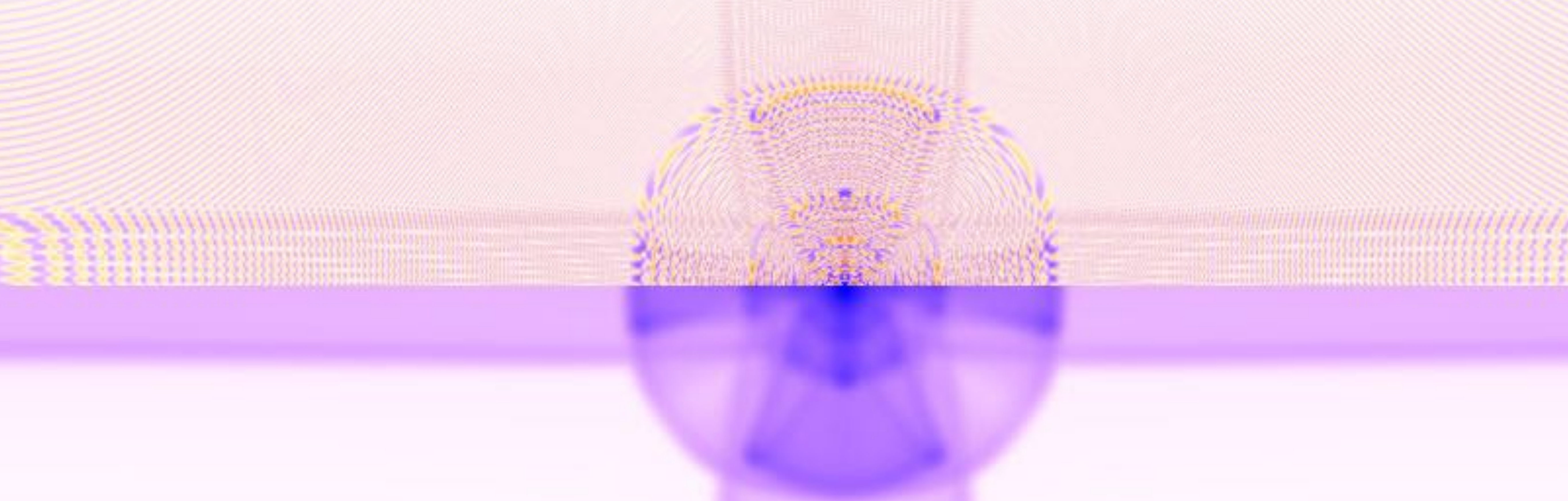
Relation between Vlasov and Schrödinger-Poisson

as unifying dynamical description for dark matter and massive neutrinos



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Michael Kopp

CEICO, Institute of Physics Prague, Czech Republic

1711.00140 with K. Vattis (Brown U.) and C. Skordis

Work in progress with T. Abel (Stanford U.)

Hot topics in Modern Cosmology.
Spontaneous Workshop XIII 5 - 11 May 2019, Cargese

1. Introduction/Motivation

Vlasov equation, Closing Boltzmann hierarchy

2. Comparison of cold DM formulations

coarse grained Vlasov and
Schrödinger method (ScM) with

a) 2D simulations (*visually*)
b) General formulas and 1D examples
c) 2D simulations (*quantitatively*)

3. Discussion of CDM

Vorticity, Effective dark matter equation of state

4. Comparison of warm DM formulations

5. Summary & Outlook

Vlasov equation: Preview and definitions

Continuous phase space distribution function $f(t, \boldsymbol{x}, \boldsymbol{u})$

- ensemble average of Klimontovich f_N, the N-body problem.
- dropping gravitational collision terms ~1/N
- moments

Gilbert (ApJ 152, 1968)
Binney, Tremaine (1987)
Bertschinger astro-ph/9503125

$$M^{(n)}_{i_1 \ldots i_n}(\boldsymbol{x}) \equiv \int \mathrm{d}^3u \;\; u_{i_1} \cdots u_{i_n} f(\boldsymbol{x}, \boldsymbol{u})$$

- density $n(\boldsymbol{x}) = M^{(0)} = e^{C^{(0)}}$
- velocity $u_i(\boldsymbol{x}) = C^{(1)}_i = M^{(1)}_i / n$
- velocity dispersion $C^{(2)}_{ij}(\boldsymbol{x}) = M^{(2)}_{ij} / n - u_i u_j$

