Stueckelberg massive electromagnetism in de Sitter and anti-de Sitter spacetimes: Two-point functions and renormalized stress-energy tensors

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Outline

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Quantum field theory in curved spacetime

- The construction of the *quantum theory of gravitation* and the achievement of its *unification* with the other *fundamental interactions* are not an easy task, principally, because of the following conceptual and technical reasons:
  - while the other quantum fields propagate on spacetime, gravitation is spacetime geometry itself;
  - quantum theory of gravitation treated perturbatively with the methods of QFT is not *renormalizable*.
- However, the low energy consequences of *quantum gravity* can be studied by considering its semiclassical approximation defined in the following sense:
  - we treat classically the spacetime metric $g_{\mu\nu}$,
  - we consider from a quantum point of view all the other fields including the graviton field to at least one-loop order for reasons of consistency.
- Such an approach is called *QFT in curved spacetime*. It allows physicists to discover, e.g.,
  - particle creation in expanding universes [Parker (1969)],
  - the black hole radiance [Hawking (1975)].

Semiclassical Einstein equations and stress-energy-tensor operator $\widehat{T}_{\mu\nu}$

- In QFT in curved spacetime, it is conjectured that the *backreaction* of a quantum field in a normalized quantum state $|\psi\rangle$ on the spacetime geometry is governed by the *semiclassical Einstein equations*

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle.$$

  - $G_{\mu\nu}$ is the Einstein tensor $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}$ or its some higher-order generalization,
  - $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ denotes the *expectation value* of the *SET operator* $\widehat{T}_{\mu\nu}$ constructed from the quantum fields.

- Here, it is important to discuss the quantity $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$:
  - From the mathematical point of view, $\widehat{T}_{\mu\nu}$ is an *operator-valued distribution*.
  - $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ is ill-defined and formally infinite due to the "pathological" short-distance behavior of the *Green functions* associated with quantum fields.
  - It is necessary to extract from $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ a finite and physically acceptable contribution.
    - This can be done by *regularizing* it and then *renormalizing* all the coupling constants of the theory.
  - The corresponding quantity $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ which denotes the *renormalized expectation value* is of fundamental importance because
    - it acts as the source in the semiclassical Einstein equations,
    - it permits us to analyze the quantum state $|\psi\rangle$ without any reference to its particle content.

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[*] $\widehat{T}_{\mu\nu}$ is an operator quadratic in the quantum fields.

Semiclassical Einstein equations and stress-energy-tensor operator $\widehat{T}_{\mu\nu}$

- In QFT in curved spacetime, it is conjectured that the *backreaction* of a quantum field in a normalized quantum state $|\psi\rangle$ on the spacetime geometry is governed by the *semiclassical Einstein equations*

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  - $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ denotes the *expectation value* of the *SET operator* $\widehat{T}_{\mu\nu}$ constructed from the quantum fields.

- The *semiclassical Einstein equations* have been used
  - by Starobinsky (1980) to show that, after the Planck era, quantum effects lead to an inflationary universe, i.e., a universe with an exponentially expanding de Sitter phase,
  - by several authors to analyze the dynamics of evaporating black holes due to Hawking radiation [see, e.g., Bardeen (1981), Hiscock (1981), etc.],
  - to explain the acceleration of the expansion of the universe [see, e.g., Parker and Vanzella (2004), Wang, Zhu and Unruh (2017)].

Remarks relative to regularization and renormalization

- *Regularization* and *renormalization* in curved spacetime are necessarily based on *representations of Green functions* in *coordinate space*.
  - Indeed, in an arbitrary gravitational background, the lack of symmetries as well as spacetime curvature prevent us from working within the framework of the Fourier transform.
- Currently, there exits various techniques of *regularization* and *renormalization* developed in the context of *QFT in curved spacetime*, e.g.,
  - *adiabatic regularization*,
  - *dimensional regularization*,
  - ζ*-function approach*,
  - *DeWitt-Schwinger approximation*,
  - *point-splitting method* and its extension to the so-called *Hadamard renormalization*.
- In the context of this presentation, we use *Hadamard renormalization*.

Remarks relative to regularization and renormalization

- In the context of *Hadamard renormalization*, we need
  - the concepts of *biscalars*, *bivectors* and, more generally, *bitensors*,
  - their *covariant Taylor series expansions*, e.g.,

$$\Delta^{1/2} = 1 + \frac{1}{12} R_{ab}\sigma^{;a}\sigma^{;b} - \frac{1}{24} R_{ab;c}\sigma^{;a}\sigma^{;b}\sigma^{;c} + \left[\frac{1}{80} R_{ab;cd} + \frac{1}{360} R^{p}{}_{aqb} R^{q}{}_{cpd} + \frac{1}{288} R_{ab}R_{cd}\right]\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}$$
$$- \left[\frac{1}{360} R_{ab;cde} + \frac{1}{360} R^{p}{}_{aqb} R^{q}{}_{cpd;e} + \frac{1}{288} R_{ab}R_{cd;e}\right]\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d}\sigma^{;e} + O(\sigma^{3}).$$

- We also recall some definitions which are important in the context of this *renormalization*:
  - The *geodetic interval* $\sigma(x,x') = \frac{1}{2} s^2(x,x')$, where $s(x,x')$ is the *geodesic distance* between x and x', satisfies
$$2\sigma = \sigma_{;\mu}\sigma^{;\mu}.$$
  - The *Van Vleck-Morette determinant* $\Delta(x,x') = -\sqrt{-g(x)}\det\left[-\sigma_{;\mu\nu'}(x,x')\right]\sqrt{-g(x')}$, which can be interpreted as a measure of the tidal focussing and defocussing of geodesic flows in spacetime, satisfies
$$\Box_x\sigma = 4 - 2\Delta^{-1/2}\Delta^{1/2}{}_{;\mu}\sigma^{;\mu} \quad \text{with the boundary condition} \quad \lim_{x'\to x}\Delta(x,x') = 1.$$
  - The *bivector of parallel transport* $g_{\mu\nu'}(x,x')$, e.g., of a bitensor along the geodesic s from x to x', is defined by
$$g_{\mu\nu';\rho}\sigma^{;\rho} = 0 \quad \text{with the boundary condition} \quad \lim_{x'\to x} g_{\mu\nu'}(x,x') = g_{\mu\nu}(x).$$

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[*] We have $\sigma(x,x') < 0$ if x and x' are timelike related, $\sigma(x,x') = 0$ if x and x' are null related and $\sigma(x,x') > 0$ if x and x' are spacelike related.

