The Regge pole approach to black hole physics

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- *Regge pole description of scattering of gravitational waves by a Schwarzschild black hole*, Writing in progress
- *Regge pole description of scattering of scalar and electromagnetic waves by a Schwarzschild black hole*, Submitted to PRD, arXiv :1901.03965
- *An alternative description of gravitational radiation from black holes based on the Regge poles of the $\mathscr{S}$-matrix and the associated residues*, PRD 98 (2018) 064052, arXiv :1807.09056

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Introduction

- **At the heart of BH perturbation theory, the $\mathscr{S}$-matrix concept (Chandrasekhar).**

  In the mid-1970s, Chandrasekhar observed that the study of BH perturbations can be reduced to a problem of resonant scattering. This appears clearly in his monograph "*The Mathematical Theory of BHs*" published in 1982 where he recapitulates and completes the results obtained by himself and with his collaborators. This point of view puts at the heart of BH perturbation theory the $\mathscr{S}$-matrix concept.

- **The dual structure of the $\mathscr{S}$-matrix.**

  Chandrasekhar's point of view is an invitation to use systematically, in the context of BH physics, the various tools developed in resonant scattering theory and, in particular, to fully exploit the dual structure of the $\mathscr{S}$-matrix. Indeed, this matrix is a double-entry mathematical object that is a function of both the angular momentum $\ell \in \mathbb{N}$ and the frequency (energy) $\omega \in \mathbb{R}$.

  The $\mathscr{S}$-matrix can be analytically extended :

  (i) for $\ell \in \mathbb{N}$, in the complex ω plane

  (ii) for $\omega \in \mathbb{R}$, in the complex ℓ plane [the so-called complex angular momentum (CAM) plane].

  It is important to note that this duality permits us to shed light, from two different points of view, on a resonant phenomenon and to juggle with its two alternative descriptions.

Introduction

- **The analytic structure of the $\mathscr{S}$-matrix in the complex ω plane.**

It is well known that the analytic structure of the $\mathscr{S}$-matrix in the complex ω plane permits us to physically interpret the response of a BH to an external excitation.

In particular :

(i) the poles of the $\mathscr{S}$-matrix and the associated residues are, respectively, the complex frequencies and the excitation factors of the BH resonant modes [the so-called quasinormal modes (QNMs)] which are involved in the description of the BH ringdown, that part of the signal that dominates the BH response at intermediate timescales,

(ii) a branch-cut integral allows us to describe the tail of the signal, i.e., the BH response at very late times.

Such a point of view is now widely considered in the literature and is systematically used to analyze the signals generated by coalescing binaries.

Introduction

- **The analytic structure of the $\mathscr{S}$-matrix in the CAM plane.**

  On the other hand, there is very little work based on the analytic structure of the $\mathscr{S}$-matrix in the complex ℓ plane. This is really surprising. Indeed, in all the other areas of physics involving resonant scattering theory (i.e., in quantum mechanics, in electromagnetism and optics, in acoustics and seismology, and in high energy physics), it is common to analyze physical phenomena by using CAM techniques and by considering the poles of the $\mathscr{S}$-matrix in the CAM plane (the so-called Regge poles) and the associated residues.

  CAM techniques are very helpful because they permit us to extract the physical information encoded into partial wave expansions by providing :

  (i) powerful tools of resummation of these expansions,

  (ii) “semiclassical” descriptions of resonance phenomena.

Introduction

- **Our talk (we shall limit ourselves to the case of the Schwarzschild BH) :**

- In Sec. II, we introduce the concept of $\mathscr{S}$-matrix in connection with BH perturbations, discuss its extensions and properties in the complex ω plane and in the CAM plane and consider more particularly the notions of Regge pole and Regge trajectory.

- In Sec. III, we consider the scattering of massless waves (scalar, electromagnetic and gravitational) by a Schwarzschild BH. We focus more particularly on the corresponding differential scattering cross sections. We show how to construct an exact CAM representation of a cross section from its partial wave expansion and to extract a Regge pole approximation. We then consider this approximation and show that, in the short-wavelength regime, it permits us to describe with very good agreement the BH glory and the orbiting oscillations and to overcome the difficulties linked to the lake of convergence of the partial wave expansions. Moreover, we explain how to construct an analytical approximation fitting both the BH glory and the orbiting oscillations.

- In Sec. IV, we briefly recall some results obtained in revisiting from CAM techniques the problem of the gravitational radiation generated by a massive "particle" falling radially from infinity into a Schwarzschild BH.

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  - A. Folacci and M. Ould El Hadj, *Complex angular momentum description of the electromagnetic radiation generated by a charged particle falling radially into a Schwarzschild black hole*, Writing in progress.

Contents

BH perturbations and $\mathscr{S}$-matrix

This section has a pedagogical aim. We intend to explain in what manner the concept of $\mathscr{S}$-matrix arises in BH perturbation theory and to discuss the analytic extensions of the $\mathscr{S}$-matrix in the ω plane and in the CAM plane. To simplify our purpose, we will mainly focus on the case of "scalar perturbations" and briefly consider some important results concerning electromagnetic and gravitational perturbations.

Scalar perturbations of the Schwarzschild BH

- We consider that the exterior of the Schwarzschild BH is defined by the metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \tag{1}$$

where $f(r) = 1 - 2M/r$ and M is the mass of the BH while $t \in]-\infty, +\infty[$, $r \in]2M, +\infty[$, $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi]$ are the usual Schwarzschild coordinates.

- We consider a scalar perturbation $\Phi(x)$ of the BH generated by a source $\rho(x)$. In order to solve the wave equation $\Box\Phi = -\rho$ we can introduce the retarded Green function $G_{\text{ret}}(x,x')$ [here $x = (t,r,\theta,\varphi)$ and $x' = (t',r',\theta',\varphi')$] solution of

$$\Box_x G_{\text{ret}}(x,x') = -\frac{1}{\sqrt{-g(x)}}\delta^4(x-x') = -\frac{1}{r^2\sin\theta}\delta(t-t')\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi'). \tag{2}$$

- The construction of the retarded Green function exploits the fact that the Schwarzschild BH is a static spherically symmetric background. We can then write

$$G_{\text{ret}}(x,x') = \frac{1}{2\pi r r'}\sum_{\ell=0}^{+\infty}(2\ell+1)P_\ell(\cos\gamma)\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\omega\, e^{-i\omega(t-t')}G_{\omega\ell}(r,r') \tag{3}$$

where $\epsilon > 0$ (causal boundary conditions) and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$. In Eq. (3), $G_{\omega\ell}(r,r')$ denotes the radial Green function. It must satisfy the Regge-Wheeler equation with a delta source

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_\ell(r)\right]G_{\omega\ell}(r,r') = -\delta(r_* - r'_*) \tag{4}$$

Scalar perturbations of the Schwarzschild BH

and has an ingoing behavior at the horizon (i.e., for $r \to 2M$) and an outgoing behavior at spatial infinity (i.e., for $r \to +\infty$). In Eq. (4), $V_\ell(r)$ denotes the Regge-Wheeler potential given by

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right]. \tag{5}$$

Its behavior has fundamental consequences and we will discuss it more lengthly below. In Eq. (4), we have introduced the tortoise coordinate r_* given by

$$r_*(r) = r + 2M \ln\left(\frac{r}{2M} - 1\right) + \text{Cte}. \tag{6}$$

It provides a bijection from $]2M, +\infty[$ to $]-\infty, +\infty[$ and, in particular, the horizon corresponds to $r_* \to -\infty$ and spatial infinity to $r_* \to +\infty$. It is interesting to chose an integration constant in (6) such that

$$r_*(r = 3M) = 0. \tag{7}$$

Indeed, $r = 3M$ corresponds to the BH photon sphere which is defined as the hypersurface where massless particles (photons, gravitons) can orbit the BH on unstable circular null geodesics.

Scalar perturbations of the Schwarzschild BH

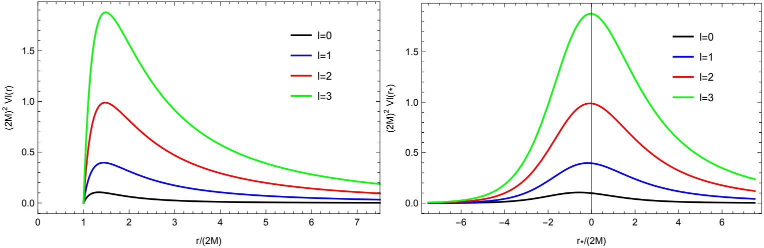

FIGURE 1 – Regge-Wheeler potential V_ℓ for the scalar field ($s=0$). We use the Schwarzschild coordinate r (left) and the tortoise coordinate r_* vanishing at $r=3M$ (right).

