Stueckelberg massive electromagnetism in curved spacetime : Hadamard renormalization of the stress-energy tensor and the Casimir effect

Andréi Belokogne *

UMR CNRS 6134 SPE, Faculté des Sciences,
Université de Corse, BP 52, F-20250 Corse, France
Projet : COMPA
Équipe physique théorique

SW10 – Hot Topics In Modern Cosmology
Cargèse, Corsica (France)

May 9-14, 2016

*. In collaboration with Antoine Folacci [Ref. Phys. Rev. D 93, 044063 (2016), arXiv:1512.06326]

Massive photon

- The electromagnetic interaction is generally assumed to be mediated by a massless photon.
- Indeed, this consideration is mainly justified by
  - the theoretical and practical successes of the classical *Maxwell's theory* of electromagnetism and its extension in the framework of quantum field theory,
  - the upper limits on the photon mass $m \leq 10^{-18}$ eV $\approx 2 \times 10^{-54}$ kg which is currently one of the most reliable results evaluated by the various terrestrial and extraterrestrial experiments.
- However, it is interesting to consider the possibility of a massive but ultralight photon for the following reasons :
  - despite of the incredibly small value mentioned above, it is not necessary that the photon mass is exactly zero ;
  - moreover, in order to test the masslessness of the photon or, more precisely, to impose experimental constraints on its mass, it is necessary to have a good understanding of the various *massive non-Maxwellian theories* ;
  - furthermore, from a theoretical point of view, massive electromagnetism can be rather easily included in the Standard Model of particle physics.
- In this work, among *massive non-Maxwellian theories*, we discuss two particularly important theories :
  - *de Broglie-Proca massive electromagnetism*,
  - *Stueckelberg massive electromagnetism*.

*. Here, we consider a four-dimensional curved spacetime $(\mathcal{M}, g)$ without boundary.

De Broglie-Proca massive electromagnetism

- *De Broglie-Proca massive electromagnetism* is the simplest generalization of *Maxwell's electromagnetism*.
  - This theory is described by a vector field A_μ of mass m.
  - Its action S, which is directly obtained from the original Maxwell Lagrangian by adding a mass contribution, is given by

$$S\left[A_\mu, g_{\mu\nu}\right] = \int_{\mathcal{M}} dx \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \right].$$

  - The extremization of S with respect to A_μ leads to the Proca equation

$$\nabla^\nu F_{\mu\nu} + m^2 A_\mu = 0.$$

- It is worth pointing out that, due to the mass term,
  - contrary to the *Maxwell's theory* which is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \nabla_\mu \Lambda$$

    for an arbitrary scalar field Λ, this gauge invariance is broken for the *de Broglie-Proca theory* ;
  - there are some important consequences when we compare, in the limit $m^2 \to 0$, the results obtained via the *de Broglie-Proca theory* with those derived from *Maxwell's theory*.

- It is also important to recall that, in general, it is the *de Broglie-Proca theory* that is used to impose experimental constraints on the photon mass.

---

*. It is interesting to note that this theory is mainly due to de Broglie but is attributed in the literature to its "PhD student" Proca.

*. The field strength is defined by $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$.

Stueckelberg massive electromagnetism

- *Stueckelberg massive electromagnetism* is the most aesthetically appealing one which, contrarily to the *de Broglie-Proca theory*, preserves the local $U(1)$ gauge invariance of *Maxwell's electromagnetism*.
  - This theory is constructed in such a way that a massive vector field A_μ is coupled appropriately with an auxiliary scalar field Φ.
  - At the classical level, its action $S_{\rm cl}$ is given by

$$S_{\rm cl}\left[A_\mu,\Phi,g_{\mu\nu}\right]=\int_{\mathcal{M}} d^4x\sqrt{-g}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}m^2\left(A^\mu+\frac{1}{m}\nabla^\mu\Phi\right)\left(A_\mu+\frac{1}{m}\nabla_\mu\Phi\right)\right].$$

  - This action is invariant under the gauge transformation

$$A_\mu\to A'_\mu=A_\mu+\nabla_\mu\Lambda,$$
$$\Phi\to\Phi'=\Phi-m\Lambda$$

    for an arbitrary scalar field Λ.
  - The extremization of $S_{\rm cl}$ with respect to A_μ and Φ leads to two coupled wave equations

$$\nabla^\nu F_{\mu\nu}+m^2A_\mu+m\nabla_\mu\Phi=0,$$
$$\Box\Phi+m\nabla^\mu A_\mu=0.$$

- It should be noted that the Stueckelberg action $S_{\rm cl}$ can be constructed from the de Broglie-Proca action S by using the substitution

$$A_\mu\to A_\mu+\frac{1}{m}\nabla_\mu\Phi.$$

*. The field strength is defined by $F_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu$.

Some remarks relative to both theories

- It is worth noting that
  - the *de Broglie-Proca theory* can be obtained from *Stueckelberg electromagnetism* by taking
  $$\Phi = 0;$$
  - therefore, the *de Broglie-Proca theory* is nothing other than the *Stueckelberg gauge theory* in this particular gauge ;
  - however, this is a "bad" choice of gauge leading to some complications ;
  - indeed, in this gauge we obtain
  $$\nabla^\mu A_\mu = 0.$$
  Due to this constraint, at the quantum level, the Feynman propagator does not admit a Hadamard representation and, as a consequence, in the *de Broglie-Proca theory*, we cannot deal directly with Hadamard quantum states.
- In order to treat these theories at the quantum level,
  - the action S of the *de Broglie-Proca theory* is directly relevant,
  - while it is necessary to add to the action S_{cl} of the *Stueckelberg theory* a gauge-breaking term and the compensating ghost contribution.

*. Applying ∇^μ to the Proca equation, we obtain the Lorenz condition $\nabla^\mu A_\mu = 0$ which is a dynamical constraint (and not a gauge condition having in mind that the *de Broglie-Proca theory* is not a gauge theory).