Vlasov (- Poisson) equation (collisionless Boltzmann)

$$\partial_t f(\boldsymbol{x}, \boldsymbol{u}) = -\frac{\boldsymbol{u}}{a^2} \nabla_x f + \nabla_x \Phi \nabla_u f$$

$$\Delta \Phi = \frac{4\pi G \rho_0}{a} \left(\int \mathrm{d}^3u \; f - 1 \right)$$

Number of particles $f \mathrm{d}^3x\, \mathrm{d}^3u$ along phase space trajectories is conserved

nonlinearity

nonlocality

$$\int_{\mathrm{vol}} \mathrm{d}^3x \int \mathrm{d}^3u \; f = \mathrm{vol}$$

Boltzmann hierarchy

$$\partial_t C^{(n)}_{i_1 \cdots i_n} = -\frac{1}{a^2} \left\{ \nabla_j C^{(n+1)}_{i_1 \cdots i_n j} + \sum_{S \in \mathcal{P}(\mathcal{I} = \{i_1, \cdots, i_n\})} C^{(n+1-|S|)}_{\mathcal{I} \setminus S \cap \{j\}} \nabla_j C^{(|S|)}_S \right\} - \delta_{n1} \nabla_{i_1} \Phi$$

Uhlemann, MK, Haugg
1403.5567

consistent truncation:
dust (pressureless perfect) **fluid**

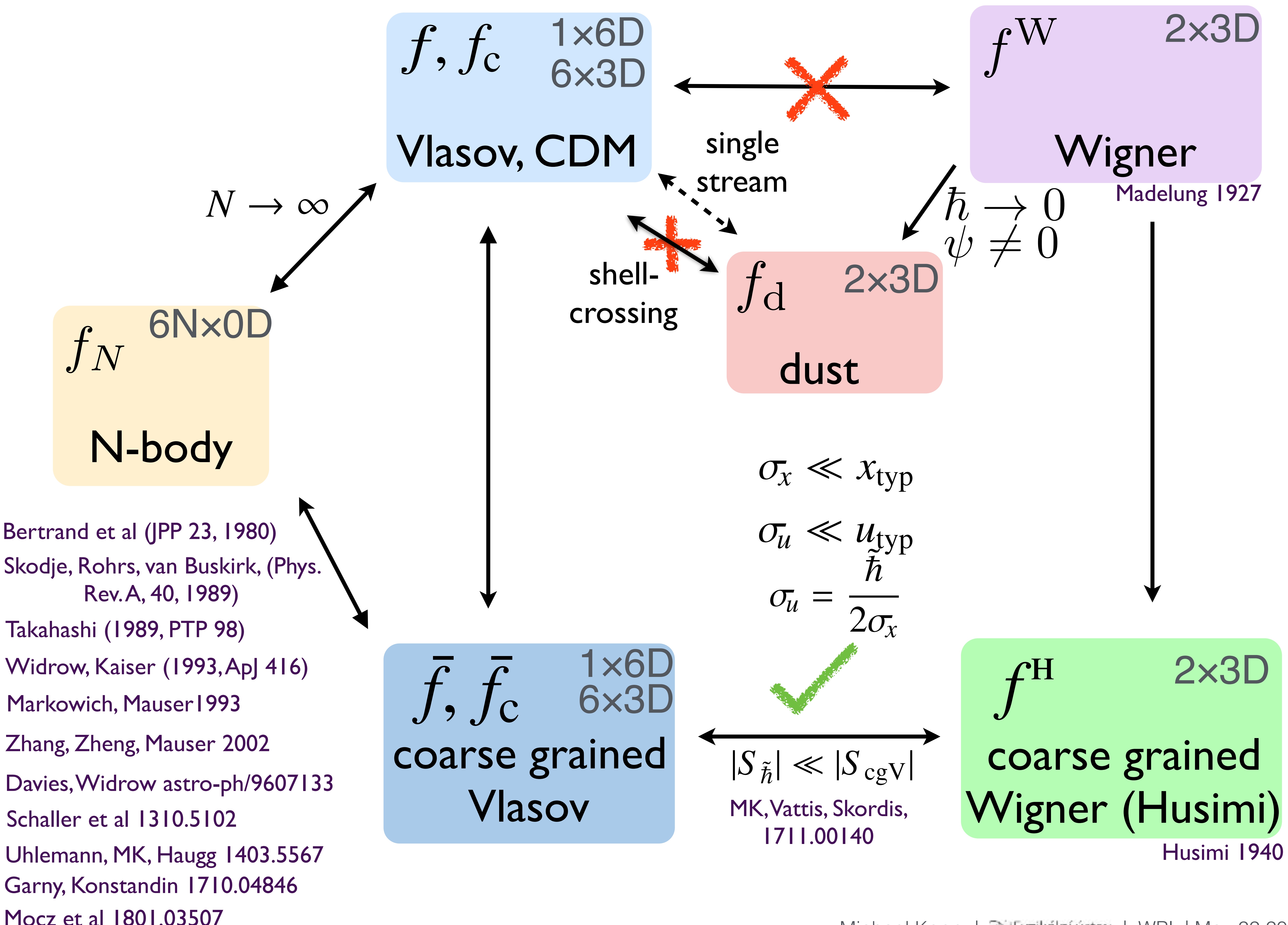
$$C^{(n \geq 2)}(\boldsymbol{x}) = 0$$

$$f_{\mathrm{d}}(t, \boldsymbol{x}, \boldsymbol{u}) = n_{\mathrm{d}}(t, \boldsymbol{x})\, \delta_{\mathrm{D}}\big(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{d}}(t, \boldsymbol{x})\big)$$

Definition of Cold Dark Matter (CDM)

$$\lim_{t \to 0} f_{\mathrm{c}}(t, \boldsymbol{x}, \boldsymbol{u}) = f_{\mathrm{d}}(t, \boldsymbol{x}, \boldsymbol{u})$$

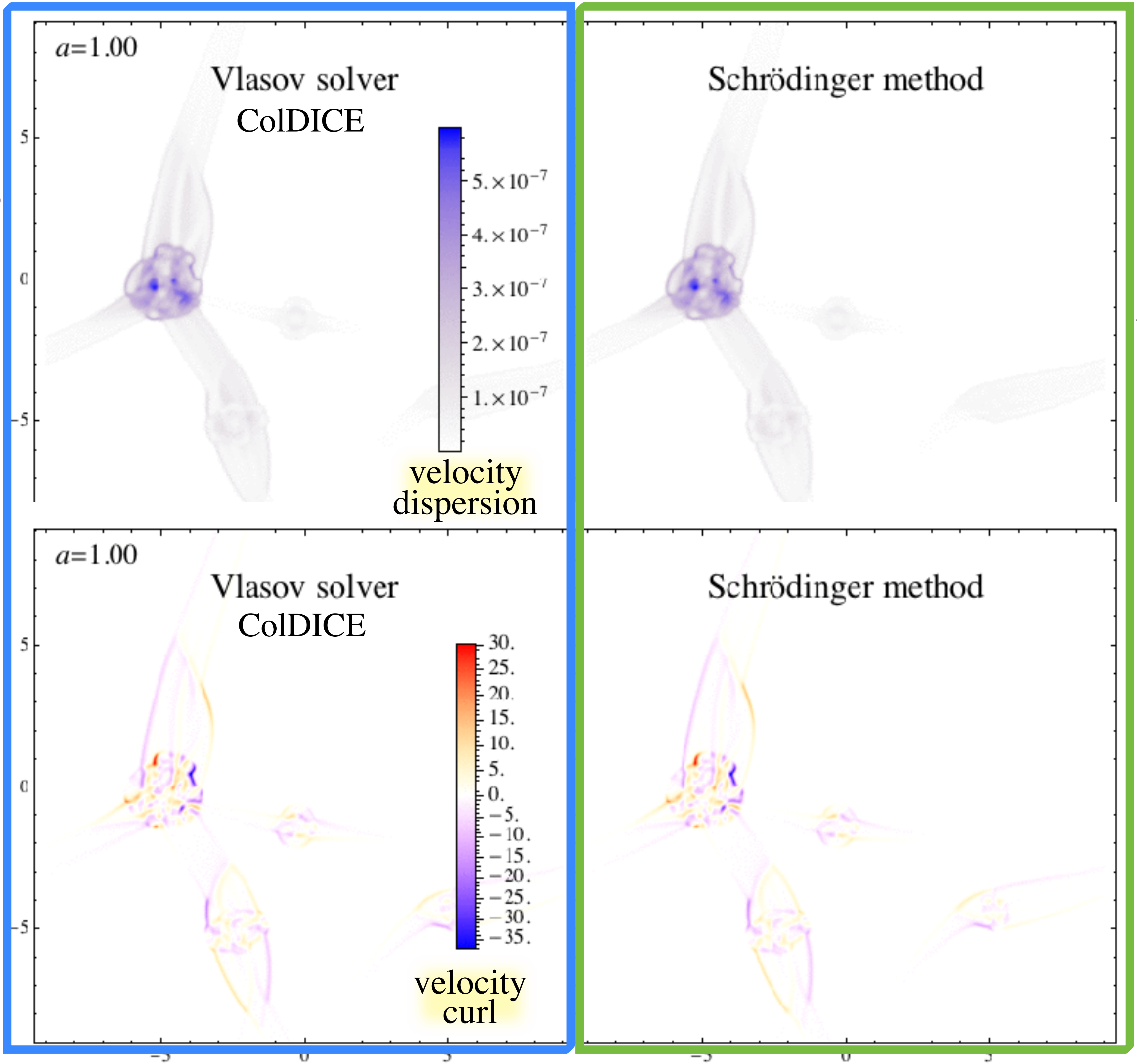
For the purpose of large scale structure formation in cosmology