- In Fig. 1, we have displayed the Regge-Wheeler potential in the variables r and r_*. It is important to note that the maximum of $V_\ell(r)$ is close to $r=3M$, i.e., to the BH photon sphere. It is also important to note that $V_\ell(r)$ vanishes more rapidly than $1/r_*$ for $r_* \to -\infty$ $(r \to 2M)$ and for $r_* \to -\infty$ $(r \to 2M)$.

Scalar perturbations of the Schwarzschild BH

- The determination of the radial Green function $G_{\omega\,\ell}(r,r')$ is achieved by considering the radial partial waves $\phi_{\omega\,\ell}(r)$ solutions of the homogeneous Regge-Wheeler equation (a one-dimensional, time-independent Schrödinger equation)

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_\ell(r)\right]\phi_{\omega\,\ell} = 0 \tag{8}$$

Due to the behavior of the Regge-Wheeler potential near the horizon and at spatial infinity, we have $\phi_{\omega\,\ell}(r) \sim \exp[\pm i\omega r_*(r)]$ for $r_* \to -\infty$ ($r \to 2M$) and for $r_* \to +\infty$ ($r \to +\infty$). We then consider (i) the solution $\phi^{\text{in}}_{\omega\,\ell}(r)$ purely ingoing at the horizon (we recall that the time-dependence is in $e^{-i\omega t}$) and (ii) the solution $\phi^{\text{up}}_{\omega\,\ell}(r)$ purely outgoing at spatial infinity. They satisfy

$$\phi^{\text{in}}_{\omega\,\ell}(r) \sim \begin{cases} e^{-i\omega r_*} & (r_* \to -\infty) \\ A^{(-)}_\ell(\omega) e^{-i\omega r_*} + A^{(+)}_\ell(\omega) e^{+i\omega r_*} & (r_* \to +\infty) \end{cases} \qquad \phi^{\text{up}}_{\omega\,\ell}(r) \sim \begin{cases} B^{(-)}_\ell(\omega) e^{-i\omega r_*} + B^{(+)}_\ell(\omega) e^{+i\omega r_*} & (r_* \to -\infty) \\ e^{+i\omega r_*} & (r_* \to +\infty). \end{cases} \tag{9}$$

These two functions are linearly independent solutions of the homogeneous Regge-Wheeler equation and the coefficients $A^{(-)}_\ell(\omega)$, $A^{(+)}_\ell(\omega)$, $B^{(-)}_\ell(\omega)$ and $B^{(+)}_\ell(\omega)$ appearing in Eq. (9) are complex amplitudes. From the functions $\phi^{\text{in}}_{\omega\ell}$ and $\phi^{\text{up}}_{\omega\ell}$ and their Wronskian $W_\ell(\omega)$, we can construct the radial Green function. We have

$$G_{\omega\ell}(r,r') = -\frac{\phi^{\text{in}}_{\omega\ell}(r_<)\,\phi^{\text{up}}_{\omega\ell}(r_>)}{W_\ell(\omega)}, \tag{10}$$

where $r_> = \max(r,r')$ and $r_< = \min(r,r')$ and with

$$W_\ell(\omega) = 2i\omega A^{(-)}_\ell(\omega) = 2i\omega B^{(+)}_\ell(\omega). \tag{11}$$

Scalar perturbations of the Schwarzschild BH

(Here, we have evaluated the Wronskian at $r_* \to -\infty$ and at $r_* \to +\infty$.) Finally, by inserting this result into Eq. (3), we have at our disposal the retarded Green function permitting us to describe the scalar perturbation $\Phi(x)$ of the BH generated by the source $\rho(x)$.

- The result (11) shows that
$$B_\ell^{(+)}(\omega) = A_\ell^{(-)}(\omega) \tag{12}$$
Similarly, the coefficient $B_\ell^{(-)}(\omega)$ can be expressed in terms of $A_\ell^{(-)}(\omega)$ and $A_\ell^{(+)}(\omega)$. Indeed, from the Wronskian of $\phi_{\omega\ell}^{\text{in}}$ and $\overline{\phi_{\omega\ell}^{\text{up}}}$ we obtain
$$B_\ell^{(-)}(\omega) = -\overline{A_\ell^{(+)}(\omega)}. \tag{13}$$
Moreover, the coefficients $A_\ell^{(-)}(\omega)$ and $A_\ell^{(+)}(\omega)$ are constraints by the normalization relation
$$|A_\ell^{(-)}(\omega)|^2 - |A_\ell^{(+)}(\omega)|^2 = 1. \tag{14}$$
which is obtained by considering the Wronskian of $\phi_{\omega\ell}^{\text{in}}$ and $\overline{\phi_{\omega\ell}^{\text{in}}}$.

- We can naturally associate a $\mathscr{S}$-matrix with the retarded Green function by considering the new modes
$$\widetilde{\phi}_{\omega\ell}^{\text{in}}(r) = \frac{\phi_{\omega\ell}^{\text{in}}(r)}{A_\ell^{(-)}(\omega)} \quad \text{and} \quad \widetilde{\phi}_{\omega\ell}^{\text{up}}(r) = \frac{\phi_{\omega\ell}^{\text{up}}(r)}{A_\ell^{(-)}(\omega)}. \tag{15}$$

Scalar perturbations of the Schwarzschild BH

These modes satisfy the boundary conditions

$$\tilde{\phi}^{\text{in}}_{\omega\ell}(r) \sim \begin{cases} T_\ell(\omega) e^{-i\omega r_*} & \text{for } r_* \to -\infty, \\ e^{-i\omega r_*} + R^{\text{in}}_\ell(\omega) e^{+i\omega r_*} & \text{for } r_* \to +\infty, \end{cases} \qquad \tilde{\phi}^{\text{up}}_{\omega\ell}(r) \sim \begin{cases} R^{\text{up}}_\ell(\omega) e^{-i\omega r_*} + e^{+i\omega r_*} & \text{for } r_* \to -\infty, \\ T_\ell(\omega) e^{+i\omega r_*} & \text{for } r_* \to +\infty, \end{cases} \tag{16}$$

where

$$T_\ell(\omega) = \frac{1}{A^{(-)}_\ell(\omega)} \quad \text{and} \quad R^{\text{in}}_\ell(\omega) = \frac{A^{(+)}_\ell(\omega)}{A^{(-)}_\ell(\omega)} \quad \text{and} \quad R^{\text{up}}_\ell(\omega) = -\frac{\overline{A^{(+)}_\ell(\omega)}}{A^{(-)}_\ell(\omega)}. \tag{17}$$

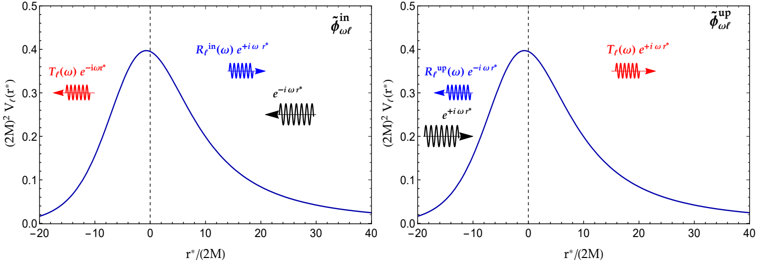

FIGURE 2 – Scattering process associated with the modes $\tilde{\phi}^{\text{in}}_{\omega\ell}(r)$ (left) and $\tilde{\phi}^{\text{up}}_{\omega\ell}(r)$ (right).

Scalar perturbations of the Schwarzschild BH

These modes have a simple physical interpretation (see Fig. 2). They are directly associated with two different scattering problems :

- The in modes correspond to scattering of waves incident on the potential barrier $V_\ell(r)$ generated by the space-time curvature and coming from the spatial infinity. $T_\ell(\omega)$ and $R_l^{\text{in}}(\omega)$ are the associated transmission and reflection coefficients,

- The up modes correspond to scattering of waves incident on the potential barrier $V_\ell(r)$ generated by the space-time curvature and coming from the horizon. $T_\ell(\omega)$ and $R_\ell^{\text{up}}(\omega)$ are the associated transmission and reflection coefficients.

It is important to note that the constraint (14), i.e., $|A_\ell^{(-)}(\omega)|^2 - |A_\ell^{(+)}(\omega)|^2 = 1$ which can be now written as

$$|R_\ell^{\text{in}}(\omega)|^2 + |T_\ell(\omega)|^2 = |R_\ell^{\text{up}}(\omega)|^2 + |T_\ell^{\text{up}}(\omega)|^2 = 1, \tag{18}$$

expresses energy conservation during the scattering process.

