Bulk Viscous Universe in Modified Gravity

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Motivation

The most challenging problem in cosmology is the accelerating expansion of the universe which is based on the recent astrophysical data explaining the universe is spatially flat and an invisible cosmic fluid called dark energy with a hugely negative pressure which is responsible for this expansion (Riess et al. 1998; Perlmutter et al. 1999).
Therefore, dark energy and related topics are important subjects to study in theoretical physics and cosmology.

Basically there are two approaches:

- One is to propose suitable forms for the energy momentum tensor $T_{\mu\nu}$ in the Einstein's equation, having a negative pressure, which culminate in the proposal of an exotic energy called dark energy.
- The second approach is to modify the geometry of the space time in the Einstein's equation.

Motivation Continued

In most of the cosmological models, the content of the universe has been considered as a perfect fluid. It is important to investigate more realistic models that take into account dissipative processes due to viscosity. In a homogeneous and isotropic universe bulk viscosity is the unique viscous effect capable to modify the background dynamics. It is known that when neutrino decoupling occurred, the matter behaved like a viscous fluid in the early stage of the universe. There are remarkable cosmological applications of viscous imperfect fluids already in (Israel and Vardalas (1970)).In the context of inflation, it has been known since long time ago that an imperfect fluid with bulk viscosity can produce an acceleration without the need of a cosmological constant or some scalar field.

In the context of inflation, many authors found that the bulk viscous fluids are capable of producing acceleration of the universe (Padmanabhan and Chitre (1987), Waga et al. (1986), Cheng (1991)). This idea was extended to explain the late acceleration of the universe

In the present work, author study FRW model with bulk viscosity in modified $f(R, T)$ gravity theory and investigate the effects of bulk viscosity in explaining the early and late time acceleration of the universe.

$f(R, T)$ gravity

Recently, the modified theory of gravity has become one of the most popular candidates to understand the problem of dark energy. The $f(R, T)$ theory is a modified theory of gravity, in which the Einstein-Hilbert Lagrangian, i.e., R is replaced by an arbitrary function of the scalar curvature R and the trace T of energy-momentum tensor. In (Harko et al. 2011), the following modification of Einstein's theory is proposed in the unit $8\pi G = 1 = c$.

The action for the modified theories of gravity takes the following form

$$S = \frac{1}{2}\int [f(R,T) + 2L_m]\sqrt{-g}\,d^4x \tag{1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $f(R,T)$ is an arbitrary function of the Ricci scalar, R, and of the trace T of the stress-energy tensor of the matter, $T_{\mu\nu}$. L_m is the matter Lagrangian density, and define the stress-energy tensor of matter as Landau and Lifshitz (1998).

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \tag{2}$$

and its trace by $T = g^{\mu\nu}T_{\mu\nu}$, respectively. By assuming that the Lagrangian density L_m of matter depends only on the metric tensor components $g_{\mu\nu}$, and not on its derivatives, we obtain

$$T_{\mu\nu} = g_{\mu\nu}L_m - 2\frac{\partial L_m}{\partial g^{\mu\nu}} \tag{3}$$

By varying the action S of the gravitational field with respect to the metric tensor components $g^{\mu\nu}$ provides the following relationship

$$\begin{aligned}\delta S &= \frac{1}{2}\int\Big[f_R(R,T)\delta R + f_T(R,T)\frac{\delta T}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \\ &- \frac{1}{2}g_{\mu\nu}f(R,T)\delta g^{\mu\nu} + 2\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}\Big]\sqrt{-g}d^4x \quad (4)\end{aligned}$$

where $f_R(R,T) = \frac{\partial f(R,T)}{\partial R}$ and $f_T(R,T) = \frac{\partial f(R,T)}{\partial T}$, respectively.