Definition of geometrical quantities useful in the context of the renormalization

- Here, we recall some important definitions concerning
  - the geodetic distance $\sigma(x,x')$ which is the one-half of the square of the geodesic distance between the points x and x' and satisfies the partial differential equation

$$2\sigma = \sigma^{;\mu}\sigma_{;\mu},$$

  - the Van Vleck-Morette determinant $\Delta(x,x')$ which is given by

$$\Delta(x,x') = -[-g(x)]^{-1/2}\det(-\sigma_{;\mu\nu'}(x,x'))[-g(x')]^{-1/2}$$

    and satisfies the partial differential equation

$$\Box_x \sigma = 4 - 2\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu}$$

    with the boundary condition

$$\lim_{x'\to x} \Delta(x,x') = 1,$$

  - the bivector of parallel transport from x to x' denoted by $g_{\mu\nu'}(x,x')$ which is defined by the partial differential equation

$$g_{\mu\nu';\rho}\sigma^{;\rho} = 0$$

    with the boundary condition

$$\lim_{x'\to x} g_{\mu\nu'}(x,x') = g_{\mu\nu}(x).$$

*. We have $\sigma(x,x') < 0$ if x and x' are timelike related, $\sigma(x,x') = 0$ if x and x' are null related and $\sigma(x,x') > 0$ if x and x' are spacelike related.

Quantum action of Stueckelberg massive electromagnetism

- The quantum action S of *Stueckelberg massive electromagnetism* is given by

$$S\left[A_\mu,\Phi,C,C^*,g_{\mu\nu}\right]=S_{\mathrm{Cl}}\left[A_\mu,\Phi,g_{\mu\nu}\right]+S_{\mathrm{GB}}\left[A_\mu,\Phi,g_{\mu\nu}\right]+S_{\mathrm{Gh}}\left[C,C^*,g_{\mu\nu}\right]$$

with

$$S_{\mathrm{cl}}\left[A_\mu,\Phi,g_{\mu\nu}\right]=\int_{\mathcal{M}} d^4x\sqrt{-g}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}m^2\left(A^\mu+\frac{1}{m}\nabla^\mu\Phi\right)\left(A_\mu+\frac{1}{m}\nabla_\mu\Phi\right)\right],$$

$$S_{\mathrm{GB}}\left[A_\mu,\Phi,g_{\mu\nu}\right]=\int_{\mathcal{M}}\mathrm{d}x\sqrt{-g}\left[-\frac{1}{2}\left(\nabla^\mu A_\mu+m\,\Phi\right)^2\right],$$

$$S_{\mathrm{Gh}}\left[C,C^*,g_{\mu\nu}\right]=\int_{\mathcal{M}}\mathrm{d}x\sqrt{-g}\left[\nabla^\mu C^*\nabla_\mu C+m^2C^*C\right].$$

- By collecting the fields in the quantum action S, its expression can be written in the form

$$S\left[A_\mu,\Phi,C,C^*,g_{\mu\nu}\right]=S_A\left[A_\mu,g_{\mu\nu}\right]+S_\Phi\left[\Phi,g_{\mu\nu}\right]+S_{\mathrm{Gh}}\left[C,C^*,g_{\mu\nu}\right],$$

where the contributions of the A_μ and Φ fields are explicitly separated and are given by

$$S_A\left[A_\mu,g_{\mu\nu}\right]=\int_{\mathcal{M}}\mathrm{d}x\sqrt{-g}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}m^2A^\mu A_\mu-\frac{1}{2}\left(\nabla^\mu A_\mu\right)^2\right],$$

$$S_\Phi\left[\Phi,g_{\mu\nu}\right]=\int_{\mathcal{M}}\mathrm{d}x\sqrt{-g}\left[-\frac{1}{2}\nabla^\mu\Phi\nabla_\mu\Phi-\frac{1}{2}m^2\Phi^2\right].$$

*. In second form of S two coupling terms $-mA^\mu\nabla_\mu\Phi$ of S_{cl} and $-m\Phi\nabla^\mu A_\mu$ of S_{GB} have disappeared because spacetime is assumed with no boundary.

Wave equations

- The vanishing of the functional derivatives with respect to the fields A_μ, Φ, C and C^* of the quantum action S provides the wave equations
  - for the massive vector field A_μ

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu} = \left[g^{\mu\nu}\Box - m^2 g^{\mu\nu} - R^{\mu\nu}\right]A_\nu = 0,$$

  - for the auxiliary scalar field Φ

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta \Phi} = \left[\Box - m^2\right]\Phi = 0,$$

  - for the ghost fields C and C^*

$$\frac{1}{\sqrt{-g}}\frac{\delta_{\mathrm{L}} S}{\delta C^*} = -\left[\Box - m^2\right]C = 0$$

and

$$\frac{1}{\sqrt{-g}}\frac{\delta_{\mathrm{R}} S}{\delta C} = -\left[\Box - m^2\right]C^* = 0.$$

*. Due to the fermionic behavior of the ghost fields C and C^*, the right functional derivative $\frac{\delta_{\mathrm{R}}}{\delta C^*}$ and the left functional derivative $\frac{\delta_{\mathrm{L}}}{\delta C}$ are introduced in order to derive the associated wave equations.

Feynman propagators and Ward identities

- From now on, we shall assume that the *Stueckelberg field theory* previously described has been quantized and is in a normalized quantum state $|\psi\rangle$.

- Feynman propagators :
  - The Feynman propagator $G^{A}_{\mu\nu'}(x,x') = i\langle\psi|TA_\mu(x)A_{\nu'}(x')|\psi\rangle$ associated with the vector field A_μ is a solution of the wave equation
  $$\left[g_\mu{}^\nu\Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] G^{A}_{\nu\rho'}(x,x') = -g_{\mu\rho'}\,\delta^4(x,x').$$
  - The Feynman propagator $G^{\Phi}(x,x') = i\langle\psi|T\Phi(x)\Phi(x')|\psi\rangle$ associated with the scalar field Φ is a solution of the wave equation
  $$\left[\Box_x - m^2\right] G^{\Phi}(x,x') = -\delta^4(x,x').$$
  - The Feynman propagator $G^{\mathrm{Gh}}(x,x') = i\langle\psi|TC^*(x)C(x')|\psi\rangle$ associated with the ghost fields C^* and C is a solution of the wave equation
  $$\left[\Box_x - m^2\right] G^{\mathrm{Gh}}(x,x') = -\delta^4(x,x').$$

- The three propagators are related by two Ward identities :
  - the first one relates the vector and ghost propagators in the form
  $$\nabla^\mu G^{A}_{\mu\nu'}(x,x') + \nabla_{\nu'} G^{\mathrm{Gh}}(x,x') = 0,$$
  - while the second provides another relation between the scalar and ghost propagators given by
  $$G^{\Phi}(x,x') - G^{\mathrm{Gh}}(x,x') = 0 \quad\Rightarrow\quad G(x,x') = G^{\Phi}(x,x') = G^{\mathrm{Gh}}(x,x').$$

---

*. T denotes the time-ordered product.

*. The Ward identities can be obtained from the wave equations by using the approach of DeWitt and Brehme or from BRST invariance.

