Non-Singular Bouncing Cosmology

and

Cosmological Perturbations

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Hot Topics in Modern Cosmology

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New Ekpyrotic Cosmology: A Brief Review

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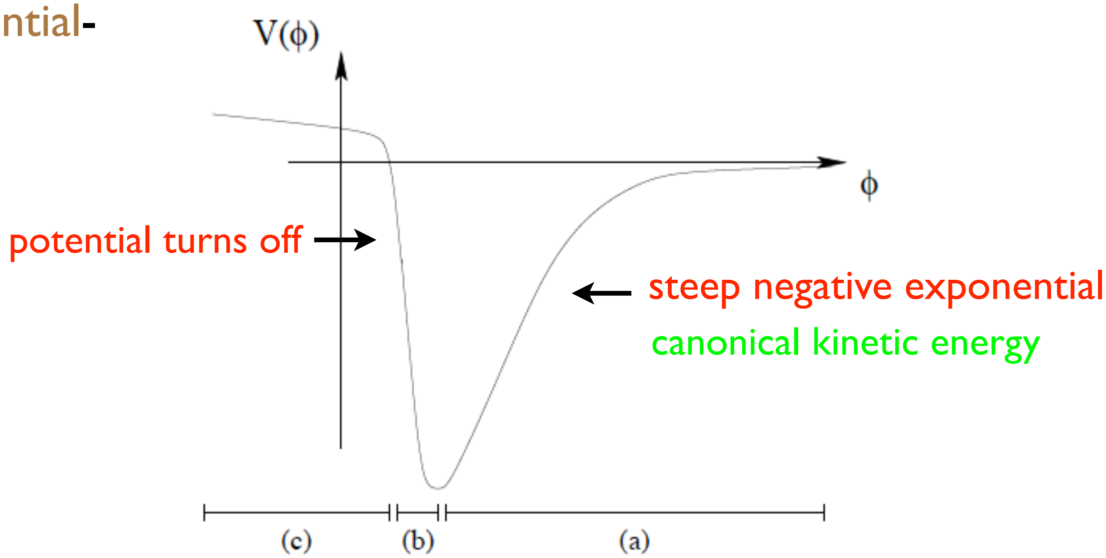
At the level of a 4d effective description, the basic ingredients of the simplest ekpyrotic scenario are essentially the same as in inflation, namely a scalar field ϕ rolling down some self-interaction potential $V(\phi)$. A key difference, however, is that while inflation requires a flat and positive potential, its ekpyrotic counterpart is *steep* and *negative*. This has a dramatic impact on the cosmological evolution. Instead of accelerated expansion, an ekpyrotic theory has slow contraction. Instead of an exponentially growing scale factor and nearly constant Hubble radius, corresponding to approximate de Sitter geometry, we now have a nearly constant scale factor and rapidly shrinking Hubble radius, corresponding to approximately flat space.

Ekpyrotic Potential-



Density Perturbations-

In the long-wavelength limit

$$\delta\phi_k^2 \sim k^{-3}$$

corresponding to a scale-invariant power spectrum. This $\Longrightarrow$ the associated gauge-invariant curvature perturbation $\mathcal{R}$ is scale invariant for large wavelengths outside the Hubble horizon.

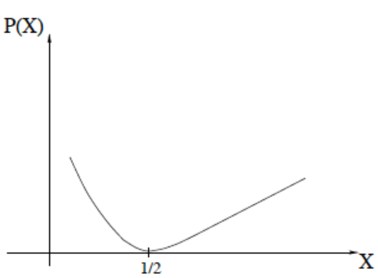
Ghost Condensation and the Bounce-

Theories of ghost condensation describe a scalar field with higher-derivative kinetic term

$$\mathcal{L} = \sqrt{-g}M^4 P(X), \qquad X = -\frac{1}{2m^4}(\partial\phi)^2$$

M and m are some arbitrary scales . Choosing P(X) to be, for example, of the form

$$P(X) = -X + X^2 \quad \Longrightarrow$$



will violate the NEC and cause the contracting universe during the Ekpyrotic phase to bounce to an expanding phase.

Question:

Will the near scale invariant long wavelength perturbations $\mathcal{R}$ go unchanged through the bounce-- or will the scale invariance be destroyed?

$\Longrightarrow$ must carefully compute the evoltion of $\mathcal{R}$ through the dynamical bouncing phase.

In natural units ($8\pi G = M_{Pl}^{-2} = 1$), the Lagrangian is given by

$$\mathcal{L} = \sqrt{-g}\left(\frac{R}{2} + P(X,\phi) + g(\phi)X\,\Box\phi\right) \, ,$$

Horndeski $\Longrightarrow$

2nd order equations of motion

where R is the Ricci scalar and

$$P(X,\phi) = k(\phi)X + \tau(\phi)X^2 - V(\phi)$$

with $X \equiv -\frac{1}{2}(\partial\phi)^2$. The explicit forms of the functions k, τ, g, V are

$$k(\phi) = 1 - \frac{2}{(1+2\kappa\phi^2)^2},$$

(chosen to allow for a simple supersymmetric extension later on)

$$\tau(\phi) = \frac{\bar{\tau}}{(1+2\kappa\phi^2)^2}, \quad g(\phi) = \frac{\bar{g}}{(1+2\kappa\phi^2)^2}$$

and

$$V(\phi) = -V_0 v(\phi) e^{-c(\phi)\phi}$$

where $v(\phi)$ is a function chosen such that the potential turns off for $\phi < \phi_{ek-end}$.

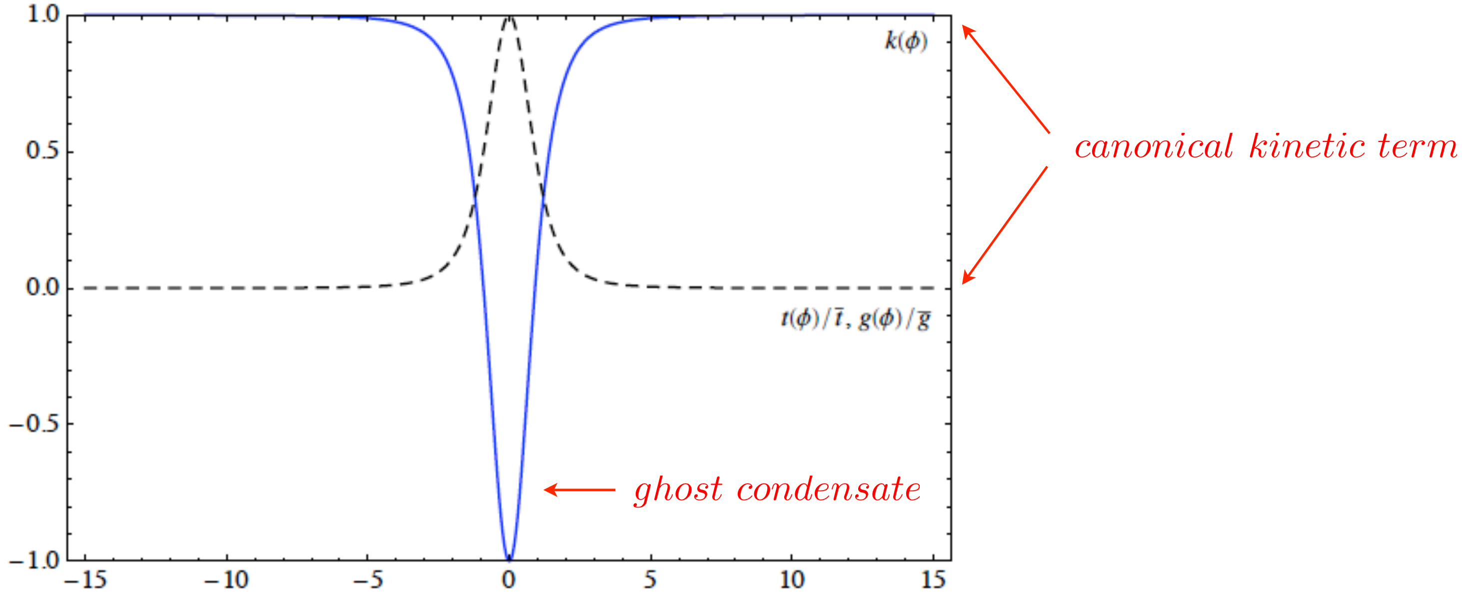


Figure 1: The solid curve shows $k(\phi)$ while the dashed curve shows the normalized functions $\tau(\phi)/\bar{\tau}, g(\phi)/\bar{g}$, all with $\kappa = 1/4$.

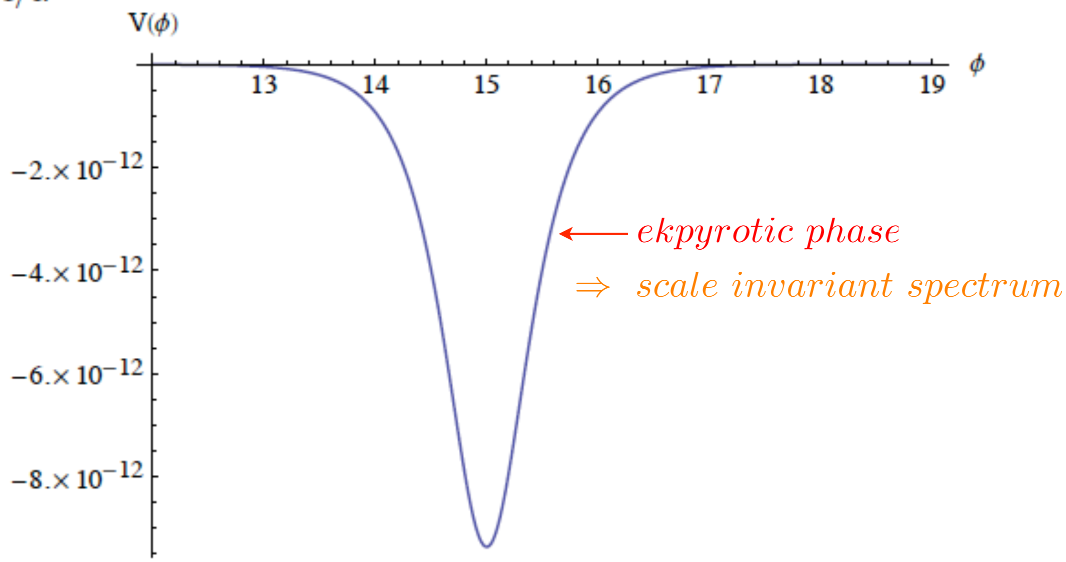


Figure 2: The ekpyrotic potential. The ekpyrotic phase starts at large positive ϕ, with the field rolling down the potential towards smaller values of the field. Around $\phi_{ek-end} \approx 15$ the potential starts to come back up to zero, and is irrelevant from then on. The bounce occurs at small values, $\phi \approx 0$.

