Massive Gravity, Lagrangian Media and Thermodynamics of Perfect Fluids

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  Φ^0 internal time of the medium
  - Internal Symmetries: shift, $SO(3)_\Phi$, Non Linear extensions
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$$\Phi^{A=0,1,2,3} \underbrace{\Rightarrow}_{\textit{shift sym.}:\Phi^a \to \Phi^A + \partial c^A} \partial \Phi^A$$

$$\underbrace{\Rightarrow}_{\textit{Lorentz Scalar}} C^{AB} = \partial_\mu \Phi^A g^{\mu\nu} \partial_\nu \Phi^B \underbrace{\rightarrow}_{SO(3)_\Phi} \underbrace{C^{00} \equiv \mathbf{S}}_{\textit{Scalar}}, \underbrace{C^{0a} \equiv \mathbf{V}}_{\textit{Vector}}, \underbrace{C^{ab} \equiv \mathbf{T}}_{\textit{Tensor}} :$$

$$\mathbf{S}, \; Tr[\mathbf{T}^{n=1,2,3}], \; \mathbf{V} \cdot \mathbf{T}^{n=0,1,2,3} \cdot \mathbf{V}$$

$$\underbrace{\Rightarrow}_{\textit{Lorentz Vector}, \; SO(3)_\Phi \; \textit{Scalar}} u^\mu \sim \epsilon^{\mu\alpha\beta\gamma} e_{abc} \partial_\alpha \phi^a \partial_\beta \phi^b \partial_\gamma \phi^c \underbrace{\rightarrow}_{\textit{Lorentz Scalar}} u^\mu \partial_\mu \Phi^0$$

$$\boxed{9 \; \textit{operators}}$$

$A, B = 0, 1, 2, 3 \,, \quad a, b, c = 1, 2, 3$

Lagrangian of 4 scalar + shift sym.

- At lowest order in derivatives: $C^{AB} = \partial_\mu \Phi^A \, g^{\mu\nu} \, \partial_\nu \Phi^A$

$$S = \int d^4x \sqrt{-g} \, \left(M_{pl}^2 \, R \, + \, U(C^{AB})\right)$$

Eqs of Motion:

$$\underbrace{\left(U_{AB} \, g^{\mu\nu} + \left(U_{AB,\,CB} + U_{AD,\,BC} \nabla^\mu \, \Phi^C \, \nabla^\nu \Phi^D\right)\right)}_{Rank[\%] = \# \; DoF} \nabla_\mu \nabla_\nu \, \Phi^B = 0 \quad \leftrightarrow \quad \nabla^\nu \, T_{\nu\mu} = 0$$

$U_{AB} = \partial U / \partial C^{AB}$, $U_{AB,CD} = \partial^2 U / (\partial C^{AB} \partial C^{CD})$

global internal spatial $SO(3)_\Phi$ symmetry

$$\Phi^0 \to \Phi^0, \quad \Phi^a \to R^a_b\, \Phi^b \qquad R\, R^T = I, \quad a = 1, 2, 3$$

$$\underline{SO(3)_\Phi \text{ Tensors}}: \quad C^{ab}, \; C^{a0}, \; C^{00}$$

Operator	Definition
C^{AB}	$g^{\mu\nu}\, \partial_\mu \Phi^A\, \partial_\nu \Phi^B\,, \quad A, B = 0, 1, 2, 3$
B^{ab}	$C^{ab} = g^{\mu\nu} \partial_\mu \Phi^a\, \partial_\mu \Phi^b\,, \quad a, b = 1, 2, 3$
Z^{ab}	$C^{a0}\, C^{b0}$
X	C^{00}
W^{ab}	$B^{ab} - Z^{ab}/X$

$SO(3)_\Phi$ Scalars

$$L.V.: \quad u^\mu = -\frac{\epsilon^{\mu\alpha\beta\gamma}}{6\, b\, \sqrt{-g}} \epsilon_{abc}\, \partial_\alpha \Phi^a\, \partial_\beta \Phi^b\, \partial_\gamma \Phi^c, \qquad \mathcal{V}^\mu = -\frac{\nabla^\mu \Phi^0}{(-X)^{1/2}}$$

L.S.:

Operator	Definition
X	C^{00}
b	$\sqrt{\det \boldsymbol{B}}$
Y	$u^\mu \partial_\mu \Phi^0$
y_n	$\mathrm{Tr}\left(\boldsymbol{B}^n \cdot \boldsymbol{Z}\right), \quad n = 0, 1, 2, 3$
τ_n	$\mathrm{Tr}\left(\boldsymbol{B}^n\right), \quad n = 1, 2, 3$
w_n	$\mathrm{Tr}\left(\boldsymbol{W}^n\right), \quad n = 0, 1, 2, 3$
$\mathcal{O}_{\alpha\beta n}$	$\left(X/Y^2\right)^\alpha \left(y_n/Y^2\right)^\beta$

$$U_{SO(3)_\Phi\ invariant} = U\underbrace{\left(X,\ Y,\ y_n,\ \tau_n\right)}_{9\ operators}$$

Shift Sym.+ $SO(3)_\Phi$ + Non Linear Sym.

Four-dimensional media		
Symmetries of the action	**LO scalar operators**	**Type of medium**
$\Phi^A \to \Phi^A + f^A, \quad \partial_\mu f^A = 0$	$X,\ Y,\ \tau_n,\ y_n$	supersolids
$\Phi^a \to \Phi^a + f^a(\Phi^0)$	$X,\ w_n$	
$\Phi^0 \to \Phi^0 + f(\Phi^a)$	$Y,\ \tau_n$	
$\Phi^0 \to \Phi^0 + f(\Phi^0)$	$\tau_n,\ w_n,\ \mathcal{O}_{\alpha\beta n}$	
$\Phi^a \to \Phi^a + f^a(\Phi^0)$ & $\Phi^0 \to \Phi^0 + f(\Phi^0)$	w_n	
V_sDiff: $\Phi^a \to \Psi^a(\Phi^b), \quad \det\|\partial\Psi^a/\partial\Phi^b\| = 1$	$b,\ Y,\ X$	superfluids
$\Phi^0 \to \Phi^0 + f(\Phi^0)$ & V_sDiff	$b,\ \mathcal{O}_\alpha$	
$\Phi^0 \to \Phi^0 + f(\Phi^a)$ & V_sDiff	$b,\ Y$	perfect fluid
$\Phi^A \to \Psi^A(\Phi^B), \quad \det\|\partial\Psi^A/\partial\Phi^B\| = 1$	$b\,Y$	Superperfect fluid $(\rho + p = 0)$

Operators with only Φ^0 or Φ^a scalars + shift + $SO(3)_\Phi$+ Non Linear Sym.

Media with reduced internal dimensionality		
Symmetries of the action	**LO scalar operators**	**Type of medium**
(Φ^a) $\Phi^a \to \Phi^a + c^a\,,\ \partial_\mu c^a = 0$	τ_n	solid
(Φ^a) V_sDiff	$b = b(\tau_n)$	perfect fluid
(Φ^0) $\Phi^0 \to \Phi^0 + c^0\,,\ \partial_\mu c^0 = 0$	X	

$$b = \frac{1}{6}(2\,\tau_3 - 3\,\tau_2\,\tau_1 + \tau_1^3)$$

Lagrangian Materials $\subset$ Massive Gravity Theories

It is possible to choose the comoving coord in such a way that the fluid system is at rest

Unitary Gauge:

If $Det|\partial_\alpha \Phi^A(x)| \neq 0$: $\boxed{\Phi^A(x) \equiv x^\mu\ \delta^A_\mu}$

$\rightarrow$ Spontaneous Breaking of spacetime symmetries ($T_X:\ x^\mu \rightarrow x^\mu + c^\mu$)
but Internal symmetry can compensate ($T_\Phi:\ \Phi^A \rightarrow \Phi^A - c^A$)
Residual symmetries: $T_x + T_\Phi$ (the same for Rotations)

$\Rightarrow$ homogeneity and isotropy of background configurations

$$h_{AB}\ \partial_\mu \Phi^A \partial_\nu \Phi^B \rightarrow h_{\mu\nu} \quad \textit{frozen metric}$$

(ex: $\eta_{\mu\nu}$ (Lorentz Inv.), $\delta^i_\mu\, \delta_{ij}\, \delta^j_\nu$, $\delta^0_\mu\, \delta_{00}\, \delta^0_\nu$ (Lorentz Br.))

