From generalized galileons to p-forms galileons

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- C.D., G. Esposito-Farese, A. Vikman
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I. Scalar Galileon

I. 1 Flat space-time Galileon in 4 D

Galileon

Originally (Nicolis, Rattazzi, Trincherini 2009) defined in flat space-time as the most general scalar theory which has **(strictly) second order fields equations**

In 4D, there is only 4 non trivial such theories

(with $\pi_\mu = \partial_\mu \pi \quad \pi_{\mu\nu} = \partial_\mu \partial_\nu \pi$)

$$\begin{aligned}
\mathcal{L}_{(2,0)} &= \pi_\mu \pi^\mu \\
\mathcal{L}_{(3,0)} &= \pi^\mu \pi_\mu \Box\pi \\
\mathcal{L}_{(4,0)} &= (\Box\pi)^2 \left(\pi_\mu \, \pi^\mu\right) - 2\left(\Box\pi\right)\left(\pi_\mu \, \pi^{\mu\nu} \, \pi_\nu\right) \\
&\quad - \left(\pi_{\mu\nu} \, \pi^{\mu\nu}\right)\left(\pi_\rho \, \pi^\rho\right) + 2\left(\pi_\mu \pi^{\mu\nu} \, \pi_{\nu\rho} \, \pi^\rho\right) \\
\mathcal{L}_{(5,0)} &= (\Box\pi)^3 \left(\pi_\mu \, \pi^\mu\right) - 3\left(\Box\pi\right)^2 \left(\pi_\mu \, \pi^{\mu\nu} \, \pi_\nu\right) - 3\left(\Box\pi\right)\left(\pi_{\mu\nu} \, \pi^{\mu\nu}\right)\left(\pi_\rho \, \pi^\rho\right) \\
&\quad +6\left(\Box\pi\right)\left(\pi_\mu \pi^{\mu\nu} \, \pi_{\nu\rho} \, \pi^\rho\right) + 2\left(\pi_\mu{}^\nu \, \pi_\nu{}^\rho \, \pi_\rho{}^\mu\right)\left(\pi_\lambda \, \pi^\lambda\right) \\
&\quad +3\left(\pi_{\mu\nu} \, \pi^{\mu\nu}\right)\left(\pi_\rho \, \pi^{\rho\lambda} \, \pi_\lambda\right) - 6\left(\pi_\mu \, \pi^{\mu\nu} \, \pi_{\nu\rho} \, \pi^{\rho\lambda} \, \pi_\lambda\right)
\end{aligned}$$

Simple rewriting of those Lagrangians with epsilon tensors (up to integrations by part):

(C.D., S.Deser, G.Esposito-Farese, 2009)

$$
\begin{aligned}
\mathcal{L}_{(2,0)} &= \epsilon^{\mu_1\lambda_1\lambda_2\lambda_3}\epsilon^{\nu_1}{}_{\lambda_1\lambda_2\lambda_3}\pi_{\mu_1}\pi_{\nu_1} \\
\mathcal{L}_{(3,0)} &= \epsilon^{\mu_1\mu_2\lambda_1\lambda_2}\epsilon^{\nu_1\nu_2}{}_{\lambda_1\lambda_2}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2} \\
\mathcal{L}_{(4,0)} &= \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3} \\
\mathcal{L}_{(5,0)} &= \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon^{\nu_1\nu_2\nu_3\nu_4}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\pi_{\mu_4\nu_4}
\end{aligned}
$$

This leads to (exactly) second order field equations

Indeed, consider e.g.

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

$$\propto (\Box\pi)^2(\pi_\mu\pi^\mu) - 2(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_\nu)$$

Varying this Lagrangian with respect to π yields (after integrating by part)

$$\delta\mathcal{L}_{(4,0)} \supset -\epsilon^{\mu_1\mu_2\ldots} \qquad - (\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) + 2(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho)$$

Second order derivative

Third order derivative…

Similarly, one also have in the field equations

$$\delta\mathcal{L}_{(4,0)} \supset \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\partial_{\nu_2}\partial_{\mu_2}\{\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_3\nu_3}\}$$

…killed by the contraction with epsilon tensor

Yields third and fourth order derivative… killed by the epsilon tensor

Hence the field equations are proportional to

$$\mathcal{E}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

Which does only contain second derivatives

NB: the field equations are linear in time derivatives (Cauchy problem ?)

I. 2 Flat space-time Galileon in arbitrary Dimension

In D dimensions, D non trivial Galileons can be defined as

$$\mathcal{L}_{(n+1,0)} = \sum_{\sigma \in S_n} \epsilon(\sigma) g^{\mu_{\sigma(1)}\nu_1} g^{\mu_{\sigma(2)}\nu_2} \dots g^{\mu_{\sigma(n)}\nu_n} (\pi_{\nu_1}\pi_{\mu_1})(\pi_{\nu_2\mu_2}\pi_{\nu_3\mu_3} \dots \pi_{\nu_n\mu_n}).$$

Only the Lagrangians with $D \geq n$ are non vanishing.

Using the tensors

$$\mathcal{A}_{(2n)}^{\mu_1\mu_2\dots\mu_{2n}} \equiv \frac{1}{(D-n)!} \varepsilon^{\mu_1\mu_3\mu_5\dots\mu_{2n-1}\,\nu_1\nu_2\dots\nu_{D-n}} \varepsilon^{\mu_2\mu_4\mu_6\dots\mu_{2n}}{}_{\nu_1\nu_2\dots\nu_{D-n}}$$

$\mathcal{A}_{(2n)}$ is antisymmetric (separately) in odd and even indices

Or to alleviate notations

$$\mathcal{A}_{(2n)}^{1234\dots} = \frac{1}{(D-n)!} \varepsilon^{135\dots\,\nu_1\nu_2\dots\nu_{D-n}} \varepsilon^{246\dots}{}_{\nu_1\nu_2\dots\nu_{D-n}}$$

One has

$$\mathcal{L}_{(n+1,0)} = -\mathcal{A}_{(2n)}(\pi_1\pi_2)(\pi_{34}\pi_{56}\pi_{78} \dots \pi_{\mu_{2n-1}\mu_{2n}})$$

All free indices are contracted with those of $\mathcal{A}_{(2n)}$

Up to total derivatives, the following Lagrangians are equivalent

$$\mathcal{L}_N^{\mathrm{Gal},1} = \left(\mathcal{A}_{(2n+2)}^{\mu_1\cdots\mu_{n+1}\nu_1\cdots\nu_{n+1}}\pi_{\mu_{n+1}}\pi_{\nu_{n+1}}\right)\pi_{\mu_1\nu_1}\ldots\pi_{\mu_n\nu_n}$$

$$\mathcal{L}_N^{\mathrm{Gal},2} = \left(\mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_{\mu_1}\pi_\lambda\pi^\lambda{}_{\nu_1}\right)\pi_{\mu_2\nu_2}\ldots\pi_{\mu_n\nu_n}$$

$$\mathcal{L}_N^{\mathrm{Gal},3} = \left(\mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\pi_\lambda\pi^\lambda\right)\pi_{\mu_1\nu_1}\ldots\pi_{\mu_n\nu_n}$$