Outline

Massive electromagnetism

- The *EM* interaction is generally assumed to be mediated by a massless photon, which is mainly justified by
  - the theoretical and practical successes of the classical *Maxwell's theory* of *EM* and its extension in the framework of *QFT*,
  - the upper limits on the photon mass $m \leq 10^{-18}$ eV $\approx 2\times 10^{-54}$ kg which is currently one of the most reliable results evaluated by the various terrestrial and extraterrestrial experiments.
- However, it is interesting to consider the possibility of a massive but ultralight photon.
  - The small value of the upper limit on m does not necessarily imply that the photon mass is exactly zero.
  - In order to test the masslessness of the photon, i.e., to impose experimental constraints on its mass, it is necessary to have a good understanding of the various *massive non-Maxwellian theories*.
  - *Massive EM* can be rather easily included in the Standard Model of particle physics.
- In the following, we discuss *de Broglie-Proca massive EM* and *Stueckelberg massive EM*.

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* In general, it is the *de Broglie-Proca theory* that is used to impose experimental constraints on the photon mass.

De Broglie-Proca massive electromagnetism

- *De Broglie-Proca massive EM* is the simplest generalization of *Maxwell's EM*.
  - This theory is described by a vector field A_μ of mass m.
  - Its action is given by

$$S[A_\mu, g_{\mu\nu}] = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A^\mu A_\mu \right].$$

  - A_μ satisfies the Proca equation

$$\nabla^\nu F_{\mu\nu} + m^2 A_\mu = 0.$$

- It is worth pointing out that, due to the mass term,
  - contrary to *Maxwell's theory* which is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \nabla_\mu \Lambda$$

    for an arbitrary scalar field Λ, this gauge invariance is broken for the *de Broglie-Proca theory*;
  - there are some important consequences when we compare, in the limit $m^2 \to 0$, the results obtained via the *de Broglie-Proca theory* with those derived from *Maxwell's theory* (e.g., discontinuities of the Green functions, ...).

Stueckelberg massive electromagnetism

- *Stueckelberg massive EM* preserves the local $U(1)$ gauge invariance of *Maxwell's EM*.
  - This theory is constructed in such a way that a massive vector field A_μ is coupled appropriately with an *auxiliary scalar field* Φ.
  - At the classical level, its action is given by

$$S_{\mathrm{cl}}\left[A_\mu,\Phi,g_{\mu\nu}\right]=\int_{\mathcal{M}}d^4x\sqrt{-g}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}m^2\left(A^\mu+\frac{1}{m}\nabla^\mu\Phi\right)\left(A_\mu+\frac{1}{m}\nabla_\mu\Phi\right)\right].$$

  - This action is invariant under the gauge transformation

$$A_\mu\to A'_\mu=A_\mu+\nabla_\mu\Lambda,$$
$$\Phi\to\Phi'=\Phi-m\Lambda$$

    for an arbitrary scalar field Λ.
  - A_μ and Φ satisfy two coupled wave equations

$$\nabla^\nu F_{\mu\nu}+m^2A_\mu+m\nabla_\mu\Phi=0,$$
$$\Box\Phi+m\nabla^\mu A_\mu=0.$$

- The Stueckelberg action can be constructed from the de Broglie-Proca action by using the substitution

$$A_\mu\to A_\mu+\frac{1}{m}\nabla_\mu\Phi.$$

Some remarks relative to both theories

- It is worth noting that
  - the *de Broglie-Proca EM* can be obtained from *Stueckelberg EM* by taking
  $$\Phi = 0;$$
  - therefore, the *de Broglie-Proca theory* is nothing other than the *Stueckelberg gauge theory* in this particular gauge;
  - however, this is a "bad" choice of gauge leading to some complications;
  - indeed, in this gauge, we obtain
  $$\nabla^\mu A_\mu = 0.$$
  Due to this constraint, at the quantum level, the Feynman propagator does not admit a Hadamard representation and, as a consequence, in the *de Broglie-Proca theory*, we cannot deal directly with Hadamard quantum states (i.e., of states mimicking in the UV regime the behavior of the Poincaré vacuum in Minkowski spacetime).

- In order to treat these theories at the quantum level,
  - the action S of the *de Broglie-Proca theory* is directly relevant,
  - while it is necessary to add to the action S_{Cl} of the *Stueckelberg theory* a gauge-breaking term S_{GB} and the compensating ghost contribution S_{Gh}.

Outline

Stueckelberg massive electromagnetism

- The quantum action S of *Stueckelberg massive EM* is given by

$$S\left[A_\mu,\Phi,C,C^*,g_{\mu\nu}\right]=S_A\left[A_\mu,g_{\mu\nu}\right]+S_\Phi\left[\Phi,g_{\mu\nu}\right]+S_{\mathrm{Gh}}\left[C,C^*,g_{\mu\nu}\right]$$

with

$$S_A\left[A_\mu,g_{\mu\nu}\right]=\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}m^2A^\mu A_\mu-\frac{1}{2}\left(\nabla^\mu A_\mu\right)^2\right],$$

$$S_\Phi\left[\Phi,g_{\mu\nu}\right]=\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}\nabla^\mu\Phi\nabla_\mu\Phi-\frac{1}{2}m^2\Phi^2\right],$$

$$S_{\mathrm{Gh}}\left[C,C^*,g_{\mu\nu}\right]=\int_{\mathcal{M}}\mathrm{d}^4x\sqrt{-g}\left[\nabla^\mu C^*\nabla_\mu C+m^2C^*C\right].$$