For the variation of Ricci scalar, one can obtain

$$\begin{aligned} \delta R &= \delta(g^{\mu\nu} R_{\mu\nu}) \\ &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}(\bigtriangledown_\lambda \delta\Gamma^\lambda_{\mu\nu} - \bigtriangledown_\nu \delta\Gamma^\lambda_{\mu\lambda}) \end{aligned} \tag{5}$$

where $\bigtriangledown_\lambda$ is the covariant derivative with respect to the symmetric connection Γ associated to the metric g. The variation of the Christoffel symbols yields

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\bigtriangledown_\mu \delta g_{\nu\alpha} + \bigtriangledown_\nu \delta g_{\mu\alpha} - \bigtriangledown_\alpha \delta g_{\nu\mu}) \tag{6}$$

and the variation of the Ricci scalar provides the expression

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \bigtriangledown_\mu \bigtriangledown_\nu \delta g^{\mu\nu} \tag{7}$$

where $\Box \equiv \bigtriangledown_\mu \bigtriangledown_\mu$ is the d'Alembert operator.

Therefor, for the variation of the action of the gravitational field is as follows

$$\begin{aligned}
\delta S &= \frac{1}{2}\int \Big[f_R(R,T) R_{\mu\nu}\delta g^{\mu\nu} + f_R(R,T) g_{\mu\nu}\Box\delta g^{\mu\nu} \\
&- f_R(R,T)\bigtriangledown_\mu \bigtriangledown_\nu \delta g^{\mu\nu} + f_T(R,T)\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \\
&- \frac{1}{2} g_{\mu\nu} f(R,T)\delta g^{\mu\nu} + 2\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}\Big]\sqrt{-g}\,d^4x \quad (8)
\end{aligned}$$

$f(R, T)$ gravity Continued

The variation of T with respect to the metric tensor as

$$\frac{\delta(g^{\alpha\beta} T_{\alpha\beta})}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu} \tag{9}$$

where

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \tag{10}$$

After partially integrating the second and third terms in equation (8), we obtain the field equations of the $f(R, T)$ gravity model as

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R, T) \\ = T_{\mu\nu} - f_T(R, T)T_{\mu\nu} - f_T(R, T)\Theta_{\mu\nu}. \quad (11)$$

Note that when $f(R, T) \equiv f(R)$, then equations (11) becomes the field equations of $f(R)$ gravity.

$f(R, T)$ gravity Continued

By contracting Eq. (11) gives the following relation between the Ricci scalar R and the trace T of the stress-energy tensor,

$$\begin{aligned} f_R(R,T)R + 3\Box f_R(R,T) - 2f(R,T) \\ = T - f_T(R,T)T - f_T(R,T)\Theta, \end{aligned} \tag{12}$$

where $\Theta = \Theta^{\mu}_{\mu}$.

By eliminating the term $\Box f_R(R, T)$ between Eqs. (11) and (12), the gravitational field equations can be written in the form

$$\begin{aligned} & f_R(R, T)\left(R_{\mu\nu} - \frac{1}{3}Rg_{\mu\nu}\right) + \frac{1}{6}f(R, T)g_{\mu\nu} \\ = & \left(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}\right) - f_T(R, T)\left(T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}\right) \\ & -f_T(R, T)\left(\Theta_{\mu\nu} - \frac{1}{3}\Theta g_{\mu\nu}\right) + \nabla_\mu \nabla_\nu f_R(R, T). \end{aligned} \tag{13}$$

Taking into account the covariant divergence of Eq. (11), with the use of the following mathematical identity

$$\nabla^{\mu}\left[f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f_R(R,T)\right] \equiv 0 \tag{14}$$

where $f(R, T)$ is an arbitrary function of the Ricci scalar R and of the trace of the stress-energy tensor T , we obtain for the divergence of the stress-energy tensor $T_{\mu\nu}$ the equation

$$\nabla^{\mu}T_{\mu\nu} = \frac{f_T(R,T)}{1-f_T(R,T)}[(T_{\mu\nu}+\Theta_{\mu\nu})\nabla^{\mu}\ln f_T(R,T) + \nabla^{\mu}\Theta_{\mu\nu}]. \tag{15}$$