- From the reflection and transmission coefficients we construct the block diagonal matrix $\mathscr{S}$ with entries given by

$$\mathscr{S}_\ell(\omega) = \begin{pmatrix} T_\ell(\omega) & R_\ell^{\text{in}}(\omega) \\ R_\ell^{\text{up}}(\omega) & T_\ell(\omega) \end{pmatrix} = \begin{pmatrix} 1/A_\ell^{(-)}(\omega) & A_\ell^{(+)}(\omega)/A_\ell^{(-)}(\omega) \\ -\overline{A_\ell^{(+)}(\omega)}/A_\ell^{(-)}(\omega) & 1/A_\ell^{(-)}(\omega) \end{pmatrix}. \tag{19}$$

Scalar perturbations of the Schwarzschild BH

It is a unitary matrix, i.e.

$$\mathscr{S}(\omega)\overline{\mathscr{S}(\omega)}^{T} = \overline{\mathscr{S}(\omega)}^{T}\mathscr{S}(\omega) = Id \tag{20}$$

[this is a consequence of Eq. (18)] and it is directly associated with all the scattering process linked to the potential barrier generated by the BH curvature. It is worth noting that, as a consequence of $\phi^{\text{in}}_{-\omega\,\ell}(r) = \overline{\phi^{\text{in}}_{\omega\,\ell}(r)}$ we have the relation $A^{(\pm)}_{\ell}(-\omega) = \overline{A^{(\pm)}_{\ell}(\omega)}$ and therefore

$$\mathscr{S}_{\ell}(-\omega) = \overline{\mathscr{S}_{\ell}(\omega)}. \tag{21}$$

• Two important remarks :

- The set of basic functions $(\Phi^{\text{in}}_{\omega\,\ell}, \overline{\Phi^{\text{in}}_{\omega\,\ell}}, \Phi^{\text{up}}_{\omega\,\ell}, \overline{\Phi^{\text{up}}_{\omega\,\ell}})$ naturally appears in all the classical and quantum physical processes involving BHs. As a consequence, the matrix $\mathscr{S}$ is ubiquitous in BH physics.

- All the powerful tools developed in scattering theory can now be introduced in BH physics and, more particularly, the complexification of the frequency (energy) ω and of the angular momentum ℓ or, in other words, the analytic continuation of the $\mathscr{S}$-matrix in the complex ω plane and in the complex ℓ plane.

Electromagnetic and gravitational perturbations of the Schwarzschild BH

- *Mutatis mutandis*, the previous discussion can be extended to the electromagnetic and gravitational perturbations of the Schwarzschild BH. These perturbations are expanded on vector harmonics (electromagnetic perturbations) and tensor harmonics (gravitational perturbations) and are divided into even/polar [notation (e)] and odd/axial [notation (o)] perturbations according they are of even or odd parity in the antipodal transformation on the unit 2-sphere S^2. The even and odd perturbations of the electromagnetic field are both governed by the Regge-Wheeler equation. As far as the gravitational perturbations are concerned, the odd ones are also described by the Regge-Wheeler equation while the even ones are governed by the Zerilli equation.

- Let us first focus on the even and odd electromagnetic perturbations or, more precisely, on the associated radial partial modes $\phi^{(e)}_{\omega\ell}$ and $\phi^{(o)}_{\omega\ell}$ (here, $\ell = 1, 2, \dots$). They are solutions of the Regge-Wheeler equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V^{(e/o)}_\ell(r)\right] \phi^{(e/o)}_{\omega\ell} = 0 \tag{22}$$

with

$$V^{(e/o)}_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2}\right] \tag{23}$$

The potentials $V^{(e)}_\ell(\omega)$ and $V^{(o)}_\ell(\omega)$ respectively associated with the even and odd perturbations are identical and, as a consequence, the $\mathscr{S}$-matrices denoted by $\mathscr{S}^{(e)}_\ell(\omega)$ and $\mathscr{S}^{(o)}_\ell(\omega)$ and respectively associated with the even and odd perturbations are also identical :

$$\mathscr{S}^{(e)}_\ell(\omega) = \mathscr{S}^{(o)}_\ell(\omega). \tag{24}$$

Electromagnetic and gravitational perturbations of the Schwarzschild BH

- Let us now consider the gravitational perturbations. The radial partial modes $\phi^{(o)}_{\omega\,\ell}$ (here, $\ell = 2, 3, \ldots$) associated with the odd perturbations are solutions of the Regge-Wheeler equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V^{(o)}_\ell(r)\right]\phi^{(o)}_{\omega\,\ell} = 0 \tag{25}$$

with

$$V^{(o)}_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right] \tag{26}$$

while the radial partial modes $\phi^{(e)}_{\omega\,\ell}$ associated with the even perturbations are solutions of the Zerilli equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V^{(e)}_\ell(r)\right]\phi_{\omega\,\ell} = 0 \tag{27}$$

with

$$V^{(e)}_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{(\ell-1)^2\ell(\ell+1)(\ell+2)^2r^3 + 6(\ell-1)^2(\ell+2)^2Mr^2 + 36(\ell-1)(\ell+2)M^2r + 72M^3}{[(\ell-1)(\ell+2)r + 6M]^2r^3}\right]. \tag{28}$$

At first sight, the situation is much more complicated because the Regge-Wheeler and Zerilli potentials are very different. But, thanks to Chandrasekhar and Detweiler, we know that they are such that

$$V^{(e/o)}_\ell = (\ell-1)\ell(\ell+1)(\ell+2)\psi + (6M)^2\psi^2 \pm 6M\frac{d\psi}{dr_*} \quad \text{where} \quad \psi(r) = \frac{f(r)}{[(\ell-1)(\ell+2)r + 6M]r} \tag{29}$$

Electromagnetic and gravitational perturbations of the Schwarzschild BH

and, as a consequence, the solutions of the homogeneous Zerilli and Regge-Wheeler equations (27) and (25) are related by the Chandrasekhar-Detweiler transformation

$$[(\ell-1)\ell(\ell+1)(\ell+2)-i(12M\omega)]\phi^{(e)}_{\omega\ell} = \left[(\ell-1)\ell(\ell+1)(\ell+2)+\frac{72M^2}{[(\ell-1)(\ell+2)r+6M]r}f(r)+12M\frac{d}{dr_*}\right]\phi^{(o)}_{\omega\ell} \quad (30a)$$

and

$$[(\ell-1)\ell(\ell+1)(\ell+2)+i(12M\omega)]\phi^{(o)}_{\omega\ell} = \left[(\ell-1)\ell(\ell+1)(\ell+2)+\frac{72M^2}{[(\ell-1)(\ell+2)r+6M]r}f(r)-12M\frac{d}{dr_*}\right]\phi^{(e)}_{\omega\ell}. \quad (30b)$$

From these results, we can show that the $\mathscr{S}$-matrices associated with the even and odd perturbations

$$\mathscr{S}^{(e/o)}_\ell(\omega) = \begin{pmatrix} \dfrac{1/A^{(-,e/o)}_\ell(\omega)}{-A^{(+,e/o)}_\ell(\omega)/A^{(-,e/o)}_\ell(\omega)} & A^{(+,e/o)}_\ell(\omega)/A^{(-,e/o)}_\ell(\omega) \\ & 1/A^{(-,e/o)}_\ell(\omega) \end{pmatrix} \quad (31)$$

have their coefficients related by

$$A^{(-,e)}_\ell(\omega) = A^{(-,o)}_\ell(\omega) \quad (32a)$$

and

$$[(\ell-1)\ell(\ell+1)(\ell+2)-i(12M\omega)]A^{(+,e)}_\ell(\omega) = [(\ell-1)\ell(\ell+1)(\ell+2)+i(12M\omega)]A^{(+,o)}_\ell(\omega). \quad (32b)$$

Analytic extension of the $\mathscr{S}$ matrix in the ω plane

- As mentioned in the introduction of our talk, the complexification of the frequency ω, or, in other words, the analytic extension of the $\mathscr{S}$-matrix in the complex ω plane permits us to physically interpret the response of a BH to an external excitation. There is an abundant literature on the subject and, here, we do not intend to discuss it at length. We shall just recall some fundamental results by focusing on the retarded Green function of the scalar field (convolution with the source ρ is not taken into account) but, of course, all these considerations are equally valid for electromagnetism and gravitational perturbations. In order to evaluate the Green function

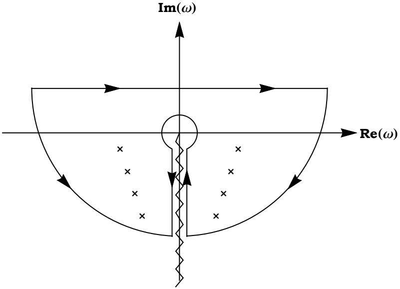

FIGURE 3 – $\mathscr{S}$-matrix structure in the complex ω plane and integration contours.

$$G_{\text{ret}}(x,x') = -\frac{1}{2\pi r r'} \sum_{\ell=0}^{+\infty} (2\ell+1) P_\ell(\cos\gamma) \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\omega\, e^{-i\omega(t-t')} \frac{\phi^{\text{in}}_{\omega\ell}(r_<)\,\phi^{\text{up}}_{\omega\ell}(r_>)}{W_\ell(\omega)}, \tag{33}$$

we can close the path of integration in the lower ω half-plane (for causality reasons) and use Cauchy's theorem (see Fig. 3). The retarded Green function $G_{\text{ret}}(x,x')$ then appears as the sum of three terms :

Analytic extension of the $\mathscr{S}$ matrix in the ω plane

(i) a sum over the poles of the $\mathscr{S}$-matrix [i.e. the zeros of the Wronskian $W_\ell(\omega) = 2i\omega A_\ell^{(-)}(\omega)$ - see Eq. (11)] and the associated residues. They are, respectively, the complex frequencies and the excitation factors of the quasinormal modes (QNMs) which are involved in the description of the BH ringdown, that part of the signal that dominates the BH response at intermediate timescales.