- The wave equations are given by
  - $\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu}=\left[g^{\mu\nu}\Box-m^2g^{\mu\nu}-R^{\mu\nu}\right]A_\nu=0$ for the massive vector field A_μ,
  - $\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\Phi}=\left[\Box-m^2\right]\Phi=0$ for the auxiliary scalar field Φ,
  - $\frac{1}{\sqrt{-g}}\frac{\delta_{\mathrm{L}}S}{\delta C^*}=-\left[\Box-m^2\right]C=0$ and $\frac{1}{\sqrt{-g}}\frac{\delta_{\mathrm{R}}S}{\delta C}=-\left[\Box-m^2\right]C^*=0$ for the ghost fields C and C^*.

Hadamard Green functions $G^{(1)}$ and Ward identities

- From now on, we shall assume that the *Stueckelberg field theory* has been quantized and is in a *normalized quantum state* $|\psi\rangle$.

- In the context of the *regularization* of the *expectation value* $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$, we use the *Hadamard Green functions* $G^{(1)}$ defined by
  - $G^{(1)A}_{\mu\nu'}(x,x') = \langle\psi|\{A_\mu(x), A_{\nu'}(x')\}|\psi\rangle$ that is a solution of the wave equation
  $$\left[g_\mu{}^\nu \Box_x - R_\mu{}^\nu - m^2 g_\mu{}^\nu\right] G^{(1)A}_{\nu\rho'}(x,x') = 0,$$
  - $G^{(1)\Phi}(x,x') = \langle\psi|\{\Phi(x), \Phi(x')\}|\psi\rangle$ that is a solution of the wave equation
  $$\left[\Box_x - m^2\right] G^{(1)\Phi}(x,x') = 0,$$
  - $G^{(1)\mathrm{Gh}}(x,x') = \langle\psi|[C^*(x), C(x')]|\psi\rangle$ that is a solution of the wave equation
  $$\left[\Box_x - m^2\right] G^{(1)\mathrm{Gh}}(x,x') = 0.$$

- These three *two-point functions* are related by two *Ward identities* given by
$$\nabla^\mu G^{(1)A}_{\mu\nu'}(x,x') + \nabla_{\nu'} G^{(1)\mathrm{Gh}}(x,x') = 0$$
and
$$G^{(1)\Phi}(x,x') - G^{(1)\mathrm{Gh}}(x,x') = 0 \quad \Rightarrow \quad G^{(1)}(x,x') \equiv G^{(1)\Phi}(x,x') = G^{(1)\mathrm{Gh}}(x,x').$$

Hadamard representation of the Green function $G^{(1)}$ associated with the scalar field or the ghost fields

- We now assume that the *quantum state* $|\psi\rangle$ is of *Hadamard type*.

- The *Hadamard form* of $G^{(1)}$ for the scalar field Φ or the ghost fields:

$$G^{(1)}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln|\sigma(x,x')| + W(x,x')\right).$$

  - $V(x,x')$ is a *symmetric* and *regular* biscalar defined
    - by the series expansions $V(x,x') = \sum_{n=0}^{+\infty} V_n(x,x')\sigma^n(x,x')$,
    - by the *recursion relations* satisfied by the *geometrical Hadamard coefficients* $V_n(x,x')$ (all these coefficients can be determined uniquely by the *recursion relations*).
  - $W(x,x')$ is a *symmetric* and *regular* biscalar defined
    - by the series expansions $W(x,x') = \sum_{n=0}^{+\infty} W_n(x,x')\sigma^n(x,x')$,
    - by the *recursion relations* satisfied by the *state-dependent Hadamard coefficients* $W_n(x,x')$ (the first coefficient $W_0(x,x')$ is unrestrained by the *recursion relations* and, therefore, can be used to encode the *quantum state*).

Hadamard representation of the Green function $G^{(1)}$ associated with the scalar field or the ghost fields

- We now assume that the *quantum state* $|\psi\rangle$ is of *Hadamard type*.
- The *Hadamard form* of $G^{(1)}$ for the scalar field Φ or the ghost fields:

$$G^{(1)}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln|\sigma(x,x')| + W(x,x')\right).$$

  - The *Hadamard representation* of $G^{(1)}$ permits us to straightforwardly identify their *singular* and *regular* parts when the coincidence limit $x' \to x$ is considered.
    - A *purely geometrical singular part* takes the form

$$G^{(1)}_{\text{sing}}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} + V(x,x')\ln|\sigma(x,x')|\right).$$

    - A *regular state-dependent part* is given by

$$G^{(1)}_{\text{reg}}(x,x') = G^{(1)}(x,x') - G^{(1)}_{\text{sing}}(x,x') = \frac{1}{4\pi^2}\, W(x,x').$$

Hadamard representation of the Green function $G^{(1)}$ associated with the vector field

- We now assume that the *quantum state* $|\psi\rangle$ is of *Hadamard type*.
- The *Hadamard form* of $G^{(1)}$ for the vector field A_μ:

$$G^{(1)A}_{\mu\nu'}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')}g_{\mu\nu'}(x,x') + V_{\mu\nu'}(x,x')\ln|\sigma(x,x')| + W_{\mu\nu'}(x,x')\right).$$

  - $V_{\mu\nu'}(x,x')$ is a *symmetric* and *regular* bivector defined
    - by the series expansions $V_{\mu\nu'}(x,x') = \sum_{n=0}^{+\infty} V_{n\,\mu\nu'}(x,x')\sigma^n(x,x')$,
    - by the *recursion relations* satisfied by the *geometrical Hadamard coefficients* $V_{n\,\mu\nu'}(x,x')$

      (all these coefficients can be determined uniquely by the *recursion relations*).
  - $W_{\mu\nu'}(x,x')$ is a *symmetric* and *regular* bivector defined
    - by the series expansions $W_{\mu\nu'}(x,x') = \sum_{n=0}^{+\infty} W_{n\,\mu\nu'}(x,x')\sigma^n(x,x')$,
    - by the *recursion relations* satisfied by the *state-dependent Hadamard coefficients* $W_{n\,\mu\nu'}(x,x')$

      (the first coefficient $W_{0\,\mu\nu'}(x,x')$ is unrestrained by the *recursion relations* and, therefore, can be used to encode the *quantum state*).

