Prolegomena to integrable de Sitter models (II)

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The first modification of GR

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \qquad (1915)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu} \qquad (1917)$$

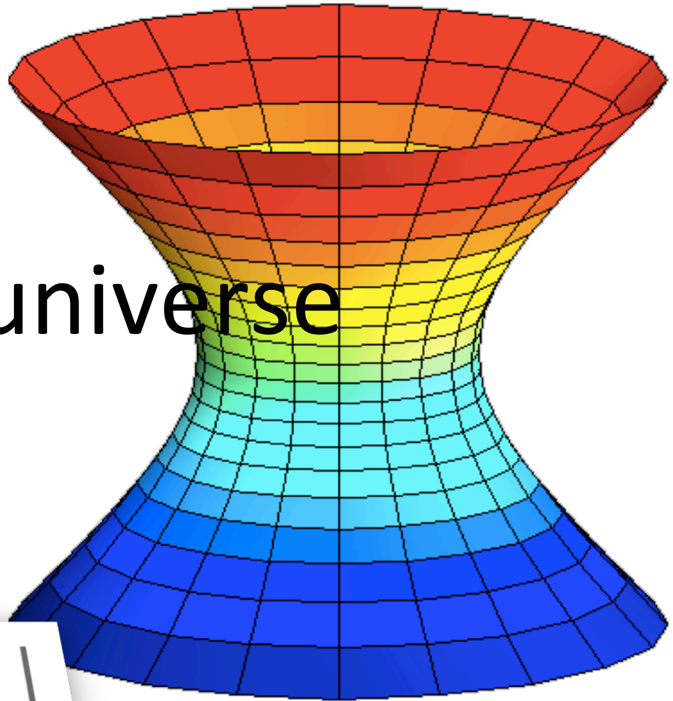
"The history of science provides many instances of discoveries which have been made for reasons which are no longer considered satisfactory. It may be that the discovery of the cosmological constant is such a case."

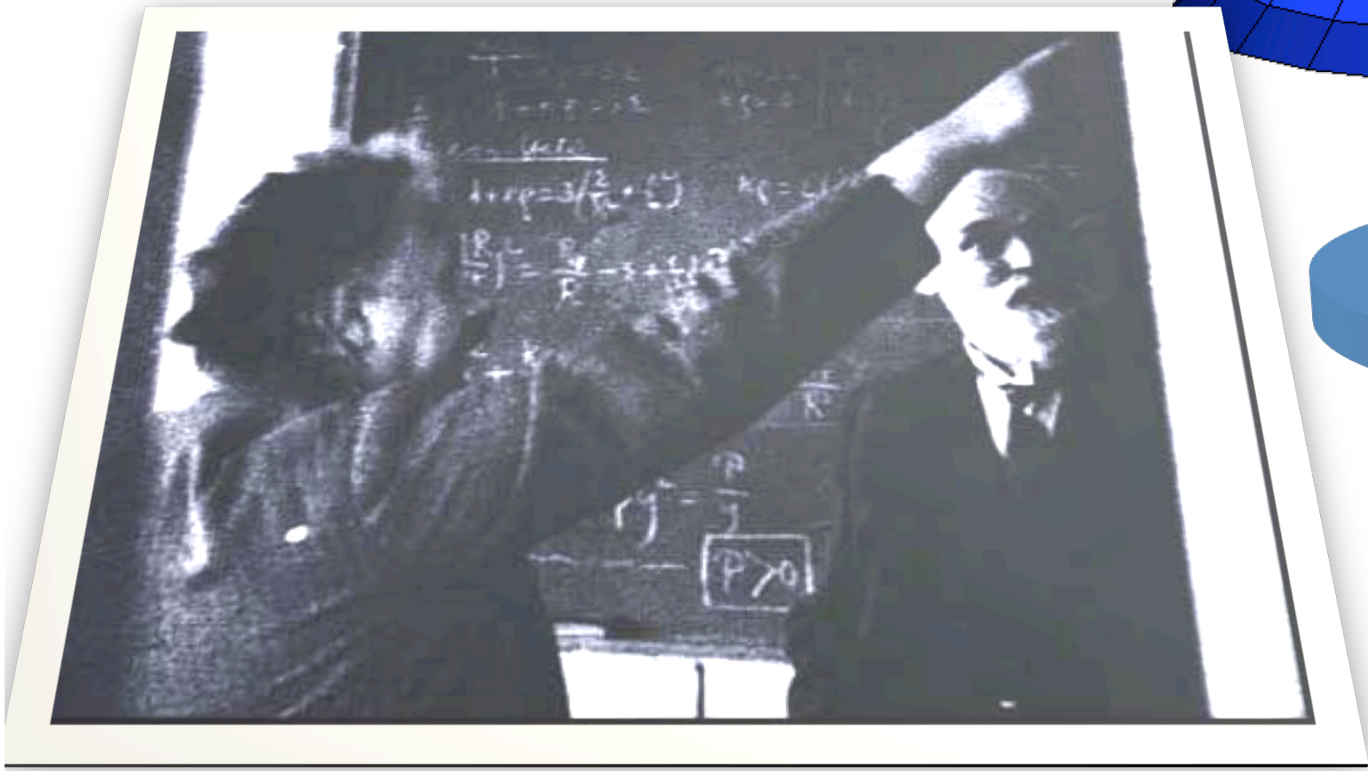
George E. Lemaître, article in the book

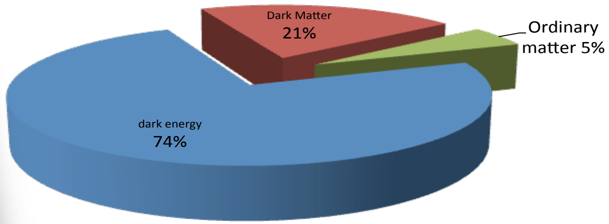
"Albert Einstein: Philosopher–Scientist", 1949

Seventy years after.

1997: The shape of the universe







de Sitter Quantum Field Theory

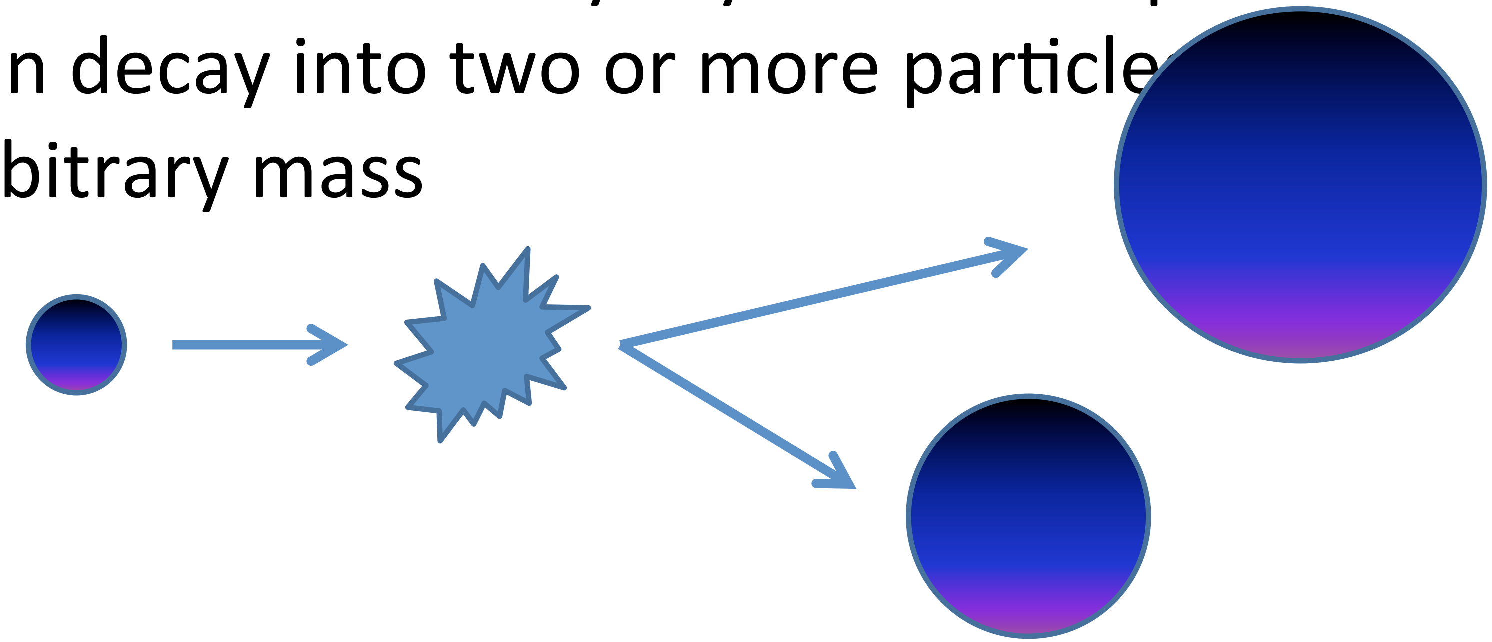
- W. Thirring. Quantum field theory in de Sitter space. Acta Physica Austriaca, suppl. IV, 1967, 269.
- Naively it was believed to be a simple example of QFT on a curved spacetime while it is plagued by a very difficult infrared problem.
- The physical importance of dS QFT increased from the eighties because of the inflationary paradigm.
- Today : dark age ...-> dS -> FLRW -> dS -> ...

