“Covariant" constraints in bimetric and massive gravity

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Outline of the talk

Basic Motivations

Linear massive spin-2 fields + History

Covariant constraints counting in Fierz-Pauli

Bimetric theory, some necessary details

Linearised theory

Covariant constraints analysis in Massive gravity

“Covariant" constraints in full bimetric theory

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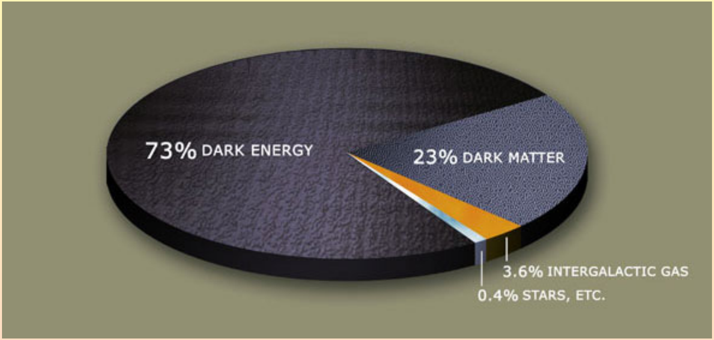
Motivation I (Original, most recently)

Cosmology and the Dark sector(s)



- Cosmological constant problem. Can a small graviton mass naturally explain the observed smallness of Λ?
- The Dark sectors. Can a modification of Einstein gravity “remedy" the inclusion of unknown energy sources?

Motivation II (Original in strict sense & more general)

To understand spin-2 interactions in field theory

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Good understanding of lower spin theories

- Spin-0: Higgs(?!), Inflaton(??), $\pi^{0,\pm}$ mesons,...
- Spin-1/2: Leptons, quarks, baryons
- Spin-1: Photon, gluons, Z, $W^{\pm}$, vector mesons,...

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  Klein-Gordon: $(\Box - m^2)\,\phi = 0$

- Spin-1/2: Leptons, quarks, baryons

  Dirac: $(i\partial\!\!\!/ - m)\,\psi = 0$

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  Maxwell, Proca: $(\Box - m^2)\,A_\mu = 0$

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What about spin-2?

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  Einstein-Hilbert (massless): $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$

But no consistent theory of massive/interacting spin-2??

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But no consistent theory of massive/interacting spin-2 ...
until recently!!

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Fierz-Pauli theory

The FP equation:

Linear massive spin-2 field $h_{\mu\nu}$ in background $\bar{g}_{\mu\nu}$

$$\mathcal{E}^{\rho\sigma}_{\mu\nu}\, h_{\rho\sigma} - \Lambda\Big(h_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu} h^{\rho}_{\rho}\Big) + \tfrac{m^2_{\mathrm{FP}}}{2}\,\big(h_{\mu\nu} - a\bar{g}_{\mu\nu} h^{\rho}_{\rho}\big) = 0$$

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[Fierz, Pauli (1939)]

$$\Big(\Box - m^2_{\rm FP}\Big)\, h_{\mu\nu} + 2R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} h_{\rho\sigma} = 0\,, \quad \nabla^{\mu} h_{\mu\nu} = 0\,,$$

$$\Big(2\Lambda - 3m^2_{\rm FP}\Big)\, \bar{g}^{\mu\nu} h_{\mu\nu} = 0$$

Fierz-Pauli theory

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$$\Big(2\Lambda - 3m^2_{\rm FP}\Big)\, \bar{g}^{\mu\nu} h_{\mu\nu} = 0$$

Problem: Nonlinear completion in terms of $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$.
Generically removes a constraint, resulting in a propagating ghost-mode. *[Boulware,Deser (1972)]*

Historical progress

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- Fully dynamical theory of interacting spin-2 shown to be ghost free nonlinearly *[Hassan, Rosen (2011-2012)]*