One has the exact relation

$$(N-2)\mathcal{L}_N^{\mathrm{Gal},2} = \mathcal{L}_N^{\mathrm{Gal},3} - \mathcal{L}_N^{\mathrm{Gal},1}$$

I. 3 Curved space-time Galileon

A naive covariantization leads to the loss of the distinctive properties of the Galileon

Indeed, consider now in curved space-time (with $\pi_\mu = \nabla_\mu \pi$ and $\pi_{\mu\nu} = \nabla_\mu \nabla_\nu \pi$)

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}$$

Variation yields in particular

$$\delta\mathcal{L}_{(4,0)} \supset \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\delta\pi\nabla_{\nu_2}\nabla_{\mu_2}\left\{\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_3\nu_3}\right\}$$

Third derivatives generate now Riemann tensors …

and fourth derivatives, derivatives of the Riemann

Indeed the (naively covariantized) $\mathcal{L}_{(4,0)}$ has the field equations

$$\begin{aligned}\mathcal{E}\quad \mathcal{E}_{(4,0)} = & -4(\Box\pi)^3 - 8\left(\pi_\mu{}^\nu\pi_\nu{}^\rho\pi_\rho{}^\mu\right) + 12(\Box\pi)\left(\pi_{\mu\nu}\pi^{\mu\nu}\right) - \left(\pi_\mu\pi^\mu\right)\left(\pi_\nu \boxed{R^{;\nu}}\right)\\ & +2\left(\pi_\mu\pi_\nu\pi_\rho \boxed{R^{\mu\nu;\rho}}\right) + 10(\Box\pi)\left(\pi_\mu R^{\mu\nu}\pi_\nu\right) - 8\left(\pi_\mu\pi^{\mu\nu}R_{\nu\rho}\pi^\rho\right)\\ & -2\left(\pi_\mu\pi^\mu\right)\left(\pi_{\nu\rho}R^{\nu\rho}\right) - 8\left(\pi_\mu\pi_\nu\pi_{\rho\sigma}R^{\mu\rho\nu\sigma}\right).\end{aligned}$$

Kinetic mixing $\quad {}^{\mu\nu})\,.$

Similarly, varying w.r.t. the metric

$$\mathcal{L}_{(4,0)} = \epsilon^{\mu_1\mu_2\mu_3\lambda_1}\,\epsilon^{\nu_1\nu_2\nu_3}{}_{\lambda_1}\,\pi_{\mu_1}\,\pi_{\nu_1}\,\underset{\substack{\cup\\ \partial g}}{\pi_{\mu_2\nu_2}}\,\pi_{\mu_3\nu_3}$$

Yields third order derivatives of the scalar π in the energy momentum tensor

Do not vanish in flat space-time !

$$\begin{aligned}
\mathcal{T}^{\mu\nu}_{(4,0)} =\; & (\pi^\mu\,\pi^\nu)\,\pi^\lambda\,\boxed{\left(2\,\pi_{\lambda\rho}{}^{\rho} - \pi^{\rho}{}_{\rho\lambda}\right)} \\
& + \left(\pi_\lambda\,\pi^\lambda\right)\pi^\mu\,\boxed{\left(\pi_\rho{}^{\rho\nu} - \pi^{\nu\rho}{}_{\rho}\right)} + \left(\pi_\lambda\,\pi^\lambda\right)\pi^\nu\,\boxed{\left(\pi_\rho{}^{\rho\mu} - \pi^{\mu\rho}{}_{\rho}\right)} \\
& - \pi^\lambda\,\pi^\rho\,\boxed{\left(\pi^\mu\pi_{\lambda\rho}{}^{\nu} + \pi^\nu\pi_{\lambda\rho}{}^{\mu}\right)} + \left(\pi_\lambda\,\pi^\lambda\right)\left(\pi_\rho\,\boxed{\pi^{\mu\nu\rho}}\right) \\
& + \left(\pi_\lambda\,\pi_\rho\,\pi_\sigma\,\boxed{\pi^{\lambda\rho\sigma}}\right) g^{\mu\nu} - \left(\pi_\lambda\,\pi^\lambda\right)\left(\pi_\rho\,\boxed{\pi_\sigma{}^{\sigma\rho}}\right) g^{\mu\nu} \\
& + (\pi^\mu\,\pi^\nu)\left[3\left(\pi_{\lambda\rho}\,\pi^{\lambda\rho}\right) - 2\,(\Box\pi)^2\right] + (\pi^{\mu\nu})\,\pi_\lambda\left(2\,\pi^{\lambda\rho}\,\pi_\rho + \pi^\lambda\,\Box\pi\right) \\
& + 3\,(\Box\pi)\,\pi_\lambda\left(\pi^{\lambda\mu}\,\pi^\nu + \pi^{\lambda\nu}\,\pi^\mu\right) - 4\,\pi_\lambda\,\pi^{\lambda\rho}\left(\pi_\rho{}^\mu\,\pi^\nu + \pi_\rho{}^\nu\,\pi^\mu\right) \\
& - 2\left(\pi_\lambda\,\pi^{\lambda\mu}\right)\left(\pi_\rho\,\pi^{\rho\nu}\right) - \frac{1}{2}\left(\pi_\lambda\,\pi^\lambda\right)\left[(\Box\pi)^2 + \left(\pi_{\rho\sigma}\,\pi^{\rho\sigma}\right)\right] g^{\mu\nu} \\
& + \pi_\lambda\,\pi_\rho\left[3\pi^{\lambda\sigma}\,\pi_\sigma{}^\rho - 2\,(\Box\pi)\,\pi^{\lambda\rho}\right] g^{\mu\nu}.
\end{aligned}$$

This can be cured by a non minimal coupling to the metric

Adding to

$$\mathcal{L}_{(4,0)} = (\Box\pi)^2 (\pi_\mu\pi^\mu) - 2(\Box\pi)(\pi_\mu\pi^{\mu\nu}\pi_\nu) - (\pi_{\mu\nu}\pi^{\mu\nu})(\pi_\rho\pi^\rho) + 2(\pi_\mu\pi^{\mu\nu}\pi_{\nu\rho}\pi^\rho)$$

The Lagrangian

$$\mathcal{L}_{(4,1)} = (\pi_\lambda\pi^\lambda)\,\pi_\mu\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right]\pi_\nu$$