Next we consider the calculation of the tensor $\Theta_{\mu\nu}$, once the matter Lagrangian is known. From Eq. (3) we obtain first

$$\begin{aligned}
\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} &= \frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} L_m + g_{\alpha\beta}\frac{\partial L_m}{\partial g^{\mu\nu}} - 2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}} \\
&= \frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} L_m + \frac{1}{2} g_{\alpha\beta} g_{\mu\nu} L_m - \frac{1}{2} g_{\alpha\beta} T_{\mu\nu} \\
&- 2\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \qquad (16)
\end{aligned}$$

From the condition $g_{\alpha\sigma}g^{\sigma\beta} = g_{\alpha}^{\beta}$, we have

$$\frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} = -g_{\alpha\sigma}g_{\beta\gamma}\delta_{\mu\nu}^{\sigma\gamma}, \tag{17}$$

where $\delta_{\mu\nu}^{\sigma\gamma} = \frac{\delta g^{\sigma\gamma}}{deltag^{\mu\nu}}$ is the generalized Kronecker symbol. Therefore, for $\Theta_{\mu\nu}$ we find

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{18}$$

$f(R, T)$ gravity Continued

The equations of $f(R, T)$ gravity are much more complicated with respect to the ones of General Relativity even for FRW metric. For this reason many possible form of $f(R, T)$, for example,

- $f(R, T) = R + 2f(T)$,
- $f(R, T) = \mu f_1(R) + \nu f_2(T)$,
- $f(R, T) = Rf(T)$

where $f_1(R)$ and $f_2(T)$ are arbitrary functions of R and T , and μ and ν are real constants, respectively, have been proposed to solve the modified field equations.

In this work we consider the following simplest particular model:

$$f(R,T) = R + 2f(T) \tag{19}$$

where $f(T)$ is an arbitrary function of the trace of the stress-energy tensor of matter. The gravitational field equations immediately follow from Eq. (11), and are given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= T_{\mu\nu} - 2f^{'}(T)T_{\mu\nu} \\ &- 2f^{'}(T)\Theta_{\mu\nu} + f(T)g_{\mu\nu}, \end{aligned} \tag{20}$$

which is considered as the field equations of $f(R,T)$ gravity.

Here, a prime stands for derivative of $f(T)$ with respect to T and $f(T) = \lambda T$, where λ is constant. If the matter source is a perfect fluid , $\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}$, then the field equations become

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= T_{\mu\nu} + 2f^{'}(T)T_{\mu\nu} \\ &+ [2pf^{'}(T) + f(T)]g_{\mu\nu}, \end{aligned} \tag{21}$$

The Metric and Field Equations Continued

Here we assume a spatially homogeneous and isotropic space time,

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (22)$$

where $a(t)$ represent the cosmic scale factor. The constant k defined curvature of space so, $k = 0, 1$ and -1 represents flat, closed and open universe respectively. Our interest in this work is the first case mainly $k = 0$.

In comoving coordinates, the components of the four-velocity vector is defined as $u^{\mu} = (1,0,0,0)$. Here we consider the source of gravitation as the bulk viscous fluid. Therefore the energy momentum tensor is given by

$$T_{\mu\nu} = (\bar{p} + \rho)u_{\mu}u_{\nu} - \bar{p}g_{\mu\nu} \tag{23}$$

and

$$\bar{p} = p - \xi\theta \tag{24}$$

or

$$\bar{p} = p - 3\xi H \tag{25}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, ρ is the energy density, ξ is the coefficient of bulk viscosity, θ is the scalar of expansion, $\bar{p}$ is the total pressure and p is the proper pressure.