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Linear massive spin-2 field $h_{\mu\nu}$ in background $\eta_{\mu\nu}$

$$\delta E_{\mu\nu} = \mathcal{E}_{\mu\nu}^{\rho\sigma}\, h_{\rho\sigma} + \frac{m^2}{2}\, \left(h_{\mu\nu} - \eta_{\mu\nu} h_\rho^\rho\right) = 0$$

$$\mathcal{E}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} = -\frac{1}{2}\Big[\delta_\mu^\rho \delta_\nu^\sigma \nabla^2 + \eta^{\rho\sigma}\nabla_\mu \nabla_\nu - \delta_\mu^\rho \nabla^\sigma \nabla_\nu - \delta_\nu^\rho \nabla^\sigma \nabla_\mu$$
$$- \eta_{\mu\nu}\eta^{\rho\sigma}\nabla^2 + \eta_{\mu\nu}\nabla^\rho \nabla^\sigma\Big] h_{\rho\sigma}$$

Fierz-Pauli theory

Divergence, $\nabla^{\mu}\delta E_{\mu\nu} = \frac{m^2}{2}\left(\nabla^{\mu}h_{\mu\nu} - \nabla_{\nu}h\right)$

implies on-shell $\nabla^{\mu}h_{\mu\nu} = \nabla_{\nu}h$

Double divergence, $\nabla^{\mu}\nabla^{\nu}\delta E_{\mu\nu} = \frac{m^2}{2}\left(\nabla^{\mu}\nabla^{\nu}h_{\mu\nu} - \nabla^2 h\right)$

and trace, $\eta^{\mu\nu}\delta E_{\mu\nu} = \nabla^2 h - \nabla^{\mu}\nabla^{\nu}h_{\mu\nu} - \frac{3m^2}{2}h$

combined, $\nabla^{\mu}\nabla^{\nu}\delta E_{\mu\nu} - \frac{m^2}{2}\eta^{\mu\nu}\delta E_{\mu\nu} = \frac{3m^4}{4}h$

implies on-shell $h = 0$

Resulting in system

$$\left(\nabla^2 - m^2\right)h_{\mu\nu} = 0\,, \quad \nabla^{\mu}h_{\mu\nu} = 0\,, \quad h = 0$$

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Ghost free bimetric theory

The basic construction:

$$\mathcal{L} = m_g^2 \sqrt{|g|} R(g) - 2m^4 \sqrt{|g|} V(S;\beta_n) + m_f^2 \sqrt{|f|} R(f)$$

$$S = \sqrt{g^{-1} f}\,, \quad \sqrt{|g|} V(S;\beta_n) = \sqrt{|f|} V(S^{-1};\beta_{d-n})$$

$$V(S;\beta_n) = \sum_{n=0}^{4} \beta_n e_n(S) = \beta_0 + \sum_{n=1}^{3} \beta_n e_n(S) + \sqrt{|g^{-1} f|}\beta_d$$

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Elementary symmetric polynomials

$$e_0(\mathbb{X}) = 1\,, \qquad e_1(\mathbb{X}) = [\mathbb{X}]\,, \qquad e_2(\mathbb{X}) = \tfrac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]),$$

$$e_3(\mathbb{X}) = \tfrac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3])\,,$$

$$e_4(\mathbb{X}) = \tfrac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4])\,,$$

$$\vdots$$

$$e_d(\mathbb{X}) = \det(\mathbb{X})$$

$$e_k(\mathbb{X}) = 0 \quad \text{for} \quad k > d\,, \quad [\ e_n(\mathbb{X}) \sim (\mathbb{X})^n\ ]$$

Ghost free bimetric theory, contd.

Equations of motion:

$$R_{\mu\nu}(g) - \tfrac{1}{2} g_{\mu\nu} R(g) + \tfrac{m^4}{m_g^2} V^g_{\mu\nu} = 0$$

$$R_{\mu\nu}(f) - \tfrac{1}{2} f_{\mu\nu} R(f) + \tfrac{m^4}{m_f^2} V^f_{\mu\nu} = 0$$

related through interchange symmetry

$$g_{\mu\nu} \leftrightarrow f_{\mu\nu}\,, \quad \beta_n \leftrightarrow \beta_{4-n}\,, \quad m_g \leftrightarrow m_f$$

Bianchi constraints (for conserved sources)**:**

$${}^g\nabla^\mu V^g_{\mu\nu} = 0 = {}^f\nabla^\mu V^f_{\mu\nu}$$

related through the covariance identity

$$\sqrt{|g|}\,{}^g\nabla^\mu V^g_{\mu\nu} = -\sqrt{|f|}\,{}^f\nabla^\mu V^f_{\mu\nu}$$

Ghost free bimetric theory, contd.