Yields second order field equations for the scalar and the metric

(but loss of the « Galilean » symmetry)

e.g. one has now the energy momentum tensor

$$\begin{aligned}
\mathcal{T}_4^{\mu\nu} = {} & 4(\Box\pi)\,\pi_\rho\left[\pi^\mu\,\pi^{\rho\nu} + \pi^\nu\,\pi^{\rho\mu}\right] - 2(\Box\pi)^2(\pi^\mu\,\pi^\nu) + 2(\Box\pi)\left(\pi_\lambda\,\pi^\lambda\right)(\pi^{\mu\nu}) \\
& +4\left(\pi_\lambda\,\pi^{\lambda\rho}\,\pi_\rho\right)(\pi^{\mu\nu}) - 4\left(\pi_\lambda\,\pi^{\lambda\mu}\right)(\pi_\rho\,\pi^{\rho\nu}) + 2\left(\pi_{\lambda\rho}\,\pi^{\lambda\rho}\right)(\pi^\mu\,\pi^\nu) \\
& -2\left(\pi_\lambda\,\pi^\lambda\right)\left(\pi^\mu{}_\rho\,\pi^{\rho\nu}\right) - 4\,\pi^\lambda\,\pi_{\lambda\rho}\left[\pi^{\rho\mu}\,\pi^\nu + \pi^{\rho\nu}\,\pi^\mu\right] - (\Box\pi)^2\left(\pi_\lambda\,\pi^\lambda\right)g^{\mu\nu} \\
& -4(\Box\pi)\left(\pi_\lambda\,\pi^{\lambda\rho}\,\pi_\rho\right)g^{\mu\nu} + 4\left(\pi_\lambda\,\pi^{\lambda\rho}\,\pi_{\rho\sigma}\,\pi^\sigma\right)g^{\mu\nu} \\
& +\left(\pi_\lambda\,\pi^\lambda\right)(\pi_{\rho\sigma}\,\pi^{\rho\sigma})\,g^{\mu\nu} + \left(\pi_\lambda\,\pi^\lambda\right)(\pi^\mu\,\pi^\nu)\,R - \frac{1}{4}\left(\pi_\lambda\,\pi^\lambda\right)(\pi_\rho\,\pi^\rho)\,g^{\mu\nu}R \\
& -2\left(\pi_\lambda\,\pi^\lambda\right)\pi_\rho\left[R^{\rho\mu}\,\pi^\nu + R^{\rho\nu}\,\pi^\mu\right] + \frac{1}{2}\left(\pi_\lambda\,\pi^\lambda\right)(\pi_\rho\,\pi^\rho)\,R^{\mu\nu} \\
& +2\left(\pi_\lambda\,\pi^\lambda\right)(\pi_\rho\,R^{\rho\sigma}\,\pi_\sigma)\,g^{\mu\nu} - 2\left(\pi_\lambda\,\pi^\lambda\right)(\pi_\rho\,\pi_\sigma\,R^{\mu\rho\nu\sigma})\,,
\end{aligned}$$

This can be generalized to arbitrary Galileons
(arbitrary number of fields and dimensions)

Introducing $\mathcal{L}_{(n+1,p)} = -\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\mathcal{S}_{(q)}$

With
$$\mathcal{R}_{(p)} \equiv (\pi_\lambda\pi^\lambda)^p \prod_{i=1}^{i=p} R_{\mu_{4i-1}\,\mu_{4i+1}\,\mu_{4i}\,\mu_{4i+2}},$$
$$\mathcal{S}_{(q)} \equiv \prod_{i=0}^{i=q-1} \pi_{\mu_{2n-1-2i}\,\mu_{2n-2i}},$$

The action
$$I = \int d^Dx\sqrt{-g}\sum_{p=0}^{p_{\max}} \mathcal{C}_{(n+1,p)}\mathcal{L}_{(n+1,p)}$$
with
$$\mathcal{C}_{(n+1,p)} = \left(-\frac{1}{8}\right)^p \frac{(n-1)!}{(n-1-2p)!\,(p!)^2} = \left(-\frac{1}{8}\right)^p \binom{n-1}{2p}\binom{2p}{p}$$

Yields second order field equations.

Heuristically, one needs to replace sucessively
pairs of twice differentiated π by Riemanns

Note that the extra term $\mathcal{L}_{(5,1)} = \frac{3}{4}\mathcal{A}_{(8)}\pi_1\pi_2\left(\pi_\lambda\pi^\lambda\right) R_{3546}\,\pi_{78}.$

Does not generate unwanted derivatives of the curvature thanks to Bianchi identity

Indeed one has
$$\begin{cases} \mathcal{A}_{(8)} R_{3546;7} = 0 \\ \mathcal{A}_{(8)} R_{3546;8} = 0 \end{cases}$$

Similarly to
$$\begin{cases} \mathcal{A}_{(8)}\partial_7\pi_{35} = 0 \\ \mathcal{A}_{(8)}\partial_8\pi_{46} = 0 \end{cases}$$

Yields an easy generalization to p-forms

I. 4 Generalization to p-forms

C.D., S.Deser, G.Esposito-Farese, arXiv 1007.5278 [gr-qc] (PRD)

E.g. consider

$$I = \int d^D x \, \varepsilon^{\mu\nu\dots} \varepsilon^{\alpha\beta\dots} \, \omega_{\mu\nu\dots} \omega_{\alpha\beta\dots} (\partial_\rho \omega_{\gamma\delta\dots} \, \dots)(\partial_\epsilon \omega_{\sigma\tau\dots} \, \dots)$$

With $A_{\mu\nu\dots}$ a *p*-form of field strength $\omega_{\lambda\mu\nu\dots} = \partial_{[\lambda} A_{\mu\nu\dots]}$

In the field equations, Bianchi identities annihilate any $\partial_\mu \partial_{[\alpha} \omega_{\beta\gamma\dots]}$

E.o.m. are (purely) second order

E.g. for a 2-form

$$I = \int d^7 x \, \varepsilon^{\mu\nu\rho\sigma\tau\varphi\chi} \varepsilon^{\alpha\beta\gamma\delta\epsilon\zeta\eta} \, \omega_{\mu\nu\rho} \, \omega_{\alpha\beta\gamma} \, \partial_\sigma \omega_{\delta\epsilon\zeta} \, \partial_\eta \omega_{\tau\varphi\chi}$$

Note that one must go to 7 dimensions (in general one has $D \geq 2p+3$)
and that this construction does not work for odd p as we show now

I.4.2. The case of (odd p)-forms

For odd p the previous construction does not work

Indeed, the action

$$I = \int d^D x\, \varepsilon^{\mu\nu\dots} \varepsilon^{\alpha\beta\dots}\, \omega_{\mu\nu\dots} \omega_{\alpha\beta\dots} (\partial_\rho \omega_{\gamma\delta\dots}\ \dots)(\partial_\epsilon \omega_{\sigma\tau\dots}\ \dots)$$

With $A_{\mu\nu\dots}$ an (odd *p*)-form of field strength $\omega_{\lambda\mu\nu\dots} = \partial_{[\lambda} A_{\mu\nu\dots]}$

Yields vanishing e.o.m. (the action is a total derivative)

Integration by part

$$I = -\int d^D x\, \varepsilon^{\mu\nu\dots} \varepsilon^{\alpha\beta\dots}\, \omega_{\mu\nu\dots} \partial_\rho (\omega_{\alpha\beta\dots}) \omega_{\gamma\delta\dots}\ \dots (\partial_\epsilon \omega_{\sigma\tau\dots}\ \dots)$$

Renumbering of an even (for odd p) number of indices

$$I = -\int d^D x\, \varepsilon^{\mu\nu\dots} \varepsilon^{\alpha\beta\dots}\, \omega_{\mu\nu\dots} \partial_\rho (\omega_{\gamma\delta\dots}) \omega_{\alpha\beta\dots}\ \dots (\partial_\epsilon \omega_{\sigma\tau\dots}\ \dots)$$

Is there an (odd p) Galileon ?

C. D. , A. E. Gumrukcuoglu, S. Mukohyama and Y.Wang, [arXiv:1312.6690 [hep-th]], JHEP 2014.

C.D., S. Mukohyama, V. Sivanesan [arXiv:1601.01287[hep-th]], PRD 2016.