Hadamard representation of the Green function $G^{(1)}$ associated with the vector field

- We now assume that the *quantum state* $|\psi\rangle$ is of *Hadamard type*.

- The *Hadamard form* of $G^{(1)}$ for the vector field A_μ:

$$G^{(1)A}_{\mu\nu'}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} g_{\mu\nu'}(x,x') + V_{\mu\nu'}(x,x')\ln|\sigma(x,x')| + W_{\mu\nu'}(x,x')\right).$$

  - The *Hadamard representation* of $G^{(1)}$ permits us to straightforwardly identify their *singular* and *regular* parts when the coincidence limit $x' \to x$ is considered.
    - A *purely geometrical singular part* takes the form

$$G^{(1)A}_{\mathrm{sing}\,\mu\nu'}(x,x') = \frac{1}{4\pi^2}\left(\frac{\Delta^{1/2}(x,x')}{\sigma(x,x')} g_{\mu\nu'}(x,x') + V_{\mu\nu'}(x,x')\ln|\sigma(x,x')|\right).$$

    - A *regular state-dependent part* is given by

$$G^{(1)A}_{\mathrm{reg}\,\mu\nu'}(x,x') = G^{(1)A}_{\mu\nu'}(x,x') - G^{(1)A}_{\mathrm{sing}\,\mu\nu'}(x,x') = \frac{1}{4\pi^2} W_{\mu\nu'}(x,x').$$

Geometrical Hadamard coefficients and their covariant Taylor series expansions

- The *geometrical Hadamard coefficients* $V_n(x,x')$ and $V_{n\,\mu\nu'}(x,x')$ can be determined explicitly from the associated *recursion relations* up to necessary order by taking their *covariant Taylor series expansions*.
  - The expansions of the *symmetric* biscalar coefficients $V_0(x,x')$ and $V_1(x,x')$ are given by

$$V_0 = v_0 - \left\{(1/2)v_{0\,;a}\right\}\sigma^{;a} + \frac{1}{2!}\,v_{0ab}\,\sigma^{;a}\sigma^{;b} + O\left(\sigma^{3/2}\right)$$

and

$$V_1 = v_1 + O\left(\sigma^{1/2}\right).$$

  - The expansions of the *symmetric* bivector coefficients $V_{0\,\mu\nu'}(x,x')$ and $V_{1\,\mu\nu'}(x,x')$ are given by

$$V_{0\,\mu\nu} = g_\nu{}^{\nu'}\,V_{0\,\mu\nu'} = v_{0(\mu\nu)} - \left\{(1/2)v_{0(\mu\nu);a} + v_{0[\mu\nu]a}\right\}\sigma^{;a} + \frac{1}{2!}\left\{v_{0(\mu\nu)ab} + v_{0[\mu\nu]a\,;b}\right\}\sigma^{;a}\sigma^{;b} + O\left(\sigma^{3/2}\right)$$

and

$$V_{1\,\mu\nu} = g_\nu{}^{\nu'}\,V_{1\,\mu\nu'} = v_{1(\mu\nu)} + O\left(\sigma^{1/2}\right).$$

  - The *Taylor coefficients* $v_{n\,\ldots}$ appearing in these series expansions are expressed in term of the *Riemann tensor* and its *covariant derivatives*, e.g., we have

$$v_{1(\mu\nu)} = (1/4)m^2R_{\mu\nu} - (1/24)\Box R_{\mu\nu} - (1/24)RR_{\mu\nu} + (1/8)R_{\mu p}R_\nu{}^p - (1/48)R_{\mu pqr}R_\nu{}^{pqr}$$
$$+g_{\mu\nu}\left\{(1/8)m^4 - (1/24)m^2R + (1/120)\Box R + (1/288)R^2 - (1/720)R_{pq}R^{pq} + (1/720)R_{pqrs}R^{pqrs}\right\}.$$

State-dependent Hadamard coefficients and their covariant Taylor series expansions

- Unlike the *geometrical Hadamard coefficients*, the *state-dependent Hadamard coefficients* $W_n(x,x')$ and $W_{n\,\mu\nu'}(x,x')$ are neither uniquely defined nor purely geometrical.

- Instead of working with the *state-dependent Hadamard coefficients*, we shall use the *covariant Taylor series expansions* of the sums $W(x,x')$ and $W_{\mu\nu'}(x,x')$ up to order $\sigma^{3/2}$.
  - The expansions of the *symmetric* biscalar $W(x,x')$ and the *symmetric* bivector $W_{\mu\nu'}(x,x')$ are given by

$$W = w - \{(1/2)w_{;a}\}\sigma^{;a} + \frac{1}{2!}w_{ab}\,\sigma^{;a}\sigma^{;b} - \frac{1}{3!}\left\{(3/2)w_{ab;c} - (1/4)w_{;abc}\right\}\sigma^{;a}\sigma^{;b}\sigma^{;c} + O(\sigma^2),$$

$$W_{\mu\nu} = g_{\nu}{}^{\nu'}\,W_{\mu\nu'} = s_{\mu\nu} - \{(1/2)s_{\mu\nu;a} + a_{\mu\nu a}\}\sigma^{;a} + \frac{1}{2!}\left\{s_{\mu\nu ab} + a_{\mu\nu a;b}\right\}\sigma^{;a}\sigma^{;b}$$
$$- \frac{1}{3!}\left\{(3/2)s_{\mu\nu ab;c} - (1/4)s_{\mu\nu;abc} + a_{\mu\nu abc}\right\}\sigma^{;a}\sigma^{;b}\sigma^{;c} + O(\sigma^2).$$

