UV renormalization, locality and global hyperbolicity

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Motivation

Usually in QFT one assumes that

- UV phenomena are local
- UV renormalization can be done via analytical continuation from Euclidian to Minkowskian signature
- QFT in curved space–time is full of surprises:
  dS — in IR, AdS — in UV.

UV renormalization in x–space

- We consider a scalar field theory:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right].$$

- The one loop contribution to the effective action of the theory:

$$\Gamma^{(4)} = -i\,\frac{3\lambda^2}{2(4\pi^2)^2}\int d^4x \int d^4y\, \phi^2(x)\phi^2(y)\, F^2\,(x-y)\,.$$

- Here

$$F(x) \approx \frac{1}{4\pi^2}\frac{i}{x^2 - i\epsilon}$$

is the most singular part of the Feynman propagator in position space when $x^2 \to 0$.

- We may extract the leading divergent contribution by changing the variables $x^\mu = X^\mu + \frac{z^\mu}{2}$, $y^\mu = X^\mu - \frac{z^\mu}{2}$, $\mu = 0, 1, 2, 3$ and by diagonally expanding $\phi^2\left(X + z/2\right)\, \phi^2\left(X - z/2\right)$
- In fact,

$$\Gamma^{(4)} = -i\, \frac{3\lambda^2}{2\,(4\pi^2)^2} \int d^4X\, \phi^4(X) \int d^4z \frac{1}{\left(z^2 - i\epsilon\right)^2} + \text{finite terms}$$

  The z–integral provides the standard logarithmic UV divergence.
- Note the importance of the proper $i\,\epsilon$ prescription. If one replaces the Feynman propagator with the Wightman function, then:

$$\int d^4z \frac{1}{\left[\left(z_0 - i\epsilon\right)^2 - \vec{z}^2\right]^2} = 0.$$

A simple example

- We continue by considering the above theory in flat space-time but in the presence of an ideal mirror placed at $x_3 = 0$. The ideal mirror reflects all the modes equally well, irrespectively of their momenta. This is expressed by the boundary condition $\phi|_{x_3=0} = 0$ at the mirror.
- A real physical mirror is definitely transparent to very high energy modes. On general physical grounds one can expect that, if a is a characteristic interatomic distance of the material of the mirror, a mode whose wavelength k is much larger than $1/a$ will not see the mirror at all.
- A real physical mirror can be modeled by a potential barrier which reflects some of the modes and is transparent to the other ones, e.g.:

$$\left[\Box + m^2\right] \phi = \alpha\, \delta(x_3)\, \phi.$$

A simple example

- The most singular part of the Feynman propagator in presence of an ideal mirror is the following distribution

$$F_{mir}(x,y) \approx \frac{i}{4\pi^2}\frac{1}{s - i\epsilon} - \frac{i}{4\pi^2}\frac{1}{\bar{s} - i\epsilon},$$

  where $s = (x-y)^2$ and $\bar{s} = (x-\bar{y})^2$ and $\bar{y}$ is the mirror image of the source point y.

- In Euclidean signature, s vanishes only when $x = y$ and $\bar{s}$ only when $x = \bar{y}$. But the point $\bar{y}$ does not belong to the portion of space-time that we are considering, $x_3 > 0$ and $y_3 > 0$. Hence, inside the loops in Euclidean signature $\bar{y}$ plays no role.
- In Lorentzian signature, s and $\bar{s}$ vanish on the light–cones whose tips are y and, respectively, $\bar{y}$. Therefore, even though $\bar{y}$ does not belong to the space-time manifold its light–cone penetrates into it.

- The first singularity in $F_{mir}(x,y)$ provides the same contribution as in empty space. As regards the second term we have to do the following diagonal expansion: $x^\mu - \bar{y}^\mu = z^\mu$,$x^\mu + \bar{y}^\mu = 2\,X^\mu,\ \mu = 0,1,2,3$.
- Then, the effective action contains the following term:

$$\Gamma^{(4)}_{\bar{s}=0} = -i\,\frac{3\lambda^2}{2\,(4\pi^2)^2}\int d^4X \int_{z_3\geq|X_3|} d^4z\,\frac{\phi^2(X+z/2)\,\phi^2(\bar{X}-\bar{z}/2)}{(z^2-i\epsilon)^2}.$$

- Even though z^2 may vanish the components of four–vector z are generically not small and the diagonal expansion of $\phi^2(X+z/2)\,\phi^2(\bar{X}-\bar{z}/2)$ cannot be performed.
- Both the singularities of $F_{mir}(x,y)$ at $s=0$ and $\bar{s}=0$ do contribute to the UV divergence of the integral on the right hand side, while the mixed terms contribute finite expressions.