LO self-gravitating media	Map	**Massive gravity**
$\mathcal{L}(C^{AB}, g_{\mu\nu})$	Unitary gauge $\longrightarrow$	$\mathcal{L}(h_{\mu\nu}, g^{\mu\nu})$
$\mathcal{O}_{LO}:\ X,\ Y,\ \tau_n,\ y_n$	$\longleftarrow$ Stuckelberg trick	ADM: N, N^i, γ_{ij}

Postulated Fundamental Relation

$$s = s(\rho, n) \;\text{ or in the energy representation }\; \rho = \rho(s,\, n)$$

I Law of Thermodynamics

$$\boxed{d\rho = T\, ds + \mu\, dn} \to T \equiv \left.\frac{\partial \rho}{\partial s}\right|_n, \quad \mu \equiv \left.\frac{\partial \rho}{\partial n}\right|_s.$$

Euler relation

$$\boxed{\rho + p = T\; s + \mu\; n},$$

Gibbs-Duhem relation

$$\boxed{dp = s\, dT + n\, d\mu}$$

Hydrodynamical Equation of Motion

Hydrodynamic equations express conservation laws

- Diff. invariance:

$$\nabla_\nu T^{\mu\nu} = 0$$

Space time translations. Rotations and boost are identically conserved.

- Global U(1) symmetry:

$$\nabla_\mu n^\mu = 0\,;$$

Perfect Fluids

variables: u^μ $(u^2 = -1)$, n, ρ, p

$$T_{\mu\nu} = \underbrace{(\rho + p)}_{Euler:\ T\ s+\mu\ n} u_\mu\ u_\nu + p\ g_{\mu\nu} = (T\ s_\mu + \mu\ n_\mu) u_\nu + p\ g_{\mu\nu}$$

$$n^\mu = n\ u^\mu, \qquad s^\mu = s\ u^\mu$$

$$EoM \quad : \quad \dot{\rho} + \theta(\rho + p) = 0 \to \quad \underbrace{T\ \nabla^\alpha s_\alpha + \mu\ \nabla^\alpha n_\alpha}_{0 \leftrightarrow 0} = 0\,,$$

$$(\rho + p)\dot{u}_\mu + (\delta^\nu_\mu + u_\mu u^\nu)\nabla_\nu p = 0$$

$$\nabla^\alpha n_\alpha = 0 \quad \Rightarrow \quad \frac{\dot{s}}{s} = \frac{\dot{n}}{n} = -\theta \quad \Rightarrow \quad u^\mu \nabla_\mu \left(\frac{s}{n}\right) \equiv \dot{\sigma} = 0\,;$$

$(\theta = \nabla_\mu u^\mu\,,\ \dot{f} = u^\mu \nabla_\mu f)$ entropy per particle $\boxed{\sigma = s/n}$ is conserved along the flow lines (adiabatic fluid).

Perfect Fluids Lagrangian

V_sDiff: $\Phi^a \to \Psi^a(\Phi^b)$, $\det|\partial\Psi^a/\partial\Phi^b| = 1$

$$S_{Fluid} = \int d^4x \sqrt{-g}\, U(\underbrace{\overbrace{b}^{\Phi^0\to 0}, Y}_{\Phi^0\to\Phi^0+f(\Phi^a)}; \overbrace{X}^{\Phi^a\to 0})$$

$$u^\mu = -\frac{\epsilon^{\mu\alpha\beta\gamma}}{6\, b\sqrt{-g}}\epsilon_{abc}\, \partial_\alpha\Phi^a\, \partial_\beta\Phi^b\, \partial_\gamma\Phi^c, \quad \mathcal{V}^\mu = -\frac{\nabla^\mu\Phi^0}{(-X)^{1/2}}$$

$$\underline{b} = det|\partial_\mu\Phi^a\, \partial^\mu\Phi^b|^{1/2}, \quad \underline{Y} = u^\mu\, \partial_\mu\Phi^0, \quad \underline{X} = \partial_\mu\Phi^0\partial^\mu\Phi^0$$

$$U(b,\, Y) \quad \to \quad T_{\mu\nu} = (U - b\, U_b)\; g_{\mu\nu} + (Y\, U_Y - b\, U_b)\; u_\mu\, u_\nu$$

$$U(X) \quad \to \quad T_{\mu\nu} = U\, g_{\mu\nu} + 2\, X\, U_X\, \mathcal{V}_\mu\, \mathcal{V}_\nu$$