Hadamard representation of the Feynman propagator associated with the scalar field Φ or the ghost fields

- We now assume that the quantum state $|\psi\rangle$ is of Hadamard type.

- The Hadamard form of the Feynman propagator for the scalar field Φ or the ghost fields is given by

$$G(x,x') = \frac{i}{8\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon} + V(x,x')\ln[\sigma(x,x')+i\epsilon] + W(x,x')\right).$$

  Here, $\sigma(x,x')$ is the geodetic distance, and $\Delta(x,x')$ is the Van Vleck-Morette determinant, while $V(x,x')$ and $W(x,x')$ are *symmetric* and *regular* biscalars given by the series expansions

$$V(x,x') = \sum_{n=0}^{+\infty} V_n(x,x')\sigma^n(x,x') \quad \text{and} \quad W(x,x') = \sum_{n=0}^{+\infty} W_n(x,x')\sigma^n(x,x'),$$

  where the Hadamard coefficients $V_n(x,x')$ and $W_n(x,x')$ are defined by recursion relations which permit us to prove that this representation of the Feynman propagator solves the wave equation associated with the scalar field Φ or the ghost fields.

---

*. The factor $i\epsilon$ with $\epsilon \to 0_+$ is introduced to give a singularity structure that is consistent with the definition of the Feynman propagator as a time-ordered product.

*. The coefficients $V_n(x,x')$ can be determined uniquely and are purely geometrical objects satisfying

$$\begin{cases} 2(n+1)(n+2)V_{n+1} + 2(n+1)V_{n+1;a}\sigma^{;a} - 2(n+1)V_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + \left[\Box_x - m^2\right]V_n = 0 & \text{(for } n\in\mathbb{N}) \\ 2V_0 + 2V_{0;a}\sigma^{;a} - 2V_0\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + \left[\Box_x - m^2\right]\Delta^{1/2} = 0 & \text{(boundary condition)} \end{cases}.$$

*. The coefficients $W_n(x,x')$ satisfy

$$\begin{cases} 2(n+1)(n+2)W_{n+1} + 2(n+1)W_{n+1;a}\sigma^{;a} - 2(n+1)W_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + 2(2n+3)V_{n+1} \\ \quad + 2V_{n+1;a}\sigma^{;a} - 2V_{n+1}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + \left[\Box_x - m^2\right]W_n = 0 & \text{(for } n\in\mathbb{N}) \end{cases}.$$

$W_0(x,x')$ is unrestrained by the recursion relation and can be used to encode the quantum state.

Hadamard representation of the Feynman propagator associated with the vector field A_μ

- The Hadamard form of the Feynman propagator for the vector field A_μ is given by

$$G^A_{\mu\nu'}(x,x') = \frac{i}{8\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon}g_{\mu\nu'}(x,x') + V_{\mu\nu'}(x,x')\ln[\sigma(x,x')+i\epsilon] + W_{\mu\nu'}(x,x')\right).$$

Here, $V_{\mu\nu'}(x,x')$ and $W_{\mu\nu'}(x,x')$ are *symmetric* and *regular* bivectors given by the series expansions

$$V_{\mu\nu'}(x,x') = \sum_{n=0}^{+\infty} V_{n\,\mu\nu'}(x,x')\sigma^n(x,x') \quad \text{and} \quad W_{\mu\nu'}(x,x') = \sum_{n=0}^{+\infty} W_{n\,\mu\nu'}(x,x')\sigma^n(x,x'),$$

where the Hadamard coefficients $V_{n\,\mu\nu'}(x,x')$ and $W_{n\,\mu\nu'}(x,x')$ are defined by recursion relations which permit us to prove that this representation of the Feynman propagator solves the wave equation associated with the vector field A_μ.

---

*. The coefficients $V_{n\,\mu\nu'}(x,x')$ can be determined uniquely and are purely geometrical objects satisfying

$$\begin{cases} 2(n+1)(n+2)V_{n+1\,\mu\nu'} + 2(n+1)V_{n+1\,\mu\nu';a}\sigma^{;a} - 2(n+1)V_{n+1\,\mu\nu'}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + \left[g_\mu{}^\rho\Box_x - R_\mu{}^\rho - m^2 g_\mu{}^\rho\right]V_{n\,\rho\nu'} = 0 & \text{(for } n\in\mathbb{N}\text{)} \\ 2V_{0\,\mu\nu'} + 2V_{0\,\mu\nu';a}\sigma^{;a} - 2V_{0\,\mu\nu'}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + \left[g_\mu{}^\rho\Box_x - R_\mu{}^\rho - m^2 g_\mu{}^\rho\right]\left(g_{\rho\nu'}\Delta^{1/2}\right) = 0 & \text{(boundary condition)} \end{cases}.$$

*. The coefficients $W_{n\,\mu\nu'}(x,x')$ satisfy

$$\begin{cases} 2(n+1)(n+2)W_{n+1\,\mu\nu'} + 2(n+1)W_{n+1\,\mu\nu';a}\sigma^{;a} - 2(n+1)W_{n+1\,\mu\nu'}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + 2(2n+3)V_{n+1\,\mu\nu'} \\ \quad + 2V_{n+1\,\mu\nu';a}\sigma^{;a} - 2V_{n+1\,\mu\nu'}\Delta^{-1/2}\Delta^{1/2}{}_{;a}\sigma^{;a} + \left[g_\mu{}^\nu\Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right]W_{n\,\rho\nu'} = 0 \qquad \text{(for } n\in\mathbb{N}\text{)} \end{cases}.$$

$W_{0\,\mu\nu'}(x,x')$ is unrestrained by the recursion relation and can be used to encode the quantum state.

Singular and regular parts of the Feynman propagators represented in the Hadamard form

- The Hadamard representation of the Feynman propagators permits us to straightforwardly identify their singular and regular parts when the coincidence limit $x' \to x$ is considered.