Equations of motion:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \tfrac{m^4}{m_g^2} V_{\mu\nu} = 0$$

$$\tilde{R}_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu} R + \tfrac{m^4}{m_f^2} \tilde{V}_{\mu\nu} = 0$$

related through interchange symmetry

$$g_{\mu\nu} \leftrightarrow f_{\mu\nu}\,, \quad \beta_n \leftrightarrow \beta_{4-n}\,, \quad m_g \leftrightarrow m_f$$

Bianchi constraints (for conserved sources)**:**

$$\nabla^\mu V_{\mu\nu} = 0 = \tilde{\nabla}^\mu \tilde{V}_{\mu\nu}$$

related through the covariance identity

$$\sqrt{|g|}\, \nabla^\mu V_{\mu\nu} = -\sqrt{|f|}\, \tilde{\nabla}^\mu \tilde{V}_{\mu\nu}$$

Ghost free bimetric theory, contd.

For example

$$V_{\mu\nu} \equiv -\frac{2}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|}V)}{\partial g^{\rho\sigma}} \delta g^{\rho\sigma}$$

Matrix polynomial in $S = \sqrt{g^{-1}f}$

$$\begin{aligned} V_{\mu\nu} =& g_{\mu\rho}\Big[\beta_0 \delta^\rho_\nu - \beta_1 \left(S^\rho{}_\nu - e_1 \delta^\rho_\nu\right) + \beta_2 \left([S^2]^\rho{}_\nu - e_1 S^\rho{}_\nu + e_2 \delta^\rho_\nu\right) \\ & - \beta_3 \left([S^3]^\rho{}_\nu - e_1 [S^2]^\rho{}_\nu + e_2 S^\rho{}_\nu - e_3 \delta^\rho_\nu\right)\Big] \end{aligned}$$

Massive gravity limit of bimetric theory

Define $m^4 \equiv m_g^2 m^2$ and define $\alpha \equiv m_f/m_g$

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + m^2 V^g_{\mu\nu} = 0$$

$$R_{\mu\nu}(f) - \frac{1}{2}f_{\mu\nu}R(f) + \frac{m^2}{\alpha^2} V^f_{\mu\nu} = 0$$

In the limit $\alpha \to \infty$ and $\beta_4 = \alpha^2 \Lambda_f / m^2$

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + m^2 V^g_{\mu\nu} = 0$$

$$R_{\mu\nu}(f) - \frac{1}{2}f_{\mu\nu}R(f) + \Lambda_f f_{\mu\nu} = 0$$

Outline of the talk

Linearisation, general outline

Linearised equations of motion

$$\delta E_{\mu\nu} = \delta_g \mathcal{G}_{\mu\nu} + m^2 \delta_g V_{\mu\nu} + m^2 \delta_f V_{\mu\nu} = 0$$

$$\delta \tilde{E}_{\mu\nu} = \delta_f \tilde{\mathcal{G}}_{\mu\nu} + \frac{m^2}{\alpha^2} \delta_f \tilde{V}_{\mu\nu} + \frac{m^2}{\alpha^2} \delta_g \tilde{V}_{\mu\nu} = 0$$

Linearisation, general outline

Linearised equations of motion

$$\delta E_{\mu\nu} = \delta_g \mathcal{G}_{\mu\nu} + m^2 \delta_g V_{\mu\nu} + m^2 \delta_f V_{\mu\nu} = 0$$

$$\delta \tilde{E}_{\mu\nu} = \delta_f \tilde{\mathcal{G}}_{\mu\nu} + m^2 \delta_f \tilde{V}_{\mu\nu} + m^2 \delta_g \tilde{V}_{\mu\nu} = 0$$