  - With practical applications in mind, it is interesting to express some of the *Taylor coefficients* in term of the bitensors $W(x,x')$ and $W_{\mu\nu'}(x,x')$ by inverting the associated *Taylor expansions*.

$$w(x) = \lim_{x'\to x} W(x,x'),$$
$$w_{ab}(x) = \lim_{x'\to x} W_{;(a'b')}(x,x')$$

and

$$s_{\mu\nu}(x) = \lim_{x'\to x} W_{\mu\nu'}(x,x'),$$
$$a_{\mu\nu a}(x) = \frac{1}{2}\lim_{x'\to x}\left[W_{\mu\nu';a'}(x,x') - W_{\mu\nu';a}(x,x')\right],$$
$$s_{\mu\nu ab}(x) = \frac{1}{2}\lim_{x'\to x}\left[W_{\mu\nu';(a'b')}(x,x') + W_{\mu\nu';(ab)}(x,x')\right].$$

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*We adopt the following notations for $s_{\mu\nu a_1\cdots a_p} \equiv w_{(\mu\nu)a_1\cdots a_p}$ and $a_{\mu\nu a_1\cdots a_p} \equiv w_{[\mu\nu]a_1\cdots a_p}$.

Outline

Stress-energy tensor

- The *SET* associated with the quantum action S of the *Stueckelberg theory* is defined by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S\left[A_\mu, \Phi, C, C^*, g_{\mu\nu}\right].$$

- Its explicit expression is given by

$$T^{\mu\nu} = T_A^{\mu\nu} + T_\Phi^{\mu\nu} + T_{\text{Gh}}^{\mu\nu},$$

where three field contributions take the forms

$$T_A^{\mu\nu} = F^{\mu}{}_{\rho} F^{\nu\rho} + m^2 A^\mu A^\nu - 2A^{(\mu}\nabla^{\nu)}\nabla_\rho A^\rho - (1/4) g^{\mu\nu}\left\{F_{\rho\tau}F^{\rho\tau} + 2m^2 A_\rho A^\rho - 4A_\rho \nabla^\rho \nabla_\tau A^\tau - 2\left(\nabla_\rho A^\rho\right)^2\right\},$$

$$T_\Phi^{\mu\nu} = \nabla^\mu \Phi \nabla^\nu \Phi - (1/2) g^{\mu\nu}\left\{\nabla_\rho \Phi \nabla^\rho \Phi + m^2 \Phi^2\right\},$$

$$T_{\text{Gh}}^{\mu\nu} = -2\nabla^{(\mu|} C^* \nabla^{|\nu)} C + g^{\mu\nu}\left\{\nabla_\rho C^* \nabla^\rho C + m^2 C^* C\right\}.$$

- By construction, the *SET* is conserved, i.e., $\nabla_\nu T^{\mu\nu} = 0$.

Expectation value of the stress-energy-tensor operator

- It is necessary to recall that, at the quantum level,
  - all fields as well as the associated *SET* are operators: we denote by $\widehat{T}_{\mu\nu}$ the *SET operator*,
  - its *expectation value* with respect to the Hadamard quantum state $|\psi\rangle$ is denoted by $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$.
- The *expectation value* $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ can be constructed by using *point-splitting method*.
  - $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ corresponding to the expression $T_{\mu\nu}$ becomes

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle = \langle\psi|\widehat{T}^{A}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\Phi}_{\mu\nu}|\psi\rangle + \langle\psi|\widehat{T}^{\mathrm{Gh}}_{\mu\nu}|\psi\rangle,$$

where the three contributions are given by

$$\langle\psi|\widehat{T}^{A}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathscr{T}^{A\;\rho\sigma'}_{\mu\nu}(x,x')\Big[G^{(1)A}_{\rho\sigma'}(x,x')\Big],$$

$$\langle\psi|\widehat{T}^{\Phi}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathscr{T}^{\Phi}_{\mu\nu}(x,x')\Big[G^{(1)\Phi}(x,x')\Big],$$

$$\langle\psi|\widehat{T}^{\mathrm{Gh}}_{\mu\nu}(x)|\psi\rangle = \frac{1}{2}\lim_{x'\to x} \mathscr{T}^{\mathrm{Gh}}_{\mu\nu}(x,x')\Big[G^{(1)\mathrm{Gh}}(x,x')\Big].$$

Here, $\mathscr{T}^{A\;\rho\sigma'}_{\mu\nu}$, $\mathscr{T}^{\Phi}_{\mu\nu}$ and $\mathscr{T}^{\mathrm{Gh}}_{\mu\nu}$ are the *differential operators* constructed by *point splitting* from the formal expressions $T^{\mu\nu}_{A}$, $T^{\mu\nu}_{\phi}$ and $T^{\mu\nu}_{\mathrm{Gh}}$.

- $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle$ is divergent due to the *singular short-distance behavior* of the *Green functions*.

Renormalized expectation value of the stress-energy-tensor operator

- The *renormalized expectation value* $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ can be constructed by using *Hadamard renormalization*, which consists of using the prescription proposed by Wald, i.e.,
  - to discard the *singular contributions*, i.e., to make the replacements $G^{(1)} \to G^{(1)}_{\text{reg}} = \frac{1}{4\pi^2} W$,
  - to add to the result a *state-independent tensor* $\widetilde{\Theta}_{\mu\nu}$ which
    - only depends on the mass parameter and on the local geometry,
    - ensures the conservation of the final expression.