For the background, we will assume a flat Friedmann-Lemaitre universe. In the "physical" time coordinate t_p, this is given by

$$ds^2 = -\mathrm{d}t_p^2 + a^2(t_p)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

However, we will transform to a "harmonic" time t

$$dt_p = a(t)^3\, dt$$

It follows that the Friedman-Lemaitre metric becomes

$$ds^2 = -a^6(t)\,\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\ .$$

We note that, with this coordinate choice, the metric satisfies

$$\Gamma^\mu = g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} = 0\ .$$

This is a useful property when we calculate metric and scalar perturbations.

For a given metric, coordinates satisfying this are called "harmonic coordinates".

Harmonic coordinates are not unique–if x^μ are harmonic, then so are $y^\mu = x^\mu + \xi^\mu(x)$ as long as

$$-(\xi^t)'' + a^4\nabla^2\xi^t = 0$$
$$-\xi'' + a^4\nabla^2\xi = 0$$

In the background

$$ds^2 = -a^6(t)\,\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \ ,$$
$$\phi = \phi(t)$$

the equations of motion become

$$3\mathcal{H}^2 = -a^6 P + P_{,X}\phi'^2 - 3g\mathcal{H}\frac{\phi'^3}{a^6} + \frac{1}{2}g_{,\phi}\frac{\phi'^4}{a^6}$$
$$-2\mathcal{H}' + 3\mathcal{H}^2 = a^6 P + g\frac{\phi'^2}{a^6}\left(\phi'' - 3\mathcal{H}\phi'\right) + \frac{1}{2}g_{,\phi}\frac{\phi'^4}{a^6}$$
$$0 = P_{,XX}\phi'^2\left(\phi'' - 3\mathcal{H}\phi'\right) + a^6 P_{,X\phi}\phi'^2 + a^6 P_{,X}\phi'' - a^{12}P_{,\phi}$$
$$-3g\phi'\left[\mathcal{H}(2\phi'' - 6\mathcal{H}\phi') + \mathcal{H}'\phi'\right] + 2g_{,\phi}\phi'^2(\phi'' - 3\mathcal{H}\phi') + \frac{1}{2}g_{,\phi\phi}\phi'^4$$

where $' = \frac{d}{dt}$ and $\mathcal{H} = \frac{a'}{a}$. Solving these with the sample conditions $\kappa = 1/4,\ \bar{\tau} = 1,\ \bar{g} = 1/100$ and initial conditions $a = 1,\ \phi = \frac{17}{2},\ \phi' = -10^{-5}$ at some time t_0 $\Longrightarrow$

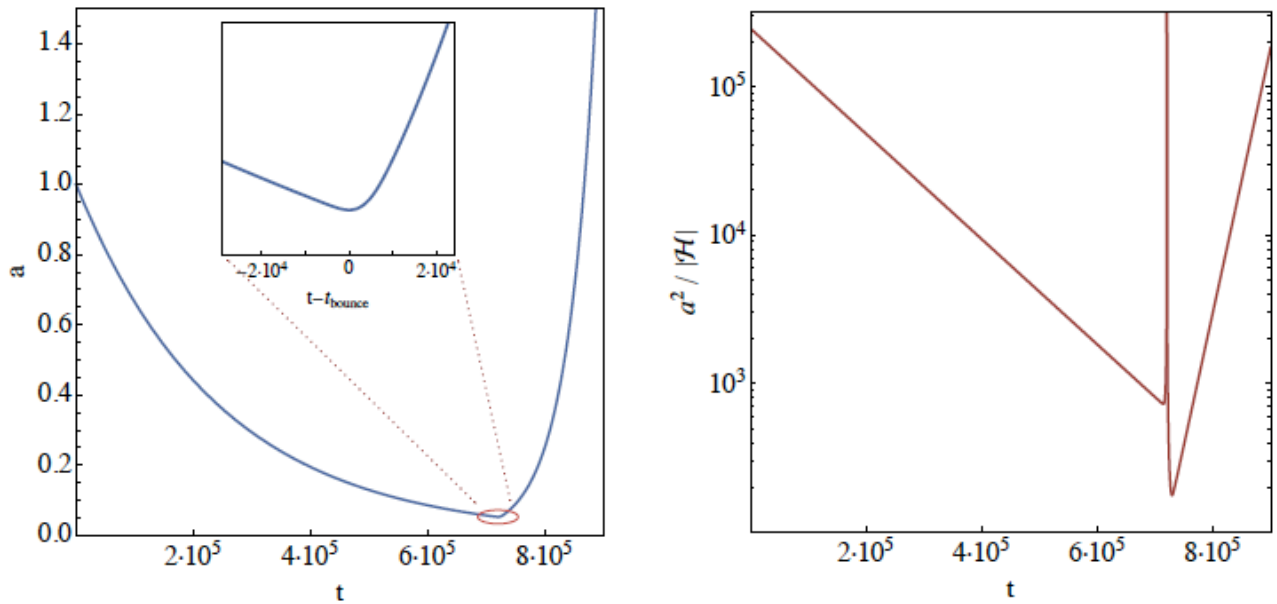
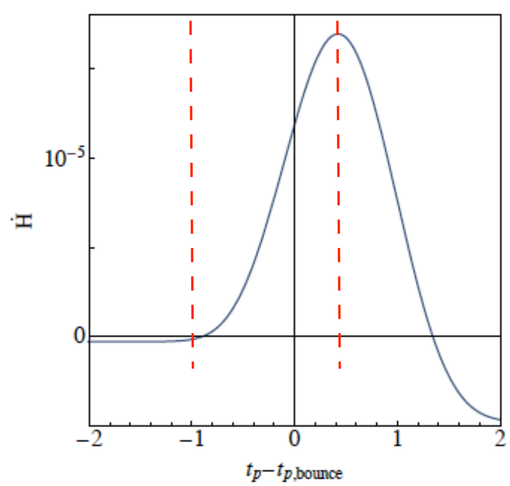


Figure 3: Scale factor and comoving Hubble length for the bounce solution with parameter values $\kappa = 1/4$, $\bar{\tau} = 1$, $\bar{g} = 1/100$ and with initial conditions $a = 1$, $\phi = 17/2$, $\phi' = -10^{-5}$.

Note that the bounce is completely smooth. Furthermore, going to physical time we find



$\Longrightarrow$ the NEC is violated for a period of order 1~2 times the ghost condensate scale.

Choosing the X^2 mass parameter to be $\Lambda \sim 10^{17} GeV$

$\Rightarrow$

$$\Delta t_{\rm bounce} \sim 10^{2-3} \frac{1}{M_P}$$

$\Longrightarrow$ completely classical

The (gauge–invariant) co-moving curvature perturbation $\mathcal{R}$ satisfies the closed equation

$$\frac{d^2\mathcal{R}}{d\tau^2} + \frac{2}{z}\frac{dz}{d\tau}\frac{d\mathcal{R}}{d\tau} + c_s^2 k^2 \mathcal{R} = 0 ,$$

to linear order in Fourier space where $d\tau \equiv a^2 dt$ is conformal time. The coefficients are given by

$$z^2 = \frac{1}{2}\frac{a^2\phi'^2\mathcal{P}}{\left(\mathcal{H} + \frac{1}{2}g(\phi)\frac{\phi'^3}{a^6}\right)^2} ,$$

$$\mathcal{P} = P_{,X} - 6g(\phi)\mathcal{H}\frac{\phi'}{a^6} + \frac{3}{2}g^2(\phi)\frac{\phi'^4}{a^{12}} + 2g_{,\phi}\frac{\phi'^2}{a^6} + P_{,XX}\frac{\phi'^2}{a^6} ,$$

$$c_s^2 = \frac{1}{\mathcal{P}}\left(P_{,X} + 2g(\phi)\mathcal{H}\frac{\phi'}{a^6} - \frac{1}{2}g^2(\phi)\frac{\phi'^4}{a^{12}} - 2g(\phi)\frac{\phi''}{a^6}\right) .$$

Also, the quadratic action for the perturbations takes the simple form

$$S_2 = \int dt\, \frac{z^2}{a^2}\left(\frac{1}{2}\mathcal{R}'^2 - \frac{1}{2}c_s^2 k^2 a^4 \mathcal{R}^2\right) .$$

With respect to harmonic time and the above intital parameters, the quantities c_s^2 and z^2 are found to be

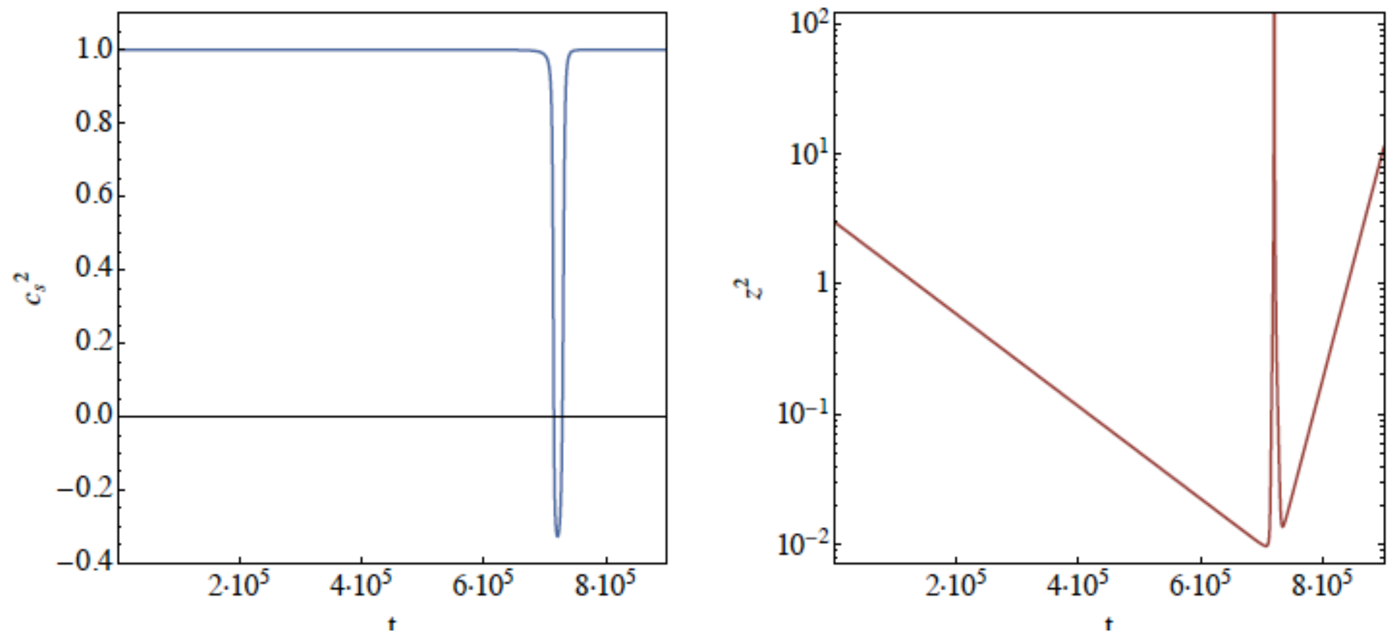


Figure 4: Evolution of the speed of sound squared and of z^2 in the non-singular bounce background. The positivity of z^2 demonstrates the absence of perturbative ghost fluctuations, while the brief period over which c_s^2 becomes negative indicates the presence of a gradient instability.