Example: particle decays

- There are no stable particle of mass

$$m^2 \geq \frac{(d-1)^2}{4R^2}$$

- Perturbation theory says that such particle can decay into two or more particle arbitrary mass



What if such difficulties were artifacts of perturbation theory?

- What about non perturbative physics on dS?
- There several approaches to nonperturbative QFT on flat space
- One is the study of exactly solvable two-dimensional models of QFT
- So: why not try to explore solvable (?) two-dimensional models in de Sitter?

Two interesting models: 1) The Thirring Model

- Classical field equations

$$i\gamma^\mu \partial_\mu \psi(x) = -gJ^\mu(x)\gamma_\mu \psi(x)$$

$$J^\mu(x) = \overline{\psi}(x)\gamma^\mu \psi(x)$$

$$\partial_\mu J^\mu(x) = 0$$

- At the quantum level the field equations need a renormalization (!)

2) Schwinger Model (Two-dim massless QED)

- Field equations

$$i\gamma^\mu \partial_\mu \psi(x) = -e\gamma^\mu A_\mu(x)\psi(x)$$

$$\partial^\nu F_{\mu\nu}(x) = -eJ_\mu(x) + \mathcal{A}_\mu(x)$$

- Point-splitting renormalization $\epsilon^2 < 0$

$$A_\mu(x)\psi(x) = \frac{1}{2}\lim\left[A_\mu(x+\epsilon)\psi(x) + A_\mu(x)\psi(x-\epsilon)\right]$$

$$J_\mu(x) = \lim\left[\bar{\psi}(x+\epsilon)\gamma_\mu\psi(x) - \langle\bar{\psi}(x+\epsilon)\gamma_\mu\psi(x)\rangle\right](1 - ie\,\epsilon^\mu A_\mu(x))$$

Two apparently elementary questions

- What is going to replace the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

  on the de Sitter manifold?

- What is the meaning of de Sitter covariance for spinor fields ?

The Fock-Ivanenko construction (1929)

- Curved space gamma matrices $\alpha^i = e_a^i \gamma^a$
- Spin connection $\omega_{jab} = e_a^i \nabla_j e_{b\,i}.$
- Fock-Ivanenko coefficients $\Gamma_j = \frac{1}{2} \Sigma^{ab} \omega_{jab}$
- Curved space Dirac equation:

$$i\alpha^j (\partial_j + \Gamma_j)\phi - m\phi = 0$$

A reminder (see Bjorken and Drell)

- Meaning of the Lorentz covariance of the Dirac's equation:

$$x' = \Lambda x \qquad \Lambda^{\mu}{}_{\nu} = \eta^{\mu}{}_{\nu} + \Delta\omega^{\mu}{}_{\nu}$$

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \qquad (i\gamma^{\mu'}\partial_{\mu'} - m)\psi'(x') = 0$$

$$\psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x')$$

$$S = 1 + \frac{1}{4}[\gamma_{\mu}, \gamma_{\nu}]\Delta\omega^{\mu\nu} = 1 - \frac{i}{8}\sigma_{\mu\nu}\Delta\omega^{\mu\nu}$$

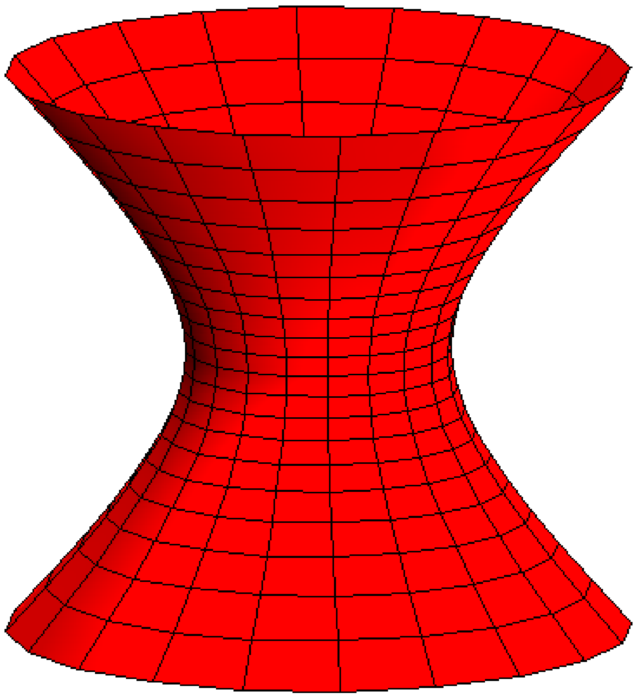
$SL(2, C)$ is the double cover of the Lorentz group

$$SL(2, C) \ni g \to \Lambda(g) \in SO_0(1, 3)$$

$$\psi'(x') = \psi'(\Lambda(g)x) = g\psi(x) = g\psi(\Lambda^{-1}(g)x')$$

De Sitter universe

$$dS_2 = \{x \in \mathbb{R}^3 : \ (X^0)^2 - (X^1)^2 - (X^2)^2 = -1\}$$



Relativity group: $SO_0(1,2)$

One parameter subgroups

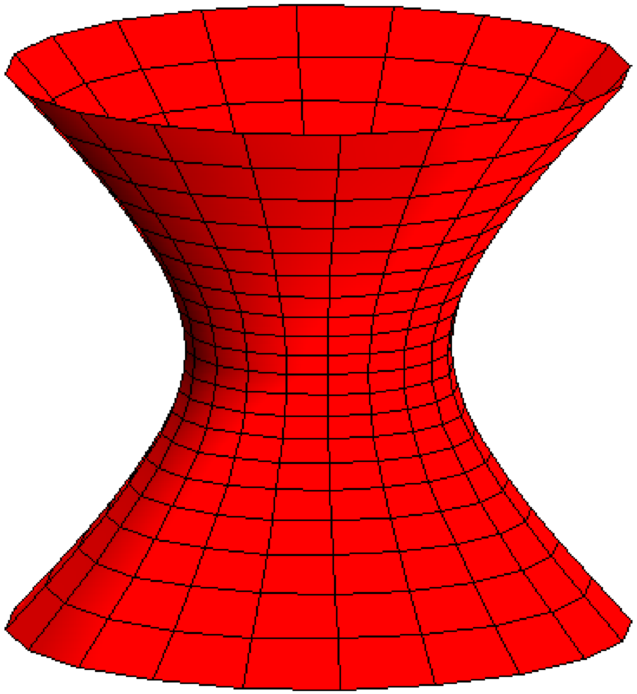
$$\begin{pmatrix} \cosh\zeta & 0 & \sinh\zeta \\ 0 & 1 & 0 \\ \sinh\zeta & 0 & \cosh\zeta \end{pmatrix}$$

$$\begin{pmatrix} \cosh\chi & \sinh\chi & 0 \\ \sinh\chi & \cosh\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Conformal cylindrical coordinates

$$dS_2 = \{x \in \mathbb{R}^3 : \ (X^0)^2 - (X^1)^2 - (X^2)^2 = -1\}$$



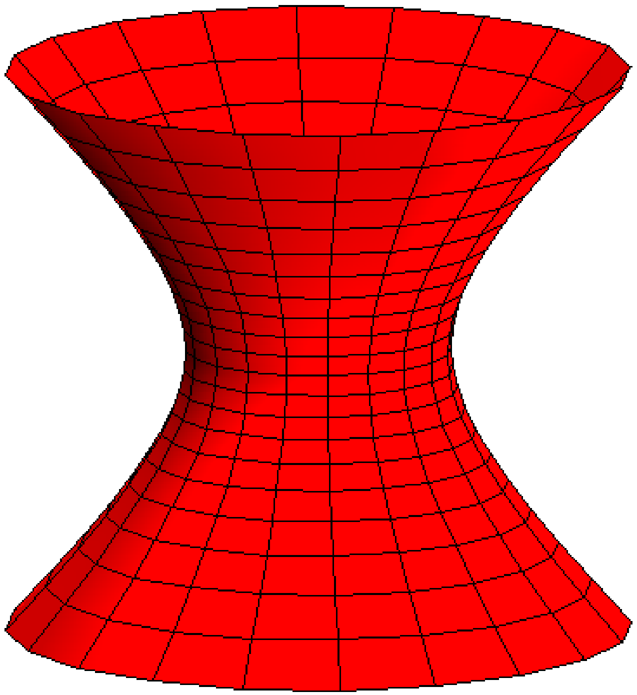
$$\begin{cases} X^0 = r \tan t \\ X^1 = r \sin\theta / \cos t \\ X^2 = r \cos\theta / \cos t \end{cases}$$

