# Reconstruction of inflationary models from the power spectrum of primordial density perturbations

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# Inflation

The inflationary scenario is based on the two main cornerstone ideas (hypothesis):

1. Existence of inflation (or, quasi-de Sitter stage) – a stage of accelerated, close to exponential expansion of our Universe in the past preceding the hot Big Bang with decelerated, power-law expansion.

2. The origin of all inhomogeneities in the present Universe is the effect of gravitational creation of particles and field fluctuations during inflation from the adiabatic vacuum (no-particle) state for Fourier modes covering all observable range of scales (and possibly somewhat beyond).

# Present status of inflation

Now we have numbers: P. A. R. Ade et al., arXiv:1502.01589

The primordial spectrum of scalar perturbations has been measured and its deviation from the flat spectrum $n_s = 1$ in the first order in $|n_s - 1| \sim N^{-1}$ has been discovered (using the multipole range $\ell > 40$):

$$< \zeta^2(\mathbf{r}) >= \int \frac{P_\zeta(k)}{k}\, dk, \quad P_\zeta(k) = \left(2.21^{+0.07}_{-0.08}\right) 10^{-9} \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$$k_0 = 0.05 \mathrm{Mpc}^{-1}, \quad n_s - 1 = -0.035 \pm 0.005$$

Two fundamental observational constants of cosmology in addition to the three known ones (baryon-to-photon ratio, baryon-to-matter density and the cosmological constant). Existing inflationary models can predict (and predicted, in fact) one of them, namely $n_s - 1$, relating it finally to $\ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$.

# From "proving" inflation to using it as a tool

Simple (one-parameter, in particular) models may be good in the first approximation (indeed so), but it is difficult to expect them to be absolutely exact, small corrections due to new physics should exist (indeed so).

Present status of inflation: transition from "proving" it in general and testing some of its simplest models to applying the inflationary paradigm to investigate particle physics at super-high energies and the actual history of the Universe in the remote past using real observational data on $n_s(k) - 1$ and $r(k)$.

The reconstruction approach – determining curvature and inflaton potential from observational data.

The most important quantities:
1) for classical gravity – $H, \dot{H}$
2) for super-high energy particle physics – $m^2_{infl}$.

# Physical scales related to inflation

"Naive" estimate where I use the reduced Planck mass $\tilde{M}_{Pl} = (8\pi G)^{-1}$.

I. Curvature scale

$$H \sim \sqrt{P_\zeta}\tilde{M}_{Pl} \sim 10^{14}\mathrm{GeV}$$

II. Inflaton mass scale

$$|m_{infl}| \sim H\sqrt{|1-n_s|} \sim 10^{13}\mathrm{GeV}$$

New range of mass scales significantly less than the GUT scale.

# Outcome of inflation

In the super-Hubble regime in the coordinate representation:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dx^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\zeta(\mathbf{r})\delta_{lm} + \sum_{a=1}^{2} g^{(a)}(\mathbf{r})\, e_{lm}^{(a)}$$

$$e_l^{l(a)} = 0, \; g^{(a)}_{\;,l}\, e_m^{l(a)} = 0, \; e_{lm}^{(a)}\, e^{lm(a)} = 1$$

$\zeta$ describes primordial scalar perturbations, $g$ – primordial tensor perturbations (primordial gravitational waves (GW)).

In fact, metric perturbations $h_{lm}$ are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in $\zeta,\ g$).

Remaining quantum coherence: deterministic correlation between $\mathbf{k}$ and $-\mathbf{k}$ modes - shows itself in the appearance of acoustic oscillations (primordial oscillations in case of GW).

