Bihyperbolic Spacetimes and Inflationary Magnetogenesis

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Hot Topics in Modern Cosmology

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2. Physics seeks predictive theories, i.e. a well-posed Cauchy problem with time-orientation $\rightarrow$ bihyperbolicity
3. How can this be ensured, or checked, in modified theories? What are the observational implications?

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2. Predictive kinematics: bihyperbolicity
3. Generalizing the Maxwell Lagrangian: the constitutive tensor and its cosmological observational implications
4. Application to inflationary magnetogenesis

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Spacetime geometry is probed by test matter, with linear field equations. The most general such PDE in $(\mathcal{U}, x)$ is

$$\left[\sum_{d=1}^{k}(D^{\bar{\mu}}_{\bar{\lambda}})(G)^{\nu_1\ldots\nu_d}\frac{\partial}{\partial x^{\nu_1}}\cdots\frac{\partial}{\partial x^{\nu_d}}\right]F_{\bar{\mu}} = 0, \quad (*)$$

$\bar{\mu}$ multi-index of test matter fields components,
$\nu_i \in \{0, \ldots, 3\}$ generalized spacetime coordinates,
$i \in \{1, \ldots, d\}$, $d \in \{1, \ldots, k\}$ partial derivative order (k highest),
$D^{\bar{\mu}}_{\bar{\lambda}}$ square matrix of derivative coefficients.

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Hence, from $(*)$,

$$e^{i\frac{S(x)}{\lambda}} \left(\frac{i}{\lambda}\right)^k \left[D^{\bar{\mu}\nu_1\dots\nu_k}_{\bar{\lambda}}(x)\frac{\partial S}{\partial x^{\nu_1}} \dots \frac{\partial S}{\partial x^{\nu_k}}\right] F_{\bar{\mu}0}(x) + \text{lower terms in } \frac{1}{\lambda} = 0.$$

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Lower order terms are negligible as $\lambda \to 0$, leaving the first term.

By linear algebra, this has a non-trivial solution for $F_{\bar{\mu}0}$ if S satisfies

$$\det\left[D_{\bar{\lambda}}^{\bar{\mu}\nu_1\ldots\nu_k}(x)\frac{\partial S}{\partial x^{\nu_1}}\cdots\frac{\partial S}{\partial x^{\nu_k}}\right]=0,$$

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For any $p\in T^*\mathcal{M}$, this defines the principal polynomial of $(*)$.

The principal polynomial of $(*)$ is $P : T^*\mathcal{M} \to \mathbb{R}$ such that

$$P(x,p) = \omega_G \det\left[D_{\bar{\lambda}}^{\bar{\mu}\nu_1\dots\nu_k}(x)p_{\nu_1}\dots p_{\nu_k}\right] = P^{\nu_1\dots\nu_{\deg P}}p_{\nu_1}\dots p_{\nu_{\deg P}},$$

where we have
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2. $\deg P \neq k$, the highest derivative order, in general.

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1. N_x is independent of the choice of ω_G;
2. if P is reducible, $P = \prod_i P_i^{n_i}, n_i \geq 1$, one removes repeated factors to obtain the reduced (irreducible) principal polynomial $\bar{P} = \prod_i P_i$. Again, N_x remains unaffected.

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Then necessarily $(\Rightarrow)$, P is hyperbolic. [Gårding (1959)]

Note: By contraposition, this hyperbolicity criterion can be used to check theories for causality.

Definition:

A polynomial $P : T^*\mathcal{M} \to \mathbb{R}$ homogeneous of deg P is hyperbolic if $\exists\ h \in T^*\mathcal{M}, h \neq 0$, such that $\forall\ p \in T^*\mathcal{M}$,

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Dual polynomial

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If P is hyperbolic, then the dual polynomial $P^\sharp : T\mathcal{M} \to \mathbb{R}$ exists,

$$P^\sharp(x, X) = P^\sharp(x)_{\nu_1 \dots \nu_{\deg P^\sharp}} X^{\nu_1} \dots X^{\nu_{\deg P^\sharp}}, \; X \in T_x\mathcal{M},$$

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3. hyperbolicity of P does not imply hyperbolicity of $P^\sharp$.

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Property: Lorentzian metrics are bihyperbolic.