- For the scalar field Φ or the ghost fields
  - a purely geometrical singular part takes the form
$$G_{\text{sing}}(x,x') = \frac{i}{8\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon} + V(x,x')\ln[\sigma(x,x')+i\epsilon]\right),$$
  - while a regular state-dependent part is given by
$$G_{\text{reg}}(x,x') = \frac{i}{8\pi^2}\, W(x,x').$$

- For the vector field A_μ
  - a purely geometrical singular part takes the form
$$G^{A}_{\text{sing}\,\mu\nu'}(x,x') = \frac{i}{8\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')+i\epsilon}\, g_{\mu\nu'}(x,x') + V_{\mu\nu'}(x,x')\ln[\sigma(x,x')+i\epsilon]\right),$$
  - while a regular state-dependent part is given by
$$G^{A}_{\text{reg}\,\mu\nu'}(x,x') = \frac{i}{8\pi^2}\, W_{\mu\nu'}(x,x').$$

Hadamard Green functions

- In the context of the regularization of the stress-energy-tensor operator, instead of working with the Feynman propagators, it is more convenient to use the associated Hadamard Green functions.
- The Feynman propagator G can be split into the average of the retarded and advanced Green functions $\overline{G}$ and the Hadamard Green functions $G^{(1)}$:
  - For the scalar field Φ or the ghost fields we have

$$G(x,x') = \overline{G}(x,x') + \frac{i}{2} G^{(1)}(x,x')$$

with

$$\overline{G}(x,x') = \frac{1}{8\pi}\left(\Delta^{1/2}(x,x')\delta[\sigma(x,x')] - V(x,x')\Theta[\sigma(x,x')]\right),$$

$$G^{(1)}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln|\sigma(x,x')| + W(x,x')\right)$$

$$= G^{(1)}_{\text{sing}}(x,x') + G^{(1)}_{\text{reg}}(x,x').$$

  - For the vector field A_μ we have

$$G^A_{\mu\nu'}(x,x') = \overline{G}^A_{\mu\nu'}(x,x') + \frac{i}{2} G^{(1)A}_{\mu\nu'}(x,x')$$

with

$$\overline{G}^A_{\mu\nu'}(x,x') = \frac{1}{8\pi}\left(\Delta^{1/2}(x,x')g_{\mu\nu'}(x,x')\delta[\sigma(x,x')] - V_{\mu\nu'}(x,x')\Theta[\sigma(x,x')]\right),$$

$$G^{(1)A}_{\mu\nu'}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} g_{\mu\nu'}(x,x') + V_{\mu\nu'}(x,x')\ln|\sigma(x,x')| + W_{\mu\nu'}(x,x')\right)$$

$$= G^{(1)A}_{\text{sing}\,\mu\nu'}(x,x') + G^{(1)A}_{\text{reg}\ \mu\nu'}(x,x').$$

*. The splitting of G into $\overline{G}$ and $G^{(1)}$ can be straightforwardly achieved by using the formal identities $(\sigma + i\epsilon)^{-1} = \mathscr{P}\sigma^{-1} - i\pi\delta(\sigma)$ and $\ln(\sigma + i\epsilon) = \ln|\sigma| + i\pi\Theta(-\sigma)$.

Hadamard coefficients and their covariant Taylor series expansions

- The *geometrical Hadamard coefficients* can be determined explicitly from the associated recursion relations up to necessary order by taking their covariant Taylor series expansions.
  - The expansions of the *symmetric* biscalar coefficients $V_0(x,x')$ and $V_1(x,x')$ are given by
  $$V_0 = v_0 - \left\{(1/2)v_{0;a}\right\}\sigma^{;a} + \frac{1}{2!}\, v_{0ab}\,\sigma^{;a}\sigma^{;b} + O\big(\sigma^{3/2}\big) \quad \text{and} \quad V_1 = v_1 + O\big(\sigma^{1/2}\big).$$
  - The expansions of the *symmetric* bivector coefficients $V_{0\,\mu\nu'}(x,x')$ and $V_{1\,\mu\nu'}(x,x')$ are given by
  $$V_{0\,\mu\nu} = v_{0(\mu\nu)} - \left\{(1/2)v_{0(\mu\nu);a} + v_{0[\mu\nu]a}\right\}\sigma^{;a} + \frac{1}{2!}\left\{v_{0(\mu\nu)ab} + v_{0[\mu\nu]a;b}\right\}\sigma^{;a}\sigma^{;b} + O\big(\sigma^{3/2}\big) \quad \text{and} \quad V_{1\,\mu\nu} = v_{1(\mu\nu)} + O\big(\sigma^{1/2}\big).$$
- The *state-dependent Hadamard coefficients* :
  - Their first coefficients $W_{0\,\mu\nu'}(x,x')$ and $W_0(x,x')$ are unrestrained by the recursion relations ;
  - This arbitrariness can be used to encode the quantum state dependence of the theory ;
  - Instead of working with these coefficients, we shall consider their sums $W_{\mu\nu'}(x,x')$ and $W(x,x')$;
  - Their covariant Taylor series expansions up to necessary order are given by
  $$W = w - \left\{(1/2)w_{;a}\right\}\sigma^{;a} + \frac{1}{2!}\, w_{ab}\,\sigma^{;a}\sigma^{;b} - \frac{1}{3!}\left\{(3/2)w_{ab;c} - (1/4)w_{;abc}\right\}\sigma^{;a}\sigma^{;b}\sigma^{;c} + O(\sigma^2),$$
  for the *symmetric* biscalar $W(x,x')$ and
  $$W_{\mu\nu} = s_{\mu\nu} - \left\{(1/2)s_{\mu\nu;a} + a_{\mu\nu a}\right\}\sigma^{;a} + \frac{1}{2!}\left\{s_{\mu\nu ab} + a_{\mu\nu a;b}\right\}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}\left\{(3/2)s_{\mu\nu ab;c} - (1/4)s_{\mu\nu;abc} + a_{\mu\nu abc}\right\}\sigma^{;a}\sigma^{;b}\sigma^{;c} + O\big(\sigma^2\big)$$
  with $s_{\mu\nu a_1\cdots a_p} \equiv w_{(\mu\nu)a_1\cdots a_p}$ and $a_{\mu\nu a_1\cdots a_p} \equiv w_{[\mu\nu]a_1\cdots a_p}$ for the *symmetric* bivector $W_{\mu\nu'}(x,x')$.

Taylor coefficients of the state-dependent Hadamard coefficients

- By using the Hadamard representation of the Green functions, we can rewrite, up to order $\sigma^{1/2}$ needed to establish some relations between the *Taylor coefficients* that will be useful to simplify the renormalized expectation value of the stress-energy-tensor operator,
  - the wave equation associated with the Green function of the scalar field Φ or the ghost fields in the form
    $$\left[\Box_x - m^2\right] W = -6V_1 - 2V_{1;a}\sigma^{;a} + O(\sigma),$$
  - the wave equation associated with the Green function of the vector field A_μ in the form
    $$g_\rho{}^{\rho'}\left[g_\mu{}^\nu \Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] W_{\nu\rho'} = -6V_{1\mu\rho} - 2g_\rho{}^{\rho'} V_{1\mu\rho';a}\sigma^{;a} + O(\sigma),$$
  - the Ward identity linking the Green fonctions associated with the vector field A_μ and the ghost fields in the form
    $$g_\nu{}^{\nu'}\left[W_{\mu\nu'}{}^{;\mu} + W_{;\nu'}\right] = -V_{1\mu\nu}\sigma^{;\mu} + V_1\sigma_{;\nu} + O(\sigma),$$
  - while, by recalling that $G(x,x') = G^{\Phi}(x,x') = G^{\mathrm{Gh}}(x,x')$, the Ward identity linking the Green fonctions associated with the scalar field Φ and the ghost fields provides trivial equalities.

