Tereza Vardanyan

Università di Bologna

The quantum Fokker-Planck equation of stochastic inflation

H. Collins, R. Holman and T. Vardanyan,
arXiv:1706.07805 [hep-th].

JHEP **1711** (2017) 065

Motivations

- It has been long know that certain quantum field theories in an expanding background that involve massless minimally coupled scalar fields, e.g. a simple scalar theory with a quartic interaction or scalar electrodynamics, have infrared divergences in the loop-corrections to their Green's functions.

- Infrared divergences that we encounter in theories that are set in the expanding background signal the breakdown of the perturbation theory past a certain point in time.

- Starobinsky (Starobinsky, et al) had argued that the dynamics of the long-wavelength modes can be described by the classical stochastic field, whose probability distribution satisfies a Fokker-Planck type equation. He showed that this equation has a late-time static solution, that can be used to calculate correlation functions in the late-time limit.

Motivations

- Despite the compelling simplicity of the stochastic picture, it would appear to be very difficult to see how it could emerge by following the full quantum evolution of the theory. A recent approach (Burgess, et al) to this problem has been to consider the quantum evolution of the theory from a different perspective by working in the Schrödinger picture. Stochastic picture was derived for a non-interacting Gaussian theory.

- In our work we use the Schrödinger picture to derive the stochastic picture for a genuinely interacting field theory. In particular, we consider here a massless scalar field with a quartic interaction in de Sitter space, and solve for the full time dependence of its density matrix perturbatively in the self-coupling of the field.

Scalar field in the de Sitter background

Expectation values of the product of n fields $\langle \Phi^n(t,\vec{x}) \rangle \equiv \langle \Omega | \Phi^n(t,\vec{x}) | \Omega \rangle$

$|\Omega\rangle$ denotes the state that we have chosen for our quantum field, which we shall take to be the Bunch-Davies state — the de Sitter invariant state matching the standard Minkowski space vacuum at very short distances.

We are only interested in the behavior of the long-wavelength part of the field

$$\langle \Phi_L^n(t,\vec{x}) \rangle$$

The metric of the de Sitter space $ds^2 = dt^2 - a^2(t)\,\delta_{ij}\,dx^i dx^j, \quad a(t) = e^{Ht}$

long wavelength (L): $k_{phys} = k/a(t) < \varepsilon H$

short wavelength (S): $k_{phys} = k/a(t) > \varepsilon H$

$$\varepsilon \ll 1$$

$$\Phi(t,\vec{x}) = \Phi_L(t,\vec{x}) + \Phi_S(t,\vec{x}) = \int_{k<\varepsilon a(t)H} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \Phi_{\vec{k}}(t) + \int_{k>\varepsilon a(t)H} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \Phi_{\vec{k}}(t)$$

Classical Fokker-Planck equation

Suppose we knew all of $\langle \Phi_L^n(t, \vec{x}) \rangle$

We could then introduce a classical variable φ, together with a probability distribution function $p(t, \varphi)$, such that together they reproduce all the information contained in the functions $\langle \Phi_L^n(t, \vec{x}) \rangle$. The weighted average of a power of this variable is defined by the following integral,

$$\langle \varphi^n \rangle(t) \equiv \int_{-\infty}^{\infty} d\varphi \, \varphi^n p(t, \varphi).$$

$$\langle \varphi^n \rangle(t) = \langle \Phi_L^n(t, \vec{x}) \rangle$$

The stochastic theory of inflation (Starobinsky:1986, Starobinsky:1994) argues that the probability function for this classical variable should satisfy a Fokker-Planck equation of the form

$$\frac{\partial p}{\partial t} = N \frac{\partial^2 p}{\partial \varphi^2} + D \frac{\partial}{\partial \varphi} \left(\frac{\partial V}{\partial \varphi} p(t, \varphi) \right).$$

The coefficients N and D are called the 'noise' and the 'drift' of this stochastic theory.

Assumption about the potential V: $V(\Phi(t, \vec{x})) \xrightarrow{\Phi(t,\vec{x}) \to \varphi} V(\varphi)$

For the massless theory with quartic interaction $D = \dfrac{1}{3H}$ $\quad N = \dfrac{H^3}{8\pi^2}$

Classical Fokker-Planck equation

Any solution to the Fokker-Planck equation approaches time-independent (static) solution at the late-time limit.

$$\frac{\partial p}{\partial t} = N\frac{\partial^2 p}{\partial \varphi^2} + D\frac{\partial}{\partial \varphi}\left(\frac{\partial V}{\partial \varphi}p(t,\varphi)\right)$$

$$\text{As } t \to \infty \; p(\varphi, t) \to p(\varphi), \partial p/\partial t = 0$$