Example: vacuum electromagnetism

Standard theory: in $\mathcal{M} = \mathbb{R}^4$, $x^\nu = (t, \mathbf{x}) = (t, x^1, x^2, x^3)$, Maxwell's equations in vacuum are $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$ and

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} &= 0, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0. \end{aligned}$$

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$$D_{\bar{\lambda}}^{\bar{\mu}\nu} \frac{\partial F_{\bar{\mu}}}{\partial x^\nu} = 0.$$

Example: principal polynomial

Now the corresponding principal polynomial is proportional to

$$\det[D_{\bar{\lambda}}^{\bar{\mu}\nu} p_\nu] = \det \begin{bmatrix} -p_0 & 0 & 0 & 0 & p_3 & -p_2 \\ 0 & -p_0 & 0 & -p_3 & 0 & p_1 \\ 0 & 0 & -p_0 & p_2 & -p_1 & 0 \\ 0 & p_3 & -p_2 & p_0 & 0 & 0 \\ -p_3 & 0 & p_1 & 0 & p_0 & 0 \\ p_2 & -p_1 & 0 & 0 & 0 & p_0 \end{bmatrix}$$

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$$= p_0^2(-p_0^2 + p_1^2 + p_2^2 + p_3^2)^2.$$

For $p_0 \neq 0$, we can read off a reduced principal polynomial with $\deg \bar{P} = 2$,

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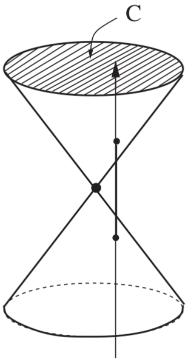
At every $x \in \mathbb{R}^4$, the null cone is given by $N = \{p \ : \ \eta^{\mu\nu} p_\mu p_\nu = 0\}$, and the hyperbolicity cone is $C = \{h \ : \ h \text{ timelike}, \ \eta^{\mu\nu} h_\mu h_\nu < 0\}$.



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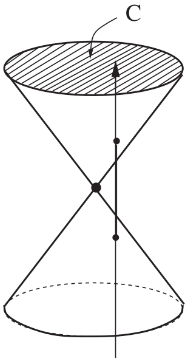

Moreover, $\bar{P}^\sharp_{\mu\nu} = \eta_{\mu\nu}$. Thus, we have bihyperbolic $(\mathbb{R}^4, G, F)$ with $G = \eta$.

The Lagrangian of Minkowski vacuum electromagnetism is

$$\mathcal{L}_{M,vac} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{8}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma})F_{\mu\nu}F_{\rho\sigma}$$

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and the corresponding constitutive tensor in vacuum,

$$\chi^{\bar{\mu}\bar{\nu}}_{M,vac} = \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & I \end{array}\right],$$

where I is the 3×3 identity.

More generally, in a dielectric medium, the Maxwell action becomes [e.g. Post (1962)]

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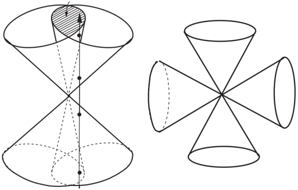
and the Petrov form

$$\chi_M^{\bar{\mu}\bar{\nu}} = \left[\begin{array}{c|c} -\epsilon & \phi \\ \hline \phi^T & \mu^{-1} \end{array}\right],$$

with 3×3 matrix blocks, where ϵ denotes electrical permittivity, μ magnetic permeability and ϕ contains the Fresnel-Fizeau effect (tracefree part) and the axion (trace part).

The principal polynomial for a general χ_M is quartic, which may or may not be (bi)hyperbolic:



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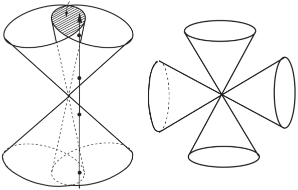

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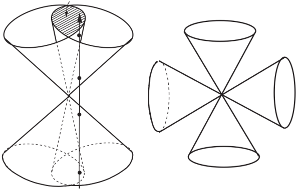

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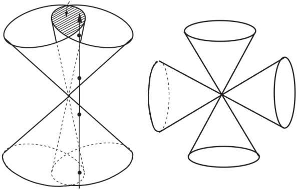

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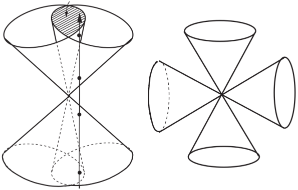

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Alternatively, it may be regarded as effective, modelling optical effects e.g. of fundamental scalar fields or HEP effects.

