The screening Horndeski cosmologies

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Contents

Horndeski theory

Horndeski /1974/

$$L_{\mathrm{H}} = \sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5),$$

where

$$\begin{aligned}
\mathcal{L}_2 &= G_2(X,\Phi)\,, \\
\mathcal{L}_3 &= G_3(X,\Phi)\,\Box\Phi\,, \\
\mathcal{L}_4 &= G_4(X,\Phi)\,R + \partial_X G_4(X,\Phi)\,\delta^{\mu\nu}_{\alpha\beta}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi\,, \\
\mathcal{L}_5 &= G_5(X,\Phi)\,G_{\mu\nu}\nabla^{\mu\nu}\Phi - \frac{1}{6}\,\partial_X G_5(X,\Phi)\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,\nabla^\alpha_\mu\Phi\nabla^\beta_\nu\Phi\nabla^\gamma_\rho\Phi\,,
\end{aligned}$$

with $X \equiv -\frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi$, $\quad \delta^{\lambda\rho}_{\nu\alpha} = 2!\,\delta^\lambda_{[\nu}\delta^\rho_{\alpha]}$, $\quad \delta^{\lambda\rho\sigma}_{\nu\alpha\beta} = 3!\,\delta^\lambda_{[\nu}\delta^\rho_\alpha\delta^\sigma_{\beta]}$.

The most general theory with second order field equations.
Contains all studied models with a scalar field – Brans-Dicke, quintessence, k-essence, $F(R)$, etc. (recently – beyond Horndeski, extended Horndeski.)

Fab Four (F4)

$$L_{\rm F4} = \sqrt{-g}\,(\mathcal{L}_J + \mathcal{L}_P + \mathcal{L}_G + \mathcal{L}_R - 2\Lambda)$$

with

$$\begin{aligned}
\mathcal{L}_J &= V_J(\Phi)\, G_{\mu\nu} \nabla^\mu\Phi\nabla^\nu\Phi\,, \\
\mathcal{L}_P &= V_P(\Phi)\, P_{\mu\nu\rho\sigma} \nabla^\mu\Phi\nabla^\rho\Phi\nabla^{\nu\sigma}\Phi\,, \\
\mathcal{L}_G &= V_G(\Phi)\, R\,, \\
\mathcal{L}_R &= V_R(\Phi)\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2),
\end{aligned}$$

the dual of Riemann $P^{\mu\nu}{}_{\alpha\beta} = -\frac{1}{4}\,\delta^{\mu\nu\gamma\delta}_{\sigma\lambda\alpha\beta}\,R^{\sigma\lambda}{}_{\gamma\delta}, \quad P^{\mu\alpha}{}_{\nu\alpha} = G^\mu_\nu$

The most general Horndeski subset where flat space is a solution, despite $\Lambda \neq 0 \Rightarrow$ screening of the cosmological constant.

/Charmousis, Copeland, Padilla, Saffin, 2012/

Fab Five (F5)

$V_J = -\alpha,\ V_P = V_R = 0,\ V_G = M_{\rm Pl}^2$, also scalar kinetic term

$$S = \int \left(M_{\rm Pl}^2\, R - (\alpha\, G_{\mu\nu} + \varepsilon\, g_{\mu\nu})\nabla^\mu\Phi\nabla^\nu\Phi - 2\Lambda\right)\sqrt{-g}\, d^4x + S_{\rm m}$$

- Static, spherically symmetric sector is completely integrable (black holes, solitons, stars)
- Cosmologies with the early and late acceleration phases and with the Hubble rate determined not by Λ but by ε/α (cosmological constant problem is solved ?)
- We wish to study these cosmologies in more detail

FLRW cosmologies

Field equations

$$\delta S = \int (E_{\mu\nu}\,\delta g^{\mu\nu} + E_\Phi\,\delta\Phi)\sqrt{-g}\,d^4x = 0$$