# Dynamical origin of scalar perturbations

Local duration of inflation in terms of $N_{tot} = \ln\left(\frac{a(t_{fin})}{a(t_{in})}\right)$ is different in different points of space: $N_{tot} = N_{tot}(\mathbf{r})$. Then

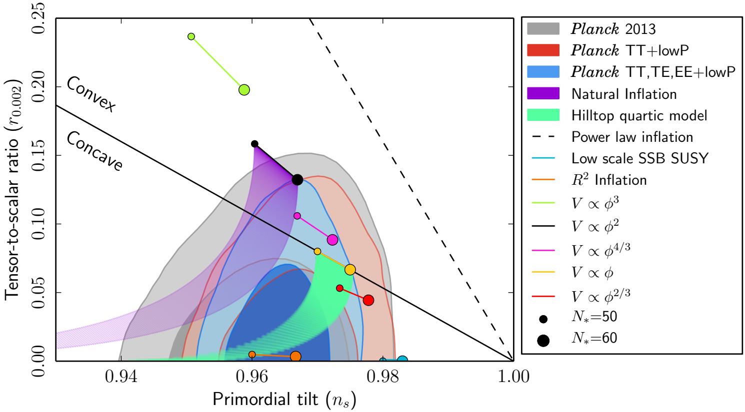
$$\zeta(\mathbf{r}) = \delta N_{tot}(\mathbf{r})$$

Correct generalization to the non-linear case: the space-time metric after the end of inflation at super-Hubble scales

$$ds^2 = dt^2 - a^2(t)e^{2N_{tot}(\mathbf{r})}(dx^2 + dy^2 + dz^2)$$

First derived in A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982) in the case of one-field inflation.

# Comparison with some simple models



# FLRW dynamics with a scalar field

In the absence of spatial curvature and other matter:

$$H^2 = \frac{\kappa^2}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

where $\kappa^2 = 8\pi G$ ($\hbar = c = 1$).

# Reduction to the first order equation

It can be reduced to the first order Hamilton-Jacobi-like equation for $H(\phi)$. From the equation for $\dot{H}$, $\frac{dH}{d\phi} = -\frac{\kappa^2}{2}\dot{\phi}$. Inserting this into the equation for $H^2$, we get

$$\frac{2}{3\kappa^2}\left(\frac{dH}{d\phi}\right)^2 = H^2 - \frac{\kappa^2}{3}V(\phi)$$

Time dependence is determined using the relation

$$t = -\frac{\kappa^2}{2}\int\left(\frac{dH}{d\phi}\right)^{-1} d\phi$$

However, during oscillations of $\phi$, $H(\phi)$ acquires non-analytic behaviour of the type $const + \mathcal{O}(|\phi - \phi_1|^{3/2})$ at the points where $\dot{\phi} = 0$, and then the correct matching with another solution is needed.

# Inflationary slow-roll dynamics

Slow-roll occurs if: $|\ddot{\phi}| \ll H|\dot{\phi}|$, $\dot{\phi}^2 \ll V$, and then $|\dot{H}| \ll H^2$.

Necessary conditions: $|V'| \ll \kappa V$, $|V''| \ll \kappa^2 V$. Then

$$H^2 \approx \frac{\kappa^2 V}{3}, \quad \dot{\phi} \approx -\frac{V'}{3H}, \quad N \equiv \ln \frac{a_f}{a} \approx \kappa^2 \int_{\phi_f}^{\phi} \frac{V}{V'}\, d\phi$$

First obtained in A. A. Starobinsky, Sov. Astron. Lett. 4, 82 (1978) in the $V = \frac{m^2\phi^2}{2}$ case and for a bouncing model.

# Spectral predictions of the one-field inflationary scenario in GR

Scalar (adiabatic) perturbations:

$$P_\zeta(k) = \frac{H_k^4}{4\pi^2\dot{\phi}^2} = \frac{GH_k^4}{\pi|\dot{H}|_k} = \frac{128\pi G^3 V_k^3}{3V_k'^2}$$

where the index $k$ means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$. Through this relation, the number of e-folds from the end of inflation back in time $N(t)$ transforms to $N(k) = \ln\frac{k_f}{k}$ where $k_f = a(t_f)H(t_f)$, $t_f$ denotes the end of inflation.