The static solution for the theory with quartic interaction $V(\varphi) = \frac{1}{4!}\lambda\varphi^4$

$$p(\varphi) \propto e^{-\frac{\bar{\lambda} D}{24N}\varphi^4}$$

The static solution can be used to determine $\langle\varphi^n\rangle(t) = \langle\Phi_L^n(t,\vec{x})\rangle$ at the late-time limit

Quantum description in the Schrödinger picture

The closest analogue of the probability function in the stochastic description is the density matrix — or rather, its diagonal components — associated with the state that we have chosen

$$P[\phi] = P[\phi_L, \phi_S] = \Psi[\phi]\Psi^*[\phi].$$

The dynamics of the long-wavelength part of the field can be described by the coarse-grained version of the density matrix

$$P_\Omega[\phi_L] = \int_S \mathcal{D}\phi_{\vec{p}}\, P[\phi_L, \phi_S].$$

The expectation values of the products of fields can be found as follows

$$\langle \Phi_L^n(t, \vec{x}) \rangle = \int \mathcal{D}\phi_L \;\; \phi_L \cdots \phi_L P_\Omega[\phi_L].$$

Note: in the Schrödinger picture fields are time-independent, so $P[\phi]$ and $P_\Omega[\phi_L]$ are time-dependent.

Scalar field in the de Sitter background

$$S[\Phi] = \int dt\, L[\Phi] = \int d^4x\, \sqrt{-g}\, \{\tfrac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \tfrac{1}{24} \lambda \Phi^4\}$$

The metric of the de Sitter space $ds^2 = dt^2 - a^2(t)\, \delta_{ij}\, dx^i dx^j = a^2(\eta) \left[d\eta^2 - \delta_{ij}\, dx^i dx^j \right]$

'cosmological' time coordinate $t \in (-\infty, \infty)$
'conformal' one $\eta \in (-\infty, 0)$

$a(t) = e^{Ht}$ or $a(\eta) = -\dfrac{1}{H\eta}$

$$L[\Phi] = \int d^3\vec{x}\, \{\tfrac{1}{2} a^3 \dot{\Phi}^2 - \tfrac{1}{2} a \delta^{ij} \partial_i \Phi \partial_j \Phi - \tfrac{1}{24} a^3 \lambda \Phi^4\}$$

The canonical momenta $\Pi(t, \vec{x}) = \dfrac{\delta L}{\delta \dot{\Phi}(t, \vec{x})} = a^3 \dot{\Phi}(t, \vec{x})$

$$H = \int d^3\vec{x}\, \{\Pi \dot{\Phi}\} - L = \int d^3\vec{x}\, \{\tfrac{1}{2} a^{-3} \Pi^2 + \tfrac{1}{2} a \delta^{ij} \partial_i \Phi \partial_j \Phi + \tfrac{1}{24} a^3 \lambda \Phi^4\}$$

Schrödinger picture

The canonical momenta $$\pi(\vec{x}) = i\frac{\delta}{\delta\phi(\vec{x})}$$

Fourier components of the field $$\phi_{\vec{k}} = \int d^3\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\phi(\vec{x})$$

Hamiltonian in the momentum space

$$\begin{aligned} H \quad = \quad & \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2)\left\{-\frac{1}{2}\frac{1}{a^3}\frac{\delta}{\delta\phi_{\vec{k}_1}}\frac{\delta}{\delta\phi_{\vec{k}_2}}+\frac{1}{2}ak_1^2\phi_{\vec{k}_1}\phi_{\vec{k}_2}\right\} \\ & +\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\frac{d^3k_4}{(2\pi)^3}\,(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\left\{\frac{1}{24}a^3\lambda\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4}\right\} \end{aligned}$$

The wave-functional

$$\Psi[\phi] = N e^{-a^3\,\Gamma[\phi]}$$

$$\Gamma[\phi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3}\,(2\pi)^3 \delta^3(\vec{k}_1 + \cdots + \vec{k}_n)\,\Gamma_n(t; \vec{k}_1, \ldots, \vec{k}_n)\,\phi_{\vec{k}_1} \cdots \phi_{\vec{k}_n}$$

N is a normalization fixed by the condition $\int \mathcal{D}\phi_{\vec{k}}\,\Psi[\phi]\Psi^*[\phi] = 1$

Symmetry under permutations of momenta

$$\Gamma_n(t; \vec{k}_1, \ldots, \vec{k}_i, \ldots, \vec{k}_j, \ldots, \vec{k}_n) = \Gamma_n(t; \vec{k}_1, \ldots, \vec{k}_j, \ldots, \vec{k}_i, \ldots, \vec{k}_n)$$

Vacuum state should have the same symmetry as the potential

$$\phi \leftrightarrow -\phi \text{ symmetry} \longrightarrow \Gamma_n(t; \vec{k}_1, \ldots, \vec{k}_n) = 0 \qquad \text{for } n \in \text{odd}$$