$\Rightarrow$ gravitational equations,

$$E_{\mu\nu} \equiv M_{\rm Pl}^2\,G_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha\,\mathcal{T}_{\mu\nu} - \varepsilon\,T^{(\Phi)}_{\mu\nu} - T^{(\rm m)}_{\mu\nu} = 0,$$

$$\begin{aligned}
\mathcal{T}_{\mu\nu} &= P_{\alpha\mu\nu\beta}\nabla^\alpha\Phi\nabla^\beta\Phi + \frac{1}{2}\,g_{\mu\lambda}\,\delta^{\lambda\rho\sigma}_{\nu\alpha\beta}\,\nabla^\alpha_\rho\Phi\nabla^\beta_\sigma\Phi - XG_{\mu\nu}\,,\\
T^{(\Phi)}_{\mu\nu} &= \nabla_\mu\Phi\nabla_\nu\Phi + Xg_{\mu\nu}\,,\\
T^{(\rm m)}_{\mu\nu} &= (\rho+p)U_\mu U_\mu + pg_{\mu\nu}\,,
\end{aligned}$$

and the scalar equation

$$E_\Phi \equiv \nabla_\mu((\alpha G^{\mu\nu} + \varepsilon g^{\mu\nu})\nabla_\nu\Phi) = 0 \quad \Rightarrow \quad \Phi \to \Phi + \Phi_0$$

$\Rightarrow$ Φ sees effective "optical" metric $\mathcal{M}_{\mu\nu} = \alpha G_{\mu\nu} + \varepsilon g_{\mu\nu}$.

FLRW

$$ds^2 = -dt^2 + \mathrm{a}^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right],$$

$K = 0, \pm 1$. Hubble rate $H = \dot{\mathrm{a}}/\mathrm{a}$, $\psi = \dot{\Phi} \Rightarrow$ Friedmann+scalar

$$3M_{\mathrm{Pl}}^2\left(H^2 + \frac{K}{\mathrm{a}^2}\right) = \frac{1}{2}\,\varepsilon\,\psi^2 - \frac{3}{2}\,\alpha\,\psi^2\left(3H^2 + \frac{K}{\mathrm{a}^2}\right) + \Lambda + \rho,$$

$$\frac{1}{\mathrm{a}^3}\frac{d}{dt}\left(\mathrm{a}^3\left(3\alpha\left(H^2 + \frac{K}{\mathrm{a}^2}\right) - \varepsilon\right)\psi\right) = 0,$$

$\Rightarrow$ first integral

$$\mathrm{a}^3\left(3\alpha\left(H^2 + \frac{K}{\mathrm{a}^2}\right) - \varepsilon\right)\psi = C$$

Solutions with $C \neq 0$ approach $C = 0$ solutions as $\mathrm{a} \to \infty$.

$C = 0$ solutions

GR branch

$$\psi = 0, \qquad H^2 + \frac{K}{\mathrm{a}^2} = \frac{\Lambda + \rho}{3M_{\rm Pl}^2}$$

The screening branch

$$H^2 + \frac{K}{\mathrm{a}^2} = \frac{\varepsilon}{3\alpha}, \qquad \psi^2 = \frac{\alpha\,(\Lambda + \rho) - \varepsilon\, M_{\rm Pl}^2}{\alpha\,(\varepsilon - 3\alpha\, K/\mathrm{a}^2)}$$

cosmological term is $\varepsilon/3\alpha$ while the Λ is screened (ρ as well).
Flat space is obtained for $\varepsilon = 0$,

$$K = -1, \qquad \mathrm{a} = t, \qquad \psi^2 = \frac{\Lambda + \rho}{3\alpha}\, t^2\,.$$

$C \neq 0$ solutions

Scalar field

$$\psi = \frac{C}{\mathrm{a}^3 \left[3\alpha \left(H^2 + \frac{K}{\mathrm{a}^2}\right) - \varepsilon\right]},$$

$\Rightarrow$ everything reduces to algebraic equation for $H(a)$

$$3M_{\mathrm{Pl}}^2 \left(H^2 + \frac{K}{\mathrm{a}^2}\right) = \frac{C^2 \left[\varepsilon - 3\alpha \left(3H^2 + \frac{K}{\mathrm{a}^2}\right)\right]}{2a^6 \left[\varepsilon - 3\alpha \left(H^2 + \frac{K}{\mathrm{a}^2}\right)\right]^2} + \Lambda + \rho.$$

Solution gives

$$t = \int \frac{d\mathrm{a}}{\mathrm{a}H(\mathrm{a})}$$

There are several solution branches, some of them show ghost.

No-ghost conditions

Ghost

Generic perturbations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, $\Phi \to \Phi + \delta\Phi$ give rise to

$$\begin{aligned} \delta E_{\mu\nu} &\equiv \delta(M_{\rm Pl}^2\, G_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha\, \mathcal{T}_{\mu\nu} - \varepsilon\, T_{\mu\nu}^{(\Phi)} - T_{\mu\nu}^{(\rm m)}), \\ \delta E_{\Phi} &\equiv \delta(\nabla_\mu((\alpha G^{\mu\nu} + \varepsilon g^{\mu\nu})\nabla_\nu \Phi)), \end{aligned}$$

which is used to compute the second varition

$$\delta^2 S = \int (\delta E_{\mu\nu}\, \delta g^{\mu\nu} + \delta E_\Phi\, \delta\Phi)\sqrt{-g}\, d^4x \equiv \int \delta^2 L\, d^4x.$$

The kinetic part of $\delta^2 L$ is a quadratic form in $\delta\dot{g}_{\mu\nu}$, $\delta\dot{\Phi}$, it should be positive definite. It splits into scalar $\otimes$ vector $\otimes$ tensor sectors.