- With practical applications in mind, it is interesting to express some of the *Taylor coefficients* in term of the bitensors $W_{\mu\nu'}(x,x')$ and $W(x,x')$ by inverting the associated Taylor expansions.

$$w(x) = \lim_{x'\to x} W(x,x'),$$
$$w_{ab}(x) = \lim_{x'\to x} W_{;(a'b')}(x,x')$$

and

$$s_{\mu\nu}(x) = \lim_{x'\to x} W_{\mu\nu'}(x,x'),$$
$$a_{\mu\nu a}(x) = \frac{1}{2}\lim_{x'\to x}\left[W_{\mu\nu';a'}(x,x') - W_{\mu\nu';a}(x,x')\right],$$
$$s_{\mu\nu ab}(x) = \frac{1}{2}\lim_{x'\to x}\left[W_{\mu\nu';(a'b')}(x,x') + W_{\mu\nu';(ab)}(x,x')\right].$$

Stress-energy tensor

- The stress-energy tensor $T_{\mu\nu}$ associated with the quantum action S of the *Stueckelberg theory* is defined by
$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}} S\left[A_\mu, \Phi, C, C^*, g_{\mu\nu}\right].$$
- Its explicit expression is given by
$$T^{\mu\nu} = T^{\mu\nu}_{\text{cl}} + T^{\mu\nu}_{\text{GB}} + T^{\mu\nu}_{\text{Gh}},$$
where the contributions of the classical and gauge-breaking parts as well as that associated with the ghost fields take the forms
$$T^{\mu\nu}_{\text{cl}} = F^{\mu}{}_{\rho}F^{\nu\rho} + m^2 A^\mu A^\nu + \nabla^\mu\Phi\nabla^\nu\Phi + 2mA^{(\mu}\nabla^{\nu)}\Phi - (1/4)g^{\mu\nu}\left\{F_{\rho\tau}F^{\rho\tau} + 2m^2 A_\rho A^\rho + 2\nabla_\rho\Phi\nabla^\rho\Phi + 4mA_\rho\nabla^\rho\Phi\right\},$$
$$T^{\mu\nu}_{\text{GB}} = -2A^{(\mu}\nabla^{\nu)}\nabla_\rho A^\rho - 2mA^{(\mu}\nabla^{\nu)}\Phi - (1/2)g^{\mu\nu}\left\{-2A_\rho\nabla^\rho\nabla_\tau A^\tau - \left(\nabla_\rho A^\rho\right)^2 + m^2\Phi^2 - 2mA_\rho\nabla^\rho\Phi\right\},$$
$$T^{\mu\nu}_{\text{Gh}} = -2\nabla^{(\mu|}C^*\nabla^{|\nu)}C + g^{\mu\nu}\left\{\nabla_\rho C^*\nabla^\rho C + m^2 C^* C\right\}.$$
- Another alternative expression can be written as follows :
$$T^{\mu\nu} = T^{\mu\nu}_{A} + T^{\mu\nu}_{\Phi} + T^{\mu\nu}_{\text{Gh}}$$
where the contributions associated with the vector field A_μ and the scalar field Φ take the forms
$$T^{\mu\nu}_{A} = F^{\mu}{}_{\rho}F^{\nu\rho} + m^2 A^\mu A^\nu - 2A^{(\mu}\nabla^{\nu)}\nabla_\rho A^\rho - (1/4)g^{\mu\nu}\left\{F_{\rho\tau}F^{\rho\tau} + 2m^2 A_\rho A^\rho - 4A_\rho\nabla^\rho\nabla_\tau A^\tau - 2\left(\nabla_\rho A^\rho\right)^2\right\},$$
$$T^{\mu\nu}_{\Phi} = \nabla^\mu\Phi\nabla^\nu\Phi - (1/2)g^{\mu\nu}\left\{\nabla_\rho\Phi\nabla^\rho\Phi + m^2\Phi^2\right\}.$$
- We can note the existence of terms coupling the fields A_μ and Φ in the expressions of $T^{\mu\nu}_{\text{cl}}$ and $T^{\mu\nu}_{\text{GB}}$, while their summation eliminates any reference to them.
- By construction, the stress-energy tensor is conserved, i.e., $\nabla_\nu T^{\mu\nu} = 0$.

Expectation value of the stress-energy-tensor operator with respect to the Hadamard quantum state

- At the quantum level, all the fields involved in the *Stueckelberg theory* as well as the associated stress-energy tensor are operators. We denote
  - the stress-energy-tensor operator by $\widehat{T}_{\mu\nu}$,
  - its expectation value with respect to the Hadamard quantum state $|\psi\rangle$ by $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$.
- The expectation value $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ can be constructed from one of the previous expressions of $T_{\mu\nu}$ as follows :

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle = \langle\psi|\widehat{T}^{\rm cl}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\rm GB}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\rm Gh}_{\mu\nu}|\psi\rangle,$$

where the three contributions given by

$$\langle\psi|\widehat{T}^{\rm cl}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x}\mathscr{T}^{{\rm cl}A\,\rho\sigma'}_{\mu\nu}(x,x')\left[G^{(1)A}_{\rho\sigma'}(x,x')\right] + \frac{1}{2}\lim_{x'\to x}\mathscr{T}^{{\rm cl}\Phi}_{\mu\nu}(x,x')\left[G^{(1)\Phi}(x,x')\right],$$

$$\langle\psi|\widehat{T}^{\rm GB}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x}\mathscr{T}^{{\rm GB}A\,\rho\sigma'}_{\mu\nu}(x,x')\left[G^{(1)A}_{\rho\sigma'(x,x')}\right] + \frac{1}{2}\lim_{x'\to x}\mathscr{T}^{{\rm GB}\Phi}_{\mu\nu}(x,x')\left[G^{(1)\Phi}(x,x')\right],$$

$$\langle\psi|\widehat{T}^{\rm Gh}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x}\mathscr{T}^{\rm Gh}_{\mu\nu}(x,x')\left[G^{(1){\rm Gh}}(x,x')\right]$$