f, f_c 1×6D 6×3D
Vlasov, CDM
f^W 2×3D
Wigner
Madelung 1927
single stream
shell-crossing
$N \to \infty$
$\hbar \to 0$
$\psi \neq 0$
f_d 2×3D
dust
f_N 6N×0D
N-body
$\sigma_x \ll x_{typ}$
$\sigma_u \ll u_{typ}$
$\sigma_u = \frac{\tilde{\hbar}}{2\sigma_x}$
$\bar{f}, \bar{f}_c$ 1×6D 6×3D
coarse grained Vlasov
$|S_{\tilde{\hbar}}| \ll |S_{cgV}|$
MK, Vattis, Skordis, 1711.00140
f^H 2×3D
coarse grained Wigner (Husimi)
Husimi 1940
Bertrand et al (JPP 23, 1980)
Skodje, Rohrs, van Buskirk, (Phys. Rev. A, 40, 1989)
Takahashi (1989, PTP 98)
Widrow, Kaiser (1993, ApJ 416)
Markowich, Mauser 1993
Zhang, Zheng, Mauser 2002
Davies, Widrow astro-ph/9607133
Schaller et al 1310.5102
Uhlemann, MK, Haugg 1403.5567
Garny, Konstandin 1710.04846
Mocz et al 1801.03507

2a) Comparison
of
2D cosmological
simulations by eye

for coarse grained Vlasov and
Schrödinger method (ScM)

Sousbie, Colombi, (JCPH,321, 644, 2016) 1509.07720



MK, Vattis, Skordis, 1711.00140

$$M^{w(2)}_{ij} = \frac{\tilde{\hbar}^2}{2}\mathfrak{R}\{\psi_{,i}\bar{\psi}_{,j} - \psi_{,ij}\bar{\psi}\}$$