Result in the scalar sector is K-dependent

$$\delta^2 S = \delta^2 \left(\int \left(M_{\rm Pl}^2\, R - (\alpha\, \underline{G_{\mu\nu}} + \varepsilon\, g_{\mu\nu})\nabla^\mu\Phi\nabla^\nu\Phi - 2\Lambda \right) \sqrt{-g}\, d^4x \right)$$

Anisotropic deformations of $K = 0$ solutions

$$ds^2 = -N^2\,dt^2 + a_1^2\,dx^2 + a_2^2\,dx_2^2 + a_3^2\,dx_3^2$$

$$L = -\left(\frac{2M_{\rm Pl}^2}{N} + \frac{\alpha\,\dot{\Phi}^2}{N^3}\right) Q + \left(\frac{\varepsilon\,\dot{\Phi}^2}{N} - 2N\Lambda\right) a_1\,a_2\,a_3$$

with $Q = a_1\,\dot{a}_2\,\dot{a}_3 + a_2\,\dot{a}_1\,\dot{a}_3 + a_3\,\dot{a}_1\,\dot{a}_2$. Constraint

$$\mathcal{C} = \frac{\partial L}{\partial N} = \left(\frac{2M_{\rm Pl}^2}{N^2} + \frac{3\alpha\,\dot{\Phi}^2}{N^4}\right) Q - \left(\frac{\varepsilon\,\dot{\Phi}^2}{N^2} + 2\Lambda\right) a_1\,a_2\,a_3 = 0.$$

perturbations

$$a_k = a + \delta a_k, \qquad \Phi = \int \psi\,dt + \delta\Phi.$$

$\delta\mathcal{C} = 0 \Rightarrow \delta\dot{\Phi} \sim (\delta\dot{a}_1 + \delta\dot{a}_2 + \delta\dot{a}_3) + \ldots \Rightarrow \delta^2 L$ is a quadratic in $\delta\dot{a}_k$ form. Its eigenvalues should be positive-definite $\Rightarrow$

No-ghost conditions for $K = 0$

$$\delta^2 L = M_{ik}\dot{a}_i\dot{a}_k + \dots$$

Positivity of M_{ik} requires in the scalar sector

$$\begin{aligned} & [18\alpha H^2\psi^2 + 6M_{\rm Pl}^2 H^2 - \varepsilon\,\psi^2] \\ \times \quad & [9\alpha^2 H^2\psi^2 - 6\alpha M_{\rm Pl}^2 H^2 + \varepsilon\,(\alpha\psi^2 + 2M_{\rm Pl}^2)] > 0 \end{aligned}$$

while in the tensor and vector sectors

$$2M_{\rm Pl}^2 + \alpha\psi^2 > 0$$

This is confirmed by the analysis of generic perturbations.
The scalar condition can also be obtained by considering only the isotropic deformations of the FLRW metric

$$a \to a + \delta a, \qquad \Phi \to \Phi + \delta\Phi$$

which can be repeated also for $K \neq 0$.

No-ghost conditions for all K

In the scalar sector

$$
\begin{aligned}
& [18\alpha H^2\psi^2 + 6M_{\rm Pl}^2 H^2 - \epsilon\,\psi^2] \\
\times \quad & [9\alpha^2 H^2\psi^2 - 6\alpha M_{\rm Pl}^2 H^2 + \epsilon\,(\alpha\psi^2 + 2M_{\rm Pl}^2)] > 0
\end{aligned}
$$

$$\epsilon = \varepsilon - \frac{3\alpha K}{a^2}\,.$$

in the tensor and vector sectors

$$2M_{\rm Pl}^2 + \alpha\psi^2 > 0 \qquad \Rightarrow \qquad \alpha > 0$$

These are used to select stable and unstable solution branches.