Now $\alpha \to \infty$ implies $\delta f_{\mu\nu} \to 0$

Linearisation, kinetic terms

$$\delta\mathcal{G}_{\mu\nu} = \mathcal{E}_{\mu\nu}{}^{\rho\sigma}\delta g_{\rho\sigma} + \frac{m^2}{4}\left[g_{\mu\nu}g^{\rho\sigma}g^{\lambda\omega}V_{\lambda\omega} - 2g_{\mu\nu}V^{\rho\sigma} - 2\delta^\rho_\mu\delta^\sigma_\nu g^{\lambda\omega}V_{\lambda\omega}\right]\delta g_{\rho\sigma}$$

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma}\delta g_{\rho\sigma} = -\frac{1}{2}\Big[\delta^\rho_\mu\delta^\sigma_\nu\nabla^2 + g^{\rho\sigma}\nabla_\mu\nabla_\nu - \delta^\rho_\mu\nabla^\sigma\nabla_\nu - \delta^\rho_\nu\nabla^\sigma\nabla_\mu - g_{\mu\nu}g^{\rho\sigma}\nabla^2 + g_{\mu\nu}\nabla^\rho\nabla^\sigma\Big]\delta g_{\rho\sigma}$$

Linearised Bianchi identity $\delta\left(g^{\mu\rho}\nabla_\rho\mathcal{G}_{\mu\nu}\right) = 0$ imply the following linear identity

$$\nabla^\mu\delta\mathcal{G}_{\mu\nu} = \delta g_{\mu\rho}\nabla^\rho\mathcal{G}^\mu{}_\nu + \mathcal{G}^\sigma{}_\nu\nabla^\rho\delta g_{\sigma\rho} - \frac{1}{2}g^{\lambda\rho}\mathcal{G}_{\nu\sigma}\nabla^\sigma\delta g_{\lambda\rho} + \frac{1}{2}\mathcal{G}^{\sigma\mu}\nabla_\nu\delta g_{\mu\sigma}$$

Linearisation, interactions

$$\begin{aligned}\delta V_{\mu\nu} &= \delta_g V_{\mu\nu} + \delta_f V_{\mu\nu} \\ &= g^{\rho\sigma} V_{\sigma\nu} \delta_g g_{\rho\nu} - g_{\mu\rho} \sum_{n=1}^{3} (-1)^n \beta_n \\ &\quad \times \sum_{k=1}^{n} (-1)^k \Big\{ \left[S^{n-k} \right]^{\rho}{}_{\nu} \sum_{m=1}^{k} (-1)^m e_{k-m}(S) \left[S^{m-1} \delta S \right]^{\sigma}{}_{\sigma} \\ &\qquad + e_{k-1}(S) \sum_{m=0}^{n-k} \left[S^m \delta S S^{n-k-m} \right]^{\rho}{}_{\nu} \Big\}\end{aligned}$$

$$\delta \tilde{V}_{\mu\nu} = \delta V_{\mu\nu} (g_{\mu\nu} \to f_{\mu\nu}, S \to S^{-1}, \beta_n \to \beta_{4-n}, V_{\mu\nu} \to \tilde{V}_{\mu\nu})$$

The square root matrix and its linearisation

How compute the variation of a square root matrix in closed form? From the relation $S^2 = g^{-f}$ we know

$$\delta S^2 = -g^{-1}\delta g S^2 + g^{-1}\delta f$$

Approach I: The Cayley-Hamilton theorem

Approach II: The Sylvester matrix equation

The square root matrix and its linearisation

Approach I: The Cayley-Hamilton theorem

$$S^4 - e_1 S^3 + e_2 S^2 - e_3 S + e_4 \mathbb{1} = 0$$

Provides the solution

$$\delta S = \mathbb{X}^{-1} \Big[\delta S^2 [S^2 - e_1 S] + [S^2 + e_2 \mathbb{1}] \delta S^2 - \frac{1}{2} \sum_{m=1}^{4} \sum_{k=1}^{m} (-1)^{m+k} e_{m-k} S^{4-m} \, \mathrm{Tr}[S^{k-2} \delta S^2] \Big]$$

with

$$\mathbb{X}^{-1} = \frac{(e_3 - e_1 e_2)\, \mathbb{1} + e_1^2 S - e_1 S^2}{e_1^2 e_4 + e_3^2 - e_1 e_2 e_3}$$