The Metric and Field Equations Continued

Therefore, the Lagrangian density may be chosen as $L_m = -\bar{p}$, and the tensor $\Theta_{\mu\nu} = -2T_{\mu\nu} - \bar{p}g_{\mu\nu}$.
Hence, the field equations (21) for bulk viscous fluid become

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= T_{\mu\nu} + 2f^{'}(T)T_{\mu\nu} \\ &+ [2\bar{p}f^{'}(T) + f(T)]g_{\mu\nu} \end{aligned} \tag{26}$$

The gravitational field equations are given by

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho + 2\lambda(\rho + \bar{p}) + \lambda T \tag{27}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\bar{p} + \lambda T \tag{28}$$

where $T = \rho - 3\bar{p}$.
The equation of continuity is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \bar{p}) = 0 \tag{29}$$

Exact Solution of the Field Equations

The field equations (27) to (29) (by substituting $H = \frac{\dot{a}}{a}$ becomes,

$$3H^2 = \rho + 2\lambda(\rho + \bar{p}) + \lambda T \tag{30}$$

$$2\dot{H} + 3H^2 = -\bar{p} + \lambda T \tag{31}$$

and

$$\dot{\rho} + 3H(\rho + \bar{p}) = 0 \tag{32}$$

Subtract (30) from (31), yields

$$2\dot{H} + (1 + 2\lambda)(p + \rho) - 3(1 + 2\lambda)\xi H = 0 \tag{33}$$

The equation of state (EoS) connecting p and ρ can be chosen in the following form

$$p = (\gamma - 1)\rho \tag{34}$$

where γ is constant known as the EoS parameter lying in the rage $0 \leq \gamma \leq 2$. we assume the general form of bulk viscous coefficient (Ren and Meng 2006)

$$\begin{aligned} \xi &= \xi_0 + \xi_1 \frac{\dot{a}}{a} + \xi_2 \frac{\ddot{a}}{\dot{a}} \\ &= \xi_0 + \xi_1 H + \xi_2 \left(\frac{\dot{H}}{H} + H \right) \end{aligned} \tag{35}$$

where ξ_0, ξ_1 and ξ_2 are constants.

- **Case-I**
  $\xi_1 = \xi_2 = 0$, so $\xi = \xi_0$, a constant.
- **Case-II**
  $\xi_2 = 0$, so $\xi = \xi_0 + \xi_1 \frac{\dot{a}}{a}$, depending only on velocity of the expansion of the universe and not on its acceleration.
- **Case-III**
  ξ_0, ξ_1, ξ_2 all are non zero, so that $\xi = \xi_0 + \xi_1 \frac{\dot{a}}{a} + \xi_2 \frac{\ddot{a}}{\dot{a}}$, depending on both the velocity and acceleration of the expansion of the universe.

Exact Solution of the Field Equations Continued

Using (25), (34) and (35) into (30), we get

$$\rho = \frac{3H\left[(1-\lambda(\xi_1+\xi_2))H - \lambda\xi_2\frac{\dot{H}}{H} - \lambda\xi_0\right]}{1+4\lambda-\lambda\gamma} \tag{36}$$

Using (34) and (36) into (33), we get

$$\begin{aligned}&\left[2-\frac{3\lambda\xi_0(1+2\lambda)}{(1+4\lambda-\lambda\gamma)}-3\lambda\xi_2\right]\dot{H}\\ &-\left[\frac{3\lambda\xi_0\gamma(1+2\lambda)}{(1+4\lambda-\lambda\gamma)}+3\xi_0(1+2\lambda)\right]H\\ &+\left[\frac{(1+2\lambda)3\gamma}{(1+4\lambda-\lambda\gamma)}(1-\lambda(\xi_1+\xi_2))\right.\\ &\left.-3(1+2\lambda)(\xi_1+\xi_2)\right]H^2=0\end{aligned} \qquad (37)$$