No-ghost conditions for flat space

FLRW in F4

$$ds^2 = -N^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right],$$

$\Rightarrow$ reduced Lagrangian

$$\begin{aligned} L_{\mathrm{F4}} &= \frac{3aV_J(\Phi)}{N^3}\dot{\Phi}^2(\dot{a}^2 + KN^2) - \frac{3\dot{a}V_P(\Phi)}{N^5}\dot{\Phi}^3(\dot{a}^2 + KN^2) \\ &+ \frac{6a}{N}\Big(V_G(\Phi)(KN^2 - \dot{a}^2) - V_G'(\Phi)\dot{\Phi}\,a\dot{a}\Big) \\ &- \frac{8\dot{a}\dot{\Phi}V_R'(\Phi)}{N^3}(\dot{a}^2 + 3KN^2) - 2\Lambda N a^3 \end{aligned}$$

Varying gives the field equations. For any $V_P(\Phi)$, $V_J(\Phi)$, $V_G(\Phi)$, $V_R(\Phi)$ (unless $V_P = V_J = V_G' = 0$) the flat space solution

$$N = 1, \qquad K = -1, \qquad a = t,$$

$$V_P\,\dot{\Phi}^3 - t\,V_J\,\dot{\Phi}^2 + t^2\,V_G'\,\dot{\Phi} = \frac{\Lambda}{3}t^3.$$

No-ghost conditions for flat space

$$9V_P\,\psi^3 - 5tV_J\,\psi^2 + (2t^2V_G' + 8V_R')\psi + 2tV_G > 0,$$
$$3V_P\,\psi^3 - tV_J\,\psi^2 + 8V_R'\psi + 2tV_G > 0$$

$\psi = \dot{\Phi}$. Fulfilled within the $\varepsilon \to 0$ limit of F5

$$V_P = V_R = 0, \quad V_J = -\alpha < 0, \qquad V_G = M_{\rm Pl}^2,$$

In general impose non-trivial conditions on the coefficient functions $V_J(\Phi), \ldots, V_R(\Phi) \Rightarrow$

flat space within the full F4 theory can have ghost, unless the coefficients $V_J, \ldots, V_R$ are properly chosen.

Ghost-free cosmologies

Master equation

$$3M_{\rm Pl}^2\left(H^2+\frac{K}{\rm a^2}\right)=\frac{C^2\left[\varepsilon-3\alpha\left(3H^2+\frac{K}{\rm a^2}\right)\right]}{2a^6\left[\varepsilon-3\alpha\left(H^2+\frac{K}{\rm a^2}\right)\right]^2}+\Lambda+\rho.$$

determines $H({\rm a})$. Dimensionless parameters

$$H^2=H_0^2\,y,\qquad {\rm a}={\rm a}_0\,a\,,\qquad \rho_{\rm cr}=3M_{\rm Pl}^2H_0^2\,,$$

$$\Omega_0=\frac{\Lambda}{\rho_{\rm cr}},\qquad \Omega_2=-\frac{K}{H_0^2{\rm a}_0^2},\qquad \Omega_6=\frac{C^2}{6\alpha\,{\rm a}_0^6\,H_0^2\,\rho_{\rm cr}},\qquad \zeta=\frac{\varepsilon}{3\alpha\,H_0^2},$$

$$\rho=\rho_{\rm cr}\left(\frac{\Omega_4}{a^4}+\frac{\Omega_3}{a^3}\right)\qquad = \text{ radiation+dust}$$

Dimensionless master equation, $y = (H/H_0)^2$

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6 \left[\zeta - 3y + \frac{\Omega_2}{a^2}\right]}{a^6 \left[\zeta - y + \frac{\Omega_2}{a^2}\right]^2}$$

$(\zeta \sim \varepsilon/\alpha)$ is equivalent to
$P(a, y) = c_3(a)\, y^3 + c_2(a)\, y^2 + c_1(a)\, y + c_0(a) = 0$
No-ghost conditions are

$$\Omega_6 > 0,$$
$$\left[y\ (y_* - y)^2\, a^6 + \Omega_6\ (6y - y_*)\right] \left[(y_* - y)^3 a^6 + \Omega_6\, (3y + y_*)\right] > 0,$$

with $y_* = \zeta + \Omega_2/a^2$.

Late time solutions, $a \to \infty$

$\zeta > \Omega_0$: GR branch, stable (=ghost free)

$$y = \Omega_0 + \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{(\zeta - 3\,\Omega_0)\,\Omega_6}{(\,\Omega_0 - \zeta)^2\, a^6} + \mathcal{O}\left(\frac{1}{a^7}\right),$$

$0 < \zeta < \Omega_0$: Unstable GR branch + stable screening branches

$$y_\pm = \zeta + \frac{\Omega_2}{a^2} \pm \frac{\chi}{(\,\Omega_0 - \zeta)\, a^3} \pm \frac{\Omega_2 \Omega_6}{\chi\, a^5} + \mathcal{O}\left(\frac{1}{a^7}\right),$$

with $\chi = \sqrt{2\,\zeta\;\Omega_6(\,\Omega_0 - \zeta)}$

$\zeta < 0$: Only the unstable GR branch.