1. Birefringence

If the diagonalized permittivity matrix in χ_M is

$$\epsilon = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \text{ without } \epsilon_1 = \epsilon_2 = \epsilon_3,$$

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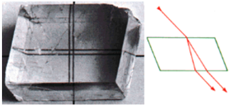

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There are indications of a violation, e.g. $\tau_{0.35} - \tau_{0.20} = -0.30 \pm 0.26$ at 95% [More, Bovy & Hogg (2009)], to be modelled by a cosmological constitutive tensor χ. [Schneider & Werner (2016), in prep.]

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$$\mathcal{L} = -\frac{\sqrt{-g}}{16\pi}\left(\lambda(\varphi,\psi)F_{\mu\nu}F^{\mu\nu} + \mathcal{M}^{\rho}_{\sigma}(\varphi)F_{\rho\alpha}F^{\sigma\alpha} - \mathcal{N}^{\rho}_{\sigma}(\psi)\tilde{F}_{\rho\alpha}\tilde{F}^{\sigma\alpha}\right),$$

with scalar fields φ, ψ, field tensor $F_{\mu\nu}$ and dual $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Cosmological magnetogenesis

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Here, we are interested in studying the bihyperbolicity properties of such theories, by identifying the corresponding cosmological constitutive tensor, [Vikman & Werner (2016), in prep.]

$$\mathcal{L} = -\frac{1}{8}\chi^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

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Thus, with the neutral system described by φ, and metric g,

$$\mathcal{L} = -\sqrt{-g}\left(\alpha_1 \partial_\alpha \varphi \partial_\beta \varphi F^{\alpha\rho} F_\rho^\beta + \alpha_2 \varphi^2 F^{\mu\nu} F_{\mu\nu}\right),$$

where constants α_1, α_2 depend on α_E, α_B. [c.f. Itzykson & Zuber (1980)]

Now, on a conformally flat cosmological background with $g_{\mu\nu}$ $= a(t)^2\eta_{\mu\nu}$, and $\varphi = \varphi(t), \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$, Now, on a conformally flat cosmological background with $g_{\mu\nu} = a(t)^2\eta_{\mu\nu}$, and $\varphi = \varphi(t), \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$, the cosmological constitutive tensor χ can be identified from

$$\chi^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 8\sqrt{-g}\left(\alpha_1 \partial_\alpha\varphi \partial_\beta\varphi F^{\alpha\rho} F_\rho^\beta + \alpha_2 \varphi^2 F^{\mu\nu} F_{\mu\nu}\right),$$

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whence we obtain, in Petrov notation,

$$\chi^{\bar{\mu}\bar{\nu}} = \left[\begin{array}{c|c} (\frac{2\alpha_1}{a^2}\dot{\varphi}^2 - 4\alpha_2\varphi^2)I & 0 \\ \hline 0 & 4\alpha_2\varphi^2 I \end{array}\right] \equiv \left[\begin{array}{c|c} \varphi_1 I & 0 \\ \hline 0 & \varphi_2 I \end{array}\right],$$

where I is again the 3×3 identity.

The eight generalized Maxwell field equations corresponding to cosmological van der Waals interactions are

$$\begin{aligned} \partial_\nu(\chi^{\mu\nu\rho\sigma} F_{\rho\sigma}) &= 0, \\ \partial_{[\nu} F_{\rho\sigma]} &= 0, \end{aligned}$$

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which need to be recast in the form discussed before,

$$D^{\bar{\mu}\nu}_{\bar{\lambda}} \partial_\nu F_{\bar{\mu}} + \tilde{D}^{\bar{\mu}}_{\bar{\lambda}} F_{\bar{\mu}} = 0,$$

in order to read off the principal polynomial determined only by the matrix D of highest derivative order,

$$P(x, p) \propto \det[D^{\bar{\mu}\nu}_{\bar{\lambda}} p_\nu].$$

Thus, the first set of generalized Maxwell equations yields four equations.