(ii) a branch-cut integral which allows us to describe the tail of the signal, i.e., the BH response at very late times.

(iii) the contribution from the two quarter-circles at infinity, which corresponds to a precursor signal.

- Remarks concerning the poles of the matrix $\mathscr{S}_\ell(\omega)$:

  - For a given value of the angular momentum ℓ, the poles of the matrix $\mathscr{S}_\ell(\omega)$ are the (simple) zeros of the coefficient $A_\ell^{(-)}(\omega)$. They lie in the third and fourth quadrant of the complex ω plane, symmetrically distributed with respect to the imaginary ω axis. They constitute the so-called complex quasinormal frequencies of the BH. We denote by $\omega_{\ell n}$ with $n = 1,2,3,\ldots$ the quasinormal frequencies lying in the fourth quadrant. They satisfy

$$A_\ell^{(-)}(\omega_{\ell n}) = 0 \tag{34}$$

and $-\overline{\omega_{\ell n}}$ is the pole symmetric of $\omega_{\ell n}$ lying in the third quadrant.

  - The BH ringing involves the associated residues or, more precisely, the excitation factors defined by

$$\mathscr{B}_{\ell n} = \left[\frac{1}{2\omega} \frac{A_\ell^{(+)}(\omega)}{\frac{d}{d\omega} A_\ell^{(-)}(\omega)} \right]_{\omega=\omega_{\ell n}} . \tag{35}$$

  - The QNM associated with the quasinormal frequency $\omega_{\ell n}$ is a resonant mode of the BH. $\mathrm{Re}\left[\omega_{\ell n}\right]$ is its frequency of oscillation while $\mathrm{Im}\left[\omega_{\ell n}\right]$ represents its damping.

Analytic extension of the $\mathscr{S}$ matrix in the CAM plane

- For $\omega > 0$, we now consider $\mathscr{S}_{\lambda-1/2}(\omega)$ which is "the" analytic extension of $\mathscr{S}_\ell(\omega)$ into the complex λ plane (the CAM plane) :

$$\mathscr{S}_\ell(\omega) \quad \text{with} \quad \ell \in \mathbb{N} \qquad \longrightarrow \qquad \mathscr{S}_{\lambda-1/2}(\omega) \quad \text{with} \quad \lambda = \ell + 1/2 \in \mathbb{C}. \tag{36}$$

It is constructed from the analytic extensions $A^{(-)}_{\lambda-1/2}(\omega)$ and $A^{(+)}_{\lambda-1/2}(\omega)$ of the coefficients $A^{(-)}_\ell(\omega)$ and $A^{(+)}_\ell(\omega)$. Its (simple) poles lie in the first and third quadrant of the CAM plane. They are symmetrically distributed with respect to the origin and they constitute the so-called Regge poles of the BH $\mathscr{S}$-matrix. The poles lying in the first quadrant are the zeros $\lambda_n(\omega)$ with $n = 1, 2, \ldots$ of $A^{(-)}_\ell(\omega)$. They therefore satisfy

$$A^{(-)}_{\lambda_n(\omega)-1/2}(\omega) = 0. \tag{37}$$

- Numerical determination of Regge poles. It can be achieved by extending the Leaver's method (1985) traditionally used for the determination of the quasinormal frequencies and by combining it with the Hill-determinant method :

  - A preliminary remark. Let us recall the boundary conditions for the partial waves $\phi^{\text{in}}_{\omega\ell}$. We have [see Eq. (9)])

$$\phi^{\text{in}}_{\omega\ell}(r) \sim \begin{cases} e^{-i\omega r_*} & (r_* \to -\infty) \\ A^{(-)}_\ell(\omega) e^{-i\omega r_*} + A^{(+)}_\ell(\omega) e^{+i\omega r_*} & (r_* \to +\infty). \end{cases} \tag{38}$$

For a Regge pole, the coefficient $A^{(-)}_\ell(\omega)$ vanishes. As a consequence, the Regge poles are associated with partial waves $\phi^{\text{in}}_{\omega\ell}$ which are ingoing for $r_* \to -\infty$ and outgoing for $r_* \to +\infty$.

Analytic extension of the $\mathscr{S}$ matrix in the CAM plane

- Such partial waves or Regge modes can be constructed in the form

$$\phi^{\text{in}}_{\omega\ell}(r) = (r-2M)^{\rho}\left(\frac{2M}{r}\right)^{2\rho}\exp\left[-\rho\left(\frac{r-2M}{2M}\right)\right]\sum_{k=0}^{\infty} a_k\left(\frac{r-2M}{r}\right)^{k} \tag{39}$$

with $\rho = -i2M\omega$. In order they satisfy the Regge-Wheeler equation (8) for the potential (5), the a_k must satisfy the recurrence relations $\alpha_k a_{k+1} + \beta_k a_k + \gamma_k a_{k-1} = 0$ with $\alpha_k = k^2 + 2k(\rho+1) + 2\rho + 1$, $\beta_k = -[2k^2 + 2k(4\rho+1) + 8\rho^2 + 4\rho + \ell(\ell+1) + 1]$ et $\gamma_k = k^2 + 4k\rho + 4\rho^2$.

- The Regge poles $\lambda_n(\omega) = \ell_n(\omega) - 1/2$ exist when the sum $\sum a_k$ converges or, equivalently, when the Hill determinant

$$D = \begin{vmatrix} \beta_0 & \alpha_0 & \cdot & \cdot & \cdots \\ \gamma_1 & \beta_1 & \alpha_1 & \cdot & \cdots \\ \cdot & \gamma_2 & \beta_2 & \alpha_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix} \tag{40}$$

vanishes.

- The method can be extended to the electromagnetic and gravitational perturbations of the Schwarzschild BH. In that case, the coefficients α_k, β_k and γ_k appearing in the recursion relation are given in terms of the spin s by $\alpha_k = k^2 + 2k(\rho+1) + 2\rho + 1$, $\beta_k = -[2k^2 + 2k(4\rho+1) + 8\rho^2 + 4\rho + \ell(\ell+1) + 1 - s^2]$ et $\gamma_k = k^2 + 4k\rho + 4\rho^2 - s^2$.

Analytic extension of the $\mathscr{S}$ matrix in the CAM plane

- Regge pole spectrum and Regge trajectories. In Fig. 4, we have displayed the Regge pole spectrum for two different frequencies. In fact, it is in addition important to follow the evolution of the Regge poles with the frequency. The curves traced out in the CAM plane by the Regge poles as a function of the frequency ω are the so-called Regge trajectories. In Fig. 5, we have displayed the Regge trajectories for the lowest Regge poles of the scalar field.

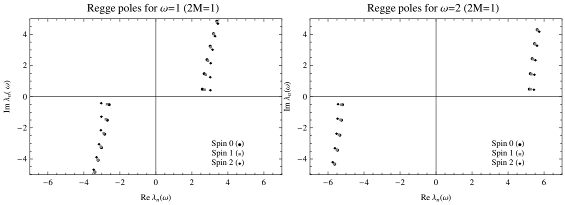

FIGURE 4 – Regge pole spectra for $\omega = 1$ and $\omega = 2$ ($2M = 1$). Spins 0, 1 et 2.

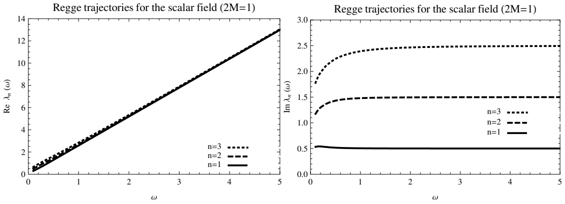

FIGURE 5 – Regge trajectories for the scalar field ($2M = 1$).