with the differential operators constructed by point splitting in the form

$$\mathscr{T}^{{\rm cl}A\,\rho\sigma'}_{\mu\nu} = g_\nu{}^{\alpha'}g^{\rho\sigma'}\nabla_\mu\nabla_{\alpha'} + g_\mu{}^{\rho}g_\nu{}^{\sigma'}g^{\alpha\beta'}\nabla_\alpha\nabla_{\beta'}$$
$$-2g_\mu{}^{\rho}g_\nu{}^{\alpha'}g^{\beta\sigma'}\nabla_\beta\nabla_{\alpha'} + m^2 g_\mu{}^{\rho}g_\nu{}^{\sigma'}$$
$$-\frac{1}{2}g_{\mu\nu}\left\{g^{\rho\sigma'}g^{\alpha\beta'}\nabla_\alpha\nabla_{\beta'} - g^{\rho\alpha'}g^{\beta\sigma'}\nabla_\beta\nabla_{\alpha'} + m^2 g^{\rho\sigma'}\right\},$$

$$\mathscr{T}^{{\rm cl}\Phi}_{\mu\nu} = g_\nu{}^{\nu'}\nabla_\mu\nabla_{\nu'} - \frac{1}{2}g_{\mu\nu}\left\{g^{\alpha\beta'}\nabla_\alpha\nabla_{\beta'}\right\},$$

$$\mathscr{T}^{{\rm GB}A\,\rho\sigma'}_{\mu\nu} = -2g_\mu{}^{\rho}g_\nu{}^{\alpha'}\nabla_{\alpha'}\nabla^{\sigma'}$$
$$-\frac{1}{2}g_{\mu\nu}\left\{-\nabla^\rho\nabla^{\sigma'} - 2g^{\rho\alpha'}\nabla_{\alpha'}\nabla^{\sigma'}\right\},$$

$$\mathscr{T}^{{\rm GB}\Phi}_{\mu\nu} = -\frac{1}{2}m^2 g_{\mu\nu},$$

$$\mathscr{T}^{\rm Gh}_{\mu\nu} = -2g_\nu{}^{\nu'}\nabla_\mu\nabla_{\nu'} + g_{\mu\nu}\left\{g^{\alpha\beta'}\nabla_\alpha\nabla_{\beta'} + m^2\right\}.$$

- It should be noted that the terms coupling A_μ and Φ are not present because two-point correlation functions involving both A_μ and Φ vanish identically.

Renormalized expectation value of the stress-energy-tensor operator

- The expectation value of $\widehat{T}_{\mu\nu}$ is divergent due to the singular short-distance behavior of the Green functions.
- It is possible to construct the *renormalized expectation value of the stress-energy-tensor operator with respect to the Hadamard quantum state* $|\psi\rangle$ by using the prescription proposed by Wald which consists
  - to discard the singular contributions, i.e., to make the replacements

$$G^{(1)A}_{\mu\nu'}(x,x') \to G^{(1)A}_{\text{reg}\ \mu\nu'}(x,x') = \frac{1}{4\pi^2} W^A_{\mu\nu'}(x,x'),$$

$$G^{(1)\Phi}(x,x') \to G^{(1)\Phi}_{\text{reg}}(x,x') = \frac{1}{4\pi^2} W^{\Phi}(x,x'),$$

$$G^{(1)\text{Gh}}(x,x') \to G^{(1)\text{Gh}}_{\text{reg}}(x,x') = \frac{1}{4\pi^2} W^{\text{Gh}}(x,x'),$$

  - to add to the result a state-independent tensor $\widetilde{\Theta}_{\mu\nu}$ which only depends on the mass parameter and on the local geometry and which ensures the conservation of the final expression.
- The renormalized expectation value of $\widehat{T}_{\mu\nu}$ is given by

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2}\left\{\lim_{x'\to x} \mathcal{T}^{\text{cl}A\ \rho\sigma'}_{\mu\nu}(x,x')\Big[W^A_{\rho\sigma'}(x,x')\Big] + \lim_{x'\to x} \mathcal{T}^{\text{cl}\Phi}_{\mu\nu}(x,x')\Big[W^{\Phi}(x,x')\Big]\right\}$$
$$+ \frac{1}{8\pi^2}\left\{\lim_{x'\to x} \mathcal{T}^{\text{GB}A\ \rho\sigma'}_{\mu\nu}(x,x')\Big[W^A_{\rho\sigma'}(x,x')\Big] + \lim_{x'\to x} \mathcal{T}^{\text{GB}\Phi}_{\mu\nu}(x,x')\Big[W^{\Phi}(x,x')\Big]\right\} + \frac{1}{8\pi^2}\lim_{x'\to x} \mathcal{T}^{\text{Gh}}_{\mu\nu}(x,x')\Big[W^{\text{Gh}}(x,x')\Big] + \widetilde{\Theta}_{\mu\nu}.$$

- In the order to ensure the conservation of the regular terms, it is suitable to redefine the purely geometrical tensor $\widetilde{\Theta}_{\mu\nu}$ by introducing a new local conserved tensor $\Theta_{\mu\nu}$ which the general expression is of the form

$$\Theta_{\mu\nu} = \frac{1}{8\pi^2}\left\{\alpha m^4 g_{\mu\nu} + \beta m^2\left[R_{\mu\nu} - (1/2)R g_{\mu\nu}\right] + \gamma_1\, {}^{(1)}H_{\mu\nu} + \gamma_2\, {}^{(2)}H_{\mu\nu}\right\},$$

where the constants α, β, γ_1 and γ_2 can be fixed by imposing additional physical conditions on $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$.

*. The tensor $\widetilde{\Theta}_{\mu\nu}$ is redefined by $\Theta_{\mu\nu} - \frac{1}{8\pi^2}\left\{6v_{1\,\mu\nu} - 2g_{\mu\nu}v_{1\,\rho}{}^{\rho} + 2g_{\mu\nu}v_1\right\}$.