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$$\begin{aligned}
\mu = 0: \quad \chi^{0\nu\bar{\mu}}\partial_\nu F_{\bar{\mu}} &= \varphi_1(\partial_1 F_{01} + \partial_2 F_{02} + \partial_2 F_{03}), \\
\mu = 1: \quad \chi^{1\nu\bar{\mu}}\partial_\nu F_{\bar{\mu}} &= -\varphi_1\partial_0 F_{01} + \varphi_2\partial_2 F_{12} - \varphi_2\partial_3 F_{31} \\
&= D_{01}^{010}\partial_0 F_{01} + D_{01}^{122}\partial_2 F_{12} + D_{01}^{313}\partial_3 F_{31}, \\
\mu = 2: \quad \chi^{2\nu\bar{\mu}}\partial_\nu F_{\bar{\mu}} &= -\varphi_1\partial_0 F_{02} - \varphi_2\partial_1 F_{12} + \varphi_2\partial_3 F_{23} \\
&= D_{02}^{020}\partial_0 F_{02} + D_{02}^{121}\partial_1 F_{12} + D_{02}^{233}\partial_3 F_{23}, \\
\mu = 3: \quad \chi^{3\nu\bar{\mu}}\partial_\nu F_{\bar{\mu}} &= -\varphi_1\partial_0 F_{03} + \varphi_2\partial_1 F_{31} - \varphi_2\partial_2 F_{23} \\
&= D_{03}^{030}\partial_0 F_{03} + D_{03}^{311}\partial_1 F_{31} + D_{03}^{232}\partial_2 F_{23}.
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Checking bihyperbolicity

The second set of generalized Maxwell equations, $\partial_{[\nu} F_{\rho\sigma]} = 0$, also gives $\binom{4}{3} = 4$ equations, again three of whom contribute to D:

$$\begin{aligned}
\nu = 0,\ \rho = 1,\ \sigma = 2: \quad 0 &= \partial_0 F_{12} - \partial_1 F_{02} + \partial_2 F_{01} \\
&= D_{12}^{120} \partial_0 F_{12} + D_{12}^{021} \partial_1 F_{02} + D_{12}^{012} \partial_2 F_{01}, \\
\nu = 0,\ \rho = 1,\ \sigma = 3: \quad 0 &= -\partial_0 F_{31} - \partial_1 F_{03} + \partial_3 F_{01} \\
&= D_{31}^{310} \partial_0 F_{31} + D_{31}^{031} \partial_1 F_{03} + D_{31}^{013} \partial_3 F_{01}, \\
\nu = 0,\ \rho = 2,\ \sigma = 3: \quad 0 &= \partial_0 F_{23} - \partial_2 F_{03} + \partial_3 F_{02} \\
&= D_{23}^{230} \partial_0 F_{23} + D_{23}^{032} \partial_2 F_{03} + D_{23}^{023} \partial_3 F_{02}, \\
\nu = 1,\ \rho = 2,\ \sigma = 3: \quad 0 &= \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}.
\end{aligned}$$

Overall, therefore, we obtain the 6×6 matrix

$$D^{\bar{\mu}\nu}_{\bar{\lambda}} p_\nu = \begin{bmatrix} \varphi_1 p_0 & 0 & 0 & 0 & \varphi_2 p_3 & -\varphi_2 p_2 \\ 0 & \varphi_1 p_0 & 0 & -\varphi_2 p_3 & 0 & \varphi_2 p_1 \\ 0 & 0 & \varphi_1 p_0 & \varphi_2 p_2 & -\varphi_2 p_1 & 0 \\ 0 & p_3 & -p_2 & p_0 & 0 & 0 \\ -p_3 & 0 & p_1 & 0 & p_0 & 0 \\ p_2 & -p_1 & 0 & 0 & 0 & p_0 \end{bmatrix}.$$

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whence the reduced principal polynomial becomes

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Thus, the cosmological van der Waals interaction is Lorentzian and hence bihyperbolic, albeit with a metric different from the cosmological background.

Concluding remarks

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Concluding remarks

1. Bihyperbolicity is a useful criterion to study the predictivity of modified theories.
2. The constitutive tensor is convenient to interpret modified electromagnetic theories in terms of optical effects, such as birefringence.
3. The cosmological van der Waals interaction for inflationary magnetogenesis is bihyperbolic, but dynamical extensions should be investigated.
4. Finally, using geometrodynamics, it is also possible to construct bihyperbolic gravitational dynamics from the kinematics. [cf. Giesel, Schuller, Witte & Wohlfarth (2012)]