Analytic extension of the $\mathscr{S}$ matrix in the CAM plane

- In practice, when we collect the Regge poles in the CAM plane, we introduce naturally the associated residues. For scattering problems, it is interesting to take the residues in the form

$$r_n(\omega) = e^{i\pi[\lambda_n(\omega)+1/2]} \left[\frac{A^{(+)}_{\lambda-1/2}(\omega)}{\frac{d}{d\lambda} A^{(-)}_{\lambda-1/2}(\omega)} \right]_{\lambda=\lambda_n(\omega)} . \tag{41}$$

while, for radiation problems, it is more convenient to introduce the excitation factors for the Regge modes

$$\beta_n(\omega) = \left[\frac{1}{2\omega} \frac{A^{(+)}_{\lambda-1/2}(\omega)}{\frac{d}{d\lambda} A^{(-)}_{\lambda-1/2}(\omega)} \right]_{\lambda=\lambda_n(\omega)} . \tag{42}$$

Both expressions involve the residue of the function $1/A^{(-)}_{\lambda-1/2}(\omega)]$ at $\lambda = \lambda_n(\omega)$.

- Important remarks concerning Regge poles, their physical interpretation and their relation with quasinormal frequencies (details in references given in the introduction) :

- It is possible to derive analytical approximations for the lowest Regge poles by using a WKB approach or, more precisely, by extending the WKB approach developed by Iyer, Schutz and Will (1985-1986) for the determination of the quasinormal frequencies. We obtain for the Regge poles

$$\lambda_n(\omega) = \left[3\sqrt{3}M\omega + \frac{\sqrt{3}a_n}{18M\omega} \right] + i(n-1/2)\left[1 + \frac{b_n}{27(M\omega)^2} \right] + \underset{\omega\to+\infty}{\mathcal{O}} \left(\frac{1}{(2M\omega)^3} \right) \tag{43}$$

where

$$a_n = \frac{2}{3} \left[\frac{5}{12}(n-1/2)^2 + \frac{115}{144} - 1 + s^2 \right] \quad \text{et} \quad b_n = -\frac{4}{9} \left[\frac{305}{1728}(n-1/2)^2 + \frac{5555}{6912} - 1 + s^2 \right] . \tag{44}$$

(43) with (44) provides very accurate results for $2M\omega > 0.5$. Let us note the non-linearity of $\lambda_n(\omega)$.

Analytic extension of the $\mathscr{S}$ matrix in the CAM plane

- The previous results permit us to associate the lowest Regge poles with "surface waves" propagating close to the photon sphere. Such waves are dispersive and damped. Moreover, this association permits us to connect partially the Regge pole spectrum and the quasinormal frequency spectrum. More precisely, from the Regge trajectories, we can semiclassically obtain the complex frequencies $\omega_{\ell n} = \omega^{(0)}_{\ell n} - i\Gamma_{\ell n}/2$ of the weakly damped QNMs by considering Bohr-Sommerfeld-type quantization conditions. There exists a first semiclassical formula (a Bohr-Sommerfeld-type quantization condition) which provides the location of the excitation frequencies $\omega^{(0)}_{\ell n}$ of the resonances generated by nth Regge pole :

$$\operatorname{Re}\lambda_n\left(\omega^{(0)}_{\ell n}\right) = \ell + 1/2, \qquad \ell \in \mathbb{N}. \tag{45}$$

There exists a second semiclassical formula which provides the widths of these resonances

$$\frac{\Gamma_{\ell n}}{2} = \left.\frac{\operatorname{Im}\lambda_n(\omega)[d/d\omega \operatorname{Re}\lambda_n(\omega)]}{[d/d\omega \operatorname{Re}\lambda_n(\omega)]^2 + [d/d\omega \operatorname{Im}\lambda_n(\omega)]^2}\right|_{\omega=\omega^{(0)}_{\ell n}} \tag{46}$$

and which reduces, in the frequency range where the condition $|d/d\omega \operatorname{Re}\lambda_n(\omega)| \gg |d/d\omega \operatorname{Im}\lambda_n(\omega)|$ is satisfied, to

$$\frac{\Gamma_{\ell n}}{2} = \left.\frac{\operatorname{Im}\lambda_n(\omega)}{d/d\omega \ \operatorname{Re}\lambda_n(\omega)}\right|_{\omega=\omega^{(0)}_{\ell n}}. \tag{47}$$

From (43), (45) and (47) we obtain for the quasinormal frequencies the asymptotic behavior :

$$\omega^{(o)}_{\ell n} = \frac{1}{3\sqrt{3}M}\left[(\ell+1/2) - \frac{a_n}{2\ell} + \frac{a_n}{4\ell^2} + \underset{\ell\to+\infty}{\mathcal{O}}\left(\frac{1}{\ell^3}\right)\right] \tag{48a}$$

$$\frac{\Gamma_{\ell n}}{2} = \frac{n-1/2}{3\sqrt{3}M}\left[1 + \frac{a_n + 2b_n}{2\ell^2} + \underset{\ell\to+\infty}{\mathcal{O}}\left(\frac{1}{\ell^3}\right)\right] \tag{48b}$$

with $\ell \in \mathbb{N}$ and $n = 1, 2, \ldots$.

Contents

Regge pole description of scattering of massless waves by a Schwarzschild BH

In this section, we consider the scattering of massless waves (scalar, electromagnetic and gravitational) by a Schwarzschild BH. We focus more particularly on the corresponding differential scattering cross sections, i.e., the ratio of the scattered flux per unit solid angle (at large distance from the BH) to the incident flux. We show how to construct an exact CAM representation of a cross section from its partial wave expansion and to extract a Regge pole approximation. We then consider this approximation and show that, in the shortwavelength regime, it permits us to describe with very good agreement the BH glory and the orbiting oscillations and to overcome the difficulties linked to the lake of convergence of the partial wave expansions. Moreover, we explain how to construct an analytical approximation fitting both the BH glory and the orbiting oscillations.

Scattering of scalar waves by a Schwarzschild BH

- We recall that, for a scalar field, the differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\omega,\theta)|^2 \tag{49}$$

where

$$f(\omega,\theta) = \frac{1}{2i\omega}\sum_{\ell=0}^{\infty}(2\ell+1)[S_\ell(\omega)-1]P_\ell(\cos\theta) \tag{50}$$

denotes the scattering amplitude. In Eq. (50), the functions $P_\ell(\cos\theta)$ are the Legendre polynomials and the S-matrix elements $S_\ell(\omega)$ are given in terms of the complex coefficients $A_\ell^{(-)}(\omega)$ and $A_\ell^{(+)}(\omega)$ by

$$S_\ell(\omega) = e^{i(\ell+1)\pi}\,\frac{A_\ell^{(+)}(\omega)}{A_\ell^{(-)}(\omega)}. \tag{51}$$

The S-matrix elements $S_\ell(\omega)$ we consider in the context of scattering are different of the $\mathscr{S}$-matrix elements $\mathscr{S}_\ell(\omega)$ previously studied. They are in fact proportional to the coefficients $R_\ell^{\text{in}}(\omega)$ because the scattering amplitude $f(\omega,\theta)$ is constructed only in terms of the modes $\phi_{\omega\ell}^{\text{in}}$. As a consequence, the S-matrix is not unitary (absorption of waves by the BH).

Scattering of scalar waves by a Schwarzschild BH

- We can replace the discrete sum over integer values of the angular momentum ℓ appearing in (50) by a contour integral in the complex $\lambda = \ell - 1/2$ plane (the CAM plane). This is achieved by means of the Sommerfeld-Watson transformation which permits us to write

$$\sum_{\ell=0}^{+\infty} (-1)^{\ell} F(\ell) = \frac{i}{2} \int_{\mathscr{C}} d\lambda \, \frac{F(\lambda - 1/2)}{\cos(\pi\lambda)} \tag{52}$$

for a function F without any singularities on the real λ axis. By noting that $P_\ell(\cos\theta) = (-1)^\ell P_\ell(-\cos\theta)$, we obtain

$$f(\omega,\theta) = \frac{1}{2\omega} \int_{\mathscr{C}} d\lambda \, \frac{\lambda}{\cos(\pi\lambda)} \left[S_{\lambda-1/2}(\omega) - 1 \right] P_{\lambda-1/2}(-\cos\theta). \tag{53}$$

In Eqs. (52) and (53), the integration contour encircles counterclockwise the positive real axis of the complex λ plane, i.e., we take $\mathscr{C} =]+\infty + i\epsilon, +i\epsilon] \cup [+i\epsilon, -i\epsilon] \cup [-i\epsilon, +\infty - i\epsilon[$ with $\epsilon \to 0_+$ (see Fig. 6). We can recover (50) from (53) by using Cauchy's theorem and by noting that the poles of the integrand in (53) that are enclosed into $\mathscr{C}$ are the zeros of $\cos(\pi\lambda)$, i.e., the semi-integers $\lambda = \ell + 1/2$ with $\ell \in \mathbb{N}$. It should be recalled that, in Eq. (53), the Legendre function of first kind $P_{\lambda-1/2}(z)$ denotes the analytic extension of the Legendre polynomials $P_\ell(z)$. It is defined in terms of hypergeometric functions by

$$P_{\lambda-1/2}(z) = F[1/2 - \lambda, 1/2 + \lambda; 1; (1-z)/2]. \tag{54}$$

In Eq. (53), $S_{\lambda-1/2}(\omega)$ denotes "the" analytic extension of $S_\ell(\omega)$. It is given by [see Eq. (51)]

$$S_{\lambda-1/2}(\omega) = e^{i(\lambda+1/2)\pi} \frac{A^{(+)}_{\lambda-1/2}(\omega)}{A^{(-)}_{\lambda-1/2}(\omega)} \tag{55}$$

where the complex amplitudes $A^{(-)}_{\lambda-1/2}(\omega)$ and $A^{(+)}_{\lambda-1/2}(\omega)$ are the analytic extension of the amplitudes $A^{(-)}_{\ell}(\omega)$ and $A^{(+)}_{\ell}(\omega)$.