$$T_{\mu\nu}^{Perfect\ Fluid} = \rho\, g_{\mu\nu} + (\rho + p)\, v_\mu\, v_\nu$$

Density, pressure and conserved currents for Perfect fluid Lagrangians

Lagrangian	ρ	p	Cons. Currents
$U(b)$	$-U$	$U - b\,U_b$	$J^\mu = b\,u^\mu$
$U(Y)$	$-U + Y\,U_Y$	U	$J^\mu_{(1)} = U_Y\,u^\mu$
$U(X)$	$-U + 2\,X\,U_X$	U	$J^\mu_{(2)} = -2\,(-X)^{1/2}\,U_X\,\mathcal{V}^\mu$
$U(b,\,Y)$	$-U + Y\,U_Y$	$U - b\,U_b$	$J^\mu = b\,u^\mu,\ J^\mu_{(1)} = U_Y\,u^\mu$

Matching Thermodynamics - Perfect Fluid Lagrangian

$$\underbrace{\Big(\overbrace{\rho,\ p}^{\rho(\mathcal{I},\mathcal{D}),\,p(\mathcal{I},\mathcal{D})}\ ,\ \overbrace{s,\ T,\ n,\ \mu}^{2\ indep.\mathcal{I}+2\ dep.\mathcal{D}(\mathcal{I})}\Big)}_{Thermo\ variables} \Leftrightarrow U(\underbrace{\mathcal{O}}_{b,\ Y,\ X,\ (b,Y)}),\quad T_{\mu\nu} \to p(\mathcal{O}),\ \rho(\mathcal{O})$$

ex: $\mathcal{I} = (n, s),\ \mathcal{D} = (T, \mu),\ \mathcal{O} = b$ then I promote $\mathcal{O} \to \mathcal{O}(\mathcal{I})$

$U(b)$: $\boxed{\rho(n,s) = \rho(b) = -U(b)}$, $p = U - b\, U_b$ with $b = b(n,s)$:

$b_s \equiv \partial_s b|_n,\ \ b_n = \partial_n b|_s$

$$\rho = \rho(n,s):\qquad T(n,s) \equiv \left.\frac{\partial\rho}{\partial s}\right|_n = -\,U_b\ b_s,$$

$$\mu(n,s) \equiv \left.\frac{\partial\rho}{\partial n}\right|_s = -\,U_b\ b_n$$

$$Euler:\ \rho + p = s\ T + n\ \mu\ \ (It\ has\ to\ be\ valid\ for\ any\ U)$$

$$\Longrightarrow -b\ U_b = -\,s\ U_b\ b_s -\ n\ U_b\ b_n \to U_b(b - s\,b_s - n\,b_n) = 0$$

$$b = s\,f\left(\frac{n}{s}\right)\ and\ T = -U_b(f - \frac{n}{s}f'),\ \mu = -U_b\,f'$$

Matching Thermodynamics - Perfect Fluid Lagrangian

$$\underbrace{\left(\overbrace{\rho,\ p}^{\rho(\mathcal{I},\mathcal{D}),\,p(\mathcal{I},\mathcal{D})}\quad,\quad\overbrace{s,\ T,\ n,\ \mu}^{2\ indep.\mathcal{I}+2\ dep.\mathcal{D}(\mathcal{I})}\right)}_{Thermo\ variables} \Leftrightarrow U(\underbrace{\mathcal{O}}_{b,\,Y,\,X,\,(b,Y)}),\quad T_{\mu\nu}\to p(\mathcal{O}),\ \rho(\mathcal{O})$$

We postulate $\mathcal{O} = \mathcal{O}(\mathcal{I})$ such that: It has to be valid for any U

$$\rho(\mathcal{I},\,\mathcal{D}(\mathcal{I})) = \rho(\mathcal{O}(\mathcal{I})),\ p(\mathcal{I},\,\mathcal{D}(\mathcal{I})) = p(\mathcal{O}(\mathcal{I}))$$

$$\underbrace{I\ Law\ of\ Th.}_{d\rho=Tds+\mu dn},\ \underbrace{Euler\ eq.}_{\rho+p=Ts+\mu n},\ \underbrace{Gibbs\ Duhem}_{dp=sdT+nd\mu} \Leftrightarrow Conservation\ Law : \nabla T = 0$$

es: $\mathcal{I} = (n,\ T) \to \mathcal{D}(\mathcal{I}) = (\mu(n,\ T),\ s(n,\ T))$ and $\mathcal{O} = (b,\ Y),\ \ p = U - b\,U_b,\ \ \rho = -U + Y\,U_Y$

$$d\rho(\mathcal{O}(\mathcal{I})) = Tds(\mathcal{I}) + \mu(\mathcal{I})dn$$
$$\rho(\mathcal{O}(\mathcal{I})) + p(\mathcal{O}(\mathcal{I})) = T\,s(\mathcal{O}(\mathcal{I})) + \mu\,n(\mathcal{O}(\mathcal{I}))$$