Schrödinger equation $i\frac{\partial \Psi}{\partial t} = H\Psi$

Matching terms with no fields $\frac{\dot{N}}{N} = -\frac{i}{2}(2\pi)^3\delta^3(\vec{0})\int \frac{d^3\vec{p}}{(2\pi)^3}\,\alpha_p(t)$

Matching terms with two fields

$$\frac{\partial \alpha_k}{\partial t} + 3\frac{\dot{a}}{a}\alpha_k = i\left\{\frac{k^2}{a^2} - \alpha_k^2 + \frac{1}{2}\frac{1}{a^3}\int \frac{d^3\vec{p}}{(2\pi)^3}\,\Gamma_4(t;\vec{k},-\vec{k},\vec{p},-\vec{p})\right\} \qquad \alpha_k(t) \equiv \Gamma_2(t;\vec{k},-\vec{k})$$

Matching terms with four fields

$$\begin{aligned}\frac{\partial \Gamma_4}{\partial t} + 3\frac{\dot{a}}{a}\Gamma_4(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) &= i\lambda - i\left[\alpha_{k_1}+\alpha_{k_2}+\alpha_{k_3}+\alpha_{k_4}\right]\Gamma_4(t,\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) \\ &\quad + \frac{i}{2}\frac{1}{a^3}\int \frac{d^3\vec{p}}{(2\pi)^3}\,\Gamma_6(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{p},-\vec{p}),\end{aligned}$$

Perturbation theory

Each function in the wave-functional can be expanded in the powers of λ

$$\alpha_k(t) = \sum_{n=0}^{\infty} \alpha_k^{(n)}(t)$$

$$\Gamma_4(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = \sum_{n=1}^{\infty} \Gamma_4^{(n)}(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$$

$$\Gamma_6(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{k}_6) = \sum_{n=2}^{\infty} \Gamma_6^{(n)}(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{k}_5,\vec{k}_6)$$

$$\alpha_k^{(n)}, \Gamma_4^{(n)}, \Gamma_6^{(n)}, \ldots \propto \lambda^n$$

Quadratic part starts at zero-th order in λ

$$\alpha_k(t) = \bar{\alpha}_k(t) + \beta_k(t) + \mathcal{O}(\lambda^2)$$

Quartic part starts at first order in λ

$$\Gamma_4(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = \Gamma_4^{(1)}(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) + \mathcal{O}(\lambda^2)$$

Leading part of $\alpha_k(t)$ (Gaussian theory)

$$\frac{\partial \alpha_k}{\partial t} + 3\frac{\dot{a}}{a}\alpha_k = i\left\{\frac{k^2}{a^2} - \alpha_k^2 + \frac{1}{2}\frac{1}{a^3}\int \frac{d^3\vec{p}}{(2\pi)^3}\,\Gamma_4(t;\vec{k},-\vec{k},\vec{p},-\vec{p})\right\}$$

At zero-th order in λ $\longrightarrow$ $$\frac{\partial \bar{\alpha}_k}{\partial t} + 3\frac{\dot{a}}{a}\bar{\alpha}_k = i\left\{\frac{k^2}{a^2} - \bar{\alpha}_k^2\right\},$$

$$\bar{\alpha}_k(t) = -i\frac{\dot{u}_k(t)}{u_k(t)} \longrightarrow \ddot{u}_k + 3\frac{\dot{a}}{a}\dot{u}_k + \frac{k^2}{a^2}u_k = 0$$

$$u_k(t) = \frac{v_k(t)}{a(t)} \longrightarrow \frac{d^2 v_k}{d\eta^2} + k^2\left(1 - \frac{2}{k^2\eta^2}\right)v = 0$$

The general solution $v_k(\eta) = \sqrt{-k\eta}\left(C_1 H^{(1)}_{3/2}(-k\eta) + C_2 H^{(2)}_{3/2}(-k\eta)\right)$

Leading part of $\alpha_k(t)$ (Gaussian theory)

$$v_k(\eta) = \sqrt{-k\eta}\left(C_1 H^{(1)}_{3/2}(-k\eta) + C_2 H^{(2)}_{3/2}(-k\eta)\right)$$

$$H^{(1)}_{3/2}(-k\eta) = i\sqrt{\frac{2}{-k^3\eta^3\pi}}(1+ik\eta)e^{-ik\eta} \qquad H^{(2)}_{3/2}(-k\eta) = -i\sqrt{\frac{2}{-k^3\eta^3\pi}}(1-ik\eta)e^{ik\eta}$$

How do we choose C_1 and C_2?