C.D., S. Mukohyama, V. Sivanesan In preparation

We start from the field equations

$$\mathcal{E}^A \equiv \frac{\delta \mathcal{S}}{\delta \mathcal{A}_A} = 0 \qquad \text{with} \qquad \mathcal{E}^A = \mathcal{E}^A(\mathcal{A}_B; \mathcal{A}_{B,a}; \mathcal{A}_{B,ab})$$

For a p-form $\boldsymbol{\mathcal{A}} \in \bigwedge^{\boldsymbol{p}}$ with components $\mathcal{A}_{a[p]}$

with $a[p] \equiv A$ p antisymmetric indices

We ask these field equations

(i) To derive from an action

$$\mathcal{S} = \int d^D x \; \mathcal{L}[\mathcal{A}_B; \partial_a \mathcal{A}_B; \partial_a \ldots \partial_b \mathcal{A}_B]$$

(ii) To depend only on second derivatives

(iii) to be gauge invariant

$$\boldsymbol{\mathcal{A}} \rightarrow \boldsymbol{\mathcal{A}} + d\boldsymbol{\mathcal{C}} \equiv \boldsymbol{\mathcal{A}}' \qquad \boldsymbol{\mathcal{C}} \in \bigwedge^{\boldsymbol{p-1}}$$

Hypothesis (i) (that field equations derive from an action *S*) leads to

$$\left[\frac{\delta}{\delta \mathcal{A}_B(y)}, \frac{\delta}{\delta \mathcal{A}_A(x)}\right] \mathcal{S} = 0$$

Which upon using $\frac{\delta}{\delta \mathcal{A}_B(y)} \frac{\delta}{\delta \mathcal{A} \ldots} \ldots \frac{\ldots}{\delta \mathcal{A}_B(y)}$

Vanish as a consequence of Hypothesis (ii) (that field equations only depend on second order derivatives)

Leads to the « integrability conditions »

$$\left\{\begin{array}{l} \mathcal{E}^{A|B} - \mathcal{E}^{B|A} + \left(\mathcal{E}^{B|A,c}\right)_{,c} - \left(\mathcal{E}^{B} \ldots\right. \\ \mathcal{E}^{A|B,c} + \mathcal{E}^{B|A,c} \ldots 2\partial_d \mathcal{E} \ldots \\ \mathcal{E}^{A|B,cd} - \mathcal{E}^{B|A,cd} = 0 \end{array}\right.$$

Vanish as a consequence of Hypothesis (ii) (that field equations only depend on second order derivatives)

Where
$$\left\{\begin{array}{rcl} \mathcal{E}^{A|B} & \equiv & \mathcal{E}^{a[p]|b[b]} \equiv \dfrac{\partial \mathcal{E}^{a[p]}}{\partial \mathcal{A}_{b[p]}} \equiv \dfrac{\partial \mathcal{E}^{A}}{\partial \mathcal{A}_{B}}, \\ \mathcal{E}^{A|B,c} & \equiv & \mathcal{E}^{a[p]|b[p],c} \equiv \dfrac{\partial \mathcal{E}^{a[p]}}{\partial\left(\partial_c \mathcal{A}_{b[p]}\right)} \equiv \dfrac{\partial \mathcal{E}^{a[p]}}{\partial \mathcal{A}_{b[p],c}} \equiv \dfrac{\partial \mathcal{E}^{A}}{\partial \mathcal{A}_{B,c}}, \\ \mathcal{E}^{A|B,cd} & \equiv & \mathcal{E}^{a[p]|b[p],cd} \equiv \dfrac{\partial \mathcal{E}^{a[p]}}{\partial\left(\partial_c \partial_d \mathcal{A}_{b[p]}\right)} \equiv \dfrac{\partial \mathcal{E}^{a[p]}}{\partial \mathcal{A}_{b[p],cd}} \equiv \dfrac{\partial \mathcal{E}^{A}}{\partial \mathcal{A}_{B,cd}}. \end{array}\right.$$

Hypothesis (iii) (gauge invariance of the field equations)

Lead to
$$\left\{\begin{array}{l} \mathcal{E}^{A|B} = 0 \\ \mathcal{E}^{A|b[p-1](b_p,c)} = 0 \\ \mathcal{E}^{A|b[p-1](b_p,cd)} = 0 \end{array}\right.$$

These symmetries extend in particular to derivatives of the field equations

Defining then

$$(\boldsymbol{\mathcal{E}}^{\boldsymbol{m}})^{AB_1c_1d_1\ldots B_{m-1}c_{m-1}d_{m-1}} = \mathcal{E}^{A|B_1,c_1d_1|\ldots|B_{m-1},c_{m-1}d_{m-1}}$$
$$\equiv \frac{\partial^{m-1}\mathcal{E}^A}{\partial\mathcal{A}_{B_1,c_1d_1}\ldots\partial\mathcal{A}_{B_{m-1},c_{m-1}d_{m-1}}}$$

Hypothesis (i) (ii) and (iii) lead to the following symmetries for the tensor $\boldsymbol{\mathcal{E}}^{\boldsymbol{m}}$

1.1. Antisymmetry within each group of p indices A and B_i
1.2. Invariance under interchange of groups of indices B_i and B_j, as well as A and any B_i

2.1. Symmetry of each pair of indices (c_i, d_i)
2.2. Invariance under interchange (c_i, d_i) and (c_j, d_j)

3. Symmetrizing over any 3 indices yields zero

NB: $\left(\boldsymbol{\mathcal{E}^m}\right)^{AB_1c_1d_1\ldots B_{m-1}c_{m-1}d_{m-1}}$

Is a *(pm + 2(m-1))* tensor

For p=1,

Conditions 1.1, 1.2, 2.1, 2.2 and 3

are enough to show that $\boldsymbol{\mathcal{E}^3} = \mathcal{E}^{A|B,ab|C,cd}$ vanishes identically
(i.e. field equations are at most linear into the second derivatives)

No vector Galileons (with gauge invariance)

C. D. , A. E. Gumrukcuoglu, S. Mukohyama and Y.Wang, [arXiv:1312.6690 [hep-th]].

For (odd) p > 1 ?

C.D., S. Mukohyama, V. Sivanesan, 1601.01287 [hep-th]

Decompose $\left(\mathcal{E}^m\right)^{AB_1c_1d_1\ldots B_{m-1}c_{m-1}d_{m-1}}$

into components belonging to irreducible representations of the symmetric group $S_{pm+2(m-1)} = S_n$

Using

1.1. Antisymmetry w
1.2. Invariance unde
well as A and any B_i

2.1. Symmetry of each p... ...ces (c_i, d_i)
2.2. Invariance ... interchange (c_i, d_i) and (c_j, d_j)

3. Symmetrizing over any 3 indices yields zero

Condition 3. implies that only Young Tableaux with 1 or 2 column will yield a non zero component of $\mathcal{E}^m$

This amounts to act with the Young symmetrizers $y^{anti}_{\lambda_k}$ / $y^{sym}_{\lambda_k}$ appearing in the decomposition of the group algebra of S_n into irreducibles given by

$$\mathbb{R}[S_n] = \bigoplus_{\lambda\vdash n} \bigoplus_{\lambda_k \in ST_\lambda} \mathbb{R}[S_n]\, y^{sym/anti}_{\lambda_k}$$