Early time solutions, $a \to 0$

GR solution is always unstable

$$y = \frac{\Omega_4}{a^4} + \frac{\Omega_3}{a^3} + \frac{\Omega_2\Omega_4 - 3\Omega_6}{\Omega_4 a^2} + \frac{3\Omega_3\Omega_6}{\Omega_4 a} + \mathcal{O}(1)$$

Screening solutions exist if $\sigma^2 = \Omega_6(8\Omega_2\Omega_4 + 9\Omega_6) > 0$,

$$y_\pm = \frac{2\Omega_2\Omega_4 + 3\Omega_6 \pm \sigma}{2\Omega_4 a^2} \mp \frac{\Omega_3\Omega_6(4\Omega_2\Omega_4 + 9\Omega_6 \pm 3\sigma)}{2\sigma\Omega_4^2\, a} + \mathcal{O}(1)$$

stable if $(5\Omega_2\Omega_4 \pm 3\sigma + 9\Omega_6)(8\Omega_2\Omega_4 \pm 3\sigma + 9\Omega_6) > 0$.

If $\Omega_2 = K = 0$ then the GR remains unstable, while the screening branches are both stable,

$$\begin{aligned} y_+ &= \frac{3\Omega_6}{\Omega_4\, a^2} - \frac{3\Omega_3\Omega_6}{\Omega_4^2\, a} + \frac{5}{3}\zeta + \frac{3\Omega_6\Omega_3^2 + 9\Omega_6^2}{\Omega_4^3} + \mathcal{O}(a) \\ y_- &= \frac{\zeta}{3} + \frac{4\,\zeta^2}{27\,\Omega_6}\left(\Omega_4\, a^2 + \Omega_3\, a^3\right) + \mathcal{O}(a^4) \end{aligned}$$

Early time solutions for $\Omega_4 = 0$

There is always a stable branch

$$y = \frac{\Omega_2}{3a^2} + \frac{4\Omega_2^2\Omega_3}{27\Omega_6 a} + \frac{\zeta}{3} + \frac{8\Omega_2^3}{81\Omega_6} - \frac{16\Omega_2^3\Omega_3^3}{243\Omega_6^2} + \mathcal{O}(a).$$

Additional branches exist if $\omega^2 = \Omega_3^2 - 12\Omega_6 > 0$

$$y_\pm = \frac{\Omega_3 \pm \omega}{2a^3} + \frac{\Omega_2(\Omega_3^2 \pm \omega\Omega_3 - 16\Omega_6)}{\omega(\omega \pm \Omega_3)a^2} + \mathcal{O}\left(\frac{1}{a}\right),$$

here y_- is stable while y_+ is unstable. If $\Omega_3 = \Omega_4 = 0$ then

$$y = \frac{\Omega_2}{3a^2} + \frac{\zeta}{3} + \frac{8\Omega_2^3}{81\Omega_6} + \left(\frac{4\Omega_2^2(\Omega_0 + \zeta)}{27\Omega_6} - \frac{32\,\Omega_2^5}{729\,\Omega_6^2}\right) a^2 + \mathcal{O}(a^4),$$

and if $\Omega_2 = \Omega_3 = \Omega_4 = 0$ then there remains only

$$y = \frac{\zeta}{3} + \frac{4\zeta^2(3\Omega_0 - \zeta)}{81\Omega_6}\, a^6 + \mathcal{O}\left(a^{12}\right) \quad \text{which is stable}$$

Ghost-free solutions

Late times

$$y = \Omega_0 + \ldots, \qquad \Psi = \frac{1}{(\alpha - \Omega_0)\, a^3} + \ldots, \qquad \Omega_0 < \alpha$$

$$y = \zeta + \ldots, \qquad \Psi = \pm\sqrt{\frac{\Omega_0 - \zeta}{2\zeta\,\Omega_6}} + \ldots, \qquad 0 < \alpha < \Omega_0$$

Early times ($K = \Omega_2 = 0$). One branch

$$y = \frac{\zeta}{3} + \ldots, \qquad \Psi = \frac{3}{2\zeta a^3} + \ldots$$

Another branch

$$y = \frac{3\Omega_6}{\Omega_4 a^2} + \ldots, \qquad \Psi = -\frac{\Omega_4}{3\Omega_6 a} + \ldots,$$

if $\Omega_4 \neq 0$ and if $\Omega_4 = 0$ but $\omega^2 = \Omega_3^2 - 12\Omega_6 > 0$ then

$$y = \frac{\Omega_3 - \omega}{2\, a^3} + \ldots, \quad \Psi = \frac{2}{\Omega_3 - \omega} + \ldots,$$

The matter-dominated solution with $y \sim \Omega_4/a^4 + \ldots$ has ghost.