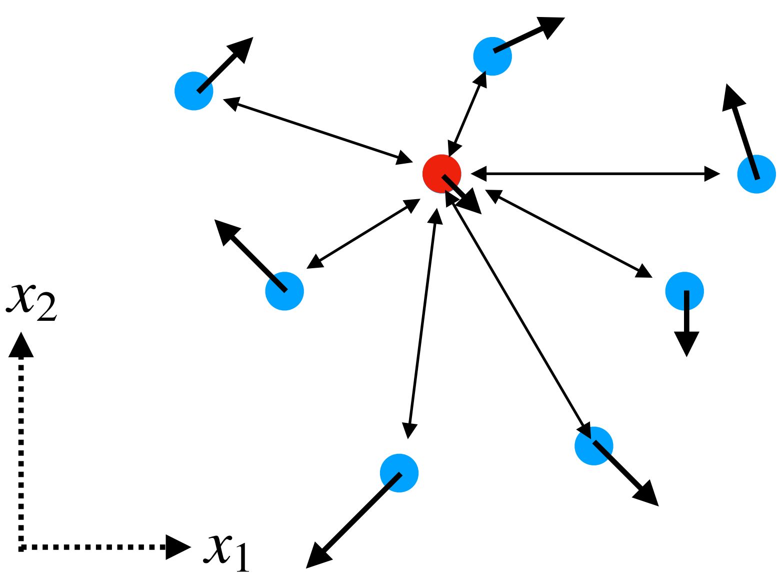
+gaussian filter with width σ_x, σ_u

$$M^{w(1)}_{i} = \tilde{\hbar}\,\mathfrak{I}\,\{\psi_{,i}\bar{\psi}\}$$

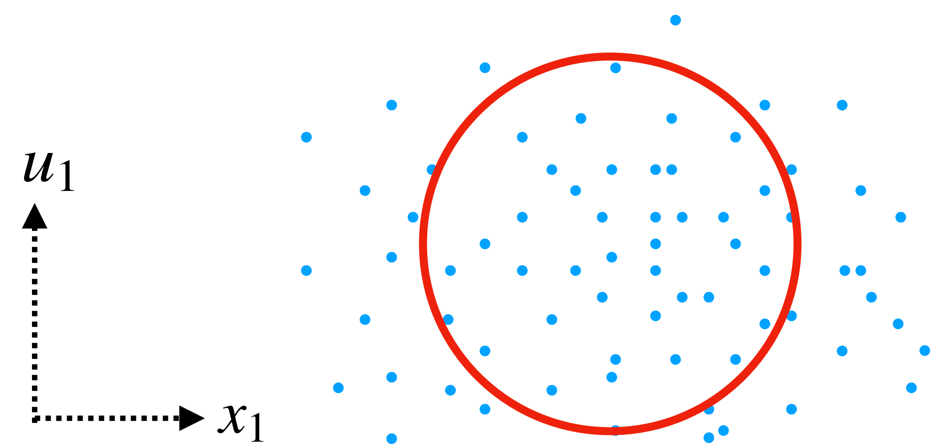
Vlasov equation: Definition

Newtonian N-body problem

Position space



Phase space



Klimontovich phase space distribution

$$f_{\mathrm{K}}(t, \boldsymbol{x}, \boldsymbol{u}) = \frac{m}{\rho_0} \sum_{i=1}^{N} \delta_{\mathrm{D}}[\boldsymbol{x} - \boldsymbol{x}_i(t)]\, \delta_{\mathrm{D}}[\boldsymbol{u} - \boldsymbol{u}_i(t)]$$

- f_{K} evolves in phase-space $(\boldsymbol{x}, \boldsymbol{u})$
- Vlasov equation: neglect all discreteness.

$$E_N = \sum_{i=1}^{N} \left(\frac{\boldsymbol{u}_i^2}{2a^2} - \frac{mG}{2a} \sum_{j=1, j\neq i}^{N} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \right)$$

$$\dot{\boldsymbol{x}}_i = \frac{\partial E_N}{\partial \boldsymbol{u}_i}, \qquad \dot{\boldsymbol{u}}_i = -\frac{\partial E_N}{\partial \boldsymbol{x}_i}$$

$N \to \infty$

$\Longleftrightarrow$

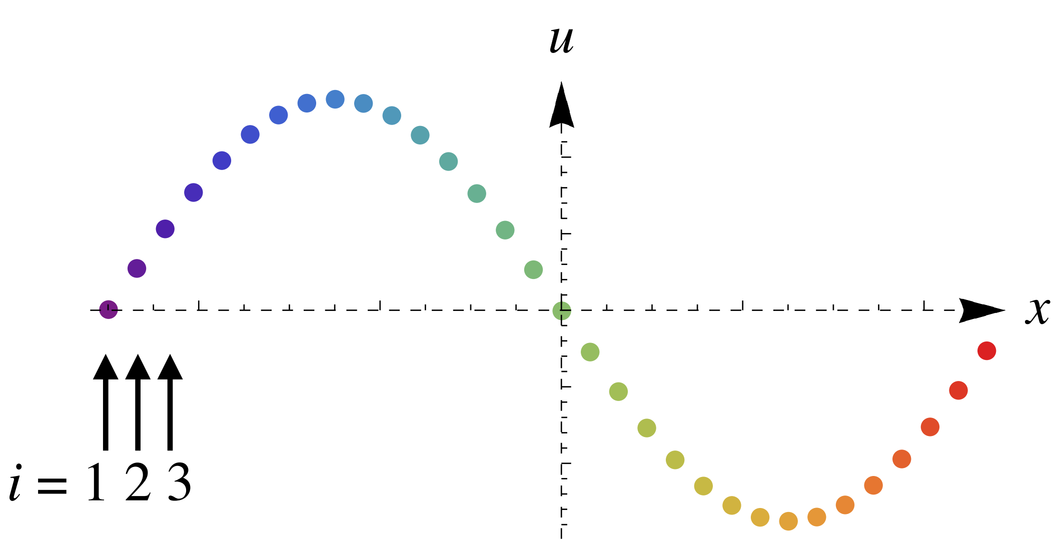
$$\partial_t f = -\frac{\boldsymbol{u}}{a^2} \cdot \boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_x \Phi \cdot \boldsymbol{\nabla}_u f$$

$$\Delta \Phi = \frac{4\pi G \rho_0}{a} \left(\int \mathrm{d}^3 u \; f - 1 \right)$$