The square root matrix and its linearisation

Approach II: The Sylvester matrix equation

$$AX + XB = C$$

Unique solution iff $\sigma(A) \cap \sigma(B) = \emptyset$

Our case

$$S\delta S + \delta S S = \delta S^2$$

has the unique solution

$$\delta S = \frac{1}{2}\mathbb{X}^{-1} \sum_{k=1}^{4} \sum_{m=0}^{k-1} (-1)^m e_{4-k}(S) S^{k-m-2} \delta S^2 S^m$$

iff $\sigma(S) \cap \sigma(-S) = \emptyset$

Outline of the talk

Redefined fluctuations

Consider new fluctuations $\delta g'$ and $\delta f'$ defined by

$$\delta S = -g^{-1}\delta g' S^2 + S^{-1} g^{-1} \delta f' S^{-1}$$

We have

$$g^{-1}\delta g = S g^{-1}\delta g' + g^{-1}\delta g' S$$

and

$$f^{-1}\delta f = S^{-1} f^{-1}\delta f' + f^{-1}\delta f' S^{-1}$$

These are Sylvester type equations. The redefinitions $\delta g'$ and $\delta f'$ exist and are uniquely invertible iff $\sigma(S) \cap \sigma(-S) = \emptyset$

Dramatically simplifies the linearised interaction terms.

Full Massive gravity analysis, $\delta f_{\mu\nu} = 0$

Construct generalised traces

$$\Phi_k^g \equiv [S^k]^\nu{}_\rho \, g^{\rho\mu} \delta E_{\mu\nu}$$

and generalised divergences

$$\Psi_k^g \equiv [S^k]^\nu{}_\rho \nabla^\rho \nabla^\mu \delta E_{\mu\nu}$$

Find linear combination

$$\mathcal{C}_g \equiv \sum_{k=0}^{3} \left(u_k^g \Phi_k^g + v_k^g \Psi_k^g \right) \sim 0$$

Only $k = 0, 1, 2, 3$ necessary to consider due to C-H theorem

$$S^4 - e_1 S^3 + e_2 S^2 - e_3 S + e_4 \mathbb{1} = 0$$

Generalised traces

The relevant ones for our purposes are given by

$$\Phi_0^g \sim -2\Big[g^{\rho\kappa}S^{\sigma\mu} - S^{\rho\sigma}g^{\kappa\mu}\Big]\nabla_\kappa\nabla_\mu\delta g'_{\rho\sigma}$$

$$\Phi_1^g \sim -\Big[[S^2]^{\rho\sigma}g^{\kappa\mu} + S^{\rho\sigma}S^{\kappa\mu} - S^{\sigma\kappa}S^{\rho\mu} - [S^2]^{\sigma\mu}g^{\rho\kappa} - e_1 S^{\rho\sigma}g^{\kappa\mu} + e_1 g^{\rho\kappa}S^{\sigma\mu}\Big]\nabla_\kappa\nabla_\mu\delta g'_{\rho\sigma}$$

$$\Phi_2^g \sim -\Big[[S^3]^{\rho\sigma}g^{\kappa\mu} + S^{\rho\sigma}[S^2]^{\kappa\mu} - S^{\sigma\kappa}[S^2]^{\rho\mu} - [S^3]^{\sigma\mu}g^{\rho\kappa} - \left(e_1^2 - 2e_2\right)S^{\rho\sigma}g^{\kappa\mu} + \left(e_1^2 - 2e_2\right)g^{\rho\kappa}S^{\sigma\mu}\Big]\nabla_\kappa\nabla_\mu\delta g'_{\rho\sigma}$$