Scattering of scalar waves by a Schwarzschild BH

- We now deform the integration contour $\mathscr{C}$ in order to collect, by using Cauchy's theorem, the contributions of the Regge poles lying in the first quadrant of the CAM plane. We use (i) the symmetry property

$$e^{i\pi\lambda} S_{-\lambda-1/2}(\omega) = e^{-i\pi\lambda} S_{\lambda-1/2}(\omega) \qquad (56)$$

of the S-matrix which can be easily obtained from its definition as well as (ii) the properties for $|\lambda| \to \infty$ of the integrand in Eq. (53). We obtain a CAM representation of the scattering amplitude given by

$$f(\omega,\theta) = f^{\mathrm{B}}(\omega,\theta) + f^{\mathrm{RP}}(\omega,\theta) \qquad (57)$$

where $f^{\mathrm{B}}(\omega,\theta)$ is a background integral contribution and where $f^{\mathrm{RP}}(\omega,\theta)$ is a sum over the Regge poles lying in the first quadrant of the CAM plane. More explicitly, the background integral is given by

$$f^{\mathrm{B}}(\omega,\theta) = \frac{1}{\pi\omega} \int_{\Gamma} d\lambda\, \lambda S_{\lambda-1/2}(\omega) Q_{\lambda-1/2}(\cos\theta + i0) \qquad (58)$$

where $Q_{\lambda-1/2}(z)$ denotes the Legendre function of the second kind which satisfies

$$Q_{\lambda-1/2}(\cos\theta + i0) = \frac{\pi}{2\cos(\pi\lambda)} \left[P_{\lambda-1/2}(-\cos\theta) - e^{-i\pi(\lambda-1/2)} P_{\lambda-1/2}(+\cos\theta) \right] \qquad (59)$$

and where the path of integration is $\Gamma =]+i\infty, 0] \cup [0, +\infty[$ (see Fig. 7) and we have for the Regge pole sum

$$f^{\mathrm{RP}}(\omega,\theta) = -\frac{i\pi}{\omega} \sum_{n=1}^{+\infty} \frac{\lambda_n(\omega) r_n(\omega)}{\cos[\pi\lambda_n(\omega)]} P_{\lambda_n(\omega)-1/2}(-\cos\theta). \qquad (60)$$

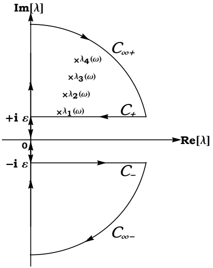

FIGURE 6 – The integration contour $\mathscr{C}$ in the complex λ plane. It defines the scattering amplitude (53) and its deformations permit us to collect, by using Cauchy's theorem, the contributions of the Regge poles $\lambda_n(\omega)$.

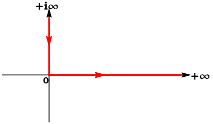

FIGURE 7 – The path of integration in the complex λ plane which defines the background integral (58).

Scattering of scalar waves by a Schwarzschild BH

Of course, Eqs. (57), (58) and (76) provide an exact representation of the scattering amplitude $f(\omega,\theta)$ for the scalar field, equivalent to the initial partial wave expansion (50). From this CAM representation, we can extract the contribution $f^{\mathrm{RP}}(\omega,\theta)$ given by (76) which, as a sum over Regge poles, is only an approximation of $f(\omega,\theta)$, and which provides us with an approximation of the differential scattering cross section (49).

Scattering of scalar waves by a Schwarzschild BH

- In Fig. 8, we compare the differential scattering cross section (49) constructed from the partial wave expansion (50) with its Regge pole approximation obtained from the Regge pole sum (76). The comparison is done for the reduced frequency $2M\omega = 0.1$ (we are working in the "long-wavelength regime") and permits us to show that the Regge pole approximation alone does not permit us to reconstruct the differential scattering cross sections but that this can be achieved by taking into account the background integral contribution (58).

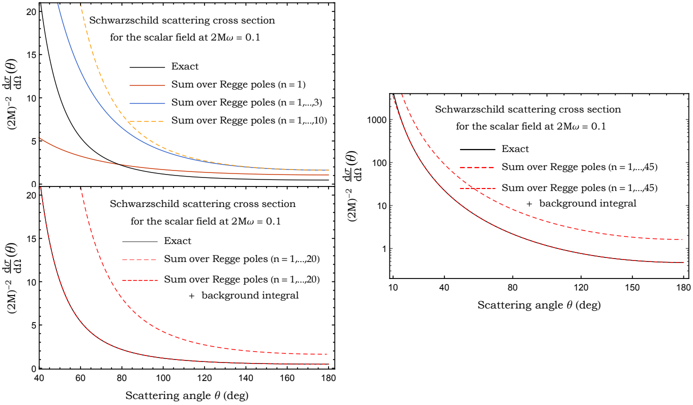

FIGURE 8 – The scalar cross section of a Schwarzschild BH for $2M\omega = 0.1$, its Regge pole approximation and the background integral contribution.

Scattering of scalar waves by a Schwarzschild BH

- In Figs. 9 and 10 where we consider the reduced frequencies $2M\omega = 3$ and 6 (we are working in the "short-wavelength regime"), we observe that the exact differential scattering cross section (49) constructed from the partial wave expansion (50) can be reproduced with very good agreement from its Regge pole approximation obtained from the Regge pole sum (76). We can show in particular that, for large values of the scattering angle, a small number of Regge poles permits us to describe the BH glory and that, by increasing the number of Regge poles, we can reconstruct very efficiently the differential scattering cross sections for small and intermediate scattering angles and therefore describe the orbiting oscillations. In fact, in this wavelength regime, the sum over Regge poles allows us to extract by resummation the physical information encoded in the partial wave expansion defining a scattering amplitude and, moreover, to overcome the difficulties linked to its lack of convergence due to the long-range nature of the fields propagating on the BH.

Scattering of scalar waves by a Schwarzschild BH

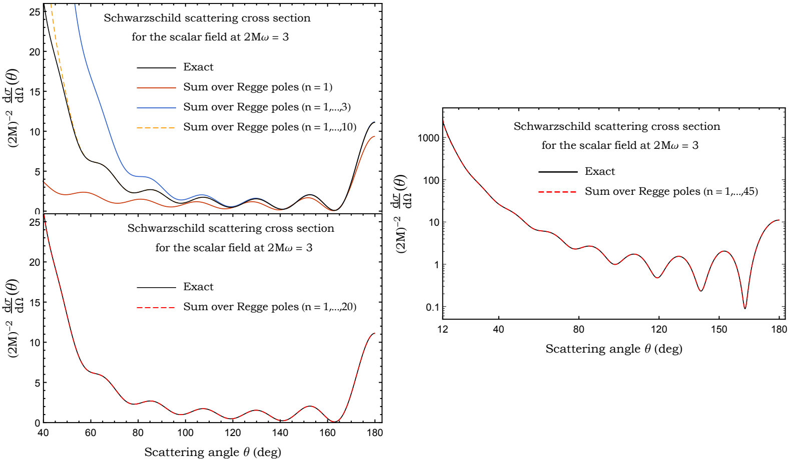

FIGURE 9 – The scalar cross section of a Schwarzschild BH for $2M\omega = 3$ and its Regge pole approximation.