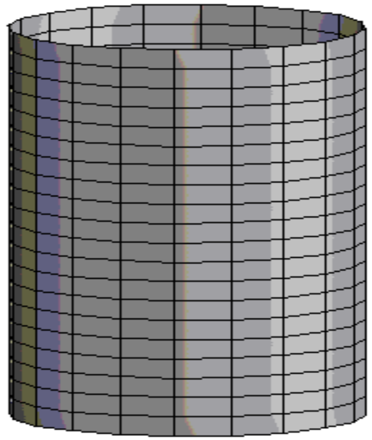
$$r = R = 1$$

$$ds^2 = \frac{1}{\cos^2 t}(dt^2 - d\theta^2)$$

$$e^t_0 = \cos t, \quad e^t_1 = 0, \quad e^\theta_0 = 0, \quad e^\theta_1 = \cos t.$$

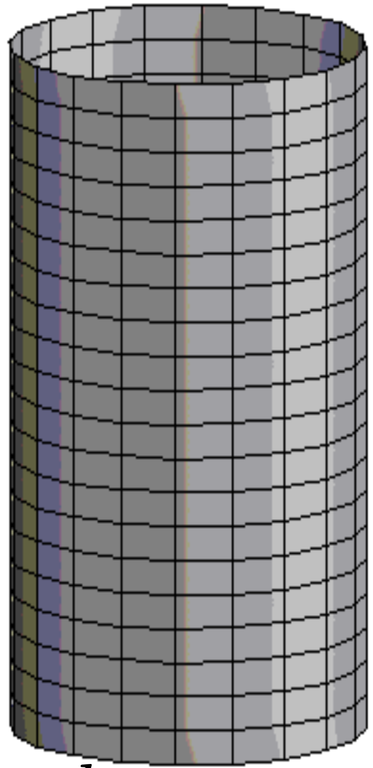
Conformal coordinates





$$ds^2 = \frac{1}{\cos^2 t}(dt^2 - d\theta^2),$$

Minkowskian cylinder



$$x^0 = t \in R, \qquad x^1 = \theta \in [0, 2\pi)$$

- Metric $$ds^2 = dt^2 - d\theta^2$$

- Clifford algebra $$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}\mathbb{I}.$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Dirac-Fock-Ivanenko construction

$$e_0^t = \cos t, \quad e_1^t = 0, \quad e_0^\theta = 0, \quad e_1^\theta = \cos t.$$

- Curved space gamma matrices $\alpha^i = e_a^i \gamma^a$

$$\alpha^t = (\cos t)\,\gamma^0, \quad \alpha^\theta = (\cos t)\,\gamma^1.$$

$$\Gamma_t = 0, \quad \Gamma_\theta = \Sigma^{01}\omega_{\theta 01} = \tan t \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$i\alpha^j(\partial_j + \Gamma_j)\phi - m\phi = 0$$

$$i\gamma^a\partial_a\phi + \frac{i}{2}\tan t\,\gamma^0\phi - m\phi = 0$$

Moving to de Sitter: conformal transformation

- Consider a spinor $\psi = (\cos t)^{-\frac{1}{2}}\phi$ on the cylinder

$$i\gamma^a \partial_a \psi = 0$$

- The spinor ϕ solves the following equation

$$i\gamma^a \partial_a \phi + \frac{i}{2}\tan t\, \gamma^0 \phi = 0$$

- This is the massless Dirac equation on the de Sitter manifold

Spin bundles on the cylinder

- There are two inequivalent spin bundles.
- Two boundary conditions:
- 1) Periodic (Ramond)

$$\psi(t,\theta) = \psi(t,\theta + 2\pi)$$

- 2) Anti-periodic (Neveu-Schwarz)

$$\psi(t,\theta) = -\psi(t,\theta + 2\pi)$$

- In both cases observables are fully well defined on the cylinder (i.e. they are periodic)

Quantum Spinor field (Ramond)

- Two sets of ladder anticommuting operators acting in a Fock space:

$$\{a_j(p),\ a_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ , \quad \{b_j(p),\ b_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ . \quad j,k = 1,2$$

$$u = t + \theta, \qquad v = t - \theta$$

$$\psi_1^{\mathrm{R}}(x) = \psi_1^{\mathrm{R}}(u) = \frac{1}{2\sqrt{\pi}}(a_1^*(0) + b_1(0)) + \frac{1}{\sqrt{2\pi}}\sum_{p>0}(a_1^*(p)e^{ipu} + b_1(p)e^{-ipu})$$

$$\psi_2^{\mathrm{R}}(x) = \psi_2^{\mathrm{R}}(v) = \frac{1}{2\sqrt{\pi}}(a_2^*(0) + b_2(0)) + \frac{1}{\sqrt{2\pi}}\sum_{p>0}(a_2^*(p)e^{ipv} + b_2(p)e^{-ipv})$$

$$w_1(x,y) = (\Omega,\ \psi_1^{\mathrm{R}}(x)\psi_1^{\mathrm{R}*}(y)\Omega) - \frac{i}{4\pi}\cot\left(\frac{u - u' - i0}{2}\right)$$

$$w_2(x,y) = (\Omega,\ \psi_2^{\mathrm{R}}(x)\psi_2^{\mathrm{R}*}(y)\Omega) = -\frac{i}{4\pi}\cot\left(\frac{v - v' - i0}{2}\right)$$

Quantum Spinor field (Neveu-Schwarz)

- Same two sets of ladder anticommuting operators acting in a Fock space:

$$\{a_j(p),\ a_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ , \quad \{b_j(p),\ b_k^*(q)\} = \delta_{j,k}\delta_{p,q}\ . \qquad j,k = 1,2$$

$$\psi_1(x) = \frac{1}{\sqrt{2\pi}} \sum_{p\geq 0} (a_1^*(p)e^{ipu+iu/2} + b_1(p)e^{-ipu-iu/2}), \quad u = x^0 + x^1$$

$$\psi_2(x) = \frac{1}{\sqrt{2\pi}} \sum_{p\geq 0} (a_2^*(p)e^{ipv+iv/2} + b_2(p)e^{-ipv-iv/2}), \quad v = x^0 - x^1$$

$$w_1(x,\ y) = (\Omega,\ \psi_1(x)\psi_1^*(y)\Omega) = \frac{e^{iu/2}}{2\pi} \sum_{p\geq 0} e^{ipu} = \frac{i}{4\pi\sin(u/2)}$$

$$w_2(x,\ y) = (\Omega,\ \psi_2(x)\psi_2^*(y)\Omega) = \frac{e^{iv/2}}{2\pi} \sum_{p\geq 0} e^{ipv} = \frac{i}{4\pi\sin(v/2)}$$

Conformal transformation of the spinors

- Given a massless Dirac (quantum) spinor field on the cylinder (either Ramond or Neveu-Schwarz)

$$i\gamma^a \partial_a \psi = 0$$

$$\phi = (\cos t)^{\frac{1}{2}} \psi$$

- is a (quantum) massless Dirac spinor field on the de Sitter manifold:

$$i\gamma^a \partial_a \phi + \frac{i}{2} \tan t \, \gamma^0 \phi = 0$$

- What about the de Sitter symmetry?
- There is a priori no reason to expect it. The spinors on the cylinder have less symmetries (space rotations + time translations)!