Note that z^2 blows up in the vicinity of the bounce, as the denominator passes through zero when $\mathcal{H} = -\frac{1}{2}g(\phi)\frac{\phi'^3}{a^6}$. This implies that at this moment the equation for $\mathcal{R}$ becomes singular. The lesson we draw from this observation is that it would be desirable to find a better, non-singular and completely reliable way to describe the evolution of perturbations across the bounce.

Following B. Xue, D. Garfinkle, F. Pretorius, and P. J. Steinhardt, Phys.Rev. **D88**, 083509 (2013), one defines "harmonic gauge" by

(a) taking generic scalar perturbations around our background given by

$$ds^2 = -a^6(1+2\mathbf{A})\mathrm{d}t^2 + 2a^4\mathbf{B}_{,i}\,\mathrm{d}t\,\mathrm{d}x^i + a^2(t)\Big[(1-2\psi)\,\delta_{ij} + 2\mathbf{E}_{,ij}\Big]\mathrm{d}x^i\mathrm{d}x^j\ ,$$

$$\phi = \phi(t) + \delta\phi(t,x)$$

(b) noting that

$$\delta\Gamma^\mu = -\frac{1}{a^6}c^\mu\ ,$$

$$c^t = \mathbf{A}' + 3\psi' - \nabla^2\left(\mathbf{E}' - a^2\mathbf{B}\right)\ ,$$

$$c^i = \Big[\left(a^2\mathbf{B}\right)' + a^4\left(\mathbf{A} - \psi - \nabla^2\mathbf{E}\right)\Big]_{,i}$$

(c) and finally choosing the $\mathbf{A}, \mathbf{B}, \mathbf{E}, \psi$ functions to satisfy

$$\delta\Gamma^\mu = 0$$

As in the unperturbed case, there is residual harmonic gauge freedom given by

$$\mathbf{A} \to \mathbf{A} - \xi^{t\prime} - 3\mathcal{H}\xi^t\ ,$$

$$\mathbf{B} \to \mathbf{B} + a^2\xi^t - \frac{1}{a^2}\xi'\ ,$$

$$\psi \to \psi + \mathcal{H}\,\xi^t\ ,$$

$$\mathbf{E} \to \mathbf{E} - \xi\ .$$

These can be used to set the simplifying initial conditions

$$\mathbf{A}(t_0) = \mathbf{B}(t_0) = \psi(t_0) = \mathbf{E}(t_0) = 0$$

The complete set of equations of motion in harmonic gauge are

$$0 = \mathbf{A}' + 3\psi' + k^2\left(\mathbf{E}' - a^2\mathbf{B}\right)$$

$$0 = \left(a^2\mathbf{B}\right)' + a^4\left(\mathbf{A} - \psi + k^2\mathbf{E}\right)$$

$$\begin{aligned}
0 = &\left(\mathcal{H}' - \frac{1}{2}P_{,XX}\frac{\phi'^4}{a^6} + 3g\mathcal{H}\frac{\phi'^3}{a^6} + \frac{1}{2}\frac{g\phi'^2\phi''}{a^6} - \frac{1}{2}g_{,\phi}\frac{\phi'^4}{a^6}\right)\mathbf{A} - k^2\left(a^2\mathcal{H} + \frac{1}{2}g\frac{\phi'^3}{a^4}\right)\mathbf{B} \\
&+3\left(\mathcal{H} + \frac{1}{2}g\frac{\phi'^3}{a^6}\right)\psi' + k^2a^4\psi + k^2\left(\mathcal{H} + \frac{1}{2}g\frac{\phi'^3}{a^6}\right)\mathbf{E}' \\
&+\frac{1}{2}\left(P_{,X}\phi' + P_{,XX}\frac{\phi'^3}{a^6} - 9g\mathcal{H}\frac{\phi'^2}{a^6} + 2g_{,\phi}\frac{\phi'^3}{a^6}\right)\delta\phi' \\
&-\frac{1}{2}\left(a^6P_{,\phi} - P_{,X\phi}\phi'^2 + k^2g\frac{\phi'^2}{a^2} + 3g_{,\phi}\mathcal{H}\frac{\phi'^3}{a^6} - \frac{1}{2}g_{,\phi\phi}\frac{\phi'^4}{a^6}\right)\delta\phi\,,
\end{aligned}$$

$$0 = \left(\mathcal{H} + \frac{1}{2}g\frac{\phi'^3}{a^6}\right)\mathbf{A} + \psi' - \frac{1}{2}g\frac{\phi'^2}{a^6}\delta\phi' - \frac{1}{2}\frac{\phi'}{a^6}\left(a^6P_{,X} - 3g\mathcal{H}\phi' + g_{,\phi}\phi'^2\right)\delta\phi\,,$$

$$0 = \frac{1}{a^4}\mathbf{E}'' + k^2\mathbf{E}\,.$$

Note that there are no $\frac{1}{\mathcal{H}}$ factors. This is the main reason for using harmonic gauge.

These equations can be solved subject to the above initial conditions.

The gauge–invariant comoving curvature perturbation is defined as

$$\mathcal{R} \equiv \psi + \frac{\mathcal{H}}{\phi'}\delta\phi \,.$$

In harmonic gauge $\mathcal{R}$ is well-defined and smooth at the bounce. For the above choices of parameters we find

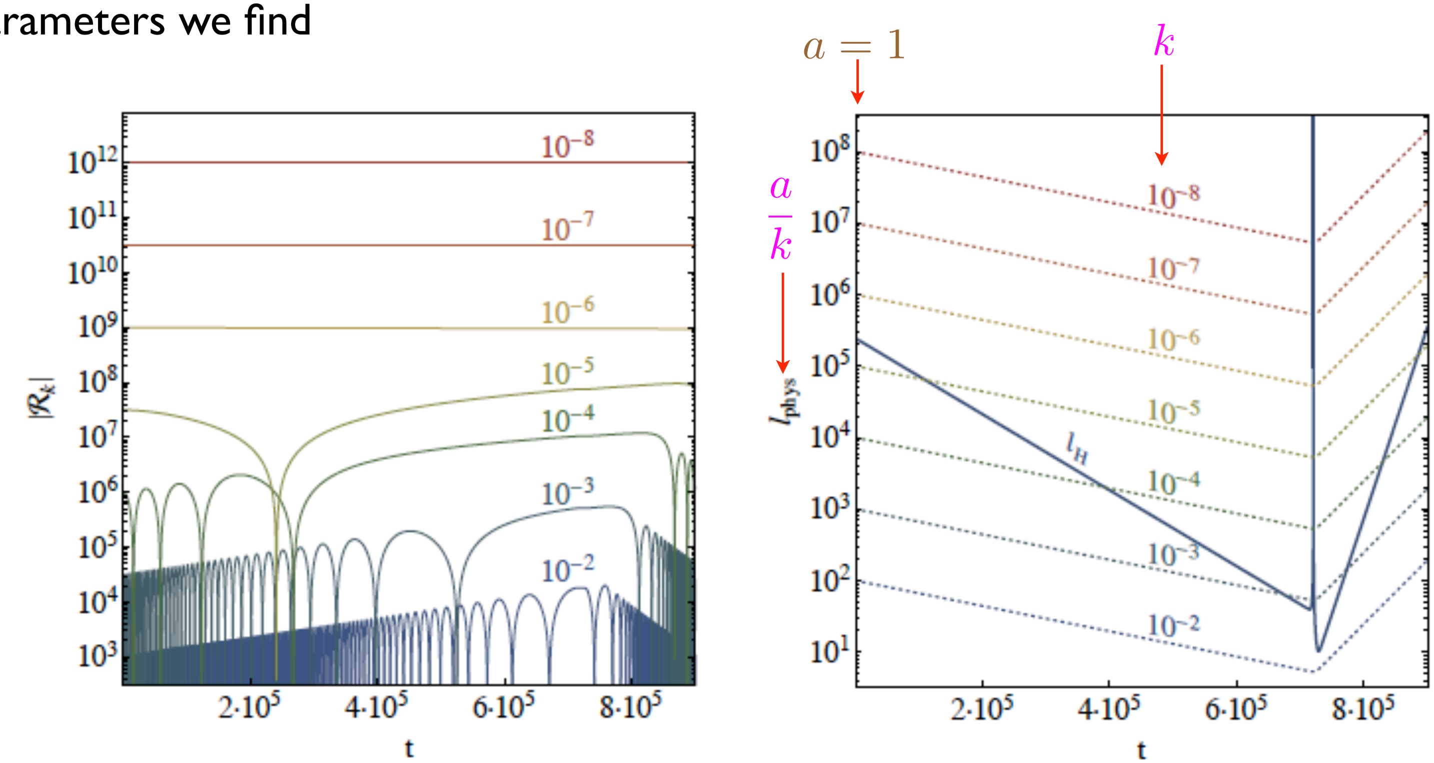


Figure 5: The left-hand panel presents the evolution of the comoving curvature perturbation for various wavelengths. Long-wavelength modes are preserved essentially unchanged across the bounce, while short, initially oscillating modes, flatten out near the bounce. The right-hand panel shows both the horizon size and the physical wavelengths a/k of the various perturbation modes as functions of harmonic time.