Global solutions, $\Omega_3 = \Omega_4 = 0$, $\zeta > \Omega_0$ or $\zeta < 0$

One local solution at small and large $a \Rightarrow$ one global solution

$$\frac{\zeta}{3} \leftarrow y \rightarrow \Omega_0$$

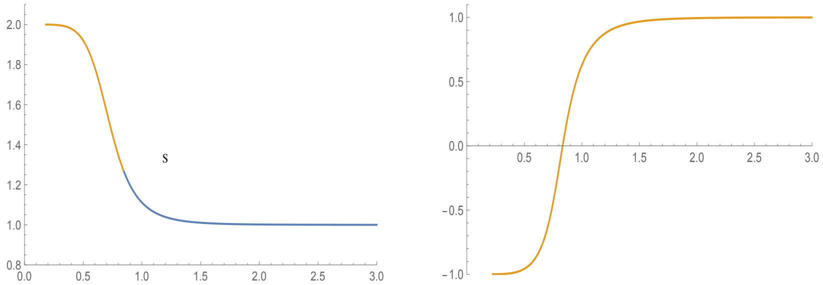

Figure: $y(a)$ for $\Omega_0 = \Omega_6 = 1$, $\Omega_2 = \Omega_3 = \Omega_4 = 0$.

Left: $\zeta = 6$, stable; Right: $\zeta = -3$, cosmological bounce, unstable

Global solutions, $\Omega_3 \neq 0$, $\Omega_4 \neq 0$, $\zeta > \Omega_0$ or $\zeta < 0$

One local solution in one limit and three in the other $\Rightarrow$ one global

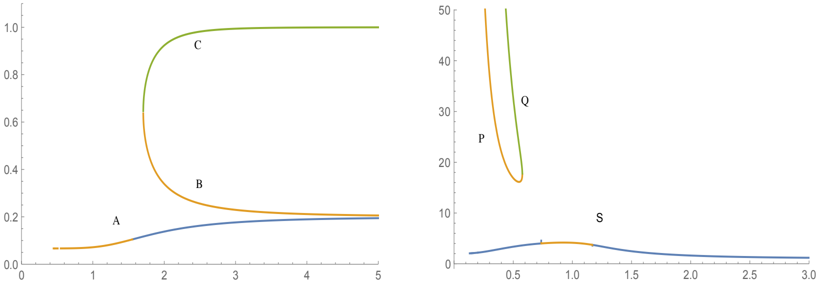

Figure: $y(a)$ for $\Omega_0 = \Omega_6 = 1$, $\Omega_2 = \Omega_3 = \Omega_4 = 0$, $\zeta = 0.2$ (left) and for $\Omega_0 = \Omega_6 = 1$, $\Omega_2 = 0$, $\Omega_3 = 5$, $\Omega_4 = 0$, $\zeta = 6$ (right).

Solutions A,S are stable, $\zeta/3 \leftarrow y \rightarrow \zeta$
Solutions B,C – "beginning of time"; P,Q – "end of time"; one of the merging branches has ghost, $a(t) = a_* + a_* \sqrt{y_*} |t - t_*| + \mathcal{O}\left((t - t_*)^{3/2}\right)$.

Global solutions, $\Omega_3 \neq 0$, $\Omega_4 \neq 0$, $0 < \zeta < \Omega_0$

Three global solutions; A,B are stable; one has for B

$$\frac{3\Omega_6}{\Omega_4 a^2} \leftarrow y \rightarrow \zeta \quad \text{or} \qquad \frac{\Omega_3 - \sqrt{\Omega_3^2 - 12\Omega_6}}{2a^3} \leftarrow y \rightarrow \zeta \quad \text{if} \quad \Omega_4 = 0.$$