Final expression of the renormalized stress-energy tensor

- The explicit expression for the renormalized expectation value of the stress-energy-tensor operator $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ associated with the *Stueckelberg theory*
  - is obtained by expanding the Hadamard coefficients in covariant Taylor series,
  - is simplified by using some relations between the Taylor coefficients involved.
- The main expression which only involves state-dependent and geometrical quantities associated with the massive vector field A_μ is given by

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2}\Big\{(1/2){s_\rho}^{\rho}{}_{;\mu\nu} + (1/2)\Box s_{\mu\nu} - {s_{\rho(\mu;\nu)}}^{\rho} + (1/2){R^\rho}_{(\mu}s_{\nu)\rho} - (1/2){a_\mu}^{\rho}{}_{(\nu;\rho)} - (1/2){a_\nu}^{\rho}{}_{(\mu;\rho)}$$
$$- {a_\mu}^{\rho}{}_{[\nu;\rho]} - {a_\nu}^{\rho}{}_{[\mu;\rho]} - {s_\rho}^{\rho}{}_{\mu\nu} + {s_{\rho(\mu\nu)}}^{\rho} - (1/2)g_{\mu\nu}\Big[(1/2)\Box {s_\rho}^{\rho} - (1/2){s_{\rho\tau}}^{;\rho\tau} - {a_{\rho\tau}}^{\rho;\tau}\Big] + v_{1\mu\nu} - g_{\mu\nu}{v_{1\rho}}^{\rho}\Big\} + \Theta_{\mu\nu}.$$

  - Here, by using the Ward identities, any reference to the auxiliary scalar field Φ can be removed.
- It is possible to split the renormalized expectation value of the stress-energy-tensor operator in the form

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = \mathscr{T}^{A}_{\mu\nu} + \mathscr{T}^{\Phi}_{\mu\nu} + \Theta_{\mu\nu},$$

where two conserved contributions associated with the vector and scalar fields are given by

$$\mathscr{T}^{A}_{\mu\nu} = \frac{1}{8\pi^2}\Big\{(1/2){s_\rho}^{\rho}{}_{;\mu\nu} + (1/2)\Box s_{\mu\nu} - {s_{\rho(\mu;\nu)}}^{\rho} - {a_\mu}^{\rho}{}_{[\nu;\rho]} - {a_\nu}^{\rho}{}_{[\mu;\rho]} - {s_\rho}^{\rho}{}_{\mu\nu} + 2{s_{\rho(\mu\nu)}}^{\rho}$$
$$- (1/2)g_{\mu\nu}\Big[(1/2)\Box {s_\rho}^{\rho} - 2{a_{\rho\tau}}^{\rho;\tau}\Big] + 2v^{A}_{1\,\mu\nu} - g_{\mu\nu}{v^{A}_{1\,\rho}}^{\rho}\Big\},$$

$$\mathscr{T}^{\Phi}_{\mu\nu} = \frac{1}{8\pi^2}\Big\{(1/2)w_{;\mu\nu} - w_{\mu\nu} - (1/4)g_{\mu\nu}\Box w - g_{\mu\nu}v_1\Big\}.$$

  - Here, in the limit $m^2 \to 0$, the term $\mathscr{T}^{A}_{\mu\nu}$ reduces to the result obtained from *Maxwell's theory*.
  - However, this is an artificial way to split the contributions of the vector and scalar fields.

*. In some sense, the auxiliary scalar field Φ plays the role of a kind of ghost field.

General considerations for the Casimir effect

- We shall consider the Casimir effect for *Stueckelberg massive electromagnetism* in the Minkowski spacetime $(\mathbb{R}^4, \eta_{\mu\nu})$ with $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$.
- We denote by (T, X, Y, Z) the coordinates of an event in this spacetime.
- We shall provide the renormalized vacuum expectation value of the stress-energy-tensor operator $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\mathrm{ren}}$ outside of a perfectly conducting medium with a plane boundary wall at $Z = 0$ separating it from free space.

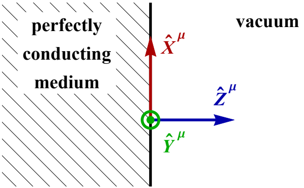

FIGURE – Geometry of the Casimir effect

- From symmetries and physical considerations, outside of the perfectly conducting medium, the renormalized stress-energy tensor takes the form

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\mathrm{ren}} = \frac{1}{3}\,\langle 0|\widehat{T}_{\rho}{}^{\rho}|0\rangle_{\mathrm{ren}}\left(\eta_{\mu\nu} - \hat{Z}_\mu \hat{Z}_\nu\right),$$

where $\hat{Z}^\mu$ is the spacelike unit vector orthogonal to the wall.

- As a consequence, it is sufficient to determine the trace of the renormalized stress-energy tensor given by

$$\langle 0|\widehat{T}_{\rho}{}^{\rho}|0\rangle_{\mathrm{ren}} = \frac{1}{8\pi^2}\left\{-m^2 s_\rho{}^\rho + s_{\rho\tau}{}^{\rho\tau} + (3/2)m^4\right\} + \Theta_\rho{}^\rho,$$

where, in the Minkowski spacetime, the term $\Theta_\rho{}^\rho$ reduces to

$$\Theta_\rho{}^\rho = \frac{1}{8\pi^2}\left\{\alpha m^4\right\}$$

with a constant α which can be fixed by imposing additional physical conditions on $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\mathrm{ren}}$.

Renormalized stress-energy tensor in the Minkowski spacetime

- The renormalized stress-energy tensor $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}$ can be evaluated by calculating the two Taylor coefficients of the regular part of the Feynman propagator $G^A_{\mu\nu}(x,x')$ corresponding to the geometry of the problem :
$$s_{\mu\nu}(x) = \lim_{x'\to x} W_{\mu\nu}(x,x') \quad \text{and} \quad s_{\mu\nu ab}(x) = \lim_{x'\to x} W_{\mu\nu;(ab)}(x,x').$$
- Let us first consider the renormalized stress-energy tensor *in the ordinary Minkowski spacetime* (i.e., without the boundary wall).
  - Due to symmetry considerations, we have $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\rm ren} = \frac{1}{4}\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}\,\eta_{\mu\nu}$ as well as we must have $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\rm ren} = 0$ which plays the role of a constraint for α.
  - The Feynman propagator $G^A_{\mu\nu}(x,x')$ associated with the vector field A_μ satisfies the wave equation
$$\left[\Box_x - m^2\right] G^A_{\mu\nu}(x,x') = -\eta_{\mu\nu}\delta^4(x,x'),$$
and its explicit expression is given in term of a Hankel function of the second kind by
$$G^A_{\mu\nu}(x,x') = -\frac{m^2}{8\pi}\frac{1}{\sqrt{-2m^2[\sigma(x,x')+i\epsilon]}}H^{(2)}_1\left[\sqrt{-2m^2[\sigma(x,x')+i\epsilon]}\right]\eta_{\mu\nu}$$
with $2\sigma(x,x') = -(T-T')^2 + (X-X')^2 + (Y-Y')^2 + (Z-Z')^2$.
  - The two Taylor coefficients involved in $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren}$ are given by
$$s_{\mu\nu}(x) = m^2\left[-1/2+\gamma+(1/2)\ln(m^2/2)\right]\eta_{\mu\nu} \quad \text{and} \quad s_{\mu\nu ab}(x) = m^4\left[-5/16+(1/4)\gamma+(1/8)\ln(m^2/2)\right]\eta_{\mu\nu}\eta_{ab}.$$
  - Then, we obtain $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\rm ren} = \frac{m^4}{8\pi^2}\left\{\alpha + 9/4 - 3\gamma - (3/2)\ln(m^2/2)\right\}$, and, necessarily, we have
$$\alpha = -9/4 + 3\gamma + (3/2)\ln(m^2/2).$$

*. In the Minkowski spacetime, the bivector of parallel transport $g_\mu{}^{\nu'}(x,x')$ is equal to the unit matrix $\delta_\mu{}^{\nu'}(x,x')$.