Momentum k is in Planck units.

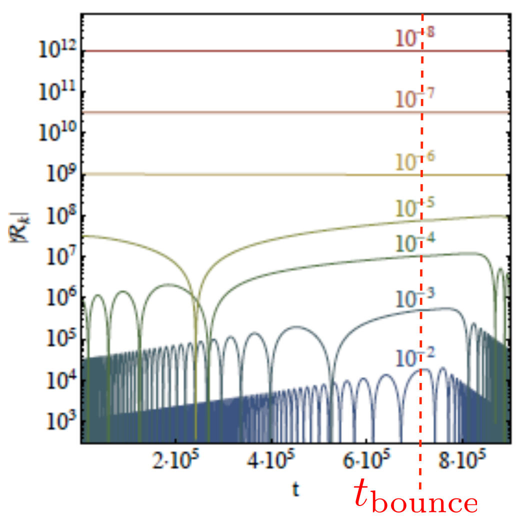
$\Rightarrow$

Main Result: non-singular ghost condensate/Galileon bounces preserve both the amplitude and spectrum of large-scale curvature perturbations across the bounce, and hence, if such perturbations are generated during the contracting phase, they will go through unmodified into the expanding phase of the universe.

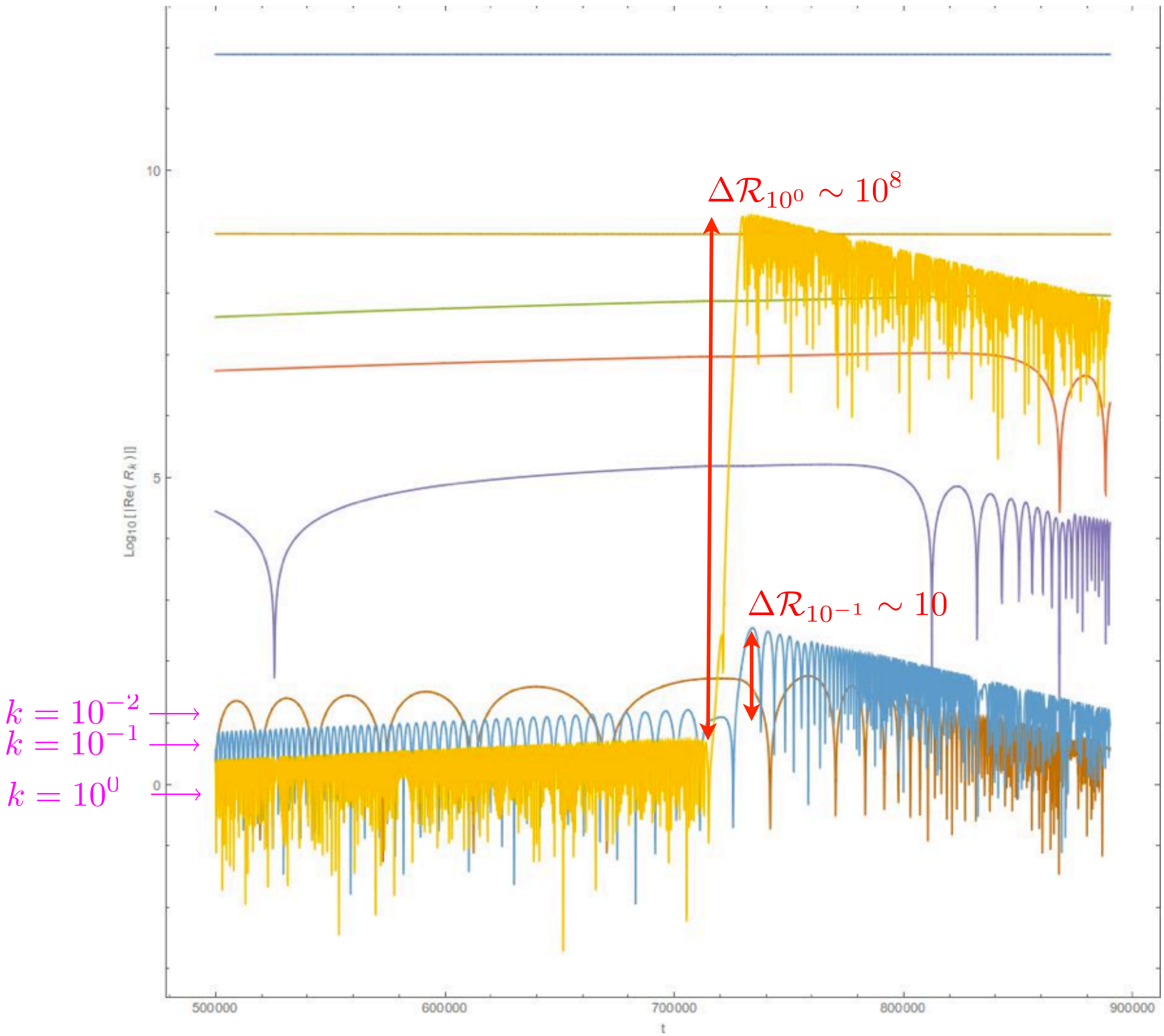
$\Rightarrow$

The nearly scale-invariant scalar perturbations generated during the Ekpyrotic contracting phase will pass essentially unchanged through a smooth bounce!

What is the behaviour of short wavelength modes--with $l_{phys} < l_H$ for all t?



Must look at $k \geq 10^{-2}$



10
5
0
500000
600000
700000
800000
900000
t
Log10 [| Re(R_k) |]
$\Delta\mathcal{R}_{10^0} \sim 10^8$
$\Delta\mathcal{R}_{10^{-1}} \sim 10$
$k = 10^{-2}$
$k = 10^{-1}$
$k = 10^0$

Does the exponential jump $\Delta\mathcal{R}_k$ at the bounce for $k \gtrsim 10^{-1}$ indicate an instability in the bounce theory? No!

To understand this, we have to compute the action for the scalar cuvature perturbation $\mathcal{R}$ to cubic order. Using both harmonic and ADM fluctuations, we find that near the bounce

$$S_{2,3} = \int \mathrm{d}t\mathrm{d}x^3 \; a^3 \left[\frac{\Sigma}{H^2}\left(\dot{\mathcal{R}}\right)^2 + \frac{-(\Sigma+2\lambda)}{H^3}\left(\dot{\mathcal{R}}\right)^3 + 3\frac{\Sigma}{H^2}\left(\dot{\mathcal{R}}\right)^2\mathcal{R} - \frac{2}{a^2}\frac{\dot{\mathcal{R}}}{H}\mathcal{R}\partial^2\mathcal{R} - \frac{1}{a^2}\frac{\dot{\mathcal{R}}}{H}(\partial\mathcal{R})^2 + \frac{1}{a^2}\mathcal{R}(\partial\mathcal{R})^2\right]$$

where $\lambda \equiv P_{,XX}X^2 + \frac{2}{3}P_{,XXX}X^3$ and $\Sigma \equiv P_{,X}X + 2P_{,XX}X^2$.

To evaluate the cut-off of the theory, we must compare the cubic action to the quadratic one, but for normalised perturbations. To this end we define

$$v \equiv \frac{\sqrt{\Sigma}}{H}\mathcal{R}$$

and obtain terms of the form

$$S_{2,3} = \int \mathrm{d}t\mathrm{d}x^3 \;\; a^3 \;\; \Big[(\dot{v})^2 \\ + \frac{-(\Sigma + 2\lambda)}{\Sigma^{3/2}} (\dot{v})^3 + 3\frac{H}{\Sigma^{1/2}} (\dot{v})^2 v + \cdots \\ - \frac{H^2}{a^2 \Sigma^{3/2}} \dot{v} \left(2\partial^2 v + (\partial v)^2 \right) + \frac{H^3}{a^2 \Sigma^{3/2}} v (\partial v)^2 \Big]$$

Since we have a time-dependent background, the coefficient functions of the various terms change over time. The most interesting time in the present context is the moment when the null energy condition starts being violated, i.e. the time of the onset of the bounce (the energy density of the background is also maximal around that time). It turns out that it is the $\dot{v}^3$ term that yields the smallest energy scale for the cut-off of the theory, and hence this is the most relevant term to consider. We rewrite this term as

$$-\int \mathrm{d}t\mathrm{d}x^3 a^3 \frac{1}{\Lambda^2_{\dot{\mathcal{R}}^3}} (\dot{v})^3 \quad \text{with} \quad \Lambda_{\dot{\mathcal{R}}^3} \equiv \frac{\Sigma^{3/4}}{(\Sigma + 2\lambda)^{1/2}}$$

Close to the bounce, the ghost condensate can be approximated by a Lagrangian function of the form

$$P(X, \phi) = -X + qX^2$$

where q has mass dimension $[-4]$ and determines the scale of the ghost condensate. At the onset of the bounce, the null energy conditions starts being violated. Thus we have $\dot{H} = -\frac{1}{2}(\rho + p) = 0$ and thus $c_s^2 = 0$ and $P_{,X} = 0$. From these conditions it follows that $X = -1/(2q)$ at that time and consequently the energy density is given by $\rho = 2P_{,X}X - P = 1/(4q)$. The cut-off scale is

$$\Lambda_{\dot{\mathcal{R}}^3} = \left(\frac{1}{4q}\right)^{1/4} \quad \text{at} \quad \dot{H} = 0$$

Taking the ghost condensate scale to be $\Lambda \sim 10^{17} GeV$ as we did previously gives

$$\Lambda_{\dot{\mathcal{R}}3} \sim 10^{-2} M_P$$

Hence, the effective theory becomes strongly coupled just where the fluctuation amplitudes begin to change dramatically.