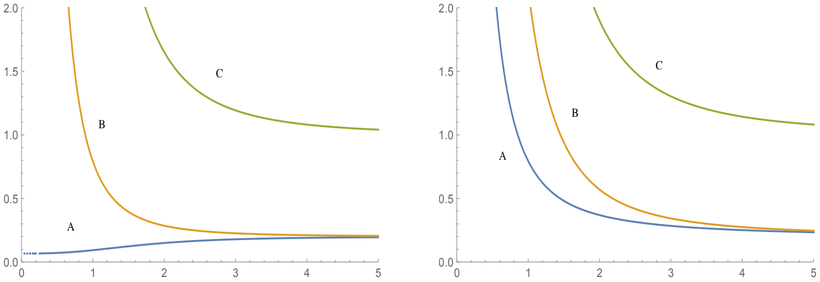

Figure: $\Omega_0 = \Omega_6 = 1$, $\Omega_3 = 5$, $\Omega_4 = 0$, $\zeta = 0.2$; $\Omega_2 = 0$ (left) and $\Omega_2 = 1$ (right).

Three types of ghost-free solutions

Solution S exists only for $\zeta > \Omega_0$, sourced by the scalar field but may also contain matter,

$$\frac{\zeta}{3} \leftarrow y \rightarrow \Omega_0$$

At late times – standard dynamic dominated by the $\Lambda +$ matter. At early times matter and Λ are screened, the Hubble rate is determined by $\zeta \sim \varepsilon/\alpha$. Can be an hierarchy between the Hubble scales. Λ is not screened at late times $\Rightarrow$ cosmological constant problem.

Three types of ghost-free solutions

Solutions A and B exist for $0 < \zeta < \Omega_0$. Solution A can exist with or without matter terms,

$$\frac{\zeta}{3} \leftarrow y \rightarrow \zeta,$$

solution B exists only with matter,

$$\frac{3\Omega_6}{\Omega_4 a^2} \leftarrow y \rightarrow \zeta \quad \text{or} \qquad \frac{\Omega_3 - \sqrt{\Omega_3^2 - 12\Omega_6}}{2a^3} \leftarrow y \rightarrow \zeta \quad \text{if} \quad \Omega_4 = 0.$$

Both show screening at late times, because their late time behaviour is controlled by $\zeta \sim \varepsilon/\alpha$ and not by $\Lambda \Rightarrow$ could in principle describe the late time acceleration while circumventing the cosmological constant problem. Cannot describe the early inflationary phase. Solution B does not show inflation at all, while for A there is no hierarchy between the two Hubble scales.

Stability of the solutions

Perturbations of FLRW with $K = 0$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \qquad \Phi = \int \psi(t)\, dt + \delta\Phi \quad \text{with}$$

$$\delta g_{0\mu} = 0, \qquad \delta g_{ik} = 2\mathrm{a}^2(t)\, h_{ik}(t) e^{i\mathbf{px}}, \qquad \delta\Phi = \phi(t) e^{i\mathbf{px}}\,,$$

the momentum is along the third axis, $\mathbf{p} = (0, 0, p)$,

$$h_{ik}(t) = \sum_{m=1}^{6} R_m(t) h^{(m)}_{ik}\,,$$

$$h^{(1)}_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h^{(2)}_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad h^{(3)}_{ik} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$h^{(4)}_{ik} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad h^{(5)}_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h^{(6)}_{ik} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Scalar sector

dimensionless $\tau = H_0 t$ and $h = \dot{a}/a$, independent amplitudes $w(\tau) \equiv R_1(\tau)$, $u(\tau) \equiv R_2(\tau)$, and $\phi(\tau)$,

$$3\Omega_6\Psi(\zeta - 3h^2)\,\dot{\phi} - 3h(3\Omega_6\Psi^2 + 1)\,\dot{w}$$
$$-\frac{P^2}{a^2}\left\{(\Omega_6\Psi^2 + 1)(w + u) + 2\Omega_6 h\Psi\,\phi\right\} = 0,$$

$$(\Omega_6\Psi^2 + 1)(\dot{w} + \dot{u}) + 2\Omega_6 h\,\dot{\phi} + 3\Omega_6\Psi(\zeta - h^2)\,\phi = 0,$$

$$(\zeta - h^2)\,\ddot{\phi} - 2h\Psi\,\ddot{w} + \left(3(\zeta - 3h^2)\Psi - 2h\dot{\Psi} - 2\Psi\dot{h}\right)\,\dot{w}$$
$$+(3\zeta - 2\dot{h} - 3h^2)\,h\,\dot{\phi} + [3(\zeta - h^2)\dot{\Psi} + 9(\zeta - h^2)h\Psi)]\,w$$
$$-\frac{2P^2}{3a^2}(\dot{\Psi} + h\Psi)\,(w + u) - \frac{P^2}{3a^2}(2\dot{h} + 3h^2 - 3\eta)\,\phi = 0.$$