Renormalized stress-energy tensor for the Casimir effect

- Let us now come back to our initial *geometry of the Casimir effect*.
  - The Feynman propagator $\widetilde{G}^A_{\mu\nu}(x,x')$ in the presence of the plane boundary wall can be constructed by the method of images if we assume, in order to simplify our problem, that this wall is a perfectly reflecting :

$$\widetilde{G}^A_{\mu\nu}(x,x') = G^A_{\mu\nu}(x,x') - q_\nu\, G^A_{\mu\nu}(x,\tilde{x}'),$$

    where $G^A_{\mu\nu}(x,x')$ is the Feynman propagator previously considered *in the ordinary Minkowski spacetime* with $x'^\mu = (T',X',Y',Z')$ and $\tilde{x}'^\mu = (T',X',Y',-Z')$, while $q_\nu = 1 - 2\delta_{3\nu}$ (the index ν is not summed).
  - The two Taylor coefficients involved in $\langle 0|\widehat{T}_\rho{}^\rho|0\rangle_{\text{ren}}$ are given in term of the modified Bessel functions of the second kind K_1 and K_2 by

$$s_{\mu\nu} = m^2\left[-1/2 + \gamma + (1/2)\ln(m^2/2)\right]\eta_{\mu\nu} - q_\nu\,(m/Z)K_1(2mZ)\eta_{\mu\nu},$$

$$s_{\mu\nu ab} = m^4\left[-5/16 + (1/4)\gamma + (1/8)\ln(m^2/2)\right]\eta_{\mu\nu}\eta_{ab}$$
$$- q_\nu\left[(m^2/Z^2)K_2(2mZ)\eta_{\mu\nu}\left(2\eta_{3a}\eta_{3b} - (1/2)\eta_{ab}\right) + (m^3/Z)K_1(2mZ)\eta_{\mu\nu}\eta_{3a}\eta_{3b}\right].$$

  - Then, we obtain the expression of the renormalized stress-energy tensor

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}} = \frac{1}{8\pi^2}\left\{\frac{m^2}{Z^2}K_2(2mZ) + \frac{m^3}{Z}K_1(2mZ)\right\}\left(\eta_{\mu\nu} - \hat{Z}_\mu\hat{Z}_\nu\right).$$

  - It is very important to note that this result coincides exactly with the result obtained by Davies and Toms in the framework of *de Broglie-Proca massive electromagnetism*.

---

*. It should be noted that the perfectly reflecting boundary condition is questionable from the physical point of view. It is logical for the transverse components of the electromagnetic field but much less natural for its longitudinal component. Indeed, for this component, we could also consider perfect transmission instead of complete reflection.

*. The term $G^A_{\mu\nu}(x,\tilde{x}')$ which is obtained by replacing x' by $\tilde{x}'$ as well as its derivatives are regular in the limit $x' \to x$.

Massless limit

- In the limit $m^2 \to 0$ and for $Z \neq 0$, the renormalized stress-energy tensor takes the form

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}} = \frac{1}{16\pi^2}\,\frac{1}{Z^4}\left(\eta_{\mu\nu} - \hat{Z}_\mu \hat{Z}_\nu\right).$$

  - In the massless limit, the renormalized stress-energy tensor associated with the *Stueckelberg theory* diverges like Z^{-4} as the boundary surface is approached.
  - This result contrasts with that obtained from *Maxwell's theory* where the renormalized stress-energy tensor vanishes identically.

- The expression of the renormalized stress-energy tensor, where we have proposed an artificial separation of the contributions associated with the vector field A_μ and the auxiliary scalar field Φ, takes the form

$$\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}} = \langle 0|\widehat{T}^{A}_{\mu\nu}|0\rangle_{\text{ren}} + \langle 0|\widehat{T}^{\Phi}_{\mu\nu}|0\rangle_{\text{ren}},$$

  where the stress-energy tensor associated with the vector field A_μ is such that

$$\langle 0|\widehat{T}^{A}_{\mu\nu}|0\rangle_{\text{ren}} = 0,$$

  while that associated with the auxiliary scalar field Φ is given by

$$\langle 0|\widehat{T}^{\Phi}_{\mu\nu}|0\rangle_{\text{ren}} = \frac{1}{8\pi^2}\left\{\frac{m^2}{Z^2}K_2(2mZ) + \frac{m^3}{Z}K_1(2mZ)\right\}\left(\eta_{\mu\nu} - \hat{Z}_\mu \hat{Z}_\nu\right).$$

  - Here, it is interesting to note that the contribution associated with the vector field A_μ vanishes identically for any value of the mass parameter m.
  - The result associated with the vector field A_μ coincides exactly with that obtained from *Maxwell's theory*.

Summary

- We have presented *Stueckelberg massive electromagnetism* on an arbitrary curved spacetime.
- We have given two alternative but equivalent expressions for the renormalized expectation value of the stress-energy-tensor operator constructed using Hadamard renormalization.
- We have also presented the results concerning Casimir effect for *Stueckelberg massive electromagnetism*.
- Bearing in mind the results obtained, we can give the following conclusions :
  - *De Broglie-Proca* and *Stueckelberg* approaches of *massive electromagnetism* are two faces of the same theory.
  - However, we can note that, with regularization and renormalization in mind, it is much more interesting to work in the framework of the *Stueckelberg* formulation of *massive electromagnetism* which permits us to have at our disposal the machinery of the Hadamard formalism.
- One of our perspectives is the application of the general formalism developed to cosmological problems and, in particular, the study of *Stueckelberg massive electromagnetism* in de Sitter spacetime and in Friedman-Lemaître-Robertson-Walker spacetimes.

THANKS FOR YOUR ATTENTION