Partitions of n

Standard Young tableaux

Young symmetrizers $y^{sym}_{\lambda_k} \equiv r_{\lambda_k} \times c_{\lambda_k}$

1.1. Antisymmetry within each group of p indices A and B_i
1.2. Invariance under interchange of groups of indices B_i and B_j, as well as A and any B_i

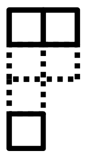
2.1. Symmetry of each pair of indices (c_i, d_i)
2.2. Invariance under interchange (c_i, d_i) and (c_j, d_j)

3. Symmetrizing over any 3 indices yields zero

Conditions 1.1,1.2., 2.1. and 2.2. then imply th $\mathcal{E}^m$ belongs to the « Plethysm »

$$\mathrm{Sym}^m(\bigwedge^p)\otimes\mathrm{Sym}^{(m-1)}(\mathrm{Sym}^2)$$

Next step: find out the content of this Plethysm.
in the representations of S_n indexed by up-to-two columns Young diagrams

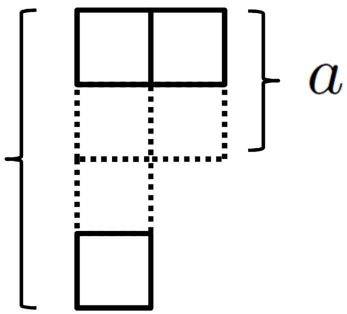


By the use of Schur functions , the multiplicity $m_{(mp+2(m-1)-a,a)}$

(which allows to count the max number of possibly non trivial theories

fixing m and allowing a to vary)

of the representations indexed by $mp+2(m-1)-a$



Inside $\mathrm{Sym}^m(\bigwedge^p)\otimes\mathrm{Sym}^{(m-1)}(\mathrm{Sym}^2)$

Is given by

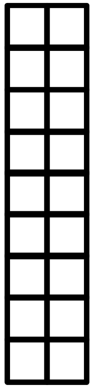
$$m_{(mp+2(m-1)-a,a)} = \begin{cases} N_{a-m+1,m} - N_{(a-m),m} & if\ p\ is\ Even \\ N^{distinct}_{a-m+1,m} - N^{distinct}_{(a-m),m} & if\ p\ is\ Odd \end{cases}$$

Where $N_{r,s}$ is the number of partitions of r into s number within (0,…,p) with repetition allowed.
$N^{distinct}_{r,s}$ is the same with repetitions not allowed.

Next step: try to construct explicit theories

E.g. for m=4, p=3 $\Rightarrow$ mp + 2(m-1) = 18

We look at Young diagrams with 18 boxes

e.g.  has multiplicity 1 inside

$$\mathrm{Sym}^4(\bigwedge{}^3)\otimes\mathrm{Sym}^3(\mathrm{Sym}^2)$$

No clear method to constuct the corresponding theory

Start with the filling

f_1	g_1
f_2	g_2
f_3	g_3
a_1	b_3
a_2	c_2
a_3	c_3
b_1	d_1
b_2	d_2
c_1	d_3

A tensor with these symmetries can be constructed from the metric corresponding to

$$\epsilon^{f_1 f_2 f_3 a_1 a_2 a_3 b_1 b_2 c_1} \epsilon^{g_1 g_2 g_3 b_3 c_2 c_3 d_1 d_2 d_3}$$

Then act on it with projectors corresponding to the symmetries of

$$\mathrm{Sym}^4(\bigwedge^3) \otimes \mathrm{Sym}^3(\mathrm{Sym}^2)$$

This gives a non trivial

$$\boldsymbol{\mathcal{E}}^{\mathbf{4}} = \mathcal{E}^{a[3]|b[3],f_1 g_1|c[3],f_2 g_2|d[3],f_3 g_3}$$

This can be integrated to yield the following action density for a 3 form (in D= 9 dimensions)

$$\int d^{\,9}x\epsilon^{a_1a_2\cdots}\ \epsilon^{b_1b_2\cdots}\ A_{a_1a_2b_1,a_3}\ A_{b_2b_3a_4,b_4}\ \partial_a\omega_B\ \partial_b\omega_A$$

To be contrasted with the p-form action constructed in

C.D., S.Deser, G.Esposito-Farese,
arXiv 1007.5278 [gr-qc] (PRD)

$$\int d^{\,9}x\epsilon^{a_1a_2\cdots}\ \epsilon^{b_1b_2\cdots}\ A_{a_1a_2a_3,a_4}\ A_{b_1b_2b_3,b_4}\ \partial_a\omega_B\ \partial_b\omega_A$$

This can be generalized …..

…… classification on the way

I. 5 From k-essence to generalized Galileons

C.D. Xian Gao, Daniele Steer, George Zahariade
arXiv:1103.3260 [hep-th] (PRD)

What is the most general **scalar theory** which has (not necessarily exactly) second order field equations in **flat space** ?

Specifically we looked for the most general scalar theory such that (in flat space-time)

i/ Its **Lagrangian contains derivatives of order 2** or less of the scalar field π

ii/ Its **Lagrangian is polynomial in second derivatives of** π (can be relaxed: Padilla, Sivanesan; Sivanesan)

iii/ The **field equations are of order 2 or lower** in derivatives

(NB: those hypothesis cover k-essence, simple Galileons ,…)

Answer: the most general such theory is given by a linear combination of the Lagrangians $\mathcal{L}_n\{f\}$

Free function of π and X

defined by
$$\begin{aligned}\mathcal{L}_n\{f\} &= f(\pi, X) \times \mathcal{L}^{\mathrm{Gal},3}_{N=n+2},\\ &= f(\pi, X) \times \left(X \mathcal{A}^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_n}_{(2n)} \pi_{\mu_1 \nu_1} \ldots \pi_{\mu_n \nu_n}\right)\end{aligned}$$

where $X \equiv \pi_\mu \pi^\mu$

Our most general flat space-time theories can easily be
« covariantized » using the previously described technology

The covariantized theory is given by a linear combination of the Lagrangians

$$\mathcal{L}_{n,p}\{f\} = \mathcal{P}_{(p)}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}\mathcal{R}_{(p)}\mathcal{S}_{(q\equiv n-2p)}$$

with
$$\begin{cases}
\mathcal{S}_{(q\equiv n-2p)} \equiv \prod_{p}^{q-1} \pi_{\mu_{n-i}\nu_{n-i}} \\
\mathcal{R}_{(p)} \equiv \prod_{i=1}^{p} R_{\mu_{2i-1}\mu_{2i}\nu_{2i-1}\nu_{2i}} \\
\mathcal{P}_{(p)} \equiv \int_{X_0}^{X} dX_1 \int_{X_0}^{X_1} dX_2 \cdots \int_{X_0}^{X_{p-1}} dX_p\, \mathcal{T}_{(2n)}^{\mu_1\mu_2\cdots\mu_n\nu_1\nu_2\cdots\nu_n}(\pi, X_1) \\
\mathcal{T}_{(2n)} = \mathcal{T}_{(2n)}(\pi, X) \\
\qquad = f(\pi, X) \times X\mathcal{A}_{(2n)}
\end{cases}$$