The spectral slope

$$n_s(k) - 1 \equiv \frac{d\ln P_\zeta(k)}{d\ln k} = \frac{1}{\kappa^2}\left(2\frac{V_k''}{V_k} - 3\left(\frac{V_k'}{V_k}\right)^2\right)$$

Tensor perturbations (A. A. Starobinsky, JETP Lett. 50, 844 (1979)):

$$P_g(k) = \frac{16GH_k^2}{\pi}; \quad n_g(k) \equiv \frac{d \ln P_g(k)}{d \ln k} = -\frac{1}{\kappa^2}\left(\frac{V'_k}{V_k}\right)^2$$

The consistency relation:

$$r(k) \equiv \frac{P_g}{P_\zeta} = \frac{16|\dot{H}_k|}{H_k^2} = 8|n_g(k)|$$

Tensor perturbations are always suppressed by at least the factor $\sim 8/N(k)$ compared to scalar ones. For the present Hubble scale, $N(k_H) = (50 - 60)$.
The latest BICEP2/Keck Array/Planck upper limit: $r < 0.07$ at 95% c.f. (P. A. R. Ade et al., arXiv:1510.09217 ).

# Potential reconstruction from scalar power spectrum

In the slow-roll approximation:

$$\frac{V^3}{V'^2} = CP_\zeta(k(t(\phi))), \quad C = \frac{12\pi^2}{\kappa^6}$$

Changing variables for $\phi$ to $N(\phi)$ and integrating, we get:

$$\frac{1}{V(N)} = -\frac{\kappa^4}{12\pi^2}\int \frac{dN}{P_\zeta(N)}$$

$$\kappa\phi = \int dN \sqrt{\frac{d\ln V}{dN}}$$

Here, $N \gg 1$ stands both for $\ln(k_f/k)$ at the present time and the number of e-folds back in time from the end of inflation.
First derived in H. M. Hodges and G. R. Blumenthal, Phys. Rev. D 42, 3329 (1990).

# "Scale-free" reconstruction

Numerical coincidence between $2/N(k_H) \sim 0.04$ and $1 - n_s$.

Let us assume that it is not a coincidence but happens for all $1 \ll N \lesssim 60$:

$$P_\zeta = P_0 N^2$$

Then:

$$V = V_0 \frac{N}{N + N_0} = V_0 \tanh^2 \frac{\kappa\phi}{2\sqrt{N_0}}$$

$$r = \frac{8N_0}{N(N + N_0)}$$

$r \sim 0.003$ for $N_0 \sim 1$. From the upper limit $r < 0.07$:

$$N_0 < \frac{0.07N^2}{8 - 0.07N}$$

. $N_0 < 57$ for $N = 57$.

Another example: $P_\zeta = P_0 N^{3/2}$.

$$V(\phi) = V_0 \frac{\phi^2 + 2\phi\phi_0}{(\phi + \phi_0)^2}$$

Not bounded from below (of course, in the region where the slow-roll approximation is not valid anymore). Crosses zero linearly.

More generally, the two "aesthetic" assumptions – "no-scale" scalar power spectrum and $V \propto \phi^{2n},\ n = 1, 2...$ at the minimum of the potential – lead to $P_\zeta = P_0 N^{n+1},\ n_s - 1 = -\frac{n+1}{N}$ unambiguously. From this, only $n = 1$ is permitted by observations. Still an additional parameter appears due to tensor power spectrum – no preferred one-parameter model (if the $V(\phi) \propto \phi^2$ model is excluded).

# Inflation in $f(R)$ gravity

The simplest model of modified gravity (= geometrical dark energy) considered as a phenomenological macroscopic theory in the fully non-linear regime and non-perturbative regime.

$$S = \frac{1}{16\pi G} \int f(R)\sqrt{-g}\, d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R^{\mu}_{\mu}$$

Here $f''(R)$ is not identically zero. Usual matter described by the action $S_m$ is minimally coupled to gravity.