To further explore the relationship between the exponential growth in the gauge invariant curvature purturbations and their wave number, we repeat the previous analysis but with a different, and more revealing, set of parameters. Again, we choose the interaction terms to be

$$\mathrm{P}(\mathrm{X},\phi) = \kappa(\phi)X + q(\phi)X^2 - V(\phi)\ , \quad g(\phi)X\Box\phi$$

Specifically, we take the coefficient of the Galileon term to be

$$\bar{g} = 0 \longleftarrow$$

while choosing

$$\kappa = \frac{1}{4}\ , \quad \bar{\tau} = 10^8 \longleftarrow$$

Additionally, we choose a potential to be in the same form as above

$$V(\phi) = -V_0 v(\phi) e^{-c(\phi)\phi}$$

but with

$$v(\phi) = \frac{1}{2}[1 + tanh(\lambda(\phi - \phi_{ek-end}))] \longleftarrow$$

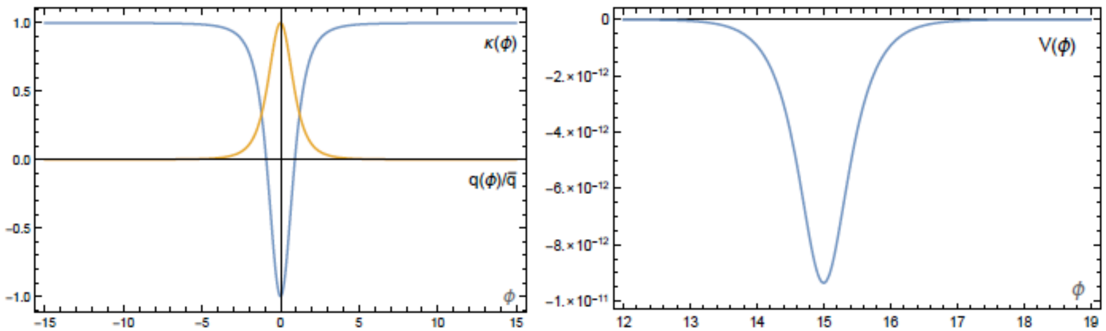
where

$$V_0 = 100, \quad \lambda = 3, \qquad \phi_{ek-end} = 15, \quad c(\phi) = 3$$

Resolving the above equations in harmonic gauge using the same initial conditions

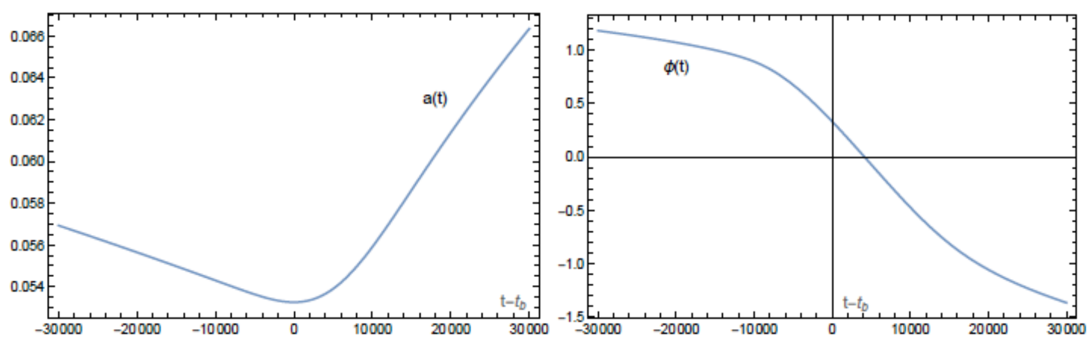
$$a_0 = 1, \quad \phi_0 = \frac{17}{2}, \quad \phi_0' = -10^{-5}$$

leads to the following results.



(a) Kinetic functions (b) Ekpyrotic Potential

Figure 6: Graphs of the functions entering the scalar field Lagrangian. (a) The blue curve shows $\kappa(\phi)$ while the yellow curve shows the normalized function $q(\phi)/\bar{q}$, both with $\bar{\kappa} = 1/4$. (b) The ekpyrotic potential (9) with $V_0 = 100, \lambda = 3, \phi_{ek-end} = 15, c(\phi) = 3$. The ekpyrotic phase starts at large positive ϕ, with the field rolling down the potential towards smaller values of the field. Around ϕ_{ek-end} the potential starts to come back up to zero, and is irrelevant from then on. In this model, the bounce occurs at small values, $\phi \approx 0$.



(a) Scale Factor (b) Scalar Field

Figure 7: (a) The scale factor around the time of the bounce as a function of physical time t minus t_b, where t_b denotes the time of the bounce ($H(t_b) \equiv 0$). Our numerical evaluation starts t at $\phi_0 = 17/2$ with $\dot{\phi}_0 = -10^{-9}$, $a_0 = 1$ and H_0 is determined by the Friedmann equation. We are using the parameters $\bar{\kappa} = 1/4$, $\bar{q} = 10^8$. The figure shows a zoom-in on the most interesting time period, namely that of the bounce. One can clearly see that the bounce is smooth. (b) The evolution of the scalar field ϕ during the bounce phase. The approximately linear evolution near $\phi = 0$ corresponds to the ghost condensate phase which is responsible for the bounce.

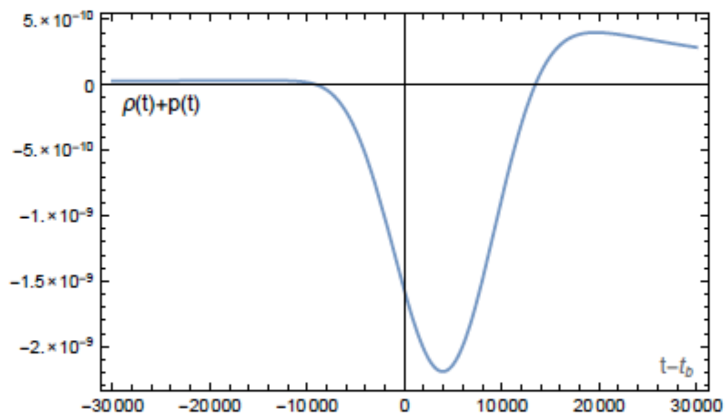
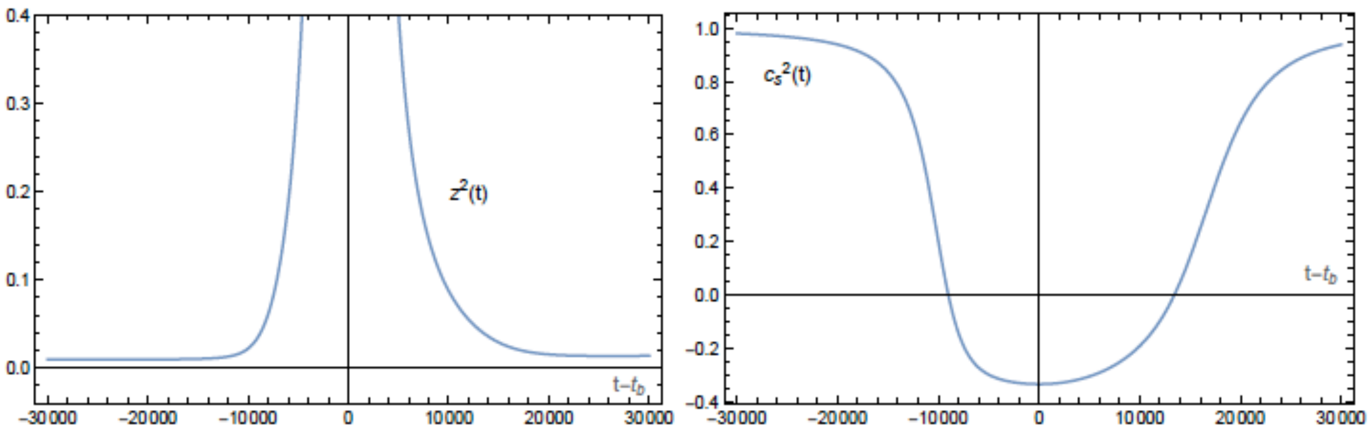


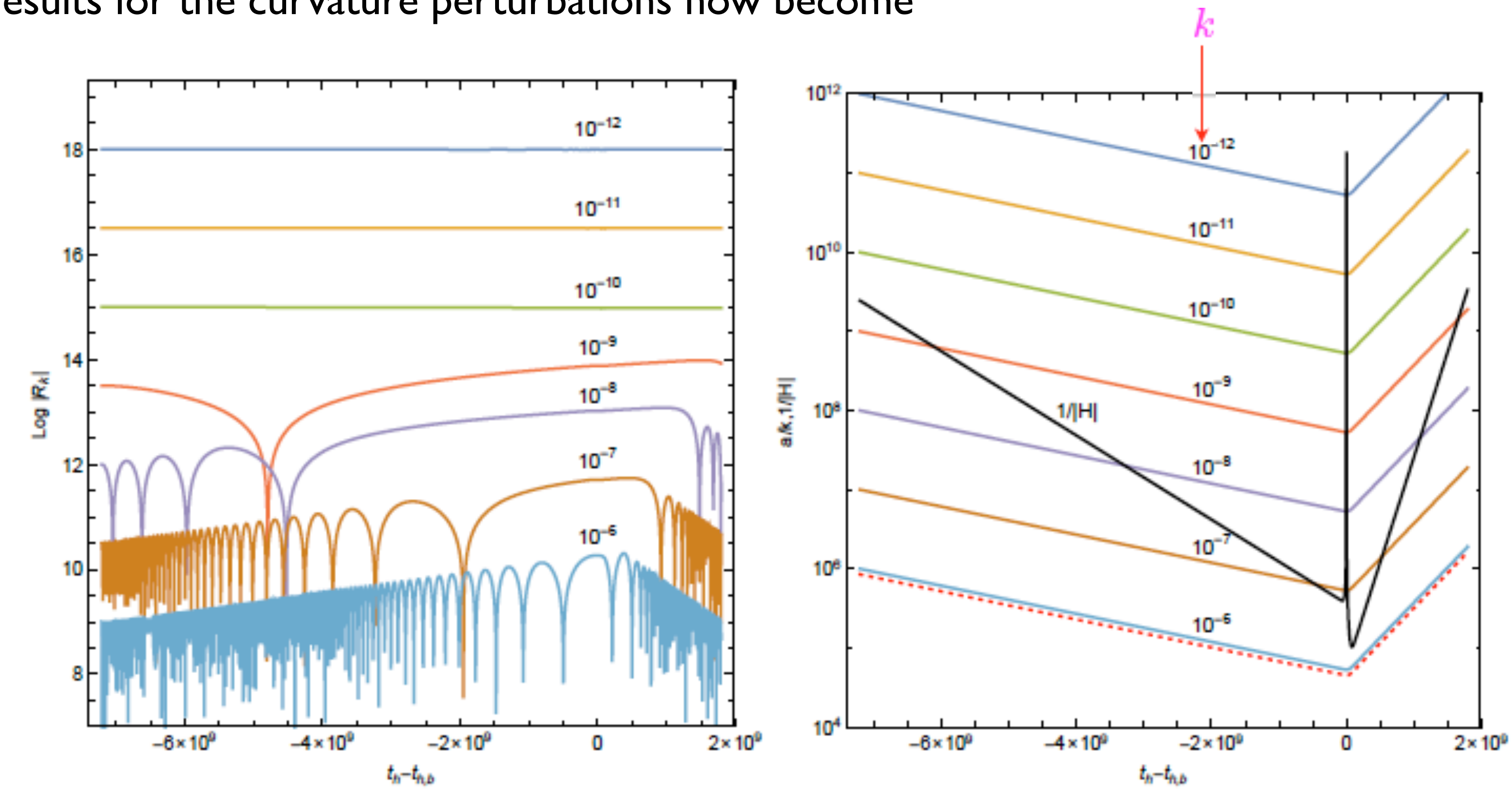
Figure 8: The sum of energy density and pressure during the bounce phase. When this quantity goes negative, the null energy condition is violated.