- **Solution with constant bulk viscous coefficient**

Assume $\xi_1 = \xi_2 = 0$, then equation (36) reduces to

$$\rho = \frac{3H^2 - 3\lambda\xi_0 H}{(1 + 4\lambda - \lambda\gamma)} \tag{38}$$

Substituting (38) and (34) in (33), we get

$$\dot{H} + \frac{3\gamma(1 + 2\lambda)H}{2(1 + 4\lambda - \lambda\gamma)}\left[H - \frac{\xi_0(1 + 4\lambda)}{\gamma}\right] = 0 \tag{39}$$

- **Case:-I Solution for $\gamma \neq 0$**

Solving (39) for $\gamma \neq 0$. We find

$$H = \frac{e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{1+4\lambda-\lambda\gamma}t}}{c_0 + \frac{\gamma}{(1+4\lambda)\xi_0} e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{1+4\lambda-\lambda\gamma}t}} \tag{40}$$

Here c_0 is a constant of integration. Using $H = \frac{\dot{a}}{a}$, the scale factor in terms of t is given by

$$a = c_1 \left[c_0 + \frac{\gamma}{(1+4\lambda)\xi_0} e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{1+4\lambda-\lambda\gamma}t} \right]^{\frac{2(1+4\lambda-\lambda\gamma)}{3\gamma(1+2\lambda)}} \tag{41}$$

where $c_1 > 0$ is integration constant. The above scale factor may be rewritten as

$$a(t) = a_0\left[1 + \frac{\gamma H_0}{(1+4\gamma)\xi_0}\left(e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{2(1+4\lambda-\lambda\gamma)}t} - 1\right)\right]^{\frac{2(1+4\lambda-\lambda\gamma)}{3\gamma(1+2\lambda)}} \tag{42}$$

The energy density can be obtained as

$$\begin{aligned}\rho \quad &= \quad \frac{3H_0}{(1+4\lambda-\lambda\gamma)}\left[\frac{e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{1+4\lambda-\lambda\gamma}t}}{1+\frac{\gamma H_0}{(1+4\gamma)\xi_0}\left(e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{2(1+4\lambda-\lambda\gamma)}t}-1\right)}\right] \\ &\times \quad \left[\frac{H_0 e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{2(1+4\lambda-\lambda\gamma)}t}}{1+\frac{\gamma H_0}{(1+4\lambda)\xi_0}\left(e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{2(1+4\lambda-\lambda\gamma)}t}-1\right)} - \lambda\xi_0\right] \end{aligned} \tag{43}$$

Exact Solution of the Field Equations Continued

For $0 \leq \gamma \leq 2$, viscous solution satisfies the dominant energy condition, i. e. $p + \rho \geq 0$. If $\gamma < 0$ we have a big rip singularity at a finite cosmic time

$$t_{br} = \frac{2(1+4\lambda-\lambda\gamma)}{3(1+2\lambda)(1+4\lambda)\xi_0} \ln\left(1 - \frac{(1+4\lambda)\xi_0}{\gamma H_0}\right) > t_0. \tag{44}$$

The energy density grows up to infinity at a finite time $t > t_0$, which leads to a big rip singularity characterized by the scale factor and Hubble parameter blowing up to infinity at this finite time. Therefore, there are cosmological models with viscous fluid which present in the development of this sudden future singularity.

The deceleration parameter is given by

$$q = \frac{\frac{3(1+2\lambda)}{2(1+4\lambda-\lambda\gamma)}\left[\gamma - \frac{(1+4\lambda)\xi_0}{H_0}\right]}{e^{\frac{3(1+2\lambda)(1+4\lambda)\xi_0}{2(1+4\lambda-\lambda\gamma)}t}} - 1 \tag{45}$$

which is time dependent in contrast to perfect fluid.

Conclusion

Thus the constant bulk viscous coefficient generates time dependent deceleration parameter (q) which may also describe the transition phases of the universe along with deceleration or acceleration of the universe.

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THANK YOU