For very short-wavelength modes $\frac{k_{\text{phys}}}{H} = \frac{k}{aH} = -k\eta \gg 1$

$$\longrightarrow \quad v_k(\eta) \to \left(C_1 e^{-ik\eta} + C_2 e^{ik\eta}\right)$$

For very short-wavelength modes the theory should behave as in flat space-time

$$\longrightarrow \quad \bar{\alpha}_k \to (k/a) \quad \longrightarrow \quad C_1 = 0$$

$$\longrightarrow \quad u_k(\eta) = \frac{v_k(\eta)}{a(\eta)} \propto (1-ik\eta)e^{ik\eta}$$ and $$\bar{\alpha}_k(\eta) = iH\frac{k^2\eta^2}{1-ik\eta}$$

Leading part of $\Gamma_4(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$

$$\frac{\partial\Gamma_4}{\partial t}+3\frac{\dot a}{a}\Gamma_4(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = i\lambda - i\big[\alpha_{k_1}+\alpha_{k_2}+\alpha_{k_3}+\alpha_{k_4}\big]\Gamma_4(t,\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$$
$$+\frac{i}{2}\frac{1}{a^3}\int\frac{d^3\vec{p}}{(2\pi)^3}\,\Gamma_6(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4,\vec{p},-\vec{p}),$$

At first order in λ $\quad \dfrac{\partial\Gamma_4^{(1)}}{\partial t}+\dfrac{\partial}{\partial t}\left[\ln\Big(a^3u_{k_1}u_{k_2}u_{k_3}u_{k_4}\Big)\right]\Gamma_4^{(1)}=i\lambda$

The general solution

$$\Gamma_4^{(1)}(\eta;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)=\frac{c_4+i\lambda\displaystyle\int_{-\infty(1-i\epsilon)}^{\eta}d\eta' a^4(\eta')u_{k_1}(\eta')u_{k_2}(\eta')u_{k_3}(\eta')u_{k_4}(\eta')}{a^3(\eta)u_{k_1}(\eta)u_{k_2}(\eta)u_{k_3}(\eta)u_{k_4}(\eta)}$$

Recall that for very short modes $\quad a(\eta)u_{k_i}(\eta)\propto e^{ik_i\eta}$

$$\longrightarrow\ \Gamma_4^{(1)}(\eta;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\approx c_4\,a(\eta)e^{-i(k_1+k_2+k_3+k_4)\eta}+\frac{a(\eta)\,\lambda}{k_1+k_2+k_3+k_4}$$

$$c_4=0$$

Leading part of $\Gamma_4(t;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$

$$\Gamma_4^{(1)}(\eta;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = \frac{i\lambda\int_{-\infty}^{\eta} d\eta' a^4(\eta')u_{k_1}(\eta')u_{k_2}(\eta')u_{k_3}(\eta')u_{k_4}(\eta')}{a^3(\eta)u_{k_1}(\eta)u_{k_2}(\eta)u_{k_3}(\eta)u_{k_4}(\eta)}$$

Plugging $u_k(\eta) = \frac{v_k(\eta)}{a(\eta)} \propto \big(1 - ik\eta\big)e^{ik\eta}$ $\longrightarrow$

$$\begin{aligned}\Gamma_4^{(1)}(\eta;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = {} & \frac{i\lambda}{3H}\frac{1 + iK\eta - \frac{1}{2}K^2\eta^2 + \frac{3}{2}\big(k_1^2+k_2^2+k_3^2+k_4^2\big)\eta^2 - \frac{3ik_1k_2k_3k_4\eta^3}{K}}{(1+ik_1\eta)(1+ik_2\eta)(1+ik_3\eta)(1+ik_4\eta)} \\ & + \frac{\lambda}{3H}\frac{\big(k_1^3+k_2^3+k_3^3+k_4^3\big)\eta^3 e^{iK\eta}\,\mathrm{Ei}(1,iK\eta)}{(1+ik_1\eta)(1+ik_2\eta)(1+ik_3\eta)(1+ik_4\eta)}.\end{aligned}$$

For modes longer than the Hubble scale

$$\lim_{k_i\eta\to 0}\Gamma_4^{(1)}(\eta;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = \frac{i}{3}\frac{\lambda}{H} + \frac{i}{2}\frac{\lambda}{H}\big(k_1^2+k_2^2+k_3^2+k_4^2\big)\eta^2 + \cdots$$