Two apparently elementary questions

- What is going to replace the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

 on the de Sitter manifold?

- What is the meaning of de Sitter covariance for spinor fields ?

Another equation by Dirac

- Clifford algebra in the ambient spacetime

$$ds^2 = (dX^0)^2 - (dX^1)^2 - (dX^2)^2$$

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\,\eta^{\alpha\beta} = 2\mathrm{diag}(1,-1,-1)$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\not{X} = \gamma^\mu X_\mu = \begin{pmatrix} -iX_2 & X_0 - X_1 \\ X_0 + X_1 & iX_2 \end{pmatrix}$$

- Generators of the de Sitter (Lorentz) group

$$L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta} = -i(X_\alpha \partial_\beta - X_\beta \partial_\alpha) - \frac{i}{4}[\gamma_\alpha, \gamma_\beta]$$

The de Sitter (Casimir)-Dirac equation

$$Q = -\frac{1}{2}L^{\alpha\beta}L_{\alpha\beta} = \left(\frac{1}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} + i\right)^2 + \frac{1}{4}.$$

- Eigenvalues of the Casimir operator

$$Q\psi = \left(\nu^2 + \frac{1}{4}\right)\psi$$

- First order equation (Dirac 1935 4-dim)

$$\left(\frac{1}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} + i + \nu\right)\psi = 0$$

Solving the Dirac-Dirac equation

$$(iD + i + \nu)\,\psi = 0 \qquad\qquad iD = \frac{1}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta}$$

The crucial identity is

$$(D+1)D = \Box$$

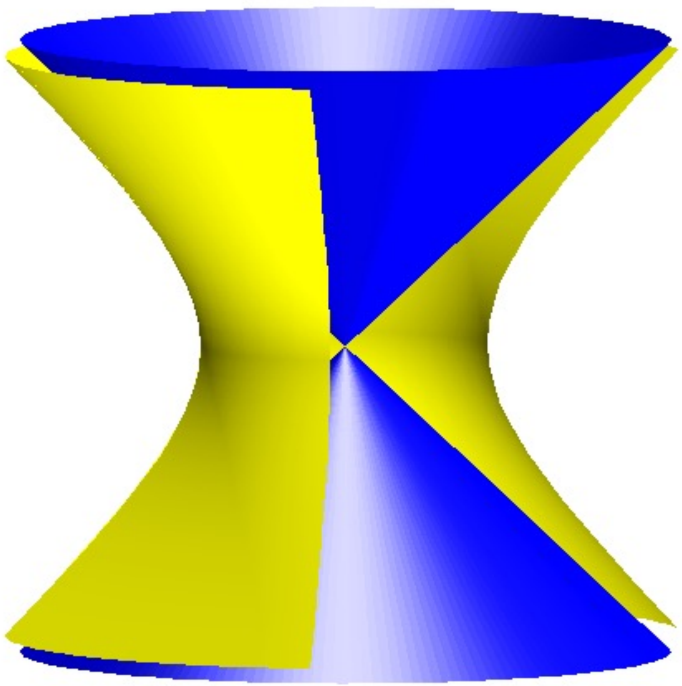
$$(iD + i + \nu)\,(-iD + \nu)\,\chi = (\Box + \mu^2)\chi = 0$$

$$\mu^2 = \nu^2 + i\nu = (-1 + i\nu)(-i\nu)$$

$$\psi = (-iD + \nu)\,\chi$$

The asymptotic cone

$$\{\xi_0^2 - \xi_1^2 - \ldots - \xi_d^2 = 0\}$$



$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1**,**-1,……,-1**)

de Sitter plane waves

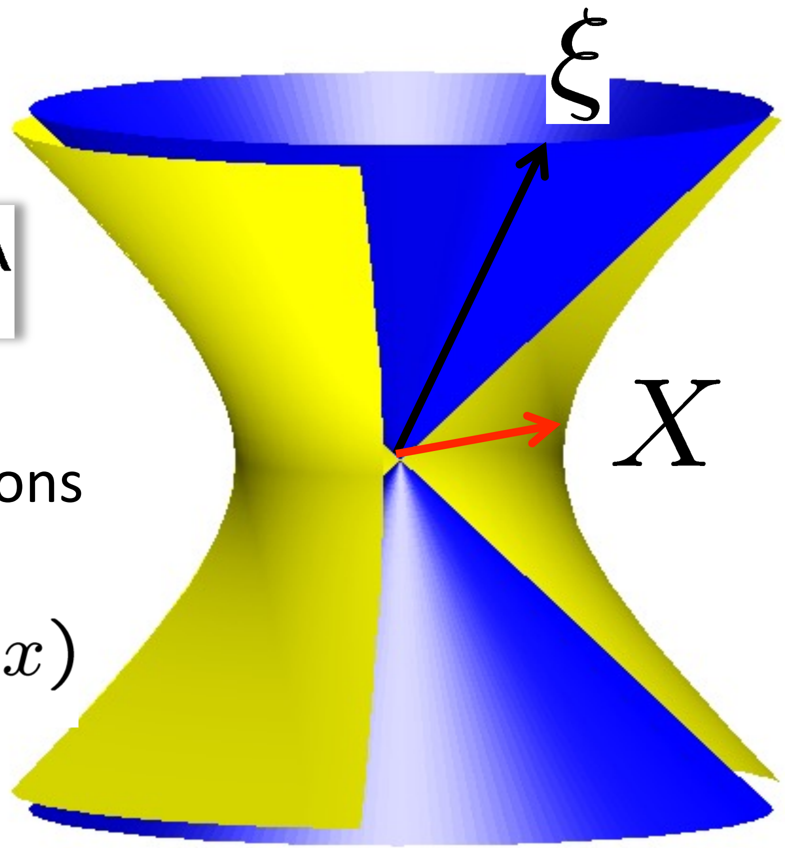
$$X \cdot \xi = X_0\xi_0 - X_1\xi_1 - \ldots - X_d\xi_d$$

$$\lambda \in \mathbf{C}, \quad \xi^2 = 0$$

$$\psi_\lambda(X, \xi) = (X \cdot \xi)^\lambda$$

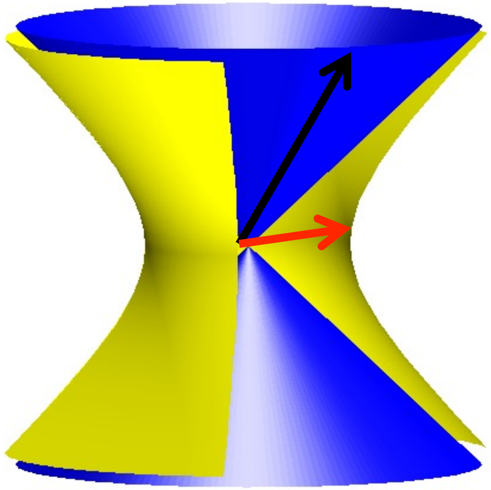
Plane waves are homogeneous functions

$$\psi(x, p) = e^{ip\cdot x} = e^{im(\widehat{p}\cdot x)}$$



de Sitter plane waves

$$\Box(X\cdot\xi)^{\lambda} = \lambda(\lambda+d-1)(X\cdot\xi)^{\lambda}$$



Involution:

$$\lambda \longrightarrow \bar{\lambda} = -\lambda-(d-1)$$

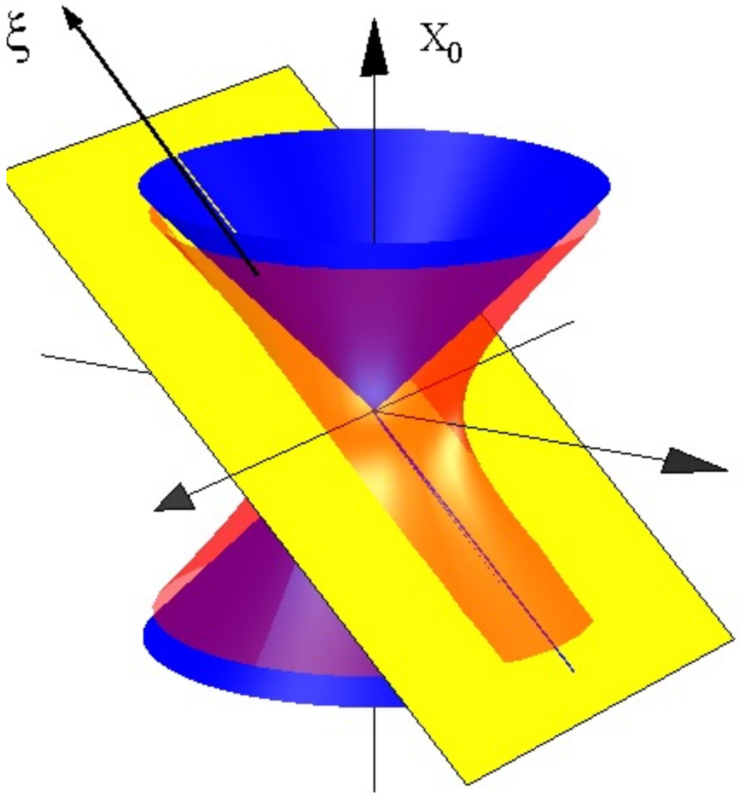
$$\lambda+\bar{\lambda} = -(d-1)$$

Scalar waves with (complex) squared mass:

$$m^2 = \lambda\bar{\lambda}$$

$$\left(\Box+\lambda\bar{\lambda}\right)(X\cdot\xi)^{\lambda} = 0, \qquad \left(\Box+\lambda\bar{\lambda}\right)(X\cdot\xi)^{\bar{\lambda}} = 0$$

