mnrs}\Big[\boldsymbol{e}^n \boldsymbol{\nabla\lambda}^{rs} + \boldsymbol{\varepsilon}^n \boldsymbol{R}^{rs}\Big] - m^2\,\boldsymbol{\tau}_m$$

$$\varepsilon^m f_m \approx 0 \approx \epsilon_{mnrs}\Big[M^{mn}\lambda^{rs} + 2(\beta_1 e^m + \beta_2 f^m)\,\varepsilon^n K^{rs}\Big]$$

$$\ldots$$

Characteristic matrix

Question

Hypersurface Σ: $\phi(x^\mu) = 0$ with normal covector $\xi_\mu = \partial_\mu \phi$

- Is the Cauchy problem for the perturbations well-defined?
- Does the system of PDEs (+constraints) determine the normal derivatives $\partial_\phi \varepsilon$, $\partial_\phi \lambda$ as a function of ε, λ and their tangential derivatives on Σ?

$\rightarrow$ Rewrite the system of PDEs (+ derivatives of constraints)

$$A^\mu \partial_\mu \begin{pmatrix} \varepsilon \\ \lambda \end{pmatrix} \equiv A^\mu \xi_\mu \partial_\phi \begin{pmatrix} \varepsilon \\ \lambda \end{pmatrix} \approx B$$

where $A^\mu(e, \omega)$, $B(e, \omega; \varepsilon, \lambda)$ are 40×40 matrices

Characteristic matrix

- $\chi := A^\mu \xi_\mu$ characteristic matrix
  - χ invertible on Σ: necessary condition for (locally) well-defined Cauchy problem
  - χ non-invertible at some point in Σ: Σ is a characteristic surface

! χ closely related to kinetic matrix for perturbations

→ May become non-invertible during time-evolution: sign of strong coupling

- Solving for ξ_μ such that $|\chi| = 0$ determines the causal structure of the theory: a cone sheet for each field component

! Hyperbolic system: real cone sheets

Characteristic matrix

- Analogous problem for first order differential equations

$$a(y,t)\dot{y} + b(y,t) = 0$$

  with $y(t_0) = y_0$

- IF $a(y_0, t_0) = 0$: impossible to evolve the differential equation for the given initial conditions at t_0!

→ Characteristic matrix invertibility properties: analogous statement of the problem for PDEs...

- Causal structure: maximal speeds of propagation for the different modes

N.B. Equivalent to speed of propagation of discontinuities/shock wave-fronts for different field components

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Rarita-Schwinger model

- Action

$$S_{3/2} = \int d^4x \overline{\psi}_\mu \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\mathcal{D}_\nu\psi_\rho$$

  where $\mathcal{D}_\nu := \partial_\nu + ieA_\nu + \frac{m}{2}\gamma_\nu$
- Equations of motion

$$\mathcal{R}^\mu := \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\mathcal{D}_\nu\psi_\rho \approx 0$$

- Constraints
  - 4 primaries: $\mu = 0$ part of the eom
  - 4 secondaries: $\mathcal{D}_\mu\mathcal{R}^\mu \approx 0$ since $[\mathcal{D}_\mu, \mathcal{D}_\nu] = ieF_{\mu\nu} + \frac{m^2}{2}\gamma_{[\mu}\gamma_{\nu]}$
- Degree of freedom count: 16-4-4=8

Velo-Zwanziger propagation analysis

- Characteristic determinant

$$|\chi| = (\xi^2)^4 \left[\xi^2 + \left(\frac{2e}{3m^2}\right)(\tilde{F}\cdot\xi)^2\right]^4$$

where $\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$

- Two different cone sheets $\xi^2 = 0$ and $\xi^2 + \left(\frac{2e}{3m^2}\right)(\tilde{F}\cdot\xi)^2 = 0$
- Is the extraordinary sheet spacelike?
  - If $\xi_\mu = (\xi, 0, 0, 0)$ then

$$|\chi| = \xi^{16}\left[1 - \left(\frac{2e}{3m^2}\right)^2 \boldsymbol{B}^2\right]^4$$

  - Zero after appropriate boosting!
- Different modes propagate on different cone sheets (some of them wider than Minkowski): crystal-like behavior

Analogy and expectations

- Analogy with perturbative mGR model

$$\psi_\mu \to (\varepsilon_\mu^m, \lambda_\mu^{mn})$$

$$A_\mu \to (e_\mu^m, \omega_\mu^{mn})$$

$$\eta_{\mu\nu} \to f_\mu^m$$

- Same propagation peculiarities as spin 3/2?
- Hyperbolic regime with different propagation speeds for different modes...

VZ-like result

- Special parameter choice: $\beta_2 = \beta_3 = 0$
- Fiducial vierbein flat: $f^m = \delta_\mu{}^m dx^\mu$
- Characteristic matrix written for $\xi_\mu = (1, 0, 0, 0)$

$$\begin{pmatrix} \mathbf{1}_{30\times 30} & 0 & 0 \\ 0 & \boldsymbol{f}_m & 0 \\ 0 & 2\epsilon_{mnrs}\boldsymbol{e}^n \times \boldsymbol{K}^{rs} & \epsilon_{mnrs}\boldsymbol{e}^r \times \boldsymbol{e}^s \\ 0 & \boldsymbol{f}^n \times \boldsymbol{K}_{nm} & \boldsymbol{f}_m \times \boldsymbol{e}_n \\ 0 & \mathcal{R}_m(\boldsymbol{e}, \boldsymbol{K}) & \mathcal{K}_{mn}(\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{K}) \end{pmatrix}$$

$\rightarrow$ Already different cone sheets seem to exists: spacelike?

VZ-like result

- Electric and magnetic part of $\boldsymbol{K}^{mn}$

$$\boldsymbol{E}^a := \boldsymbol{K}^{0a} \quad \text{and} \quad \boldsymbol{B}_a := \epsilon_{abc}\boldsymbol{K}^{bc}$$

- Background choice $\boldsymbol{e}^0 = \boldsymbol{E}^a = 0$
- Characteristic determinant ("roughly")

$$(\boldsymbol{B}^a \cdot \tilde{\mathbf{e}}^b)(\boldsymbol{B}_{[a} \cdot \tilde{\mathbf{e}}_{b]}) - \frac{1}{2}(\boldsymbol{B}^a \cdot \tilde{\mathbf{e}}_{[a})(\boldsymbol{B}^b \cdot \tilde{\mathbf{e}}_{b]}) + 6m^2_{\mathrm{FP}}\left((\boldsymbol{f}^a \cdot \tilde{\mathbf{e}}_a) - 4\right)$$

where $\tilde{\mathbf{e}}_a$ is the 3-inverse of $\boldsymbol{e}^a$

+ VZ boosting argument

Superluminality vs acausality

- If hyperbolic system i.e. real cone sheets: no local acausality!
- Depends on the choice of background

→ Closed time-like curves form for particular choices of backgrounds where hyperbolicity fails

- BUT different modes propagate at different speeds some superluminal: cf Galileons
- Birefringence properties of the vacuum for the massive graviton...

Conclusion

- dRGT covariant constraint analysis
- Characteristic matrix computation in non-linear theory = Characteristic matrix computation for linearized theory about some solution
- Analogy with Rarita-Schwinger model
- Some backgrounds: non-hyperbolicity, acausality

→ Non-hyperbolic case can be reached dynamically: strong coupling

- When hyperbolic: multiple causal cone sheets...

→ Some modes are superluminal!