Density matrix $P[\phi] = P[\phi_L, \phi_S] = \Psi[\phi]\Psi^*[\phi]$

Up to the linear order in λ

$$
\begin{aligned}
P[\phi] &= |N|^2 \exp\Big\{ -\frac{1}{2}a^3 \int \frac{d^3\vec{k}}{(2\pi)^3} (\bar{\alpha}_k + \beta_k + \bar{\alpha}_k^* + \beta_k^*)\phi_k\phi_{-k} \\
&\quad -\frac{1}{24}a^3 \int \frac{d^3\vec{k}_1}{(2\pi)^3} \cdots \frac{d^3\vec{k}_4}{(2\pi)^3} (2\pi)^3\delta^3(\vec{k}_1 + \cdots + \vec{k}_4)\, \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_4} \left(\Gamma_4^{(1)}(\vec{k}_1, \ldots, \vec{k}_4) + \Gamma_4^{(1)*}(-\vec{k}_1, \ldots, -\vec{k}_4) \right) \Big\} \\
&\approx \frac{|N|^2}{|N_0|^2} P_0[\phi] \Big\{ 1 - \frac{1}{2}a^3 \int \frac{d^3\vec{k}}{(2\pi)^3} (\beta_k + \beta_k^*)\phi_k\phi_{-k} \\
&\qquad -\frac{1}{24}a^3 \int \frac{d^3\vec{k}_1}{(2\pi)^3} \cdots \frac{d^3\vec{k}_4}{(2\pi)^3} (2\pi)^3\delta^3(\vec{k}_1 + \cdots + \vec{k}_4)\, \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_4} \left(\Gamma_4^{(1)} + \Gamma_4^{(1)*} \right) \Big\}
\end{aligned}
$$

$P_0[\phi]$ is associated with the non-interacting part of the theory. It can be factorized into the short and long parts.

$$
\begin{aligned}
P_0[\phi] &= |N_0|^2 \exp\Big\{ -\frac{1}{2}a^3 \int \frac{d^3\vec{k}}{(2\pi)^3} (\bar{\alpha}_k + \bar{\alpha}_k^*)\phi_k\phi_{-k} \Big\} \\
&= |N_0^S N_0^L|^2 \exp\Big\{ -\frac{1}{2}a^3 \int_S \frac{d^3\vec{k}}{(2\pi)^3} (\bar{\alpha}_k + \bar{\alpha}_k^*)\phi_k\phi_{-k} - \frac{1}{2}a^3 \int_L \frac{d^3\vec{k}}{(2\pi)^3} (\bar{\alpha}_k + \bar{\alpha}_k^*)\phi_k\phi_{-k} \Big\} \\
&= P_0[\phi_S] P_0[\phi_L]
\end{aligned}
$$

Coarse-grained density matrix $P_\Omega[\phi_L]$

$$P[\phi] = \frac{|N|^2}{|N_0|^2} P_0[\phi_L] P_0[\phi_S] \Big\{ 1 - \frac{1}{2} a^3 \int \frac{d^3\vec{k}}{(2\pi)^3} (\beta_k + \beta_k^*) \phi_k \phi_{-k}$$
$$- \frac{1}{24} a^3 \int \frac{d^3\vec{k}_1}{(2\pi)^3} \cdots \frac{d^3\vec{k}_4}{(2\pi)^3} \, (2\pi)^3 \delta^3(\vec{k}_1 + \cdots + \vec{k}_4) \, \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_4} \left(\Gamma_4^{(1)} + \Gamma_4^{(1)*} \right) \Big\}$$

Integrating out short modes from the above expression we arrive at the coarse grained density matrix

$$P_\Omega[\phi_L] \equiv \int_S \mathcal{D}\phi_{\vec{p}} \, P[\phi] = \int_{p \geq \varepsilon a H} \mathcal{D}\phi_{\vec{p}} \, P[\phi] \longrightarrow$$

$$P_\Omega[\phi_L] = |N_\Omega|^2 \exp\Big\{ - \frac{1}{2} a^3 \int_L \frac{d^3\vec{k}}{(2\pi)^3} (\alpha_{\Omega,k} + \alpha_{\Omega,k}^*) \phi_k \phi_{-k}$$
$$- \frac{1}{24} a^3 \int_L \frac{d^3\vec{k}_1}{(2\pi)^3} \cdots \frac{d^3\vec{k}_4}{(2\pi)^3} \, (2\pi)^3 \delta^3(\vec{k}_1 + \cdots + \vec{k}_4) \, \phi_{\vec{k}_1} \cdots \phi_{\vec{k}_4} \left(\Gamma_4^{(1)} + \Gamma_4^{(1)*} \right) \Big\}$$

The quadratic part of the density matrix receives a contribution from $\Gamma_4^{(1)}$

$$\alpha_{\Omega,k} = \bar{\alpha}_k + \beta_k + \frac{1}{2} \frac{1}{a^3} \int_S \frac{d^3\vec{p}}{(2\pi)^3} \, \frac{1}{\bar{\alpha}_p + \bar{\alpha}_p^*} \Gamma_4^{(1)}(t; \vec{k}, -\vec{k}, \vec{p}, -\vec{p}) + \mathcal{O}(\lambda^2)$$