- $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ takes the form:

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2} \lim_{x'\to x} \mathscr{T}^{A\ \rho\sigma'}_{\mu\nu}(x,x') \left[W^{A}_{\rho\sigma'}(x,x')\right] + \frac{1}{8\pi^2} \lim_{x'\to x} \mathscr{T}^{\Phi}_{\mu\nu}(x,x') \left[W^{\Phi}(x,x')\right]$$
$$+ \frac{1}{8\pi^2} \lim_{x'\to x} \mathscr{T}^{\text{Gh}}_{\mu\nu}(x,x') \left[W^{\text{Gh}}(x,x')\right] + \widetilde{\Theta}_{\mu\nu}.$$

- Its explicit expression
  - is obtained by expanding the *Hadamard coefficients* in *covariant Taylor series*,
  - is simplified by using some relations between the *Taylor coefficients* involved.

Final expression of the renormalized stress-energy tensor

- The main expression which only involves *state-dependent* and *geometrical* quantities associated with the massive vector field A_μ is given by

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2}\Big\{(1/2)s_\rho{}^\rho{}_{;\mu\nu} + (1/2)\Box s_{\mu\nu} - s_{\rho(\mu;\nu)}{}^\rho + (1/2)R^\rho{}_{(\mu}s_{\nu)\rho} - (1/2)a_\mu{}^\rho{}_{(\nu;\rho)} - (1/2)a_\nu{}^\rho{}_{(\mu;\rho)}$$
$$- a_\mu{}^\rho{}_{[\nu;\rho]} - a_\nu{}^\rho{}_{[\mu;\rho]} - s_\rho{}^\rho{}_{\mu\nu} + s_{\rho(\mu\nu)}{}^\rho - (1/2)g_{\mu\nu}\Big[(1/2)\Box s_\rho{}^\rho - (1/2)s_{\rho\tau}{}^{;\rho\tau} - a_{\rho\tau}{}^{\rho;\tau}\Big] + v_{1\mu\nu} - g_{\mu\nu}v_1{}_\rho{}^\rho\Big\} + \Theta_{\mu\nu}.$$

  - Here, by using the Ward identities, any reference to the auxiliary scalar field Φ has be removed.
  - This result does not reduce, in the limit $m^2 \to 0$, to the result obtained from *Maxwell's theory* because it involves implicitly the contribution of the auxiliary scalar field Φ.

- It should be recalled that we have

$$w(x) = \lim_{x'\to x} W(x,x'), \qquad w_{ab}(x) = \lim_{x'\to x} W_{;(a'b')}(x,x')$$

and

$$s_{\mu\nu}(x) = \lim_{x'\to x} W_{\mu\nu'}(x,x'),$$
$$a_{\mu\nu a}(x) = \frac{1}{2}\lim_{x'\to x}\Big[W_{\mu\nu';a'}(x,x') - W_{\mu\nu';a}(x,x')\Big],$$
$$s_{\mu\nu ab}(x) = \frac{1}{2}\lim_{x'\to x}\Big[W_{\mu\nu';(a'b')}(x,x') + W_{\mu\nu';(ab)}(x,x')\Big].$$

*In some sense, the auxiliary scalar field Φ plays the role of a kind of ghost field.

Final expression of the renormalized stress-energy tensor

- The main expression which only involves state-dependent and geometrical quantities associated with the massive vector field A_μ is given by

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2}\Big\{(1/2)s_{\rho}{}^{\rho}{}_{;\mu\nu} + (1/2)\Box s_{\mu\nu} - s_{\rho(\mu;\nu)}{}^{\rho} + (1/2)R^{\rho}{}_{(\mu}s_{\nu)\rho} - (1/2)a_{\mu}{}^{\rho}{}_{(\nu;\rho)} - (1/2)a_{\nu}{}^{\rho}{}_{(\mu;\rho)}$$
$$- a_{\mu}{}^{\rho}{}_{[\nu;\rho]} - a_{\nu}{}^{\rho}{}_{[\mu;\rho]} - s_{\rho}{}^{\rho}{}_{\mu\nu} + s_{\rho(\mu\nu)}{}^{\rho} - (1/2)g_{\mu\nu}\left[(1/2)\Box s_{\rho}{}^{\rho} - (1/2)s_{\rho\tau}{}^{;\rho\tau} - a_{\rho\tau}{}^{\rho;\tau}\right] + v_{1\mu\nu} - g_{\mu\nu}v_{1\rho}{}^{\rho}\Big\} + \Theta_{\mu\nu}.$$

  - Here, by using the Ward identities, any reference to the auxiliary scalar field Φ has be removed.
  - This result does not reduce, in the limit $m^2 \to 0$, to the result obtained from *Maxwell's theory* because it involves implicitly the contribution of the auxiliary scalar field Φ.

- In this result, $\Theta_{\mu\nu}$ is a local conserved tensor which can be expressed in the form

$$\Theta_{\mu\nu} = \frac{1}{8\pi^2}\left\{\alpha m^4 g_{\mu\nu} + \beta m^2\left[R_{\mu\nu} - (1/2)Rg_{\mu\nu}\right] + \gamma_1\,{}^{(1)}H_{\mu\nu} + \gamma_2\,{}^{(2)}H_{\mu\nu}\right\},$$

where the constants α, β, γ_1 and γ_2 can be fixed by imposing additional physical conditions on $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$.

  - $\Theta_{\mu\nu}$ represent the general form of the ambiguities in $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$.
  - It includes a term $\Theta_{\mu\nu}(M)$ associated with the *renormalization mass* M which is introduced in order to make dimensionless the argument of the logarithm in the *Hadamard representation* of the *Green functions*, i.e., $\ln\left|M^2\sigma(x,x')\right|$.

* In some sense, the auxiliary scalar field Φ plays the role of a kind of ghost field.