The plane waves are however irregular



$$\psi_\lambda(X,\xi) = (X \cdot \xi)^\lambda$$

$$X \in dS : (X \cdot \xi) = 0$$

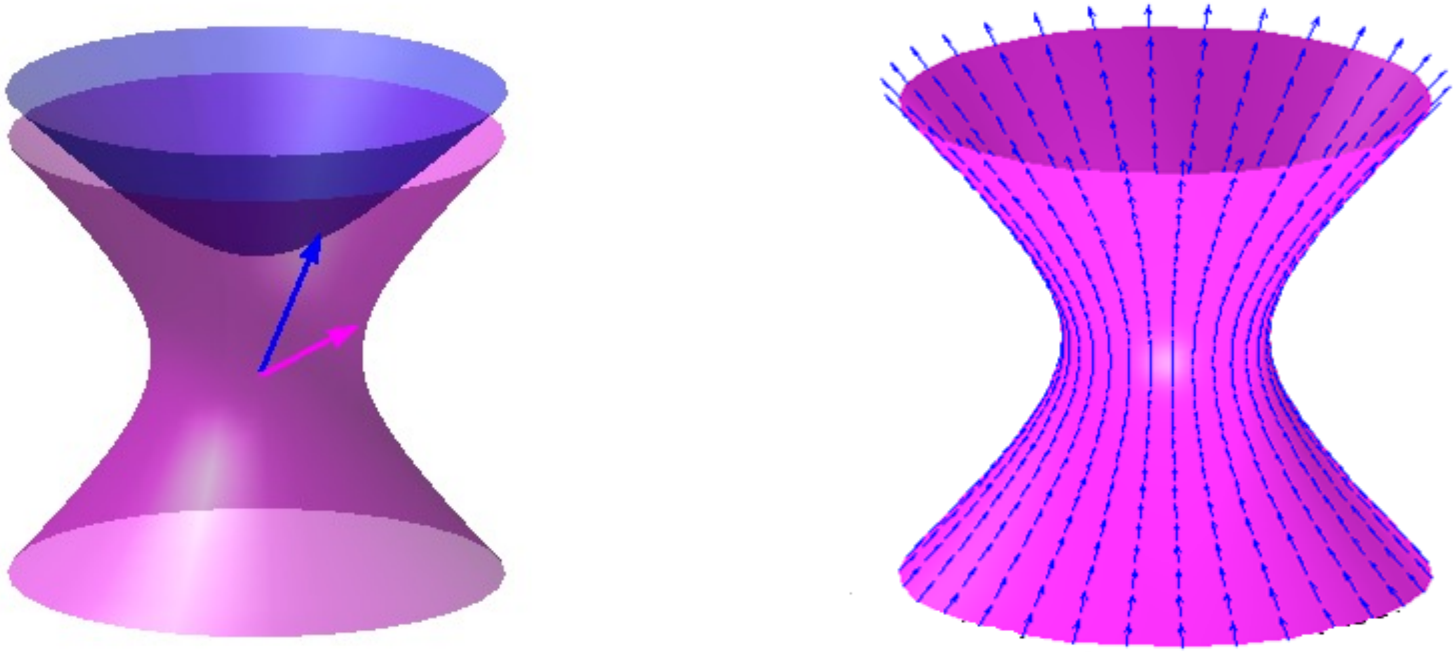
$$(X \cdot \xi)^\lambda \to |X \cdot \xi|^\lambda \, (a(\lambda)\theta(X \cdot \xi) + b(\lambda)\theta(-X \cdot \xi))$$

Geometry: de Sitter tubes

$Z = X + iY, \quad X^2 - Y^2 = -R^2 \quad X \cdot Y = 0$

$\mathcal{T}^+ = Y$ in the forward cone.

$\mathcal{T}^- = Y$ in the backward cone.

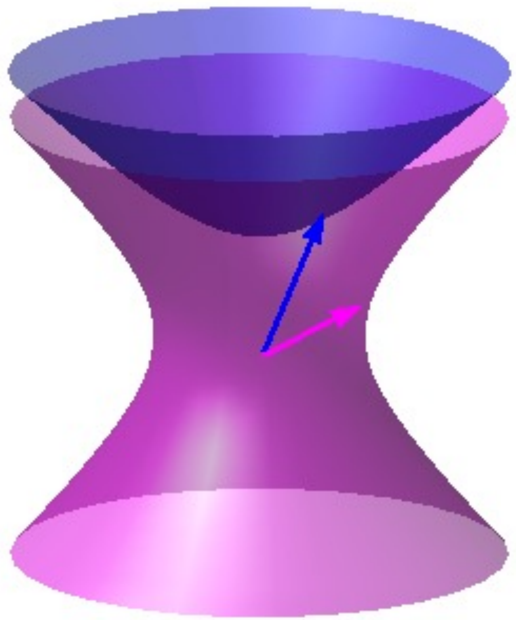


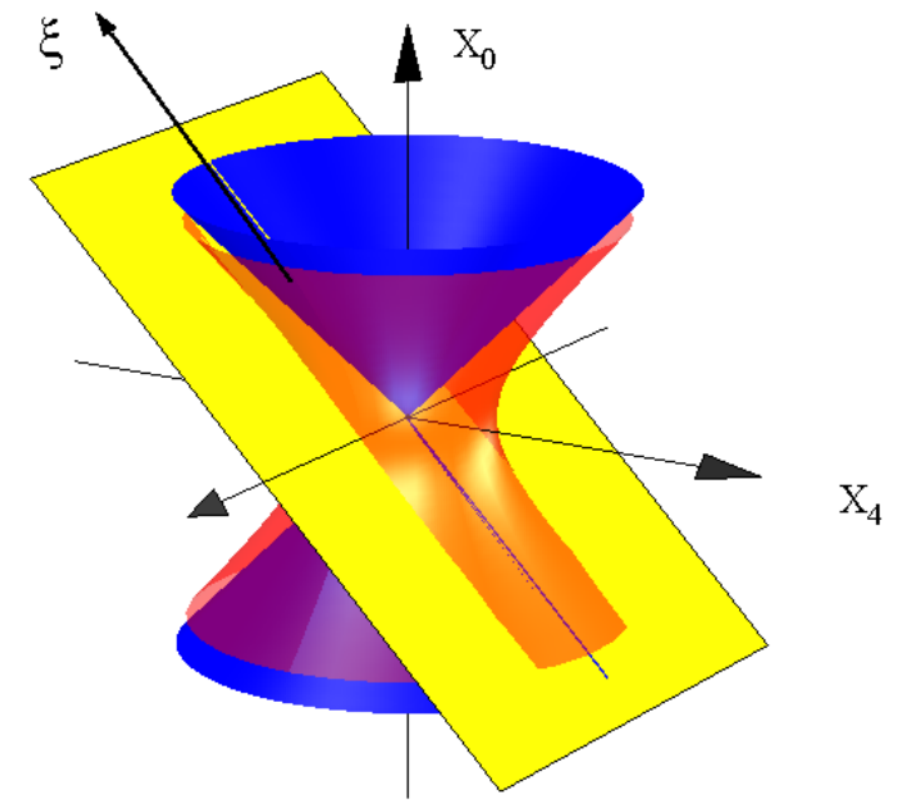
$$(Z \cdot \xi)^\lambda = (X \pm iY)^\lambda$$

$$Y^2 = (Y^0)^2 - (Y^1)^2 - (Y^2)^2 > 0, \qquad Y^0 > 0$$

$\Im Z \cdot \xi$ is positive for $Z \in \mathcal{T}^+$

$\Im Z \cdot \xi$ is negative for $Z \in \mathcal{T}^-$





Boundary values on the reals:

$$(X\cdot\xi)_{\pm}^{\lambda} \to |X\cdot\xi|^{\lambda}\Big(\theta(X\cdot\xi) + e^{\pm i\pi\lambda}\theta(-X\cdot\xi)\Big)$$