Vacuum one-loop corrections depending on $R$ only (not on its derivatives) are assumed to be included into $f(R)$. The normalization point: at laboratory values of $R$ where the scalaron mass (see below) $m_s \approx \mathrm{const}$.

Metric variation is assumed everywhere. Palatini variation leads to a different theory with a different number of degrees of freedom.

# Field equations

$$\frac{1}{8\pi G}\left(R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R\right) = -\left(T^\nu_{\mu\,(vis)} + T^\nu_{\mu\,(DM)} + T^\nu_{\mu\,(DE)}\right) \;,$$

where $G = G_0 = const$ is the Newton gravitational constant measured in laboratory and the effective energy-momentum tensor of DE is

$$8\pi G T^\nu_{\mu\,(DE)} = F'(R)\,R_\mu^\nu - \frac{1}{2}\,F(R)\delta_\mu^\nu + \left(\nabla_\mu\nabla^\nu - \delta_\mu^\nu\nabla_\gamma\nabla^\gamma\right)F'(R)\,.$$

Because of the need to describe DE, de Sitter solutions in the absence of matter are of special interest. They are given by the roots $R = R_{dS}$ of the algebraic equation

$$Rf'(R) = 2f(R)\;.$$

The special role of $f(R) \propto R^2$ gravity: admits de Sitter solutions with any curvature.

# Transformation to the Einstein frame and back

In the Einstein frame, free particles of usual matter do not follow geodesics and atomic clocks do not measure proper time.

From the Jordan (physical) frame to the Einstein one:

$$g^E_{\mu\nu} = f' g^J_{\mu\nu}, \quad \kappa\phi = \sqrt{\frac{3}{2}} \ln f', \quad V(\phi) = \frac{f'R - f}{2\kappa^2 f'^2}$$

where $\kappa^2 = 8\pi G$.

Inverse transformation:

$$R = \left( \sqrt{6}\kappa \frac{dV(\phi)}{d\phi} + 4\kappa^2 V(\phi) \right) \exp\left( \sqrt{\frac{2}{3}} \kappa\phi \right)$$

$$f(R) = \left( \sqrt{6}\kappa \frac{dV(\phi)}{d\phi} + 2\kappa^2 V(\phi) \right) \exp\left( 2\sqrt{\frac{2}{3}} \kappa\phi \right)$$

$V(\phi)$ should be at least $C^1$.

# Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\,, \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

# Reduction to the first order equation

In the absence of spatial curvature and $\rho_m = 0$, it is always possible to reduce these equations to a first order one using either the transformation to the Einstein frame and the Hamilton-Jacobi-like equation for a minimally coupled scalar field in a spatially flat FLRW metric, or by directly transforming the 0-0 equation to the equation for $R(H)$:

$$\frac{dR}{dH} = \frac{(R - 6H^2)f'(R) - f(R)}{H(R - 12H^2)f''(R)}$$

Analogues of large-field (chaotic) inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of $R$, namely

$$|A'(R)| \ll \frac{A(R)}{R}\ ,\ |A''(R)| \ll \frac{A(R)}{R^2}\ .$$

In particular,

$$f(R) \approx \frac{R^2}{6m^2 \ln^2(R/m^2)}$$

for $R \gg m^2$ to have the same $n_s, r$ as for $V = m^2\phi^2/2$.

Analogues of small-field (new) inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1}\ ,\ F''(R_1) \approx \frac{2F(R_1)}{R_1^2}\ .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of $R$.

# Inflation reconstruction in $f(R)$ gravity

$$f(R) = R^2 A(R)$$

$$A = const - \frac{\kappa^2}{96\pi^2} \int \frac{dN}{P_\zeta(N)}$$

$$\ln R = const + \int dN \sqrt{-\frac{2\, d \ln A}{3\, dN}}$$

Here, the additional assumptions that $P_\zeta \propto N^\beta$ and that the resulting $f(R)$ can be analytically continued to the region of small $R$ without introducing a new scale, and it has the linear (Einstein) behaviour there, leads to $\beta = 2$ and the $R + R^2$ inflationary model with $r = \frac{12}{N^2} = 3(n_s - 1)^2$ unambiguously.