(a) $z^2(t)$ (b) $c_s^2(t)$

Figure 9: (a) Evolution of z^2 and (b) of the speed of sound squared in the non-singular bounce background. The positivity of z^2 demonstrates the absence of perturbative ghost fluctuations, while the brief period over which c_s^2 becomes negative indicates the presence of a gradient instability.

The results for the curvature perturbations now become

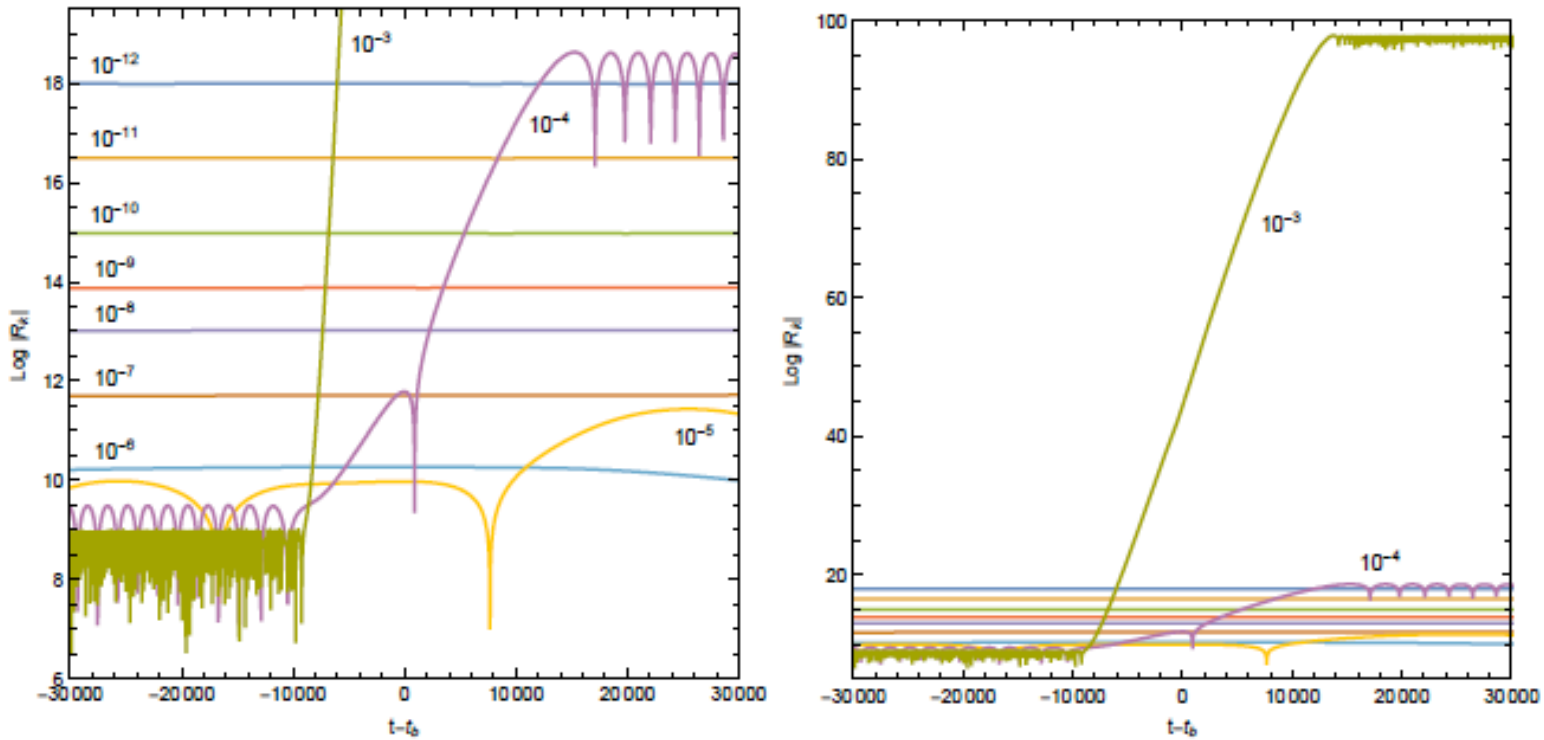


(a) Fourier Modes of the Curvature Perturbation (b) Physical Wavenumbers and Horizon Size

Figure 10: The left-hand panel presents the evolution of the comoving curvature perturbation for various wavelengths. Long-wavelength modes are preserved essentially unchanged across the bounce, while short, initially oscillating modes, flatten out near the bounce. The right-hand panel shows both the horizon size and the physical wavelengths a/k of the various perturbation modes as functions of harmonic time.

Momentum k is in Planck units.

Focussing in on the curvature perturbations for large wave number k, we find



(a) Perturbations Modes Near Bounce

(b) Amplification of Short Modes

Figure 11: (a) The curvature perturbation modes $\mathcal{R}_k$ near the time of the bounce, expressed as functions of physical time t. The initial conditions for these sub-horizon modes are taken to correspond to the early time limit of the Bunch-Davies state, in particular $\mathcal{R}_k \propto k^{-1/2}$. The period of NEC violation extends from about $t = -8000$ to $t = +14000$. During this time period short modes with wavenumber $k \geq 10^{-5}$ are seen to be amplified significantly. (b) The same plot, but with an expanded vertical scale. The mode with wavenumber $k = 10^{-3}$ (and thus with a physical wavelength more than 3 orders of magnitude smaller than the minimum horizon size) is seen to be amplified by almost 100 orders of magnitude near the bounce.

These results have a clear analytic interpretation.

Even though the numerical solutions shown in the figures were obtained via calculations in harmonic gauge, we know that the results are gauge invariant. Thus, we may obtain an estimate for the amplification by analyzing directly the equation of motion for the curvature perturbation $\mathcal{R}$, which leads to the approximate solution

$$\mathcal{R}_{\text{post-bounce}} \sim \exp\left(k \int_{c_s^2<0} \frac{|c_s|}{a} \mathrm{d}t\right) \mathcal{R}_{\text{pre-bounce}} \sim e^{k/k_\star} \mathcal{R}_{\text{pre-bounce}}$$

For the classical background considered here, numerical integration gives $k_\star \simeq 9 \times 10^{-5}$. This equation thus gives a quasi-analytic explanation for the results shown in Fig. 11. More specifically, it indicates that the amplitudes for shorter wavelength modes – that is, modes with wavelengths always smaller than the horizon (but larger than the Planck length) – grow exponentially. Naively, this dramatic growth seems to imply that the effective field theory and, hence, the bounce solution become wildly unstable at these scales – perhaps negating the validity of the non-singular classical bounce discussed above. However, this is not the case and a smooth bounce solution exists – even including its scalar and metric perturbations.

The Strong Coupling Scale:

Reintroducing mass scales, the full action we will consider is

$$S=\int \mathrm{d}t\mathrm{d}^3x\sqrt{-g}\left(\frac{\mathrm{M}_\mathrm{P}^2}{2}R+P(X,\phi)\right)\,.$$

Note that, henceforth, "t" is physical time and we have restored mass units.

It is most convenient to employ the Arnowitt-Deser-Misner (ADM) decomposition

$$\mathrm{d}s^2=-N^2\mathrm{d}t^2+h_{ij}\left(\mathrm{d}x^i+N^i\mathrm{d}t\right)\left(\mathrm{d}x^j+N^j\mathrm{d}t\right)\,,$$

where N represents the lapse function, N_i the shift and h_{ij} the metric on spatial slices of constant time. The action may then be written as

$$S=\frac{1}{2}\int \mathrm{d}t\mathrm{d}x^3\sqrt{h}\left[N\left(\mathrm{M}_\mathrm{P}^2R^{(3)}+2P(X,\phi)\right)+\frac{\mathrm{M}_\mathrm{P}^2}{N}\left(K^{ij}K_{ij}-K^2\right)\right]\,,$$

where $R^{(3)}$ is the three-dimensional Ricci scalar formed from h_{ij} and where the extrinsic curvature is defined as

$$K_{ij}=\frac{1}{2}\dot{h}_{ij}-\frac{1}{2}N_{i,j}-\frac{1}{2}N_{j,i}+\Gamma^k_{ij}N_k\,.$$

We are interested in determining the scale at which strong coupling occurs–that is, we are interested in determining the cut–off of the models under consideration, in order to assess the validity and reliability of particular solutions. We will focus on scalar perturbations.