$b = n,\ \ Y = T,\ \ \ s = U_Y,\ \ \mu = -U_b$

Thermodynamical potentials and related thermodynamical variables

Thermo Pot.	$\mathcal{I}$	Leg. Transf.	$\mathcal{D}$
energy ρ	$s,\ n$	none	$T = \frac{\partial\rho}{\partial s}\Big\rvert_n,\ \mu = \frac{\partial\rho}{\partial n}\Big\rvert_s$
free energy $\mathcal{F}$	$T,\ n$	$\mathcal{F} = \rho - T\,s$	$s = -\frac{\partial\mathcal{F}}{\partial T}\Big\rvert_n,\ \mu = \frac{\partial\mathcal{F}}{\partial n}\Big\rvert_s$
grand potential ω	$T,\ \mu$	$\omega = -p$	$s = -\frac{\partial\omega}{\partial T}\Big\rvert_\mu,\ n = -\frac{\partial\omega}{\partial\mu}\Big\rvert_T$
potential $\mathcal{I}$	$s,\ \mu$	$\mathcal{I} = \rho - \mu\,n$	$T = \frac{\partial\mathcal{I}}{\partial s}\Big\rvert_\mu,\ n = -\frac{\partial\mathcal{I}}{\partial\mu}\Big\rvert_s$

The Thermodynamical connection with Perfect Fluid Lagrangian it works when

$$\boxed{U \equiv -Thermodynamical\ Potential}$$

Dictionary: Thermo $\leftrightarrow$ Lagrangian

Potentials	$U(b)$	$U(Y)$	$U(X)$	$U(b,\ Y)$
Indep. Variables	$\rho = -U$ $p = U - b\ U_b$ $J_\mu = b\ u_\mu$	$\rho = -U + Y\ U_Y$ $p = U$ $J_\mu = U_Y\ u_\mu$	$\rho = -U + 2\,X\,U_X$ $p = U$ $J_\mu = -2\,(-X)^{1/2}\ U_X\ \mathcal{V}_\mu$	$\rho = -U + Y\ U_Y$ $p = U - b\ U_b$ $J^{(1)}_\mu = b\ u_\mu$ $J^{(2)}_\mu = U_Y\ u_\mu$
$(\mu,\ s)$ $\mathcal{I} = -U$	$b = s$ $n = 0$ $T = -U_b$	$Y = \mu$ $n = U_Y$ $T = 0$	$X = -\mu^2$ $n = -2\,U_X\,\sqrt{-X}$ $T = 0$	$b = s,\ \ Y = \mu$ $n = U_Y$ $T = -U_b$
$(n,\ T)$ $\mathcal{F} = -U$	$b = n$ $s = 0$ $\mu = -U_b$	$Y = T$ $s = U_Y$ $\mu = 0$	$X = -T^2$ $s = -2\,U_X\,\sqrt{-X}$ $\mu = 0$	$b = n,\ \ Y = T$ $s = U_Y$ $\mu = -U_b$
$(\mu,\ T)$ $\omega = -U$		$Y = T\,f\left(\frac{\mu}{T}\right)$ $n = U_Y\ f'$ $s = U_Y\,(f - \frac{\mu}{T}\ f')$	$X = -\mu^2\ f\left(\frac{T}{\mu}\right)$ $n = U_X\,(T\,f' - 2\,\mu\,f)$ $s = -U_X\,\mu\,f'$	
$(n,\ s)$ $\rho = -U$	$b = s\,f\left(\frac{n}{s}\right)$ $\mu = -U_b\ f'$ $T = -U_b\,(f - \frac{n}{s}\ f')$			

$\boxed{b \to n,\ s}$, $\boxed{Y \to \mu,\ T}$, $\boxed{X \to \mu,\ T}$

Barotropic Fluids $p = p(\rho)$

The perfect fluids $U(b)$, $U(Y)$, $U(X)$ are all barotropic.