Specifically the covariantized theory is given by

$$\mathcal{L}_n^{\mathrm{cov}}\{f\} = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{C}_{n,p}\mathcal{L}_{n,p}\{f\} \quad \text{with} \quad \mathcal{C}_{n,p} = \left(-\frac{1}{8}\right)^p \frac{n!}{(n-2p)!p!}$$

I. 6 Some previous and recent results and other approaches

• **Flat space-time** Galileons and flat-space time generalized Galileons (in the shift symmetric case) have been obtained previously by Fairlie, Govaerts and Morozov (1992) by the « Euler hierarchy » construction :

Start from a set of arbitrary functions $F_\ell = F_\ell(\pi^\mu)$

Then define the recursion relation $W_{\ell+1} = -\hat{\mathcal{E}} F_{\ell+1} W_\ell$

$\hat{\mathcal{E}}$ being the Euler-Lagrange operator (and W_0=1)

$$\hat{\mathcal{E}} = \left[\frac{\partial}{\partial \pi} - \partial_\mu \left(\frac{\partial}{\partial \pi_\mu}\right) + \partial_\mu \partial_\nu \left(\frac{\partial}{\partial \pi_{\mu\nu}}\right)\right]$$

Hence W_ℓ is the field equation of the Lagrangian $\mathcal{L}_\ell = F_\ell W_{\ell-1}$ (« Euler hierarchy »)

⟹ The hierarchy stops after at most D steps

⟹ Choosing $F_k = \pi^\mu \pi_\mu /2$, one has $\mathcal{L}_\ell = \frac{1}{2} X W_{\ell-1} = \frac{1}{2} \mathcal{L}^{\mathrm{Gal},3}_{\ell+1}$

(see e.g. the review by Curtright and Fairlie (2012))

• Horndeski (1972) obtained the most general **scalar tensor theory** in 4D which has second order field equation for the scalar and the metric

Using our notations it is given by

$$\begin{aligned}\mathcal{L}_H \;=\; & -\mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\kappa_1 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3\nu_3} - \frac{4}{3}\kappa_{1,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right) \\ & -\mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\kappa_3 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3}\pi_{\nu_3} - 4\kappa_{3,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3}\pi_{\nu_3}\right) \\ & -\mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}\left(F R_{\mu_1\mu_2\nu_1\nu_2} - 4F_{,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\right) \\ & -2\kappa_8\mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2} \\ & -3\left(2F_{,\pi} + X\kappa_8\right)X + \kappa_9,\end{aligned}$$

and is parametrized by four free functions of X and π : $\kappa_1,\ \kappa_3,\ \kappa_8,\ \kappa_9$
and one constraint $F_{,X} = \kappa_{1,\pi} - \kappa_3 - 2X\kappa_{3,X}$

⇨ First it is clear that the flat space-time restriction of Horndeski theory must be included in our generalized flat-space time Galileon

⇨ Conversely, our covariantized generalized Galileons must be included into Horndeski theory

$$
\begin{aligned}
\mathcal{L}_H \ = \ & -\mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\kappa_1 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3\nu_3} - \frac{4}{3}\kappa_{1,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right) \\
& -\mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\kappa_3 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3}\pi_{\nu_3} - 4\kappa_{3,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3}\pi_{\nu_3}\right) \\
& -\mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}\left(F R_{\mu_1\mu_2\nu_1\nu_2} - 4F_{,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\right) \\
& -2\kappa_8\mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2} \\
& -3\left(2F_{,\pi} + X\kappa_8\right)X + \kappa_9, \qquad\qquad F_{,X} = \kappa_{1,\pi} - \kappa_3 - 2X\kappa_{3,X}
\end{aligned}
$$

In fact, one can show that the two sets of theories (Horndeski and covariantized generalized Galileons) are identical in 4D (even though they start from different hypothesis)

$$
\mathcal{L}_H = \sum_{n=0}^{3} \mathcal{L}_n^{\mathrm{cov}}\{f_n\}
$$

$$
\begin{aligned}
X f_0(\pi, X) \ &= \ -\kappa_9(\pi, X) - \frac{X}{2}\int dX\left(2\kappa_8 - 4\kappa_{3,\pi}\right)_{,\pi}, \\
X f_1(\pi, X) \ &= \ X\left(4\kappa_{3,\pi} + \kappa_8\right) - \frac{1}{2}\int dX\left(2\kappa_8 - 4\kappa_{3,\pi}\right) + 6F_{,\pi} \\
X f_2(\pi, X) \ &= \ 4\left(F + X\kappa_3\right)_{,X}, \\
X f_3(\pi, X) \ &= \ \frac{4}{3}\kappa_{1,X}.
\end{aligned}
$$

C.D. Xian Gao, Daniele Steer, George Zahariade
arXiv:1103.3260 [hep-th] (PRD)

T.Kobayashi, M.Yamaguchi, J. Yokoyama
arXiv 1105.5723 [hep-th]

II. D.o.f. counting in Galileons and generalized Galileons theories

C.D., G. Esposito-Farèse, D. Steer, arXiv:1506.01974 [gr-qc], PRD

Horndeski-like theories: Cauchy problem ? Numerical studies (e.g. adressing the Vainshtein mechanism in grav. Collapse) ?

No Hamiltonian analysis so far !

Horndeski-like theories: Scalar tensor theories with second order field equations + diffeo invariance

⇒ A priori 2 (graviton) + 1 (scalar) d.o.f.

Claimed to be true in an even larger set of theories (« Beyond Horndeski » theories) !

J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, **(GLPV)** arXiv:1404.6495, arXiv:1408.1952

Our works aims at

Provides a first step toward a proper Hamiltonian treatment of Horndeski-like and beyond Horndeski theories

Rexamine the GLPV claim
(arguments of GLPV being not convincing to us)

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Provides a first step toward a proper Hamiltonian treatment of Horndeski-like and beyond Horndeski theories

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(arguments of GLPV being not convincing to us)

Along two directions

II.1. Show how in a (large) set of beyond Horndeski theories (matching the one considered by GLPV), the order in time derivatives of the field equations can indeed be reduced.

II.2. Analyze in details via Hamiltonianian formalism a simple (the simplest ?) non trivial beyond Horndeski theory

Consider one single Galileon « counterterm »: $\mathcal{L}_{(n+1,p)} = -\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\mathcal{S}_{(q)}$

With
$$\begin{cases} \mathcal{R}_{(p)} \equiv \left(\pi_\lambda\pi^\lambda\right)^p \prod_{i=1}^{i=p} R_{\mu_{4i-1}\,\mu_{4i+1}\,\mu_{4i}\,\mu_{4i+2}}, \\ \mathcal{S}_{(q)} \equiv \prod_{i=0}^{i=q-1} \pi_{\mu_{2n-1-2i}\,\mu_{2n-2i}}, \end{cases}$$

Consider one single Galileon « counterterm »: $\mathcal{L}_{(n+1,p)} = -\mathcal{A}_{(2n)}\pi_1\pi_2\mathcal{R}_{(p)}\mathcal{S}_{(q)}$

With
$$\begin{cases} \mathcal{R}_{(p)} \equiv (\pi_\lambda \pi^\lambda)^p \prod_{i=1}^{i=p} R_{\mu_{4i-1}\,\mu_{4i+1}\,\mu_{4i}\,\mu_{4i+2}}, \\ \mathcal{S}_{(q)} \equiv \prod_{i=0}^{i=q-1} \pi_{\mu_{2n-1-2i}\,\mu_{2n-2i}}, \end{cases}$$