Tensor sector

Each of the two tensor amplitudes $w(\tau) \equiv R_5(\tau)$ and $u(\tau) \equiv R_6(\tau)$ fulfills exactly the same equation

$$
\begin{aligned}
&(\Omega_6 \Psi^2 + 1)\, \ddot{w} + \left(2\Omega_6 \Psi \dot{\Psi} + 3(\Omega_6 \Psi^2 + 1) h\right) \dot{w} \\
&- \Big(2(\Omega_6 \Psi^2 + 1)(2\dot{h} + 3h^2) + 2\Omega_6 (3\zeta \Psi^2 + 4h\Psi\dot{\Psi}) - 6\Omega_0 \\
&+ \frac{P^2}{a^2} (\Omega_6 \Psi^2 - 1)\Big)\, w = 0.
\end{aligned}
$$

Solutions of linear equations can become unbounded only in the vicinity of singular points, $a = 0, \infty$.

Late time limit

GR branch:

$$y = \Omega_0 + \ldots, \qquad \Psi = \frac{1}{(\alpha - \Omega_0)\, a^3} + \ldots, \qquad \Omega_0 < \alpha$$

tensor sector: $w, u = C_1 + C_2\, e^{-3h\tau}$

scalar sector: $w = C_1, \qquad u = C_2, \qquad \phi = C_3 + C_4 e^{-3h\tau}$

perturbations are bounded. Screening branch:

$$y = \zeta + \ldots, \qquad \Psi = \pm\sqrt{\frac{\Omega_0 - \zeta}{2\zeta\,\Omega_6}} + \ldots, \qquad 0 < \alpha < \Omega_0$$

solutions are very similar to those for the GR branch.

All solutions are stable at late times $\Rightarrow$ the model is OK in this limit.

Early times, homogeneous modes with $P = 0$

First branch: $$y = \frac{\zeta}{3} + \ldots, \qquad \Psi = \frac{3}{2\zeta a^3} + \ldots,$$

perturbations are $\sim C_1 + C_2 e^{+3h\tau}$, bounded for $\tau \to -\infty$.

Second branch: $$y = \frac{3\Omega_6}{\Omega_4 a^2} + \ldots, \qquad \Psi = -\frac{\Omega_4}{3\Omega_6 a} + \ldots,$$

perturbations $$w = C_1 \cos(\sqrt{6}\ln(\tau) + C_2),$$

bounded as $\tau \to 0$ but their derivatives grow and the curvature blows up. In the scalar sector ϕ contains a piece proportional to $1/\tau^3$, hence $\dot{\phi}/\Psi$ is unbounded $\Rightarrow$ this branch is unstable

The first branch may be stable

Inhomogeneous modes with $P \neq 0$

Equation for tensor perturbations of the first branch reduces to

$$\ddot{w} - 3\dot{w} - \frac{P^2}{a^2(\tau)}\, w = 0 \qquad \text{with} \qquad a(\tau) = e^{h\tau},$$

whose solution

$$\begin{aligned} w &= C_1\, a^2(\tau)[P - ha(\tau)] \exp\left(\frac{P}{ha(\tau)}\right) \\ &+ C_2\, a^2(\tau)[P + ha(\tau)] \exp\left(-\frac{P}{ha(\tau)}\right) \end{aligned}$$

diverges as $a(\tau) \to 0$, and the divergence is very strong – it is proportional to the exponent of exponent of τ. This effect is produced by terms proportional to Ω_6, hence by the background scalar. Therefore, the screening branch is unstable as well.

Conclusions

- All isotropic ghost-free solutions in the F5 theory are unstable in the vicinity of the initial spacetime singularity. The instability is very strong – it is exponential and not power law as in GR. Therefore, the F5 theory does not have viable solutions describing the whole of the cosmological history.
- However, at late times it admits stable solutions with an accelerating phase. For $0 < \zeta < \Omega_0$ they show the screening, since their Hubble parameter is determined not by the conventional Λ-term but by $\zeta \sim \varepsilon/\alpha$, which circumvents the cosmological constant problem. Hence the model, although not completely satisfactory, could be used to explain the current cosmic acceleration.
- It could be that more realistic models can be obtained by adjusting the coefficient functions $G_k(X, \Phi)$ in the Horndeski Lagrangian. It seems therefore that the Horndeski theory may indeed offer interesting for cosmology features, but a detailed analysis is needed each time.