$p = U - b\, U_b,\ \rho = -U;$ $p = f[\rho] \to$ *Differential eq for* $\boxed{\rho(b) : -\rho + b\, \rho_b = f[\rho]}$

$\boxed{p = w\ \rho}$ with constant w

$$U(b) = b^{1+w},\ U(Y) = Y^{(1+w)/w},\ U(X) = X^{(1+w)/2w},$$

Cold Dark Matter $(w = 0)$: $\rho = m\, n \to \boxed{U = -m\, b}$ with $b = n,\ s = 0,\ \mu = m$

Radiation $(w = \frac{1}{3})$: $\rho = 3\, p = T^4$, $\boxed{U = Y^4/3}$ with $Y = T,\ s = Y^3,\ \mu = 0$

$\boxed{\textit{Chaplygin gas}}$: $p = -A/\rho$, with constant A

$$U(b) = \sqrt{A + \lambda\, b^2},\ U(Y) = \sqrt{A + \lambda\ Y^2},\ U(X) = \sqrt{A + \lambda\ X}$$

Bose-Einstein and Fermi-Dirac distributions

Relativistic limits: $\boxed{T \gg m}$, $\quad \alpha = e^{\frac{\mu}{T}}$

$$\underbrace{p = \frac{\rho}{3}}_{\textit{barotropic eq of state}} = -\epsilon \, \frac{g}{\pi^2} \, T^4 \, \mathrm{Li}_4 \left(-\epsilon \, \alpha\right),$$

$$n = -\,\epsilon \, \frac{g}{\pi^2} \, T^3 \, \mathrm{Li}_3 \left(-\epsilon \, \alpha\right), \; s = \frac{4}{3} \frac{\rho}{T} - n \, \log(\alpha) \, ;$$

$$\boxed{U(Y) = \frac{g}{3} \, Y^4}, \quad \boxed{Y = T \, f\left(\frac{\mu}{T}\right)} \; \textit{where } f\left(\frac{\mu}{T}\right) = \left[-\frac{1}{\pi^2} \, \epsilon \, \mathrm{Li}_4 \left(-\epsilon \, e^{\frac{\mu}{T}}\right)\right]^{1/4}$$

For $\frac{\mu}{T} \ll 1$ we have $Y \to T$; for $\frac{\mu}{T} \gg 1$ we have $Y \to \mu$

Non relativistic limit: $\boxed{m \gg T}$

$$n = g \left(\frac{m\,T}{2\pi}\right)^{3/2} e^{\frac{(\mu - m)}{T}}\,, \qquad \rho = n\,(m + \frac{3\,T}{2}), \qquad p = n\,T \ll \rho\,.$$

$\boxed{\text{b=n,Y=T}}$

$$\underline{U(b,\ Y)} = b\,Y \left\{1 + \mathrm{Log}\left[\frac{g}{b}\left(\frac{m\ Y}{2\pi}\right)^{3/2}\right]\right\} - b\,m\,.$$

van der Waals gas

$$\boxed{(p + \kappa\, n^2)\,(1 - \gamma\, n) = \; T\, n\,,}$$

b=n,Y=T

$$U(b,\; Y) = b \left\{ b\,\kappa + Y - m + Y \log\left[\frac{g\,(1-\gamma\, b)}{b} \left(\frac{m\, Y}{2\,\pi}\right)^{3/2} \right] \right\}\,.$$

$$\rho = b\left(m + \frac{3}{2}\,T - n\,\kappa\right)\;;$$

$$s = \frac{5\,b}{2} + b\,\log\left[\left(\frac{m\,Y}{2\,\pi}\right)^{3/2}\;\frac{g(1-b\,\gamma)}{b}\right]\;;$$

$$\mu = m - 2\,b\,\kappa + \frac{b\,Y\gamma}{1-b\,\gamma} + Y\,\log\left[\frac{b}{g(1-b\,\gamma)}\left(\frac{m\,Y}{2\,\pi}\right)^{-3/2}\right]\,.$$