Scattering of scalar waves by a Schwarzschild BH

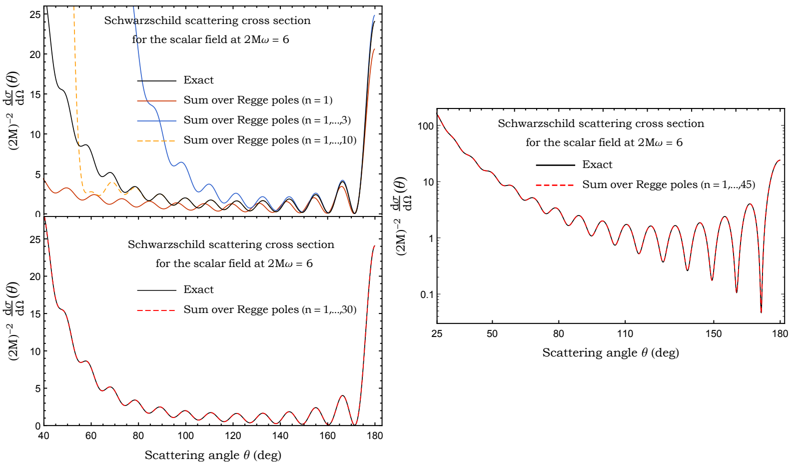

FIGURE 10 – The scalar cross section of a Schwarzschild BH for $2M\omega = 6$ and its Regge pole approximation.

Scattering of electromagnetic waves and gravitational waves by a Schwarzschild BH

- The previous calculations can be (non trivially) extended to analyze the scattering of electromagnetic waves and gravitational waves by a Schwarzschild BH. Here, we only discuss the Regge pole approximation of the differential cross sections, i.e., we only consider the short-wavelength regime. We refer to our papers for more details and to the full theory.

- For the electromagnetic field, the differential scattering cross section can be written in the form

$$\frac{d\sigma}{d\Omega} = |A(\omega,x)|^2 \tag{61}$$

where the scattering amplitude is given by

$$A(\omega,x) = \mathscr{D}_x B(\omega,x) \tag{62}$$

with

$$B(\omega,x) = \frac{1}{2i\omega}\sum_{\ell=1}^{\infty}\frac{(2\ell+1)}{\ell(\ell+1)}[S_\ell(\omega)-1]P_\ell(x) \tag{63}$$

and

$$\mathscr{D}_x = -(1+x)\frac{d}{dx}\left[(1-x)\frac{d}{dx}\right]. \tag{64}$$

In the previous expressions, the variable x is linked to the scattering angle θ by $x=\cos\theta$ and the expression (62)-(64) takes into account the two polarizations of the electromagnetic field. The CAM machinery permits us to extract from the electromagnetic scattering amplitude (62) the Regge pole approximation

$$A^{\mathrm{RP}}(\omega,x) = \mathscr{D}_x\left\{-\frac{i\pi}{\omega}\sum_{n=1}^{+\infty}\frac{\lambda_n(\omega)r_n(\omega)}{[\lambda_n(\omega)^2-1/4]\cos[\pi\lambda_n(\omega)]}P_{\lambda_n(\omega)-1/2}(-x)\right\}. \tag{65}$$

Scattering of electromagnetic waves and gravitational waves by a Schwarzschild BH

- For gravitational waves, the differential scattering cross section can be written in the form

$$\frac{d\sigma}{d\Omega} = |f^+(\omega,x)|^2 + |f^-(\omega,x)|^2 \tag{66}$$

where $f^+(\omega,x)$ and $f^-(\omega,x)$ are scattering amplitudes which are given by

$$f^{\pm}(\omega,x) = \widehat{\mathscr{L}_x^{\pm}}\, \tilde{f}^{\pm}(\omega,x) \tag{67}$$

with

$$\tilde{f}^{\pm}(\omega,x) = \frac{1}{2i\omega}\sum_{\ell=2}^{\infty}\frac{(2\ell+1)}{(\ell-1)\ell(\ell+1)(\ell+2)}\left[\frac{1}{2}\left(S_\ell^{(e)}(\omega)\pm S_\ell^{(o)}(\omega)\right)-\left(\frac{1\pm 1}{2}\right)\right]P_\ell(x). \tag{68}$$

Here, the differential operators $\widehat{\mathscr{L}_x^{\pm}}$ which act on the partial-wave series (68) are given by

$$\widehat{\mathscr{L}_x^{\pm}} = (1\pm x)^2\frac{d}{dx}\left\{(1\mp x)\frac{d^2}{dx^2}\left[(1\mp x)\frac{d}{dx}\right]\right\}. \tag{69}$$

It is interesting to recall that $f^+(\omega,x)$ and $f^-(\omega,x)$ are respectively the helicity-preserving and helicity-reversing scattering amplitudes and that they take into account the two polarizations of the gravitational field. The CAM machinery permits us to extract from the helicity-preserving and helicity-reversing scattering amplitudes (67) the Regge pole approximations

$$f^{\pm}_{\mathrm{RP}}(\omega,x) = \widehat{\mathscr{L}_x^{\pm}}\left\{-\frac{i\pi}{2\omega}\sum_{n=1}^{+\infty}\frac{\lambda_n(\omega)\left[r_n^{(e)}(\omega)\pm r_n^{(o)}(\omega)\right]}{[\lambda_n(\omega)^2-1/4][\lambda_n(\omega)^2-9/4]\cos[\pi\lambda_n(\omega)]}P_{\lambda_n(\omega)-1/2}(-x)\right\}. \tag{70}$$

Scattering of electromagnetic waves and gravitational waves by a Schwarzschild BH

- In Figs. 11, 12 and 13, we can observe that in the "short-wavelength regime" the Regge pole approximations for the electromagnetic and gravitational waves permit us to reproduce with very good agreement the differential scattering cross sections as well as the helicity-preserving and helicity reversing scattering amplitudes.

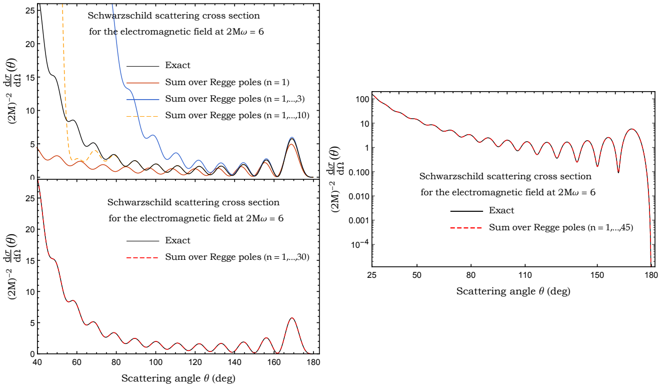

FIGURE 11 – The electromagnetic cross section of a Schwarzschild BH for $2M\omega = 6$ and its Regge pole approximation.

Scattering of electromagnetic waves and gravitational waves by a Schwarzschild BH

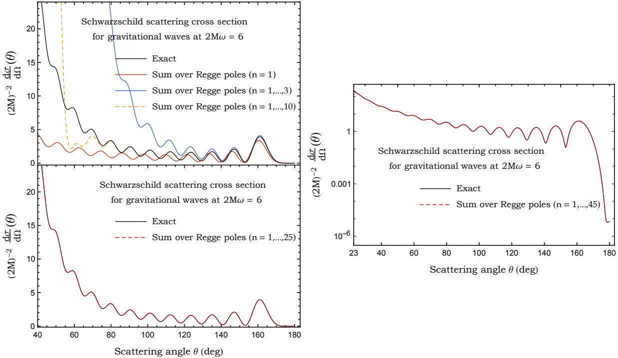

FIGURE 12 – Scattering cross section of a Schwarzschild BH for gravitational waves ($2M\omega = 6$). We compare the exact cross section defined by (66)-(68) with its Regge pole approximation constructed from (70).

Scattering of electromagnetic waves and gravitational waves by a Schwarzschild BH

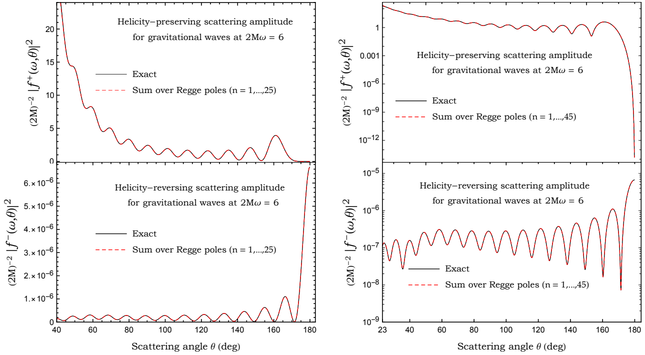

FIGURE 13 – Scattering cross section of a Schwarzschild BH for gravitational waves ($2M\omega = 6$). We emphasize the role of the helicity-preserving and helicity-reversing scattering amplitudes and we compare the exact results (67)-(68) with the corresponding Regge pole approximations (70).