Normalization of the Plane Waves

Klein Gordon product

$$(f,g) = i\int_\Sigma n^\mu (f^* \partial_\mu g - \partial_\mu f^* g)\sqrt{h}\, d^{d-1}x$$

Introduce an involution that

- generalises the complex conjugation and
- works for all complex λ

$$f_{\lambda,\xi}(X) = (X\cdot\xi)_-^{-\lambda-d+1}$$

$$f^*_{\lambda,\xi}(X) = (X\cdot\xi)_+^{\lambda}$$

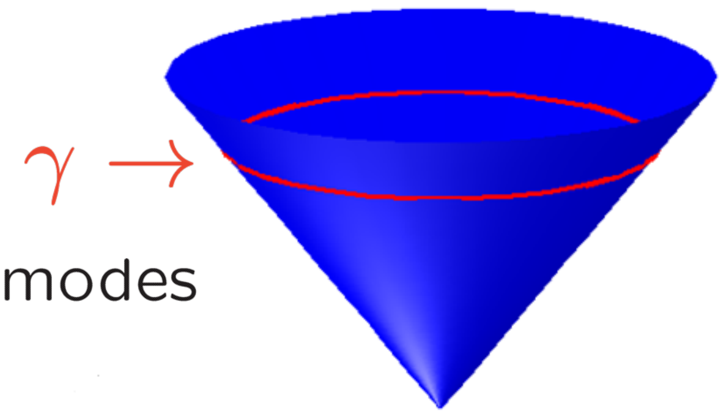
Normalization of the PW. Two-point function

$$f_{\lambda,\xi}(X) = (X\cdot\xi)_-^{-\lambda-d+1} \qquad f^*_{\lambda,\xi}(X) = (X\cdot\xi)_+^{\lambda}$$

$$(f_{\lambda,\xi}, f_{\lambda,\xi'}) = \frac{2^{d+1}\pi^d R^{d-1} e^{i\pi\left(\lambda+\frac{d-1}{2}\right)}}{\Gamma(-\lambda)\Gamma(\lambda+d-1)}\delta(\xi-\xi') = \frac{1}{c(\lambda)}\delta(\xi-\xi').$$

$$W_\lambda(X_1,X_2) = c(\lambda)\int_\gamma (X_1\cdot\xi)_-^{-\lambda-d+1}\,(X_2\cdot\xi)_+^{\lambda}\,d\sigma(\xi)$$

For $\lambda = n$ plane waves are zero modes



Solving the Dirac equation

$$(iD + i + \nu)\,\psi = 0$$

$$\psi(X;\xi) = (X \cdot \xi)^{-1+i\nu} u(\xi)$$

$$\not{\xi}\, u(\xi) = 0 = \begin{pmatrix} -i\xi_2 & \xi_0 - \xi_1 \\ \xi_0 + \xi_1 & i\xi_2 \end{pmatrix} \begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_1) \end{pmatrix}$$

$$u(\xi) = \frac{1}{\sqrt{2(\xi^0 - \xi^1)}} \begin{pmatrix} \xi^0 - \xi^1 \\ i\xi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi^0 - \xi^1} \\ i\sqrt{\xi^0 + \xi^1} \end{pmatrix}$$

Compare Cartan’s definition of a spinor

The two-point function

- Defining the adjoint spinor as usual

$$\overline{u}(\xi) = u^*(\xi)\gamma^0 \qquad u(\xi) \otimes \overline{u}(\xi) = \frac{1}{2}\not{\xi}$$

$$W_\nu(X_1, X_2) = \frac{1}{2} c_\nu \int_\gamma (X_1 \cdot \xi)_-^{-1-i\nu} (X_2 \cdot \xi)_+^{-1+i\nu} \not{\xi} d\sigma(\xi)$$

- In the massless limit

$$W_0(Z_1, Z_2) = \frac{1}{2\pi i} \frac{\not{Z}_1 - \not{Z}_2}{(Z_1 - Z_2)^2}$$

Spin group and de Sitter covariance

$$Sp(1,2) = \left\{ g \in SL(2,C) : \quad \gamma^0 g^\dagger \gamma^0 = g^{-1} \right\}.$$

$$g = \begin{pmatrix} a & i\,b \\ i\,c & d \end{pmatrix} \qquad ad + bc = 1$$

$Sp(1,2)$ is conjugated to $SL(2,R)$ in $SL(2,C)$:

$$h = \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{pmatrix} \qquad hgh^{-1} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}$$

Covering

$Sp(1,2)$ acts on the de Sitter manifold by similarity

$$\not{X} = \gamma^\mu X_\mu = \begin{pmatrix} -iX_2 & X_0 - X_1 \\ X_0 + X_1 & iX_2 \end{pmatrix} \qquad \not{X}' = g\not{X}g^{-1}$$

The covering projection $g \to \Lambda(g)$ of $Sp(1,2)$ onto $SO_0(1,2)$

$$g \to \Lambda(g)^\alpha{}_\beta = \frac{1}{2}\mathrm{tr}(\gamma^\alpha g \gamma_\beta g^{-1}) \qquad \Lambda(g) = \Lambda(-g)$$

$$\begin{pmatrix} a & ib \\ ic & d \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}\left(a^2+b^2+c^2+d^2\right) & \frac{1}{2}\left(-a^2+b^2-c^2+d^2\right) & cd-ab \\ \frac{1}{2}\left(-a^2-b^2+c^2+d^2\right) & \frac{1}{2}\left(a^2-b^2-c^2+d^2\right) & ab+cd \\ ac-bd & -ac-bd & ad-bc \end{pmatrix}$$

$$\not{X}' = g\not{X}g^{-1} = \cancel{\Lambda(g)X}$$

De Sitter covariance of the Dirac-Dirac field

$$\psi'(Z) = g\psi\big(\Lambda^{-1}(g)Z\big)$$

$$gW_0(Z_1,Z_2)g^{-1} = \frac{1}{2\pi i}\frac{g\,\not{Z}_1 g^{-1} - g\,\not{Z}_2 g^{-1}}{(Z_1-Z_2)^2} = \frac{1}{2\pi i}\frac{\not{\Lambda(g)Z_1} - \not{\Lambda(g)Z_2}}{(Z_1-Z_2)^2}.$$

$$gW_\nu\big(\Lambda^{-1}(g)X_1,\Lambda^{-1}(g)X_2\big)g^{-1} =$$

$$= \frac{1}{2}c_\nu\int_\Gamma \big(\Lambda^{-1}(g)X_1\cdot\xi\big)^{-1+i\nu}\big(\Lambda^{-1}(g)X_2\cdot\xi\big)^{-1-i\nu} g\not{\xi}g^{-1}d\mu(\xi)$$

$$= \frac{1}{2}c_\nu\int_\Gamma \big(X_1\cdot\Lambda(g)\xi\big)^{-1+i\nu}\big(X_2\cdot\Lambda(g)\xi\big)^{-1-i\nu}\gamma^\alpha(\Lambda(g)\xi)_\alpha d\mu(\xi)$$

$$= W_\nu\big(X_1,X_2\big)$$

$$\not{X}' = g\not{X}g^{-1} = \not{\Lambda(g)X}$$

The symmetric space Sp(1,2)/A

- Iwasawa decomposition $g = \begin{pmatrix} a & i\,b \\ i\,c & d \end{pmatrix} \qquad ad + bc = 1$

$$g = k(\zeta)\,n(\lambda)\,a(\chi) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{pmatrix} \begin{pmatrix} 1 & i\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{-\frac{\chi}{2}} \end{pmatrix};$$

$$\cos\frac{\zeta}{2} = \frac{a}{\sqrt{a^2+c^2}}, \quad \sin\frac{\zeta}{2} = \frac{c}{\sqrt{a^2+c^2}}, \quad \lambda = ab - cd, \quad e^{\frac{\chi}{2}} = \sqrt{a^2+c^2}$$

where $0 \leq \zeta < 4\pi$ and λ and χ are real.