Thermodynamical Stability

The entropy density $s = s(\rho,\, n)$ is a convex function, thus the Hessian matrix $|s_{ij}| \equiv |\partial^2_{ij} s|, \;\; i, j = \rho, n.$ is negative define

$$\underline{s_{\rho\rho} + s_{nn} \leq 0,} \qquad \underline{s_{\rho\rho}\; s_{nn} - s^2_{\rho n} \geq 0}$$

$$\boxed{\boxed{\omega_{\mu\mu} + \omega_{TT} \leq 0, \qquad \omega_{\mu\mu}\; \omega_{TT} - \omega^2_{\mu T} \geq 0}} \qquad \underline{\omega = -U(X),\;\; U(Y)}$$

$$\boxed{U_X + 2\, X\, U_{XX} \leq 0, \quad (f'^2 - 2f\; f'')\; U_X \geq 0}, \quad X = -T^2\, f[z],\;\; z = \frac{\mu}{T},$$

$$\boxed{U_{YY} \geq 0, \quad f''\; U_Y \geq 0}, \qquad Y = T\, f[z],\;\; z = \frac{\mu}{T},$$

$$\boxed{\boxed{\rho_{ss} + \rho_{nn} \geq 0 \quad \rho_{ss}\rho_{nn} - \rho^2_{sn} \geq 0}} \qquad \underline{\rho = -U(b)}$$

$$\boxed{U_{bb} \leq 0, \quad f''\; U_b \leq 0}, \qquad b = s\, f[z],\;\; z = \frac{n}{s},$$

$$\boxed{\boxed{\mathcal{I}_{\mu\mu} \leq 0 \quad \mathcal{I}_{ss} \geq 0}} \qquad \underline{\mathcal{I} = -U(b,\;Y)}$$

$$\boxed{U_{bb} \leq 0, \quad U_{YY} \geq 0}, \qquad b = n,\; s,\;\; Y = T,\; \mu,$$

for $f = f_{B/FD}$ we have $f'' \geq 0$ and $(f'^2 - 2f\; f'') \leq 0$ so that

$$\boxed{U_{YY},\; U_Y \geq 0}; \quad \boxed{U_{bb},\; U_b \leq 0}; \quad \boxed{U_X + 2\, X\, U_{XX},\, U_X \leq 0}$$

Goldstone Bosons fluctuations

$$\Phi^i(t,x^j) = x^i + \pi^i(t,x^j), \quad \Phi^0 = t + \pi^0$$

$\boxed{\boxed{\textit{Thermo stability}} + \boxed{(p+\rho) \geq 0} \Rightarrow \textit{Dynamical stability}}$ but $Th.s. \not\Leftrightarrow \quad (p+\rho) \geq 0 + D.s.$

$$S^{(2)}[b] = \frac{1}{2}(\rho+p)\int d^4x \left(\dot\pi^i\dot\pi^i - c_s^2\,(\partial_i\pi^i)^2\right) \,,$$

$$\boxed{U_{bb} \leq 0, \; f'' \; U_b \leq 0} \quad \leftrightarrow \quad \boxed{(\rho+p) = -b\,U_b \geq 0, \; c_s^2 = \frac{b\,U_{bb}}{U_b} \geq 0}$$

$$S^{(2)}[Y] = \frac{1}{2}(\rho+p)\int d^4x \left(\dot\pi^i\dot\pi^i - c_s^2\,(\partial_i\pi^i)^2\right)$$

$$\boxed{U_{YY} \geq 0, \; f'' \; U_Y \geq 0} \quad \leftrightarrow \quad \boxed{(\rho+p) = Y\,U_Y \geq 0, \; c_s^2 = \frac{U_Y}{Y U_{YY}} \geq 0}$$

$$S^{(2)}[X] = \frac{1}{2}(\rho+p)\int d^4x \left(c_s^{-2}(\dot\pi^0)^2 - (\partial_i\pi^0)^2\right)$$

$$\boxed{2XU_{XX} + U_X \leq 0, \; ((f')^2 - 2ff'')\; U_X} \geq 0 \quad \leftrightarrow \quad \boxed{(\rho+p) = 2X\,U_X \geq 0, \; c_s^2 = \frac{U_X}{2XU_{XX} + U_X} \geq 0}$$

$$S^{(2)}[b,\,Y] = \frac{1}{2}\int d^4x \left[(\rho+p)\dot\pi^i\dot\pi^i + \rho_Y(\dot\pi^0)^2 - p_b\,(\partial_i\pi^i)^2 - 2\,c_b^2\,\rho_Y\,\dot\pi^0\partial_i\pi^i\right]$$

$$\boxed{U_{bb} \leq 0, \; U_{YY} \geq 0}\,, \quad \boxed{(\rho+p) = Y\,U_Y - b\,U_b \geq 0, \; c_s^2 = \frac{(b\,U_{bY} + U_Y)^2 - b^2\,U_{bb}U_{YY}}{U_Y^2(\rho+p)} \geq 0}$$