Generalised divergences

$$
\begin{aligned}
\bar{\Psi} \sim m^2 \Big\{ & \beta_1 \left[g^{\rho\kappa} S^{\sigma\mu} - S^{\rho\sigma} g^{\mu\kappa} \right] \\
& + \beta_2 \Big[g^{\mu\kappa} [S^2]^{\rho\sigma} + S^{\rho\sigma} S^{\mu\kappa} - g^{\mu\rho} [S^2]^{\sigma\kappa} - S^{\mu\rho} S^{\sigma\kappa} \\
& \qquad + e_1 g^{\mu\rho} S^{\sigma\kappa} - e_1 S^{\rho\sigma} S^{\mu\kappa} \Big] \\
& + \beta_3 \Big[g^{\mu\rho} [S^3]^{\sigma\kappa} - g^{\mu\kappa} [S^3]^{\rho\sigma} - S^{\rho\sigma} [S^2]^{\mu\kappa} + S^{\mu\rho} [S^2]^{\sigma\kappa} \\
& \qquad + [S^2]^{\mu\rho} S^{\sigma\kappa} - S^{\mu\kappa} [S^2]^{\rho\sigma} \\
& \qquad + e_1 g^{\mu\kappa} [S^2]^{\rho\sigma} + e_1 S^{\rho\sigma} S^{\mu\kappa} \\
& \qquad - e_1 g^{\mu\rho} [S^2]^{\sigma\kappa} - e_1 S^{\mu\rho} S^{\sigma\kappa} \\
& \qquad + e_2 g^{\mu\rho} S^{\sigma\kappa} - e_2 S^{\rho\sigma} g^{\mu\kappa} \Big] \Big\} \nabla_\kappa \nabla_\mu \delta g'_{\rho\sigma}
\end{aligned}
$$

Problem when $\beta_3 \neq 0$

By inspection one easily finds the constraint

$$\mathcal{C}_g = \frac{1}{2}\beta_1 \Phi_0^g + \beta_2 \Phi_1^g - \beta_3 \left(\Phi_2^g - e_1 \Phi_1^g + \frac{1}{2} e_2 \Phi_0^g \right) + \frac{1}{m^2} \bar{\Psi}$$
$$\sim \beta_3 \left([S^2]^{\mu\rho} S^{\sigma\kappa} - S^{\mu\kappa} [S^2]^{\rho\sigma} \right) \nabla_\kappa \nabla_\mu \delta g'_{\rho\sigma}$$

Constraint is manifestly covariant if $\beta_3 = 0$. For β_3 it has to be checked in component form after a $3 + 1$ split.

Dramatically simplified analysis with the redefined variables.

Outline of the talk

Statement of problem & Connection to MG analysis

Construct linear combination

$$\mathcal{C} = \mathcal{C}_g + \mathcal{C}_f = \sum_{k=0}^{3} \left(u_k^g \Phi_k^g + v_k^g \Psi_k^g + u_k^f \Phi_k^f + v_k^f \Psi_k^f \right)$$

Due to Massive gravity analysis all coefficients are known and only remaining term to study is

$$[S^{-1}]^{\nu}{}_{\kappa} g^{\kappa\lambda} g^{\mu\sigma} \nabla_\lambda \nabla_\sigma \delta_f V_{\mu\nu} + S^{\nu}{}_{\kappa} f^{\kappa\lambda} f^{\mu\sigma} \tilde{\nabla}_\lambda \tilde{\nabla}_\sigma \delta_g \tilde{V}_{\mu\nu}$$

These have to vanish separately. There are no covariant constraints. Needs to be checked after $3 = 1$ split

Outline of the talk

Summary

- Full linearisation of bimetric theory and Massive gravity for arbitrary backgrounds
- Linearised problem only well-defined if $\sigma(S) \cap \sigma(-S) = \emptyset$
- Simplified linearisation in terms of redefined fluctuations, which manifestly encode $\sigma(S) \cap \sigma(-S) = \emptyset$
- Existence of a scalar constraint necessary to remove the B-D ghost
- Covariant constraint in Massive gravity if $\beta_3 = 0$
- No covariant constraint if $\beta_3 \neq 0$
- No covariant constraint in bimetric theory

The end is only the beginning . . .

Thanks for your attention!