The field equations have the following structure (as seen before)

Scalar field eom	$\mathcal{E}$	$\supset$	$\partial\partial\,\pi$
		$\supset$	$\partial\partial\partial\, g_{\mu\nu}\,(\sim \nabla\ \mathrm{Riemann})$
Energy momentum tensor	$\mathcal{T}^{\mu\nu}$	$\supset$	$\partial\partial\, g_{\mu\nu}\,(\sim\ \mathrm{Riemann})$
		$\supset$	$\partial\partial\partial\,\pi$

Concentrate on the

Energy momentum tensor	$\mathcal{T}^{\mu\nu}$	$\supset$	$\partial\partial\, g_{\mu\nu}\,(\sim\ \mathrm{Riemann})$
		$\supset$	$\partial\partial\partial\, \pi$

Concentrate on the

Energy momentum tensor	$\mathcal{T}^{\mu\nu}$	$\supset$	$\partial\partial\, g_{\mu\nu}\, (\sim \text{ Riemann})$
		$\supset$	$\partial\partial\partial\, \pi$

$$
\begin{aligned}
T^{\mu\nu} \;=\; & \left[\left(\frac{1}{2}g^{\mu\nu} + p\,\frac{\pi^{\mu}\pi^{\nu}}{\pi_{\lambda}^{2}}\right)\mathcal{A}_{(2n)} - \mathcal{A}^{\mu\nu}_{(2n+2,2n+1,2n+2)}\right]\pi_{\mu_1}\pi_{\mu_2}\mathcal{R}_{(p)}\mathcal{S}_{(q)} \\
& -q\,\mathcal{A}^{(\mu}_{(2n,4p+4)}\pi_{\mu_1}\left[\pi^{\nu)}\pi_{\mu_2}\mathcal{R}_{(p)}\mathcal{S}_{(q-1)}\right]_{;\mu_{4p+3}} \\
& +\frac{q}{2}\mathcal{A}^{\mu\nu}_{(2n,4p+3,4p+4)}\left[\pi^{\sigma}\pi_{\mu_1}\pi_{\mu_2}\mathcal{R}_{(p)}\mathcal{S}_{(q-1)}\right]_{;\sigma} \\
& +2p\,\mathcal{A}^{(\mu\nu)}_{(2n,4p+1,4p+2)}\left[\pi_{\mu_1}\pi_{\mu_2}(\pi_{\lambda}^{2})\mathcal{R}_{(p-1)}\mathcal{S}_{(q)}\right]_{;\mu_{4p}\;\mu_{4p-1}} \\
& -p\,\mathcal{A}^{(\mu}_{(2n,4p-1)}R^{\nu)}{}_{\mu_{4p+1}\;\mu_{4p}\;\mu_{4p+2}}\,\pi_{\mu_1}\pi_{\mu_2}(\pi_{\lambda}^{2})\mathcal{R}_{(p-1)}\mathcal{S}_{(q)},
\end{aligned}
$$

Where
$$
\left\{
\begin{aligned}
\mathcal{A}^{\alpha}_{(2n,i)} &\equiv \mathcal{A}^{\mu_1\mu_2\ldots\mu_{i-1}\;\alpha\;\mu_{i+1}\ldots\mu_{2n}}_{(2n)}, \\
\mathcal{A}^{\alpha\beta}_{(2n,i,j)} &\equiv \mathcal{A}^{\mu_1\mu_2\ldots\mu_{i-1}\;\alpha\;\mu_{i+1}\ldots\mu_{j-1}\;\beta\;\mu_{j+1}\ldots\mu_{2n}}_{(2n)}
\end{aligned}
\right.
$$

Concentrate on the

Energy momentum tensor	$\mathcal{T}^{\mu\nu}$	$\supset$	$\partial\partial\, g_{\mu\nu}\, (\sim \ \mathrm{Riemann})$
		$\supset$	$\partial\partial\partial\, \pi$

T^{00} and $\pi_\mu T^{\mu 0}$

1/ Do not contain any $\dddot{\pi}$

2/ Do contain the same combination of $\ddot{g}_{\mu\nu}$

3/ Do not contain any $\partial_i \ddot{\pi}$

Concentrate on the

Energy momentum tensor	$\mathcal{T}^{\mu\nu}$	$\supset$	$\partial\partial\, g_{\mu\nu}\ (\sim\ \mathrm{Riemann})$
		$\supset$	$\partial\partial\partial\, \pi$

T^{00} and $\pi_\mu T^{\mu 0}$

1/ Do not contain any $\dddot{\pi}$

2/ Do contain the same combination of $\ddot{g}_{\mu\nu}$

3/ Do not contain any $\partial_i \ddot{\pi}$

$$\pi^0 \pi_\mu T^{\mu 0} - \pi^\mu \pi_\mu T^{00}$$

Contains only $\ddot{\pi}$ as second time derivatives

Hence, the combination of the field equations

$$\pi^0 \pi_\mu \left(T^{\mu 0} - G^{\mu 0}\right) - \pi^\mu \pi_\mu \left(T^{00} - G^{00}\right)$$

Can be used on shell to extract $\ddot{\pi}$ as function of time derivatives

of **order $<$ 2** of the scalar field and the metric

Hence, the combination of the field equations

$$\pi^0 \pi_\mu \left(T^{\mu 0} - G^{\mu 0}\right) - \pi^\mu \pi_\mu \left(T^{00} - G^{00}\right)$$

Can be used on shell to extract $\ddot{\pi}$ as function of time derivatives

of <u>**order $<$ 2**</u> of the scalar field and the metric

Take then a time derivative of the obtained expression
and insert back into

Energy momentum tensor	$\mathcal{T}^{\mu\nu}$	$\supset$	$\partial\partial\, g_{\mu\nu}\, (\sim \ \mathrm{Riemann})$
		$\supset$	$\partial\partial\partial\, \pi$

Reduces the Einstein equations to a system
od PDE of second order in time

Similarly, another linear combination of the time derivatives of the field equations $\pi_\mu \left(T^{\mu 0} - G^{\mu 0}\right)$ and $\left(T^{00} - G^{00}\right)$ can be used to reduce the order of the time derivatives of the scalar field equation by extracting $\dddot{g}_{ij}$ as function of lower time derivatives

NB: in

T^{00} and $\pi_\mu T^{\mu 0}$

1/ Do not contain any $\dddot{\pi}$

2/ Do contain the same combination of $\ddot{g}_{\mu\nu}$

3/ Do not contain any $\partial_i \ddot{\pi}$

NB: in

$$T^{00} \quad \text{and} \quad \pi_\mu T^{\mu 0}$$

1/ Do not contain any $\dddot{\pi}$

2/ Do contain the same combination of $\ddot{g}_{\mu\nu}$

3/ Do not contain any $\partial_i \ddot{\pi}$

The crucial 1/ and 2/ are just consequences of

Scalar field eom	$\mathcal{E}$	$\supset$	$\partial\partial\, \pi$
		$\supset$	$\partial\partial\partial\, g_{\mu\nu}\, (\sim \nabla\ \text{Riemann})$
Energy momentum tensor	$\mathcal{T}^{\mu\nu}$	$\supset$	$\partial\partial\, g_{\mu\nu}\, (\sim\ \text{Riemann})$
		$\supset$	$\partial\partial\partial\, \pi$

And $\nabla_\mu T^{\mu\nu} = \frac{1}{2}\pi^\nu \mathcal{E}$ (from invariance under diffeo)

The found « reduction » of the order in time derivatives of the field equations

can be generalized to an arbitrary theory of the type

$$-f(\pi, X) \times \mathcal{L}_{(n+1,p)} \quad \left(= \; -f(\pi, X)\, \mathcal{A}_{(2n)} \pi_1 \pi_2 \mathcal{R}_{(p)} \mathcal{S}_{(q)}\right)$$

Each theory of this type should propagate 2 (graviton) + 1 (scalar) d.o.f.