Evolution of the coarse-grained density matrix

Hitting the coarse grained density matrix with the time-derivative we get

$$
\begin{aligned}
i\frac{\partial P_\Omega}{\partial t} = \Bigg\{ & -\frac{1}{2}a^3 \int_L \frac{d^3\vec{k}}{(2\pi)^3}\, \phi_{\vec{k}}\phi_{-\vec{k}}\big(\alpha_{\Omega,k}^2 - \alpha_{\Omega,k}^{*2}\big) - \frac{i}{2}a^3 \int_{\partial L} \frac{d^3\vec{k}}{(2\pi)^3}\, \phi_{\vec{k}}\phi_{-\vec{k}}\big(\alpha_{\Omega,k} + \alpha_{\Omega,k}^*\big) \\
& + \frac{1}{4}\int_L \frac{d^3\vec{k}}{(2\pi)^3}\, \phi_{\vec{k}}\phi_{-\vec{k}} \int_L \frac{d^3\vec{k}'}{(2\pi)^3} \Big[\Gamma_4^{(1)}(\vec{k},-\vec{k},\vec{k}',-\vec{k}') - \Gamma_4^{(1)*}(-\vec{k},\vec{k},-\vec{k}',\vec{k}')\Big] \\
& - \frac{i}{4}\int_L \frac{d^3\vec{k}}{(2\pi)^3}\, \phi_{\vec{k}}\phi_{-\vec{k}} \int_{\partial S} \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{1}{\bar{\alpha}_p + \bar{\alpha}_p^*} \Big[\Gamma_4^{(1)}(\vec{k},-\vec{k},\vec{p},-\vec{p}) + \Gamma_4^{(1)*}(-\vec{k},\vec{k},-\vec{p},\vec{p})\Big] \\
& - \frac{1}{4!}a^3 \int_L \frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\frac{d^3\vec{k}_3}{(2\pi)^3}\frac{d^3\vec{k}_4}{(2\pi)^3}\,(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4} \\
& \qquad \times\Big[\big[\bar{\alpha}_{k_1}+\bar{\alpha}_{k_2}+\bar{\alpha}_{k_3}+\bar{\alpha}_{k_4}\big]\Gamma_4^{(1)}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) \\
& \qquad\quad - \big[\bar{\alpha}_{k_1}^*+\bar{\alpha}_{k_2}^*+\bar{\alpha}_{k_3}^*+\bar{\alpha}_{k_4}^*\big]\Gamma_4^{(1)*}(-\vec{k}_1,-\vec{k}_2,-\vec{k}_3,-\vec{k}_4)\Big] \\
& - \frac{i}{4!}a^3 \int_{\partial L} \frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\frac{d^3\vec{k}_3}{(2\pi)^3}\frac{d^3\vec{k}_4}{(2\pi)^3}\,(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4} \\
& \qquad \times\Big[\Gamma_4^{(1)}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) + \Gamma_4^{(1)*}(-\vec{k}_1,-\vec{k}_2,-\vec{k}_3,-\vec{k}_4)\Big] \\
& + \frac{1}{2}(2\pi)^3\delta^3(\vec{0}) \int_L \frac{d^3\vec{k}}{(2\pi)^3}\,\big(\alpha_{\Omega,k} - \alpha_{\Omega,k}^*\big) + \frac{i}{2}(2\pi)^3\delta^3(\vec{0}) \int_{\partial L} \frac{d^3\vec{k}}{(2\pi)^3} + \mathcal{O}(\lambda^2)\Bigg\} P_\Omega .
\end{aligned}
$$

Terms containing two and zero fields can be rewritten in terms of the second functional derivative of the coarse-grained density matrix

Evolution of the coarse-grained density matrix

$$
\begin{aligned}
i\frac{\partial P_\Omega}{\partial t} = & -\frac{i}{2}\int_{\partial L}\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{a^3}\frac{1}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\frac{\delta^2 P_\Omega}{\delta\phi_{\vec{k}}\delta\phi_{-\vec{k}}} + \frac{i}{2}\int_L\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{i}{a^3}\frac{\alpha_{\Omega,k}-\alpha^*_{\Omega,k}}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\frac{\delta^2 P_\Omega}{\delta\phi_{\vec{k}}\delta\phi_{-\vec{k}}} \\
& -\frac{1}{2}\int_L\frac{d^3\vec{k}}{(2\pi)^3}\,\phi_{\vec{k}}\phi_{-\vec{k}}\int_L\frac{d^3\vec{k}'}{(2\pi)^3}\,\frac{1}{\bar{\alpha}_{k'}+\bar{\alpha}^*_{k'}}\Big[\bar{\alpha}_{k'}\Gamma_4^{(1)*}(-\vec{k},\vec{k},-\vec{k}',\vec{k}') \\
& \qquad\qquad -\bar{\alpha}^*_{k'}\Gamma_4^{(1)}(\vec{k},-\vec{k},\vec{k}',-\vec{k}')\Big]P_\Omega[\phi_L] \\
& +\frac{1}{4!}a^3\int_L\frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\frac{d^3\vec{k}_3}{(2\pi)^3}\frac{d^3\vec{k}_4}{(2\pi)^3}\,(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4} \\
& \qquad\times\Big[\big[\bar{\alpha}_{k_1}+\bar{\alpha}_{k_2}+\bar{\alpha}_{k_3}+\bar{\alpha}_{k_4}\big]\Gamma_4^{(1)*}(-\vec{k}_1,-\vec{k}_2,-\vec{k}_3,-\vec{k}_4) \\
& \qquad\quad -\big[\bar{\alpha}^*_{k_1}+\bar{\alpha}^*_{k_2}+\bar{\alpha}^*_{k_3}+\bar{\alpha}^*_{k_4}\big]\Gamma_4^{(1)}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\Big]P_\Omega[\phi_L] \\
& +\mathcal{O}(\lambda^2)
\end{aligned}
$$