For $P_\zeta = P_0 N^2$:

$$A = \frac{1}{6M^2}\left(1 + \frac{N_0}{N}\right), \quad M^2 \equiv \frac{16\pi^2 N_0 P_\zeta}{\kappa^2}$$

Two cases:

1. $N \gg N_0$ always.

$$A = \frac{1}{6M^2}\left(1 + \left(\frac{R_0}{R}\right)^{\sqrt{3/(2N_0)}}\right)$$

For $N_0 = 3/2, \ R_0 = 6M^2$ we return to the simplest $R + R^2$ inflationary model.

2. $N_0 \gg 1$.

$$A = \frac{1}{6M^2}\left(\frac{1 + \left(\frac{R_0}{R}\right)^{\sqrt{3/(2N_0)}}}{1 - \left(\frac{R_0}{R}\right)^{\sqrt{3/(2N_0)}}}\right)^2$$

# The simplest models producing the observed scalar slope

$$f(R) = R + \frac{R^2}{6M^2}$$

$$M = 2.6 \times 10^{-6} \left(\frac{55}{N}\right) M_{Pl} \approx 3.2 \times 10^{13}\,\mathrm{GeV}$$

$$n_s - 1 = -\frac{2}{N} \approx -0.036, \quad r = \frac{12}{N^2} \approx 0.004$$

$$H_{dS}(N = 55) = 1.4 \times 10^{14}\,\mathrm{GeV}$$

The same prediction from a scalar field model with $V(\phi) = \frac{\lambda\phi^4}{4}$ at large $\phi$ and strong non-minimal coupling to gravity $\xi R\phi^2$ with $\xi < 0,\ |\xi| \gg 1$, including the Brout-Englert-Higgs inflationary model.

The Lagrangian density for the simplest 1-parametric model:

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{N^2}{288\pi^2 P_\zeta(k)} R^2 = \frac{R}{16\pi G} + 5 \times 10^8 R^2$$

1. The specific case of the fourth order gravity in 4D

$$\mathcal{L} = \frac{R}{16\pi G} + AR^2 + BC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$$

for which $A \gg 1, \; A \gg |B|$.

2. Another, completely different way: a non-minimally coupled scalar field with a large negative coupling $\xi$ ($\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R \phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \; .$$

In this limit, the Higgs-like scalar tree level potential $V(\phi) = \frac{\lambda(\phi^2 - \phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi|R/\lambda$ (plus small corrections $\propto |\xi|^{-1}$).

# Conclusions

- From the scalar power spectrum $P_\zeta(k)$, it is possible to reconstruct an inflationary model both in the Einstein and $f(R)$ gravity up to one arbitrary physical constant of integration.
- Using the measured value of $n_s - 1$ and assuming a scale-free scalar power spectrum leads to the prediction that the value $r > 10^{-3}$ is well possible.
- Under the same assumptions, $r$ can be even larger and close to its present observational upper limit in two-parametric inflationary models having large, but not too large $N_0 \lesssim N$. However, this requires a moderate amount of parameter tuning.

- Even without using the observed value of $n_s - 1$, the assumptions of the absence of any new physical scale both during inflation and after it and of the model applicability up to the zero values of energy and space-time curvature distinguish the case $P_\zeta(k) \propto \ln^2(k_f/k)$ just corresponding to this slope.
- In the Einstein gravity, the simplest inflationary models permitted by observational data are two-parametric, no preferred quantitative prediction for $r$, apart from its parametric dependence on $n_s - 1$, namely, $\sim (n_s - 1)^2$ or larger.
- In the $f(R)$ gravity, the simplest model is one-parametric and has the preferred value $r = \frac{12}{N^2} = 3(n_s - 1)^2$.
- Thus, it has sense to search for primordial GW from inflation at the level $r > 10^{-3}$!