The geometry of AdS and Lobachevsky spaces

- The 4–dimensional Euclidean AdS (Lobachevsky space) space

$$EAdS_4 := \{X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = 1\},$$

is one of the sheets (say $X_0 \geq 1$) of the two–sheeted real hyperboloid embedded into:

$$ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2.$$

- The 4–dimensional (Lorentzian) AdS space:

$$AdS_4 := \{X_0^2 - X_1^2 - X_2^2 - X_3^2 + X_4^2 = 1\}.$$

embedded into

$$ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 + dX_4^2.$$

- The EAdS and AdS are related to each other via the analytic continuation $X_4 \to i\,X_4$. We set the curvatures of the hypeboloids to one: $R = 1$.

The geometry of AdS and Lobachevsky spaces

- The hyperbolic distance is defined via the invariant scalar product:

$$\xi = \eta_{\mu\nu} X^\mu Y^\nu = \cosh d(X, Y),$$

where $d(X, Y)$ is the geodesic distance.

- In EAdS the hyperbolic distance is $\xi \geq 1$, because $d(X, Y)$ is real, while in AdS ξ can take any real value.
- The Feynman propagator obeys

$$\left[\Box + m^2\right] F(X, Y) = \left[\left(1 - \xi^2\right)\, \partial_\xi^2 - 4\,\xi\,\partial_\xi + m^2\right]\, F(\xi) =$$

$$= 4\,\pi\,\delta(X, Y) + 4\,\pi\,i\,e^{-i\,\pi\,\nu}\,\delta(X, -Y).$$

The Feynman propagator in AdS and Lobachevsky spaces

- The Feynman propagator can be represented both in EAdS, in global AdS and in its covering $\widetilde{AdS}$ manifold as follows:

$$F(X,Y) = A_{+}\ {}_1F_2\left(\frac{3}{2}-\nu, \frac{3}{2}+\nu; 2; \frac{1+\xi+i\epsilon}{2}\right) + \\ + A_{-}\ {}_1F_2\left(\frac{3}{2}-\nu, \frac{3}{2}+\nu, 2; \frac{1-\xi-i\epsilon}{2}\right)$$

- $\nu = \sqrt{\frac{9}{4}+m^2}$
- When $\xi^2 \to 1$ there is the following leading singularities of the AdS Feynman propagator:

$$F(\xi) \approx -\frac{i}{8\pi^2\ (\xi - 1 + i\epsilon)} - \frac{e^{-i\pi\nu}}{8\pi^2\ (\xi + 1 + i\epsilon)}.$$

- The first singularity, at $\xi = 1$, is the same as in flat empty space. Note that:

$$(X - Y)^2 - i\epsilon = 2\,(1 - (\xi + i\epsilon))$$

- The second singularity, at $\xi = -1$, is when X sits on the light cone with the apex at $\bar{Y} = -Y$ — point antipodal to Y.
- In Lobachevsky space the second singularity is not seen, because there $\xi \geq 1$. But in AdS the second singularity is present.

Loop corrections in AdS and Lobachevsky spaces

- In global AdS the relevant part of the correction is:

$$\Gamma^{(4)} \propto \lambda^2 \int d^5X\, \delta\left(X^2-1\right) \int d^5Y\, \delta\left(Y^2-1\right)\, \phi^2\left(X\right)\, \phi^2\left(Y\right) \times \\ \times \left[\frac{1}{(X-Y)^2 - i\epsilon} - \frac{1}{(X+Y)^2 - i\epsilon}\right]^2.$$

- The first pole leads to the same renormalization as in flat spacetime. The second pole is different and leads to divergences of a new type. The cross terms lead to less singular contributions.
- Thus, we have to introduce a new counter–term into the Lagrangian:

$$\Delta\mathcal{L} = \frac{\gamma\, e^{-2\pi i \nu}}{4} \phi^2(X)\phi^2(-X), \qquad (1)$$

with a complex coefficient depending on the mass parameter and a new coupling constant γ.