- Parametrization of the coset space Sp(1,2)/A

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

Group action

- Sp(1,2) acts on the coset space by left multiplication:

$$g\,\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

Rotation (K)

$$k(a) = \begin{pmatrix} \cos\frac{a}{2} & i\sin\frac{a}{2} \\ i\sin\frac{a}{2} & \cos\frac{a}{2} \end{pmatrix} \qquad k(\alpha)\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\lambda'(\alpha) = \lambda, \qquad \zeta'(\alpha) = \zeta + \alpha.$$

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

Boost (A)

$$a(\kappa) = \left(\begin{array}{cc} e^{\frac{\kappa}{2}} & 0 \\ 0 & e^{-\frac{\kappa}{2}} \end{array} \right) \qquad a(\kappa)\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\left\{ \begin{array}{l} \lambda'(\kappa) = \lambda \cosh\kappa + \sinh\kappa(\lambda\cos\zeta + \sin\zeta), \\ \cot\frac{\zeta'(\kappa)}{2} = e^{\kappa}\cot\frac{\zeta}{2}. \end{array} \right.$$

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \left(\begin{array}{cc} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{array} \right)$$

Lightlike Boost (N)

$$n(\mu) = \begin{pmatrix} 1 & i\mu \\ 0 & 1 \end{pmatrix} \qquad n(\mu)\tilde{X}(\lambda,\zeta) \to \tilde{X}(\lambda',\zeta')$$

$$\begin{cases} \lambda'(\mu) = \lambda\left(1+\frac{1}{2}\mu^2\right) - \mu\left(\lambda+\frac{\mu}{2}\right)\sin\zeta + \mu\left(1-\frac{1}{2}\lambda\mu\right)\cos\zeta, \\ \cot\frac{\zeta'(\mu)}{2} = \cot\frac{\zeta}{2} - \mu. \end{cases}$$

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\,n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$

Maureer-Cartan metric

- The Maureer-Cartan form $dg\, g^{-1}$ gives to the symmetric space Sp(1,2)/A a natural Lorentzian metric
- There exists a inner automorphism of Sp(1,2) that leaves A invariant

$$g \to \mu(g) = -\gamma^2 g \gamma^2 \qquad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

- It may be used to construct a map from the coset space Sp(1,2)/A into the group Sp(1,2) and an induced Lorentzian metric on Sp(1,2)/A

$$g(\tilde{X}) = g\mu(g)^{-1} = -\tilde{X}\gamma^2 \tilde{X}^{-1}\gamma^2.$$

$$ds^2 = \frac{1}{2}\mathrm{Tr}(dg\, g^{-1})^2 = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

Maureer-Cartan metric

$$ds^2 = \frac{1}{2}\mathrm{Tr}(dg\,g^{-1})^2 = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

1. The metric is invariant under the group left action
2. The curvature is constant (R=-2) and the Ricci tensor is proportional to the metric:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = R_{\mu\nu} + g_{\mu\nu} = 0$$

3. The map

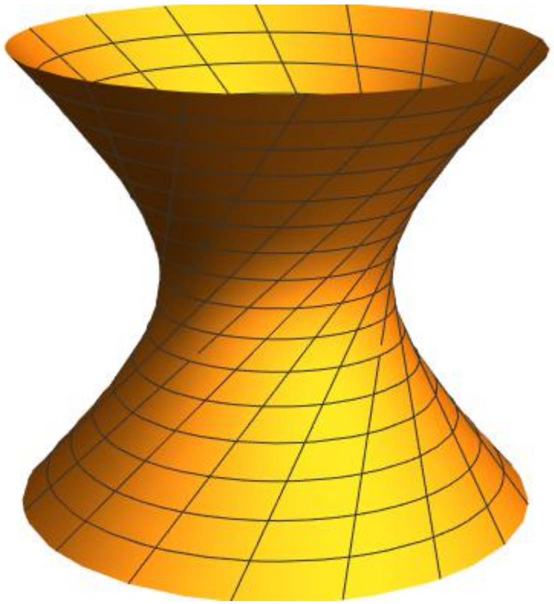
$$p : \tilde{X}(\lambda,\zeta) \to X(\lambda,\zeta) = \begin{cases} X^0 = -\lambda \\ X^1 = \lambda\cos\zeta + \sin\zeta \\ X^2 = \cos\zeta - \lambda\sin\zeta \end{cases}$$

is a covering map.

$$ds^2 = \left(dX^{0^2} - dX^{1^2} - dX^{2^2}\right)\Big|_{dS_2} = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

de Sitter

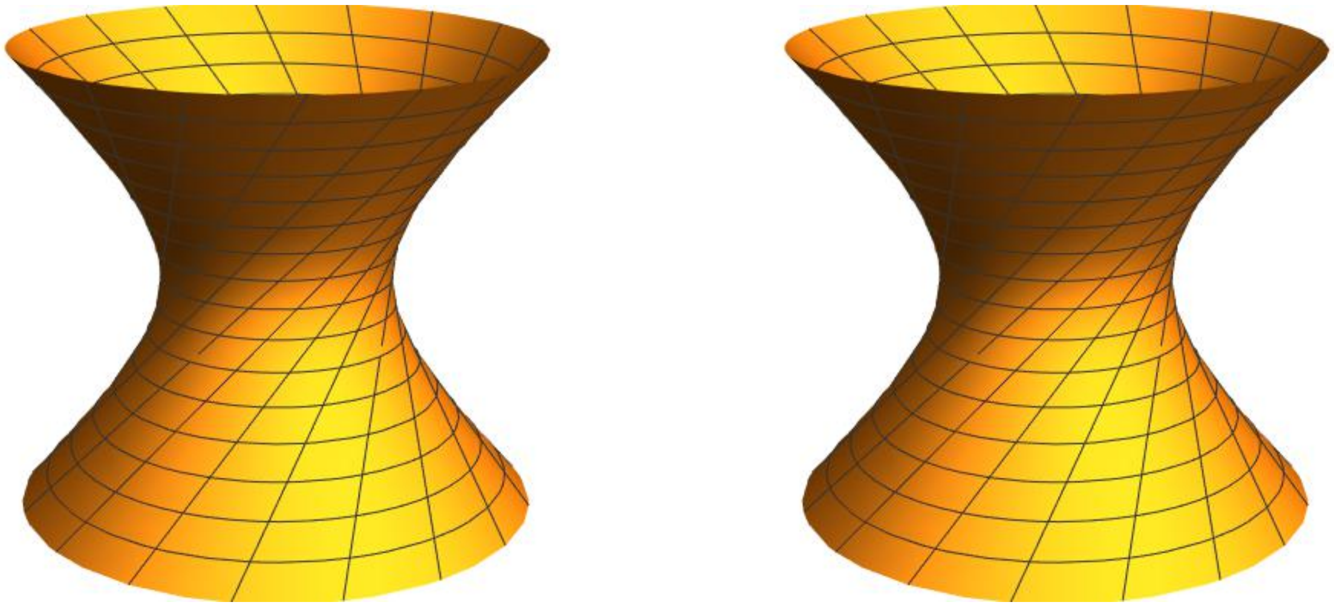
$$\begin{cases} X^0 = -\lambda \\ X^1 = \lambda \cos\zeta + \sin\zeta \\ X^2 = \cos\zeta - \lambda \sin\zeta \end{cases}$$



$$ds^2 = \left(dX^{0\,2} - dX^{1\,2} - dX^{2\,2}\right)\Big|_{dS_2} = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

Double covering of de Sitter

$$\tilde{X}(\lambda,\zeta) = k(\zeta)\, n(\lambda) = \begin{pmatrix} \cos\frac{\zeta}{2} & i\lambda\cos\frac{\zeta}{2} + i\sin\frac{\zeta}{2} \\ i\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{pmatrix}$$



$$ds^2 = \frac{1}{2}\mathrm{Tr}(dg\, g^{-1})^2 = -2d\lambda d\zeta - \left(\lambda^2+1\right) d\zeta^2$$

Double covering of de Sitter

- In conclusion: the symmetric space

$$Sp(1,2)/A = \widetilde{dS_2}$$

  may be identified with the double covering of the two dimensional de Sitter universe.
- The spin group Sp(1,2) acts directly on the covering space as a group of spacetime transformations:

$$\tilde{X} \rightarrow g\tilde{X}$$

- We were not able to find the above identification in the (enormous) literature on the group SL(2,R).