We will start our calculation in *flat gauge* where the spatial metric $h_{ij} = a(t)^2\delta_{ij}$ is kept fixed (by choosing the appropriate time and space reparameterisations of the coordinates) as the spatial section of a flat FLRW universe. The remaining scalar perturbations are defined

$$\begin{aligned}\phi &= \phi(t) + \varphi(t, x^i),\\ N &= 1 + \alpha(t, x^i),\\ N_i &= \partial_i\beta(t, x^i).\end{aligned}$$

At linear order, which is all we will need, the constraints are given by

$$\begin{aligned}\alpha &= \frac{\dot{\phi}}{2\mathrm{M}_\mathrm{P}^2 H}P_{,X}\,\varphi\\ \frac{1}{a}\partial^2\beta &= \left(\frac{1}{2\mathrm{M}_\mathrm{P}^2 H}P_{,\phi} + \frac{\dot{\phi}}{2\mathrm{M}_\mathrm{P}^4 H^2}PP_{,X} - \frac{\dot{\phi}^3}{4\mathrm{M}_\mathrm{P}^4 H^2}P_{,X}^2 - \frac{\dot{\phi}^2}{2\mathrm{M}_\mathrm{P}^2 H}P_{,X\phi} + \frac{\dot{\phi}^5}{4\mathrm{M}_\mathrm{P}^4 H^2}P_{,X}P_{,XX}\right)\varphi\\ &\quad + \left(-\frac{\dot{\phi}}{2\mathrm{M}_\mathrm{P}^2 H}P_{,X} - \frac{\dot{\phi}^3}{2\mathrm{M}_\mathrm{P}^2 H}P_{,XX}\right)\dot{\varphi},\end{aligned}$$

where $\partial^2 = \delta^{ij}\partial_i\partial_j$ is summed only over spatial indices. The action in flat gauge at quadratic order in fluctuations is given by

$$
\begin{aligned}
S^{(2)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \Big\{ & \frac{1}{2}\dot{\varphi}^2 \left[P_{,X} + P_{,XX}\dot{\phi}^2\right] - \frac{1}{2a^2}P_{,X}(\partial\varphi)^2 \\
& + \varphi^2 \Big[\frac{1}{2}P_{,\phi\phi} + \frac{3\dot{\phi}^2 P_{,X}^2}{8\mathrm{M}_{\mathrm{P}}^2} + \frac{\dot{\phi}P_{,X}P_{,\phi}}{2\mathrm{M}_{\mathrm{P}}^2 H} + \frac{\dot{\phi}^4 P_{,X}^3 + \dot{\phi}^6 P_{,X}^2 P_{,XX}}{8\mathrm{M}_{\mathrm{P}}^4 H^2} + \frac{P_{,X}^2\dot{\phi}\ddot{\phi}}{2\mathrm{M}_{\mathrm{P}}^2 H} \\
& + \frac{PP_{,X}^2\dot{\phi}^2}{8\mathrm{M}_{\mathrm{P}}^4 H^2} + \frac{3P_{,X}P_{,XX}\dot{\phi}^3\ddot{\phi}}{2\mathrm{M}_{\mathrm{P}}^2 H} + \frac{9P_{,X}P_{,XX}\dot{\phi}^4}{8\mathrm{M}_{\mathrm{P}}^2} + \frac{P_{,X}P_{,XX}P\dot{\phi}^4}{8\mathrm{M}_{\mathrm{P}}^4 H^2} \\
& + \frac{P_{,XX}^2\dot{\phi}^5\ddot{\phi}}{4\mathrm{M}_{\mathrm{P}}^2 H} + \frac{P_{,X\phi}P_{,XX}\dot{\phi}^5}{4\mathrm{M}_{\mathrm{P}}^2 H} + \frac{P_{,X}P_{,XXX}\dot{\phi}^5\ddot{\phi}}{4\mathrm{M}_{\mathrm{P}}^2 H} + \frac{P_{,X}P_{,XX\phi}\dot{\phi}^5}{4\mathrm{M}_{\mathrm{P}}^2 H} \\
& - \frac{1}{2}P_{,X\phi}\ddot{\phi} - \frac{1}{2}P_{,XX\phi}\dot{\phi}^2\ddot{\phi} - \frac{1}{2}P_{,X\phi\phi}\dot{\phi}^2 - \frac{3}{2}P_{,X\phi}H\dot{\phi}\Big]\Big\}
\end{aligned}
$$

The speed of propagation (speed of sound) c_s of the fluctuations can be read off from the ratio of spatial to time derivative terms,

$$
c_s^2 = \frac{P_{,X}}{P_{,X} + P_{,XX}\dot{\phi}^2} \, .
$$

At cubic order, the action is given by

$$
\begin{aligned}
S^{(3)} = \int \mathrm{d}t\mathrm{d}^3x a^3 \Big\{ & \dot{\varphi}^3 \left[\frac{1}{2}\dot{\phi}P_{,XX} + \frac{1}{6}\dot{\phi}^3 P_{,XXX} \right] \\
& + \dot{\varphi}^2\varphi \Big[-\frac{\dot{\phi}P_{,X}^2}{4\mathrm{M_P^2}H} - \frac{2\dot{\phi}^3 P_{,X}P_{,XX}}{\mathrm{M_P^2}H} - \frac{\dot{\phi}^5 P_{,X}P_{,XXX}}{4\mathrm{M_P^2}H} + \frac{1}{2}P_{,X\phi} + \frac{1}{2}\dot{\phi}^2 P_{,XX\phi} \Big] \\
& + \dot{\varphi}\varphi^2 \Big[\frac{\dot{\phi}^3 P_{,X}^3}{4\mathrm{M_P^4}H^2} - \frac{\dot{\phi}^2 P_{,X}P_{,X\phi}}{2\mathrm{M_P^2}H} + \frac{1}{2}\dot{\phi}P_{,X\phi\phi} + \frac{5\dot{\phi}^5 P_{,X}^2 P_{,XX}}{8\mathrm{M_P^4}H^2} \\
& \qquad - \frac{\dot{\phi}^4 P_{,X}P_{,XX\phi}}{2\mathrm{M_P^2}H} + \frac{\dot{\phi}^7 P_{,X}^2 P_{,XXX}}{8\mathrm{M_P^4}H^2} \Big] \\
& + \varphi^3 \Big[\frac{1}{6}P_{,\phi\phi\phi} + \frac{\dot{\phi}P_{,X}P_{,\phi\phi}}{4\mathrm{M_P^2}H} + \frac{3\dot{\phi}^3 P_{,X}^3}{8\mathrm{M_P^4}H} - \frac{\dot{\phi}^5 P_{,X}^4}{16\mathrm{M_P^6}H^3} + \frac{\dot{\phi}^4 P_{,X}^2 P_{,X\phi}}{8\mathrm{M_P^4}H^2} \\
& \qquad - \frac{\dot{\phi}^3 P_{,X}P_{,X\phi\phi}}{4\mathrm{M_P^2}H} - \frac{\dot{\phi}^7 P_{,X}^3 P_{,XX}}{8\mathrm{M_P^6}H^3} + \frac{\dot{\phi}^6 P_{,X}^2 P_{,XX\phi}}{8\mathrm{M_P^4}H^2} - \frac{\dot{\phi}^9 P_{,X}^3 P_{,XXX}}{48\mathrm{M_P^6}H^3} \Big] \\
& + \frac{\dot{\phi}P_{,X}}{4a^2H}\varphi\left[\partial^2\beta\partial^2\beta - \beta_{,ij}\beta^{,ij}\right] + \varphi^2\partial^2\beta\Big[\frac{\dot{\phi}^2 P_{,X}^2}{4\mathrm{M_P^2}aH} + \frac{1}{2a}\dot{\phi}P_{,X\phi} - \frac{P_{,X}P_{,XX}\dot{\phi}^4}{4\mathrm{M_P^2}aH} \Big] \\
& + \varphi(\partial\varphi)^2\Big[-\frac{\dot{\phi}P_{,X}^2}{4\mathrm{M_P^2}a^2H} - \frac{1}{2a^2}P_{,X\phi} + \frac{P_{,X}P_{,XX}\dot{\phi}^3}{4\mathrm{M_P^2}a^2H} \Big] \\
& - \dot{\varphi}\partial\varphi\partial\beta\frac{1}{a}\left[P_{,X} + \dot{\phi}^2 P_{,XX}\right] - \frac{1}{2a^2}\dot{\phi}P_{,XX}\dot{\varphi}(\partial\varphi)^2 \Big\}
\end{aligned}
$$

We are now ready to analyze various special cases of interest.

Ghost Condensate Bounces:

The onset of ghost condensations occurs when

$$P_{,X} = 0 \quad \Rightarrow \quad c_s = 0$$

Between the two times that this occurs, the NEC is violated and the bounce occurs.

During the period when the NEC is violated, the bounce energy density is small since

the Friedmann equation is

$$\rho = 3M_P^3 H^2$$

and H=0 at the exact bounce time. Also, before and after the NEC is violated, we do not expect any troublesome effects. $\Longrightarrow$ The two times at which $P_{,X} = 0$ are the most important. At these times, we find

$$\begin{aligned}
S^{(2+3)}|_{P_{,X}=0} = \int \mathrm{d}t\mathrm{d}^3x a^3 \Big\{ & \frac{1}{2}\dot{\varphi}^2 \left[P_{,XX}\dot{\phi}^2 \right] + \dot{\varphi}\varphi [\dot{\phi}P_{,X\phi}] + \varphi^2 [\frac{1}{2}P_{,\phi\phi}] \\
& + \dot{\varphi}^3 \left[\frac{1}{2}\dot{\phi}P_{,XX} + \frac{1}{6}\dot{\phi}^3 P_{,XXX} \right] \\
& + \dot{\varphi}^2\varphi \left[\frac{1}{2}P_{,X\phi} + \frac{1}{2}\dot{\phi}^2 P_{,XX\phi} \right] + \dot{\varphi}\varphi^2 \left[\frac{1}{2}\dot{\phi}P_{,X\phi\phi} \right] + \varphi^3 \left[\frac{1}{6}P_{,\phi\phi\phi} \right] \\
& + \varphi^2\partial^2\beta [\frac{2}{a}\dot{\phi}P_{,X\phi}] - \varphi(\partial\varphi)^2 [\frac{1}{2a^2}P_{,X\phi}] - \dot{\varphi}\partial\varphi\partial\beta\frac{1}{a}[\dot{\phi}^2 P_{,XX}] \\
& - \frac{1}{2a^2}\dot{\phi}P_{,XX}\dot{\varphi}(\partial\varphi)^2 \Big\}
\end{aligned}$$

while the constraint is given by

$$\frac{1}{a}\partial^2\beta\,|_{P_{,X}=0} = \left(\frac{1}{2\mathrm{M}_{\mathrm{P}}^2 H}P_{,\phi} - \frac{\dot{\phi}^2}{2\mathrm{M}_{\mathrm{P}}^2 H}P_{,X\phi}\right)\varphi - \left(\frac{\dot{\phi}^3}{2\mathrm{M}_{\mathrm{P}}^2 H}P_{,XX}\right)\dot{\varphi}$$
$$= -\frac{\rho_{,\phi}}{2\mathrm{M}_{\mathrm{P}}^2 H}\varphi - \left(\frac{\dot{\phi}^3}{2\mathrm{M}_{\mathrm{P}}^2 H}P_{,XX}\right)\dot{\varphi}\,.$$