Notice the presence of the surface term

$$
\int_{\partial L}\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{a^3}\frac{1}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\frac{\delta^2 P_\Omega}{\delta\phi_{\vec{k}}\delta\phi_{-\vec{k}}} \equiv \int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{a^3}\frac{1}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\frac{\delta^2 P_\Omega}{\delta\phi_{\vec{k}}\delta\phi_{-\vec{k}}}\frac{\partial}{\partial t}\Theta(\varepsilon aH-k)
$$

Evolution of the coarse-grained density matrix

Replacing with its long wavelength asymptotic value

$$\Gamma_4^{(1)}(\eta;\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = \frac{i}{3}\frac{\lambda}{H} + \mathcal{O}(\varepsilon^2), \quad k_i^2\eta^2 \leq \varepsilon^2$$

we arrive at

$$\begin{aligned}
\frac{\partial P_\Omega}{\partial t} = & -\frac{1}{2}\int_{\partial L}\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{a^3}\,\frac{1}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\,\frac{\delta^2 P_\Omega}{\delta\phi_{\vec{k}}\delta\phi_{-\vec{k}}} + \frac{1}{2}\int_{L}\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{i}{a^3}\,\frac{\alpha_{\Omega,k}-\alpha^*_{\Omega,k}}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\,\frac{\delta^2 P_\Omega}{\delta\phi_{\vec{k}}\delta\phi_{-\vec{k}}} \\
& +\frac{1}{2}\int_L\frac{d^3\vec{k}}{(2\pi)^3}\,\phi_{\vec{k}}\phi_{-\vec{k}}\int_L\frac{d^3\vec{k}'}{(2\pi)^3}\left[\frac{\lambda}{3H}\right]P_\Omega[\phi_L] \\
& -\frac{1}{4!}a^3\int_L\frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\frac{d^3\vec{k}_3}{(2\pi)^3}\frac{d^3\vec{k}_4}{(2\pi)^3}\,(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4} \\
& \times\left[\left(\bar{\alpha}_{k_1}+\bar{\alpha}^*_{k_1}\right)\frac{\lambda}{3H}+\left(\bar{\alpha}_{k_2}+\bar{\alpha}^*_{k_2}\right)\frac{\lambda}{3H}+\left(\bar{\alpha}_{k_3}+\bar{\alpha}^*_{k_3}\right)\frac{\lambda}{3H}+\left(\bar{\alpha}_{k_4}+\bar{\alpha}^*_{k_4}\right)\frac{\lambda}{3H}\right]P_\Omega[\phi_L] \\
& +\mathcal{O}(\lambda^2)
\end{aligned}$$

Coarsely grained potential

$$\mathcal{V}_\Omega[\phi_L] = \frac{1}{4!}\lambda \int_L \frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\frac{d^3\vec{k}_3}{(2\pi)^3}\frac{d^3\vec{k}_4}{(2\pi)^3}\,(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_3+\vec{k}_3+\vec{k}_4)\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4}$$

Consider the following expression for arbitrary D_k

$$\begin{aligned}
&\int_L \frac{d^3\vec{k}}{(2\pi)^3}\,\mathcal{D}_k \frac{\delta}{\delta\phi_{\vec{k}}}\left[\frac{\delta\mathcal{V}_\Omega}{\delta\phi_{-\vec{k}}}P_\Omega\right] \\
&\quad = \frac{1}{2}\lambda\int_L \frac{d^3\vec{k}}{(2\pi)^3}\,\phi_{\vec{k}}\phi_{-\vec{k}}\int_L \frac{d^3\vec{k}'}{(2\pi)^3}\,\mathcal{D}_{k'}P_\Omega \\
&\qquad -\frac{1}{4!}a^3\lambda\int_L \frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\frac{d^3\vec{k}_3}{(2\pi)^3}\frac{d^3\vec{k}_4}{(2\pi)^3}\,(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\,\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4} \\
&\qquad\qquad \times\Big[(\bar{\alpha}_{k_1}+\bar{\alpha}^*_{k_1})\mathcal{D}_{k_1}+(\bar{\alpha}_{k_2}+\bar{\alpha}^*_{k_2})\mathcal{D}_{k_2}+(\bar{\alpha}_{k_3}+\bar{\alpha}^*_{k_3})\mathcal{D}_{k_3}+(\bar{\alpha}_{k_4}+\bar{\alpha}^*_{k_4})\mathcal{D}_{k_4}\Big]P_\Omega
\end{aligned}$$