BH glory and orbiting oscillations

- It is interesting to recall that, in a series of papers, Cécile DeWitt-Morette and coworkers (1985 → 1992) have obtained semiclassical and by using path integration techniques, two analytic formulas describing separately the BH glory and the orbiting oscillations. They have established that, for a Schwarzschild BH, the glory scattering cross section of massless waves of spin s can be approximated by

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{glory}} = 30.752 M^3 \omega \left[J_{2s}(5.357 M\omega \sin\theta)\right]^2, \tag{71}$$

  a formula which is formally valid for $2M\omega \gg 1$ and $|\theta - \pi| \ll 1$. Here, J_{2s} is a Bessel function of the first kind. This formula is quite precise as we shall see. By contrast, it is well known that their analytic description of the orbiting oscillations is unsatisfactory.

- By assuming that, at large reduced frequencies $2M\omega$, the lowest Regge poles are in a one-to-one correspondence with "surface waves" propagating close to the unstable circular photon (graviton) orbit at $r = 3M$, i.e., near the Schwarzschild photon sphere (see above), we can derive an analytical approximation describing with good agreement both the BH glory and the orbiting oscillations. This is achieved by inserting in the Regge pole sums, asymptotic approximations for the lowest Regge poles and the associated residues.

BH glory and orbiting oscillations

- To simplify our purpose, we only consider the case of the scalar field. For the regge poles, we use [see also Eqs. (43) and (44)]

$$\lambda_n(\omega) \approx 3\sqrt{3}M\omega + i(n-1/2) + \frac{\sqrt{3}a_n}{18M\omega} \tag{72}$$

where

$$a_n = \frac{2}{3}\left[\frac{5}{12}(n-1/2)^2 + \frac{115}{144}\right] \tag{73}$$

and, for the associated residues, we consider

$$r_n(\omega) \approx \frac{\left[-i\,216(3\sqrt{3}M\omega)/\xi\right]^{n-1/2}}{\sqrt{2\pi}(n-1)!} e^{i2M\omega y} e^{i\pi\lambda_n(\omega)} \tag{74}$$

where

$$\xi = 7 + 4\sqrt{3} \quad \text{and} \quad y = 3 - 3\sqrt{3} + 4\ln 2 - 3\ln\xi, \tag{75}$$

an expression which can be obtained by solving the Regge-Wheeler equation with WKB methods and matching techniques.

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- In Fig. 14, we now compare the exact scattering cross sections with its analytical approximation. The comparisons are achieved for the reduced frequencies $2M\omega = 3$ and 6 and the summations are over the first five Regge poles. We can observe that the analytical Regge pole approximations permit us to reproduce with very good agreement both the glory cross section and a large part of the orbiting cross section. We have included the plot of the glory formula (71) in order to show the superiority of our approach.

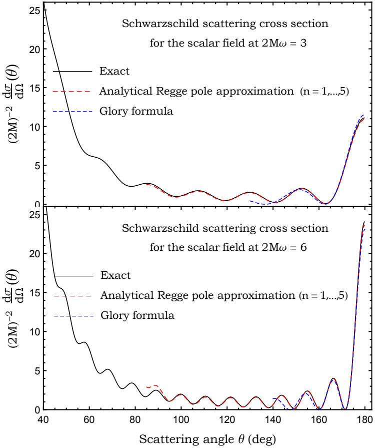

FIGURE 14 – The scalar cross section of a Schwarzschild BH for $2M\omega = 3$ and $2M\omega = 6$. We compare the exact result with the glory formula and with that obtained from the analytical Regge pole approximation.

BH glory and orbiting oscillations

- Remarks : It is possible to go even further and to interpret the BH glory and the orbiting oscillations in terms of interferences due to the surface waves propagating close to the photon sphere. We reconsider the Regge pole sum (76)

$$f^{\mathrm{RP}}(\omega,\theta) = -\frac{i\pi}{\omega}\sum_{n=1}^{+\infty}\frac{\lambda_n(\omega)r_n(\omega)}{\cos[\pi\lambda_n(\omega)]}P_{\lambda_n(\omega)-1/2}(-\cos\theta). \tag{76}$$

From the asymptotic expansion

$$P_{\lambda-1/2}(-\cos\theta) \sim \frac{e^{i\lambda(\pi-\theta)-i\pi/4}+e^{-i\lambda(\pi-\theta)+i\pi/4}}{(2\pi\lambda\sin\theta)^{1/2}}$$

valid for $|\lambda|\to\infty$ and $|\lambda|\sin\theta>1$ as well as $1/[\cos\pi\lambda] = 2i\sum_{m=0}^{+\infty}\exp[i\pi(2m+1)(\lambda-1/2)]$ which is true if Im $\lambda>0$ we obtain

$$f^{\mathrm{RP}}(\omega,\theta) = \frac{2\pi}{i\omega}\sum_{n=1}^{+\infty}\frac{\lambda_n(\omega)r_n(\omega)}{[2\pi\lambda_n(\omega)\sin\theta]^{1/2}}\sum_{m=0}^{+\infty}\left[e^{i\lambda_n(\omega)(\theta+2m\pi)-im\pi+i\pi/4}+e^{i\lambda_n(\omega)(2\pi-\theta+2m\pi)-im\pi-i\pi/4}\right]. \tag{77}$$

In (77), terms like $\exp[i\lambda_n(\omega)\theta]$ and $\exp[i\lambda_n(\omega)(2\pi-\theta)]$ correspond to "surface waves" ("Regge waves") propagating counterclockwise and clockwise around the BH [let us recall the time-dependance in $\exp(-i\omega t)$]. Re $\lambda_n(\omega)$ represent their azimuthal propagation constants and Im $\lambda_n(\omega)$ their dampings. The exponential decay $\exp[-\text{Im }\lambda_n(\omega)\theta]$ and $\exp[-\text{Im }\lambda_n(\omega)(2\pi-\theta)]$ are due to continual reradiation of energy. Moreover, the sum over m takes into account the multiple circumnavigations of the "surface waves" around the BH.

BH glory and orbiting oscillations

We can note that, for high frequencies, these "surface waves" can be considered as waves lying near the photon sphere at $r = 3M$ because we have

$$\lambda_n(\omega) \sim 3\sqrt{3}M\omega + i(n-1/2) \qquad n = 1,2,3,\dots \quad \text{for} \quad 2M\omega \gg 1. \tag{78}$$

Indeed, terms like $\exp[i(\lambda_n(\omega)\theta - \omega t)]$ are associated with waves circling the BH in time $T = 2\pi\text{Re}\ \lambda_n(\omega)/\omega \sim 2\pi(3\sqrt{3}M)$. But a massless particle on the circular orbit with constant radius R takes the time $T' = 2\pi r/(1-2M/r)^{1/2}$ to circle the BH. By equating T and T', we obtain $r = 3M$.
The previous interpretation permits us, in particular, to consider that the BH glory and the orbiting oscillations are due to constructive and destructive interferences generated by the "surface waves" diffracted by the photon sphere. Similarly, the resonant behavior of the BH (and therefore the existence of QNMs) can now be understood in terms of constructive interferences between the different components of a given 'surface waves", each component corresponding to a different number of circumnavigations.

Contents

Regge pole description of gravitational radiation generated by a massive particle falling radially from infinity into a Schwarzschild BH

- Due to lack of time, we shall not consider this subject in our talk. We refer the interested reader to our paper

  A. Folacci and M. Ould El Hadj, *An alternative description of gravitational radiation from black holes based on the Regge poles of the $\mathscr{S}$-matrix and the associated residues*, PRD 98 (2018) 064052

  We just note that, in this article, we have revisited from CAM techniques the old problem of the description of gravitational radiation generated by a massive particle falling radially from infinity into a Schwarzschild BH. We have shown that :

  - The Fourier transform of a sum over the Regge poles and their residues which can be evaluated numerically from the associated Regge trajectories permits us to reconstruct, for an arbitrary direction of observation, a large part of the multipolar waveform represented by the Weyl scalar Ψ_4. In particular, it can reproduce with very good agreement the quasinormal ringdown as well as with rather good agreement the tail of the signal. This is achieved even if we take into account only one Regge pole and, if a large number of modes are excited, the result can be improved by considering additional poles.

  - While QNM contributions do not provide physically relevant results at “early times” due to their exponentially divergent behavior as time decreases, it is not necessary to determine from physical considerations a starting time for the Regge ringdown.

Contents

Conclusion and perspectives

- Many problems we can encounter in BH physics are usually tackled from partial wave methods and analyzed by working in the complex ω plane. We are developing an alternative approach based on the analytic extension of the $\mathscr{S}$-matrix in the CAM plane and on the use of Regge poles. It allows us :
  (i) to extract by resummation the information encoded in partial wave expansions,
  (ii) to overcome the difficulties linked to their lack of convergence due to the long-range nature of the fields propagating on BHs,
  (iii) to provide nice physical interpretations of the results obtained when it is combined with high frequency asymptotic methods.

- We hope in next works to explore implications of our results in the context of strong gravitational lensing of electromagnetic and gravitational waves by BHs. A more challenging task is the extension of our study to scattering of waves by a Kerr BH. We also hope to make progress in this direction in the near future.