Final expression of the renormalized stress-energy tensor

- It is possible to split $\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}}$ in the form

$$\langle\psi|\widehat{T}_{\mu\nu}|\psi\rangle_{\text{ren}} = \langle\psi|\widehat{T}^{A}_{\mu\nu}|\psi\rangle_{\text{ren}} + \langle\psi|\widehat{T}^{\Phi}_{\mu\nu}|\psi\rangle_{\text{ren}},$$

where two conserved contributions associated with the vector and scalar fields are given by

$$\langle\psi|\widehat{T}^{A}_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2}\Big\{(1/2){s_\rho}^{\rho}{}_{;\mu\nu} + (1/2)\Box s_{\mu\nu} - {s_{\rho(\mu;\nu)}}^{\rho} - {a_\mu}^{\rho}{}_{[\nu;\rho]} - {a_\nu}^{\rho}{}_{[\mu;\rho]} - {s_\rho}^{\rho}{}_{\mu\nu} + 2{s_{\rho(\mu\nu)}}^{\rho} - (1/2)g_{\mu\nu}\Big[(1/2)\Box {s_\rho}^{\rho} - 2{a_{\rho\tau}}^{\rho;\tau}\Big] + 2v^{A}_{1\,\mu\nu} - g_{\mu\nu}{v^{A}_{1\,\rho}}^{\rho}\Big\} + \Theta^{A}_{\mu\nu},$$

$$\langle\psi|\widehat{T}^{\Phi}_{\mu\nu}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2}\Big\{(1/2)w_{;\mu\nu} - w_{\mu\nu} - (1/4)g_{\mu\nu}\Box w - g_{\mu\nu}v_1\Big\} + \Theta^{\Phi}_{\mu\nu}.$$

- Here, in the limit $m^2 \to 0$ and by assuming that $m^2 w \to 0$, the term $\langle\psi|\widehat{T}^{A}_{\mu\nu}|\psi\rangle_{\text{ren}}$ reduces to the result obtained from *Maxwell's theory* and, therefore, we recover the associated trace anomaly given by

$$\langle\psi|{{\widehat{T}^{A}}_{\rho}}^{\rho}|\psi\rangle_{\text{ren}} = \frac{1}{8\pi^2}\Big\{2{v_{1\rho}}^{\rho} - 4v_1\Big\} = \frac{1}{8\pi^2}\Big\{-(1/20)\Box R - (5/72)R^2 + (11/45)R_{pq}R^{pq} - (13/360)R_{pqrs}R^{pqrs}\Big\}.$$

- However, this is an artificial way to split the contributions of the vector and scalar fields.

Outline

General considerations

- Here, we shall provide the exact analytical expression for $\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren}}$ associated with the massive vector field propagating in
  - the four-dimensional *de Sitter spacetime* (dS^4),
  - the four-dimensional *anti-de Sitter spacetime* (AdS^4).
- Such results do not exist in the literature due to the fact that the *two-point functions* are in general constructed in the framework of the *de Broglie-Proca theory*.
- These results could have interesting implications in cosmology of the very early Universe or in the context of the AdS/CFT correspondence.
- dS^4 and AdS^4:
  - These spacetimes are locally characterized by the relations

$$R_{\mu\nu\rho\tau} = (R/12)\left(g_{\mu\rho}g_{\nu\tau} - g_{\mu\tau}g_{\nu\rho}\right), \quad R_{\mu\nu} = (R/4)g_{\mu\nu} \quad \text{and} \quad R = \begin{cases} +12H^2 & \text{for dS}^4, \\ -12K^2 & \text{for AdS}^4, \end{cases}$$

    where H and K are two positive constants of dimension $(\text{length})^{-1}$,
  - They can be realized as the four-dimensional hyperboloids

$$\eta_{ab}X^aX^b = 12/R$$

    embedded in the flat five-dimensional space $\mathbb{R}^5$ equipped with the metric $\eta_{ab} = \text{diag}(-1,+1,+1,+1,+1)$ for dS^4 and $\eta_{ab} = \text{diag}(-1,-1,+1,+1,+1)$ for AdS^4.

General considerations

- dS^4 and AdS^4:
  - Instead of working with the *geodetic interval* $\sigma(x,x')$, it is advantageous to consider

$$z(x,x') = \frac{1}{2}\left[1+(R/12)\eta_{ab}X^a(x)X^b(x')\right]$$
$$= \cos^2\sqrt{(R/24)\,\sigma(x,x')}.$$

  - With respect to the *antipodal transformation* P which sends the point x with coordinates $X^a(x)$ on the hyperboloid to its antipodal point Px with coordinates $X^a(Px) = -X^a(x)$, we have

$$z(x,Px') = 1 - z(x,x').$$

  - In order to construct the *two-point functions* of *Stueckelberg EM*,
    - we assuming that the vacuum $|0\rangle$ is a maximally symmetric quantum state;
    - we solve the wave equations by taking into account, as constraints, two Ward identities;
    - we then fix the remaining integration constants by imposing:

    (i) Hadamard-type singularities at short distance,

    (ii) in dS^4, the regularity of the solutions at Px,

    (iii) in AdS^4, that the solutions fall off as fast as possible at spatial infinity.

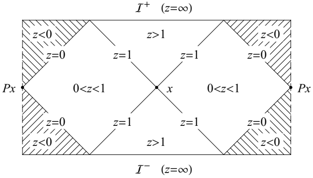

Figure: Carter-Penrose diagram of dS^4

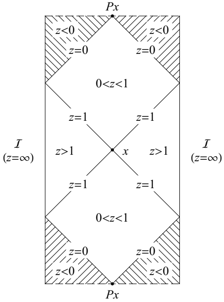

Figure: Carter-Penrose diagram of AdS^4

Renormalized stress-energy tensor in dS

- The *Hadamard Green function* associated with the massive vector field propagating in dS^4 is given in term of the *real* part of the *hypergeometric function* $F(a,b;c;z)$ on the *branch cut* denoted by $(\mathrm{Re}F)(a,b;c;z)$:

$$\begin{aligned} G^{(1)A}_{\mu\nu'}(x,x') = \frac{R}{192\pi}\Bigg[&\frac{9/4-\lambda^2}{\cos(\pi\lambda)} z(1-z)(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;z) + \frac{3}{2}\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}(1-2z)(\mathrm{Re}F)(5/2+\lambda,5/2-\lambda;3;z) \\ &- \frac{1}{2}\frac{(1/4-\kappa^2)}{\cos(\pi\kappa)}(\mathrm{Re}F)(5/2+\kappa,5/2-\kappa;3;z)\Bigg] g_{\mu\nu'} \\ + \frac{1}{16\pi}\Bigg[&\frac{9/4-\lambda^2}{\cos(\pi\lambda)}(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;z) + 3\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}(1/z)(\mathrm{Re}F)(5/2+\lambda,5/2-\lambda;3;z) \\ &- \frac{1/4-\kappa^2}{\cos(\pi\kappa)}(\mathrm{Re}F)'(5/2+\kappa,5/2-\kappa;3;z) - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}(1/z)(\mathrm{Re}F)(5/2+\kappa,5/2-\kappa;3;z)\Bigg] z_{;\mu} z_{;\nu'} \end{aligned}$$

with $\lambda = \sqrt{1/4 - 12m^2/R}$ and $\kappa = \sqrt{9/4 - 12m^2/R}$.

- The *renormalized SET* with respect to a vacuum $|0\rangle$ of Hadamard type is given by

$$\begin{aligned} \langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\mathrm{ren}\ \mathrm{dS}^4} = \frac{1}{32\pi^2}\Big\{&(\beta+17/24)m^2R + (19/1440)R^2 \\ &-\left[(3/2)m^4 + (1/4)m^2R\right]\left[\ln(R/(12m^2)) + \Psi(5/2+\lambda) + \Psi(5/2-\lambda)\right]\Big\} g_{\mu\nu}. \end{aligned}$$

  - In this expression, we have introduced the Digamma function $\Psi(z) = (d/dz)\ln\Gamma(z)$.
  - This result is not free of ambiguities due to the arbitrary coefficient β remaining in the expression. However, we can cancel the corresponding term by a finite renormalization of the Einstein-Hilbert action of the gravitational field.

Renormalized stress-energy tensor in AdS

- The *Hadamard Green function* associated with the massive vector field propagating in AdS^4 is given in term of the *real* and *imaginary* parts of the *hypergeometric function* $F(a,b;c;z)$ on the *branch cut* denoted by $(\mathrm{Re}F)(a,b;c;z)$ and $(\mathrm{Im}F)(a,b;c;z)$:

$$
\begin{aligned}
G^{(1)A}_{\mu\nu'}(x,x') = \frac{R}{192\pi}\Bigg\{ & \frac{9/4-\lambda^2}{\cos(\pi\lambda)}\, z(1-z)\Big[(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;z)+\sin(\pi\lambda)(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;1-z) \\
& -\cos(\pi\lambda)(\mathrm{Im}F)'(5/2+\lambda,5/2-\lambda;3;1-z)\Big] + \frac{3}{2}\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}(1-2z)\Big[\dots\Big] - \frac{1}{2}\,\frac{(1/4-\kappa^2)}{\cos(\pi\kappa)}\Big[\dots\Big]\Bigg\} g_{\mu\nu'} \\
& + \frac{1}{16\pi}\Bigg\{\frac{9/4-\lambda^2}{\cos(\pi\lambda)}\Big[(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;z)+\sin(\pi\lambda)(\mathrm{Re}F)'(5/2+\lambda,5/2-\lambda;3;1-z) \\
& -\cos(\pi\lambda)(\mathrm{Im}F)'(5/2+\lambda,5/2-\lambda;3;1-z)\Big] + 3\,\frac{(9/4-\lambda^2)}{\cos(\pi\lambda)}(1/z)\Big[\dots\Big] - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}\Big[\dots\Big] - \frac{1/4-\kappa^2}{\cos(\pi\kappa)}(1/z)\Big[\dots\Big]\Bigg\} z_{;\mu}z_{;\nu'}
\end{aligned}
$$

with $\lambda = \sqrt{1/4-12m^2/R}$ and $\kappa = \sqrt{9/4-12m^2/R}$.

- The *renormalized SET* with respect to a vacuum $|0\rangle$ of Hadamard type is given by

$$
\begin{aligned}
\langle 0|\widehat{T}_{\mu\nu}|0\rangle_{\text{ren AdS}^4} = \frac{1}{32\pi^2}\Big\{ & (\beta+5/24)m^2R-(11/1440)R^2 \\
& -\big[(3/2)m^4+(1/4)m^2R\big]\big[\ln(-R/(12m^2))+2\Psi(1/2+\lambda)\big]\Big\} g_{\mu\nu}.
\end{aligned}
$$

  - In this expression, we have introduced the Digamma function $\Psi(z) = (d/dz)\ln\Gamma(z)$.
  - This result is not free of ambiguities due to the arbitrary coefficient β remaining in the expression. However, we can cancel the corresponding term by a finite renormalization of the Einstein-Hilbert action of the gravitational field.

Outline

Conclusion

- We have presented *Stueckelberg massive EM* on an arbitrary curved spacetime*.
- We have given two alternative but equivalent expressions for the *renormalized expectation value* of the *SET operator* constructed using *Hadamard renormalization*.
- We have also presented the results concerning the *renormalized SET* of the massive vector field propagating in dS and AdS spacetimes*.
- It is necessary to point out that
  - (i) *de Broglie-Proca* and *Stueckelberg* approaches of *massive EM* are two faces of the same theory,
  - (ii) however, we can note that, with *regularization* and *renormalization* in mind, it is much more interesting to work in the framework of the *Stueckelberg* formulation of *massive EM* which permits us to use the *Hadamard formalism*.
- One of our perspectives is the application of the general formalism developed to cosmological problems and, in particular, the study of *Stueckelberg massive EM* in FLRW spacetimes.

THANKS FOR YOUR ATTENTION

*Phys. Rev. D 93, 044063 (2016) (arXiv:1512.06326)
*Phys. Rev. D 94, 105028 (2016) (arXiv:1610.00244)