Quantum Fokker-Planck equation

If we identify $\mathcal{D}_k = 1/3H$

$$\frac{\partial P_\Omega}{\partial t} = \int_L \frac{d^3\vec{k}}{(2\pi)^3} \left\{ \mathcal{N}_k \frac{\delta^2 P_\Omega}{\delta\phi_{\vec{k}}\delta\phi_{-\vec{k}}} + \frac{1}{3H}\frac{\delta}{\delta\phi_{\vec{k}}}\left[\frac{\delta\mathcal{V}_\Omega}{\delta\phi_{-\vec{k}}}P_\Omega\right]\right\}$$

where

$$\mathcal{N}_k = -\frac{1}{2}\frac{1}{a^3}\frac{1}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\left[\frac{\partial}{\partial t}\Theta(\varepsilon a H - k)\right] + \frac{1}{2}\frac{i}{a^3}\frac{\alpha_{\Omega,k}-\alpha^*_{\Omega,k}}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\Theta(\varepsilon a H - k)$$

Recursion relations

For classical Fokker-Planck equation

$$\longrightarrow \qquad \frac{\partial}{\partial t}\langle\varphi^n\rangle = n(n-1)\frac{H^3}{8\pi^2}\langle\varphi^{n-2}\rangle - n\frac{1}{3H}\frac{\lambda}{6}\langle\varphi^{n+2}\rangle$$

For quantum Fokker-Planck equation

$$\longrightarrow \qquad \frac{\partial}{\partial t}\langle\Phi_L^n(t,\vec{x})\rangle = n(n-1)\left(\int_L \frac{d^3\vec{k}}{(2\pi)^3}\mathcal{N}_k\right)\langle\Phi_L^{n-2}(t,\vec{x})\rangle - n\mathcal{D}\frac{\lambda}{6}\langle\Phi_L^{n+2}(t,\vec{x})\rangle$$

For these to match we need to have

$$\int_L \frac{d^3\vec{k}}{(2\pi)^3}\mathcal{N}_k = \frac{H^3}{8\pi^2} \qquad \text{and} \qquad \mathcal{D} = \frac{1}{3H}.$$

The noise at the leading order in λ

$$N = \int_L \frac{d^3\vec{k}}{(2\pi)^3}\,\mathcal{N}_k$$

$$\mathcal{N}_k = -\frac{1}{2}\frac{1}{a^3}\frac{1}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\left[\frac{\partial}{\partial t}\Theta(\varepsilon aH-k)\right] + \frac{1}{2}\frac{i}{a^3}\frac{\alpha_{\Omega,k}-\alpha^*_{\Omega,k}}{\alpha_{\Omega,k}+\alpha^*_{\Omega,k}}\,\Theta(\varepsilon aH-k)$$

$$\alpha_{\Omega,k} = \bar{\alpha}_k + \cdots\,, \qquad \bar{\alpha}_k(\eta) = iH\frac{k^2\eta^2}{1-ik\eta}$$

$$\begin{aligned}
N &= \int \frac{d^3\vec{k}}{(2\pi)^3}\left\{-\frac{1}{2}\frac{1}{a^3}\frac{1}{\bar{\alpha}_k+\bar{\alpha}^*_k}\left[\frac{\partial}{\partial t}\Theta(\varepsilon aH-k)\right] + \frac{1}{2}\frac{i}{a^3}\frac{\bar{\alpha}_k-\bar{\alpha}^*_k}{\bar{\alpha}_k+\bar{\alpha}^*_k}\,\Theta(\varepsilon aH-k)\right\} \\
&= \varepsilon a\frac{H^4}{8\pi^2}\int_0^\infty dk\left\{\frac{1+k^2\eta^2}{k}\delta(\varepsilon aH-k)\right\} - \frac{H^3\eta^2}{4\pi^2}\int_0^{\varepsilon aH} dk\,k \\
&= \frac{H^3}{8\pi^2}\left(1+\varepsilon^2\right) - \frac{H^3}{8\pi^2}\varepsilon^2 = \frac{H^3}{8\pi^2}
\end{aligned}$$

Conclusions and further applications

- We saw that the leading form of the quantum version of the Fokker-Planck equation for the effective theory of the long wavelength fluctuations exactly generates the standard Fokker-Planck equation for the stochastic theory.
- Now that we can completely follow the derivation between these two pictures, we can refine the basic picture further by evaluating the higher order corrections to noise and drift.
- We can use our approach to treat real curvature perturbations of the inflationary theories.

Thank you