II.2. Hamiltonian analysis of the quartic Galileon

Consider $$S = \int d^4x \sqrt{-g}\left[R + \mathcal{L}_{(4,0)}\right]$$

With
$$\left\{\begin{aligned}
\mathcal{L}_{(4,0)} &= (\Box\pi)^2\,(\pi_\mu\,\pi^\mu) - 2\,(\Box\pi)\,(\pi_\mu\,\pi^{\mu\nu}\,\pi_\nu) \\
&\quad - (\pi_{\mu\nu}\,\pi^{\mu\nu})\,(\pi_\rho\,\pi^\rho) + 2\,(\pi_\mu\pi^{\mu\nu}\,\pi_{\nu\rho}\,\pi^\rho) \\
&= \epsilon^{\mu_1\mu_3\mu_5\nu_1}\,\epsilon^{\mu_2\mu_4\mu_6}{}_{\nu_1}\,\pi_{\mu_1}\pi_{\mu_2}\pi_{\mu_3\mu_4}\pi_{\mu_5\mu_6}
\end{aligned}\right.$$

In the ADM parametrization, the action *S* becomes (in an arbitrary gauge)

$$\begin{aligned} S \;=\; & \int dt d^3x \, N\sqrt{\gamma}(K_{ij}K^{ij} - K^2 + {}^{(3)}R) \\ & + \int dt d^3x \, \frac{\sqrt{\gamma}}{N}\epsilon^{ijk}\epsilon^{\ell m}{}_k\big[-\dot{\pi}^2 s_{i\ell}s_{jm} - 2\pi_i\pi_\ell s_{00}s_{jm} + 2\pi_i\pi_\ell s_{0m}s_{0j} + 4\dot{\pi}\pi_\ell s_{i0}s_{jm}\big] \\ & + \int dt d^3x \, \frac{\sqrt{\gamma}}{N}\epsilon^{ijk}\epsilon^{\ell mn}N_k\big[2\dot{\pi}\pi_\ell s_{im}s_{jn} - 4\pi_i\pi_\ell s_{0m}s_{jn}\big] \\ & + \int dt d^3x \, N\sqrt{\gamma}\left(1 - \frac{N_pN^p}{N^2}\right)\epsilon^{ijk}\epsilon^{\ell mn}s_{jm}s_{kn}\pi_i\pi_\ell \end{aligned}$$

Where $s_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\pi$

In the ADM parametrization, the action *S* becomes (in an arbitrary gauge)

$$
\begin{aligned}
S \;=\; & \int dt d^3x\, N\sqrt{\gamma}(K_{ij}K^{ij} - K^2 + {}^{(3)}R) \\
& + \int dt d^3x\, \frac{\sqrt{\gamma}}{N} \epsilon^{ijk}\epsilon^{\ell m}{}_{k}\big[-\dot{\pi}^2 s_{i\ell}s_{jm} + \boxed{2\pi_i\pi_\ell s_{00}s_{jm}} + 2\pi_i\pi_\ell s_{0m}s_{0j} + 4\dot{\pi}\pi_\ell s_{i0}s_{jm}\big] \\
& + \int dt d^3x\, \frac{\sqrt{\gamma}}{N} \epsilon^{ijk}\epsilon^{\ell mn} N_k \big[2\dot{\pi}\pi_\ell s_{im}s_{jn} - 4\pi_i\pi_\ell s_{0m}s_{jn}\big] \\
& + \int dt d^3x\, N\sqrt{\gamma}\left(1 - \frac{N_p N^p}{N^2}\right)\epsilon^{ijk}\epsilon^{\ell mn} s_{jm}s_{kn}\pi_i\pi_\ell
\end{aligned}
$$

Where $\quad s_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \pi$

- Generates third order time derivatives
- Absent in the unitary gauge (used by GLPV)

In the ADM parametrization, the action *S* becomes (in an arbitrary gauge)

$$
\begin{aligned}
S \; = \; & \int dt d^3x \, N\sqrt{\gamma}(K_{ij}K^{ij} - K^2 + {}^{(3)}R) \\
+ \; & \int dt d^3x \, \frac{\sqrt{\gamma}}{N}\epsilon^{ijk}\epsilon^{\ell m}{}_k \big[-\dot{\pi}^2 s_{i\ell}s_{jm} + \boxed{2\pi_i\pi_\ell s_{00}s_{jm}} + 2\pi_i\pi_\ell s_{0m}s_{0j} + 4\dot{\pi}\pi_\ell s_{i0}s_{jm} \big] \\
+ \; & \int dt d^3x \, \frac{\sqrt{\gamma}}{N}\epsilon^{ijk}\epsilon^{\ell mn} N_k \big[2\dot{\pi}\pi_\ell s_{im}s_{jn} - 4\pi_i\pi_\ell s_{0m}s_{jn} \big] \\
+ \; & \int dt d^3x \, N\sqrt{\gamma}\left(1 - \frac{N_p N^p}{N^2}\right)\epsilon^{ijk}\epsilon^{\ell mn} s_{jm}s_{kn}\pi_i\pi_\ell
\end{aligned}
$$

Where $\quad s_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \pi$

- Generates third order time derivatives
- Absent in the unitary gauge (used by GLPV)

but S depends on $\dot{N}$, $\dot{N}^i$ and non linearly on second derivatives of π

More convenient to work with

$$\tilde{S} = S + \int d^4x \, \tilde{\lambda}^{\mu\nu} \left(s_{\mu\nu} - \nabla_\mu \nabla_\nu \pi \right)$$

31 canonical (Lagrangian) fields $N \,,\, N^i \,,\, \gamma_{ij} \,,\, \pi ,\, \lambda_{\mu\nu} \,,\, s_{\mu\nu}$

(where $\lambda^{\mu\nu} = N\sqrt{-\gamma}\tilde{\lambda}^{\mu\nu}$)

- 23 primary constraints
- 23 secondary constraint
- At least 8 of them are first class

At most $62 - (2 \times 8) - (46 - 8) = 8$ Hamiltonian d.o.f.

Further analysis shows that there exist a tertiary (and likely also a quaternary) second class constraint, hence less than 8 d.o.f.

Conclusions (of part II)

Have shown how the e.o.m. of beyond Horndeski theory can indeed be reduced in agreement with GLPV claim (but correcting a flaw in GLPV proof).

Provide a first step toward a proper Hamiltonian treatment of these theories (also supporting GLPV claim).

Various possible follow up: classification of these theories, Cauchy problem etc…

Thank you for your attention