Sound speed

$$\boxed{c_s^2 \equiv \frac{\dot{p}}{\dot{\rho}}}$$

For barotropic perfect fluids $U(b)$, $U(Y)$, $U(X)$

$$\delta p = c_s^2\, \delta\rho \to \quad c_s^2 = \frac{dp}{d\rho} : \quad \frac{b\, U_{bb}}{U_b},\ \frac{U_Y}{Y\, U_{YY}},\ \frac{U_X}{U_X + 2XU_{XX}}$$

$U(b, Y)$ is not barotropic

$$\delta p = \underbrace{c_s^2\, \delta\rho}_{adiabatic\ modes} + \underbrace{b\, Y\ \left(c_b^2 - c_s^2\right)\ \delta\sigma}_{entropic\ modes}, \quad c_b^2 = \left.\frac{\partial p}{\partial \rho}\right|_b,\ \sigma = \frac{s}{n}\ with\ \dot{\sigma} = 0$$

Conclusions

- Media classification and internal scalar symmetries
- Media Lagrangians $\subset$ Massive Gravity
- Perfect Fluid Lagrangians
- Thermodynamics of Perfect Fluid Lagrangians
- Thermodynamical and Dynamical stability

Non perturbative # DoF and structure of the Potentials

ADM variables: $g^{\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2}\,N^i \\ N^{-2}\,N^i & \gamma^{ij} - N^{-2} N^i N^j \end{pmatrix}$, $(N^A \equiv (N, N^i),\ \gamma^{ij})$

- 6 DoF: $U = U[N^A, \gamma]$: Supersolid
- 5 DoF: $U = (\mathcal{U}[\mathcal{K}] + (\mathcal{E}[\xi, \gamma] + \mathcal{U}_{\xi^i}\ \mathcal{Q}^i)/N)$

  where $N^i = N\,\xi^i + \mathcal{Q}^i$, $\mathcal{K}^{ij} = \gamma^{ij} - \xi^i\,\xi^j$ and $\mathcal{Q}^i = (\partial^2_{\xi^i\xi^j}\mathcal{U})^{-1}\partial_{\xi^j}\mathcal{E}$

  Solids $U(\tau_n)$ and Perfect fluids $U(b = b(\tau_n))$, $\mathcal{E} = 0$

- 3 DoF: $U = U[N, \gamma]$ Perfect fluid $U(X)$
- 3 DoF: $U = \left(\mathbf{U}[\gamma] + \mathbf{E}[\gamma]/N\right)$
- 2 DoF: $U = \left(\Lambda + \mathbf{E}[\gamma]/N\right)$

$$N \to \boxed{X = -\frac{1}{N^2}},\ \gamma \to \boxed{w_n = Tr[\gamma^n]},\ \mathcal{K} \to \boxed{\tau_n = Tr[(\gamma - \xi \otimes \xi)^n]},$$

$$Y = \frac{1}{N\sqrt{1-\xi^2}},\ b = \sqrt{(1-\xi^2)\,\gamma},\ y_n = \frac{\xi \cdot (\gamma - \xi \otimes \xi) \cdot \xi}{N^2},\ O_{\alpha\beta n} = (1-\xi^2)^{\alpha+\beta}\,\xi \cdot (\gamma - \xi \otimes \xi) \cdot \xi$$

$$S_{Solids} = \int d^4x \sqrt{-g}\, U(\tau_{n=1,2,3} = Tr[(B \equiv \partial\Phi^a \partial\Phi^b)^n])$$

Hook Potential

$$U(\tau_i) = \sqrt{det|B|}\ \left(\lambda\ Tr[I - B]^2 + 2\ \mu\ Tr[(I - B)^2]\right) =$$
$$b\ \left(6\ \mu + 9\ \lambda - (6\ \lambda + 4\ \mu)\ \tau_1 + \lambda\ {\tau_1}^2 + 2\ \mu\ \tau_2\right)$$

$\lambda\ \mu$: Lame' Constants, B strain tensor

$$T_{\mu\nu} = \underbrace{U}_{\rho}\ u_\mu\ u_\nu + \underbrace{2\ b\ \partial_\mu \Phi^a \left((-(6\lambda + 4\mu) + 2\lambda\tau_1)\delta_{ab} + 4\mu B_{ab}\right) \partial_\nu \Phi^b}_{p\ g_{\mu\nu} + \Pi_{\mu\nu}}$$