Gursey and Lee's trick

- What is the relation between the two Dirac's equation? Introduce the matrices β:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\beta^\mu = \frac{\partial y^\mu}{\partial X^\nu}\gamma^\nu, \quad y^\mu = (t, \theta, r)$$

$$\begin{cases} X^0 = r \tan t \\ X^1 = r \sin\theta / \cos t \\ X^2 = r \cos\theta / \cos t \end{cases}$$

$$\{\beta^i, \beta^j\} = g^{ij}, \; i, j = 0, 1$$

$$\beta^2 = -\frac{\not{X}}{r}, \qquad \{\beta^i, \beta^2\} = 0$$

Gursey and Lee's trick

$$\alpha^i = e^i_a \gamma^a \quad \{\alpha^i, \alpha^j\} = \{\beta^i, \beta^j\} = g^{ij}$$

- There should (more than one) matrix *S* such that

$$\alpha^i = S\beta^i S^{-1}$$

- The solution only exists on the covering manifold. The most convenient choice

$$S(t,\theta) = \frac{1}{\sqrt{\cos t}} \begin{pmatrix} \cos\frac{t-\theta}{2} & i\sin\frac{t-\theta}{2} \\ -i\sin\frac{t+\theta}{2} & \cos\frac{t+\theta}{2} \end{pmatrix}$$

Dressing

$$S(t,\theta) = \frac{1}{\sqrt{\cos t}} \begin{pmatrix} \cos\frac{t-\theta}{2} & i\sin\frac{t-\theta}{2} \\ -i\sin\frac{t+\theta}{2} & \cos\frac{t+\theta}{2} \end{pmatrix}$$

- Given a solution Ψ of the Dirac Dirac Equation the dressed spinor

$$\phi(t,\theta) = \frac{1}{\sqrt{2}} f(t,\theta) S(t,\theta)(1 - \not{X})\Psi(t,\theta)$$

- Solves the Fock-Iwanenko- Dirac equation

$$i\alpha^t\left(\partial_t + \Gamma_t\right)\phi + i\alpha^\theta\left(\partial_\theta + \Gamma_\theta\right)\phi - i\alpha^i(\partial_i \ln f)\phi - \nu\phi = 0$$

Remarks

$$S(t,\theta) = \frac{1}{\sqrt{\cos t}} \begin{pmatrix} \cos\frac{t-\theta}{2} & i\sin\frac{t-\theta}{2} \\ -i\sin\frac{t+\theta}{2} & \cos\frac{t+\theta}{2} \end{pmatrix}$$

- The matrix S is anti-periodic well-defined only on the double covering of the de Sitter hyperboloid.

- The map $(t,\theta) \to S(t,\theta)$ is thus a map from the double covering of the de Sitter spacetime with values in the spin group Sp(1,2)

- The dressing changes periodicities: periodic (R) fields become anti-periodic (NS) and viceversa.

Answer to the second question

- Dress the DD field and get a quantum field solving the ϕ_ν standard Dirac-Fock-Iwanenko) equation.
- The dressed field has NS antiperiodicity and therefore well-defined only on the covering of the de Sitter manifold

$$\psi'(X) = g\psi\big(\Lambda^{-1}(g)X\big)$$

$$\phi'(\tilde{X}) = \Sigma(g, \tilde{X})\,\phi(g^{-1}\tilde{X}),$$

$$\Sigma(g, \tilde{X}) = S(\tilde{X})\,g\,S(g^{-1}\tilde{X})^{-1}$$

- $\Sigma(g, \tilde{X})$ is a nontrivial cocyle of Sp(1,2)

$$\Sigma(g_1, \tilde{X})\Sigma(g_2, g_1^{-1}\tilde{X}) = \Sigma(g_1 g_2, \tilde{X}).$$

Cocyclic de Sitter Covariance

- The de Sitter covariance of the de Sitter FI Dirac NS field is thus expressed in terms of a cocycle.

$$\phi'(\tilde{X}) = \Sigma(g, \tilde{X})\,\phi(g^{-1}\tilde{X}),$$

$$\Sigma(g_1, \tilde{X})\Sigma(g_2, g_1^{-1}\tilde{X}) = \Sigma(g_1 g_2, \tilde{X}).$$

- On the other hand there is no covariant Dirac field (in the above sense) in the Ramond sector.
- The following remarkable result play an important role in the construction of the de Sitter - Thirring model :

For any g in the spin group $Sp(1,2)$ the cocycle $\Sigma(g, \tilde{X})$ is diagonal.

Rotations

For every spatial rotation $g = k(\theta)$
and every $\tilde{X} \in \widetilde{dS_2}$ the cocycle

$$\Sigma(k(\theta), \tilde{X}) = 1$$

Boosts

- For a transformation $a(\kappa)$ belonging to the abelian subgroup A

$$\Sigma(a(\kappa),\tilde{X}) = \begin{pmatrix} e^{-\kappa/2}\frac{(1+\lambda'^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta'}{2}\right)}{(1+\lambda^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta}{2}\right)} & 0 \\ 0 & e^{\kappa/2}\frac{(1+\lambda^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta}{2}\right)}{(1+\lambda'^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta'}{2}\right)} \end{pmatrix}$$

where

$$\begin{cases} \lambda' = \lambda'(-\kappa) = \lambda\cosh\kappa - \sinh\kappa(\lambda\cos\zeta + \sin\zeta), \\ \cot\frac{\zeta'}{2} = \cot\frac{\zeta'(-\kappa)}{2} = e^{-\kappa}\cot\frac{\zeta}{2}. \end{cases}$$

Lightlike Boosts

For a transformation $n(\mu)$ of the upper triangular subgroup N we have

$$\Sigma(n(\mu),\tilde{X}) = \begin{pmatrix} \frac{(1+\lambda'^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta'}{2}\right)}{(1+\lambda^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta}{2}\right)} & 0 \\ 0 & \frac{(1+\lambda^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta}{2}\right)}{(1+\lambda'^2)^{\frac{1}{4}}\ \sin\left(\frac{\zeta'}{2}\right)} \end{pmatrix}$$

where

$$\begin{cases} \lambda' = \lambda'(-\mu) = \frac{1}{2}\left(\mu(2\lambda-\mu)\sin\zeta - \mu(\lambda\mu+2)\cos\zeta + \lambda\left(\mu^2+2\right)\right) \\ \cot\frac{\zeta'}{2} = \cot\frac{\zeta'(-\mu)}{2} = \cot\frac{\zeta}{2} + \mu. \end{cases}$$

In the end: massless NS Spinors have a hidden de Sitter symmetry

$$\frac{S(t,\theta)(\Omega,\Psi_0(t,\theta)\overline{\Psi}_0(t',\theta')\Omega)S(t',\theta')^{-1}}{\sqrt{\cos t}\sqrt{\cos t'}} =$$

$$= -\frac{i}{4\pi}\begin{pmatrix} 0 & \frac{1}{\sin\left(\frac{1}{2}(u-u')\right)} \\ \frac{1}{\sin\left(\frac{1}{2}(v-v')\right)} & 0 \end{pmatrix}$$

$$\Psi_0(t,\theta) = \sqrt{\cos t}\, S(t,\theta)^{-1}\psi^{\mathrm{NS}}(t,\theta).$$

In the very end: the Thirring field

- Field Equation $i\alpha^\mu(\partial_\mu + \Gamma_\mu)\psi = -g\alpha^\mu J_\mu \psi$

- Solution $\phi(x) = e^{i\chi^+(x)}\phi_0(x)e^{i\chi^-(x)}$.

$$\chi_1^\pm(x) = \alpha j^\pm(x) - \beta\tilde{j}^\pm(x) + a_1^\pm(x)Q_1 + b_1^\pm(x)Q_2 \ ,$$
$$\chi_2^\pm(x) = \alpha j^\pm(x) + \beta\tilde{j}^\pm(x) + a_2^\pm(x)Q_2 + b_2^\pm(x)Q_1 \ .$$

- Under certain conditions it is possible to find local and de Sitter covariant solutions

Perspectives

- Opens the way to he study of integrable QFT models on the de Sitter Manifold
- Based on work in progress with Henri Epstein (IHES)