The dominant terms in the action are the $\dot{\varphi}^2$ and $\dot{\varphi}^3$ terms. With the field redefinition

$$\varphi \equiv (P_{,XX}\dot{\phi}^2)^{-1/2}\chi$$

the dominant quadratic and cubic terms can be written as

$$S^{(2+3)} \supset \int \mathrm{d}t\mathrm{d}^3x a^3\Big\{\frac{1}{2}\dot{\chi}^2 + \frac{1}{2}\frac{P_{,XX}\dot{\phi} + \frac{1}{3}P_{,XXX}\dot{\phi}^3}{(P_{,XX}\dot{\phi}^2)^{3/2}}\dot{\chi}^3 + \cdots\Big\}$$
$$\equiv \int \mathrm{d}t\mathrm{d}^3x\frac{1}{2}a^3\Big\{\dot{\chi}^2 + \frac{1}{\Lambda^2}\dot{\chi}^3 + \cdots\Big\}.$$

We can then read off the strong coupling scale Λ, with the result that

$$\Lambda = \frac{(P_{,XX})^{3/4}\dot{\phi}}{(P_{,XX} + \frac{1}{3}\dot{\phi}^2 P_{,XXX})^{1/2}} \approx (P_{,XX})^{1/4}\dot{\phi}\,.$$

This scale should now be compared to the energy density of the background at that time, which is $\rho = -P$. Using the condition that $P_{,X} = 0$, which implies $X = -\kappa(\phi)/(2q(\phi))$, it follows that

$$\Lambda^4 = \frac{2\kappa^2}{q}, \quad \rho = \frac{\kappa^2}{4q} + V(\phi)$$

where the functions κ, q and V are evaluated at ϕ for which $P_{,X} = 0$. In the absence of a potential, we find

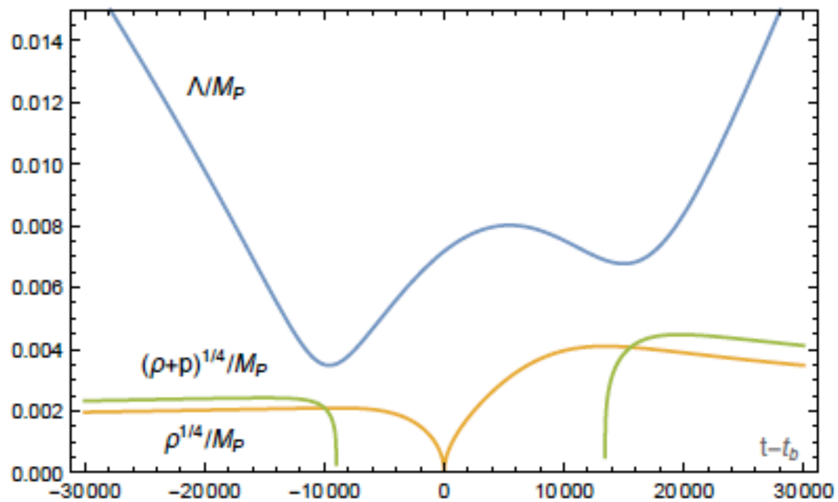
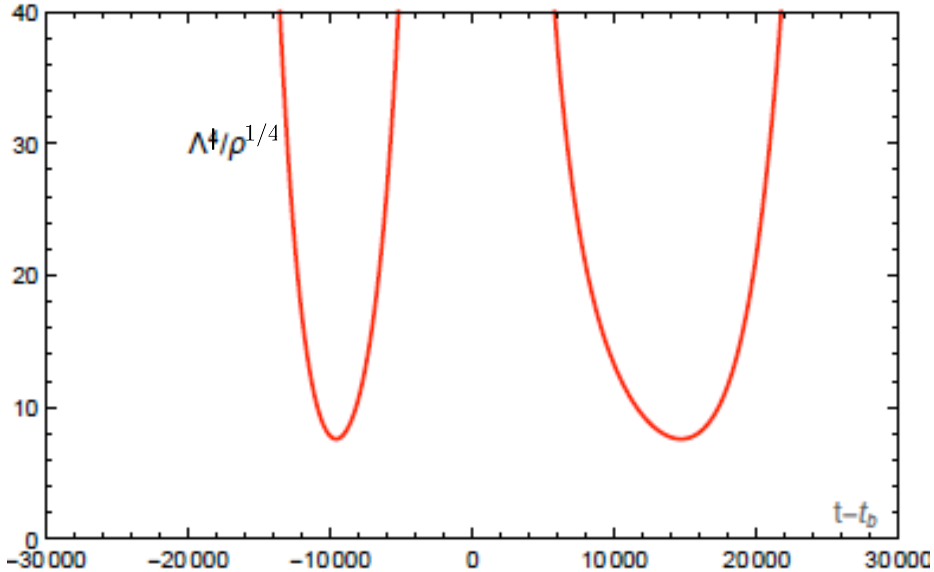


Figure 12:

Ghost condensate bounce without a potential, $V_0 = 0$. Plotted here are the strong coupling scale Λ and the energy density $\rho^{1/4}$ against physical time t, relative to the time of the bounce t_b. Also plotted is the sum of the energy density and pressure (to the quarter power). At the two moments where this quantity vanishes the null energy condition is marginally satisfied, while in the time interval in between the NEC is violated. This plot confirms that Λ and $\rho^{1/4}$ are closest to each other precisely at the moments when the NEC starts and ends being violated.



$\Rightarrow$ $\Lambda \sim 8 \times \rho^{1/4}$

$\Rightarrow$ $\Lambda \sim 8 \times 10^{-6} \ll 10^{-4}$

$\Rightarrow$ Effective field theory is no *longer valid in the region where the fluctuation amplitudes blow up*

Figure 13: Plot of the ratio of strong coupling scale to background energy density (to the quarter power) against physical time. As expected from our analytical treatment, we see that at the moments where the NEC starts and ends being violated, this ratio reduces to a factor of 8. Thus the bouncing background solution lies within the regime of validity of the effective theory, while dangerous short wavelength modes lie outside.

It is important to note, however, that

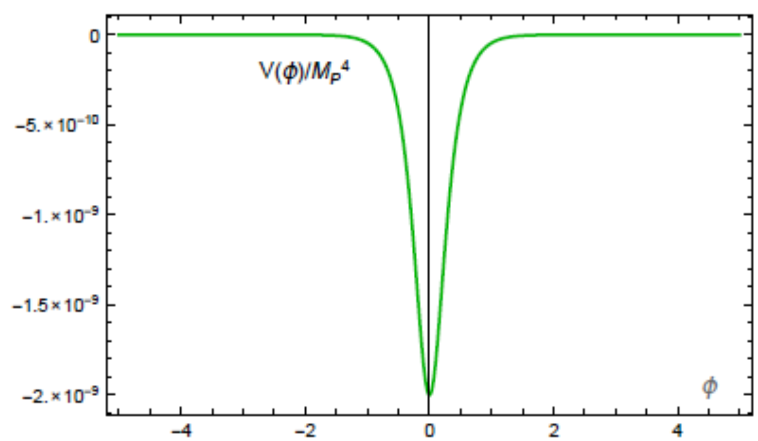
a negative potential during the bounce phase increases the separation between the energy density of the background and the strong coupling scale. The two scales can, in fact, be separated by an arbitrarily large factor – provided the potential can approach close to the minimally allowed value of $V_{min} = -\kappa^2/(4q)$.

It is interesting to note that a negative potential is natural in ekpyrotic models. Up to now it was typically assumed that this negative potential would be non-vanishing during the contracting phase–but rapidly vanish before, and be irrelevant at, the moment of the bounce. Our results suggest a new perspective, in that we see here that the potential can still play an important role during the bounce phase. This has implications for ekpyrotic model building.

As a concrete example, we leave all previous functions and coefficients the same but change the potential function to

$$V(\phi) = -\frac{2V_0}{e^{-\sqrt{20}\frac{\phi}{M_P}} + e^{\sqrt{20}\frac{\phi}{M_P}}} ,$$

where $V_0 = 0.2 \times 10^{-8}\ M_P^4$. This potential is plotted below.



Again plotting the strong coupling scale and the background energy density, but now for the above non-zero potential, we find

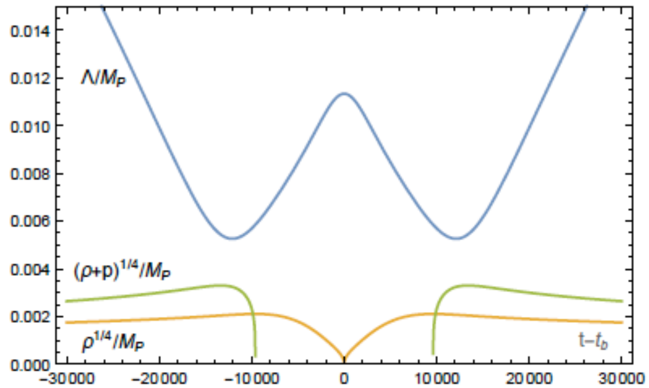
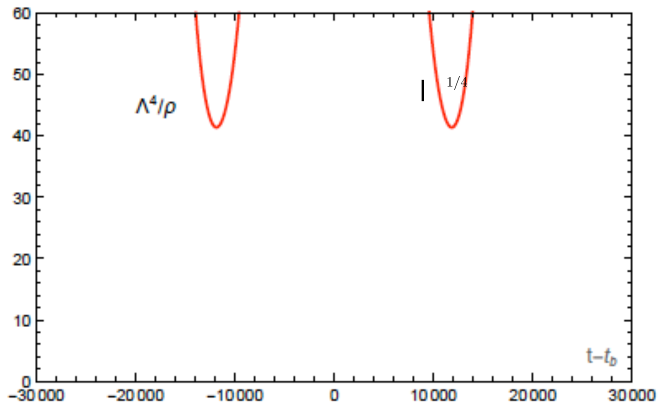


Figure 14: The zeroes of the curve plotting $(\rho + p)^{1/4}$ indicate the start and end of the NEC violating phase.



$\Rightarrow$ $\Lambda \sim 40 \times \rho^{1/4}$

$\Rightarrow$ $\Lambda \sim 4 \times 10^{-5} < 10^{-4}$

$\Rightarrow$ Effective field theory is no *longer valid in the region where the fluctuation amplitudes blow up*

Figure 15: When a negative potential is included, the background energy density and the strong coupling scale are further separated from each other. For $V_0 = 0.2 \times 10^{-8}\ M_P^4$ the ratio $\Lambda/\rho^{1/4}$ always remains above a factor of about 40. This implies that the background solution lies more comfortably inside the regime of validity of the effective theory, compared to the case where no potential is present during the bounce. But closer to blow up mass.