Vlasov equation: Definition

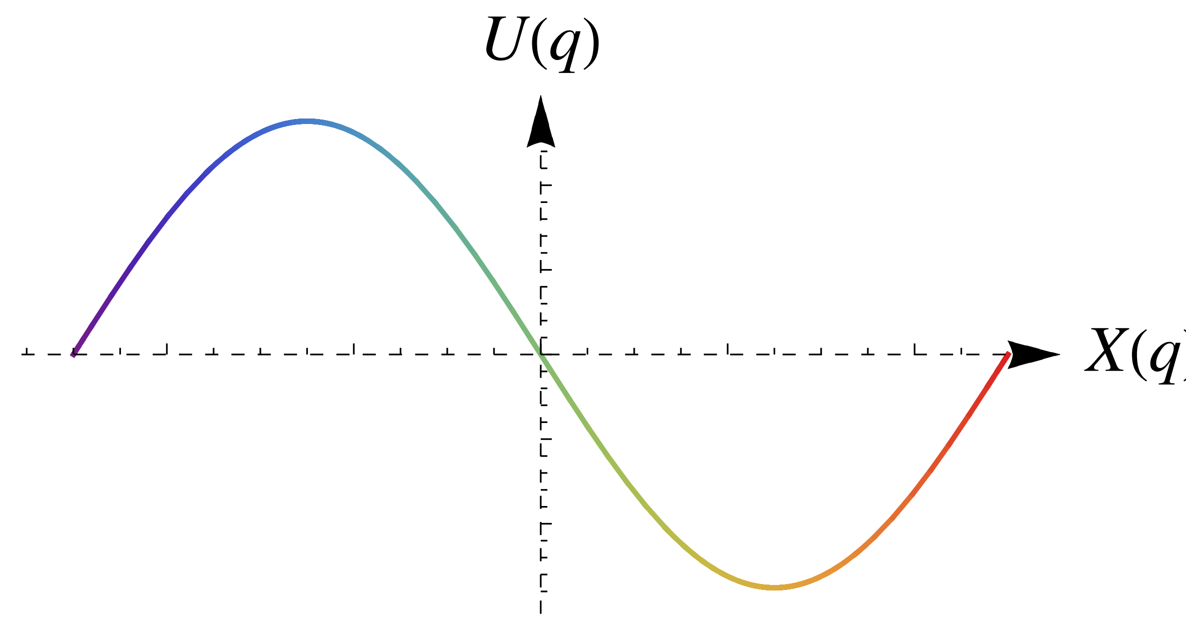
Cold dark matter: the continuum limit $\quad \boldsymbol{i} \rightarrow \boldsymbol{q} \qquad \boldsymbol{x}_i(t) \rightarrow \boldsymbol{X}(t, \boldsymbol{q})$

Klimontovich phase space distribution

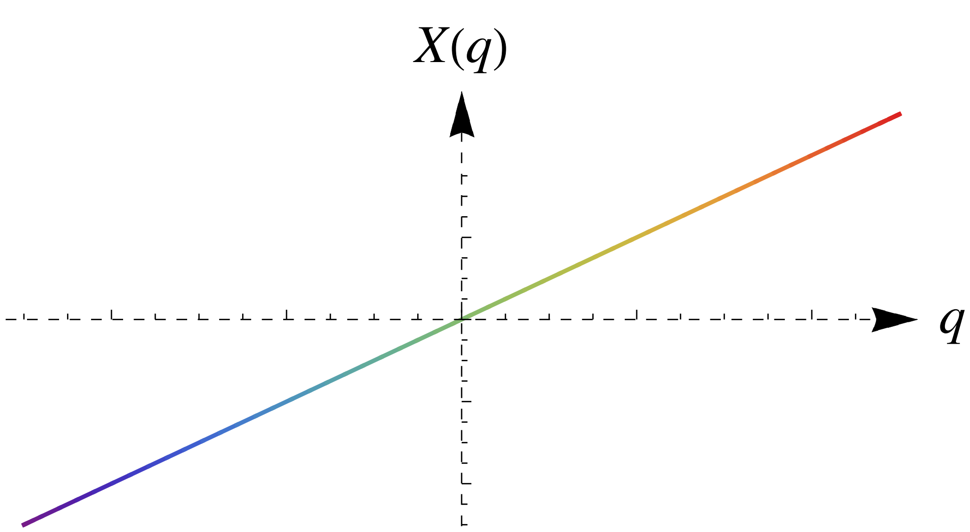


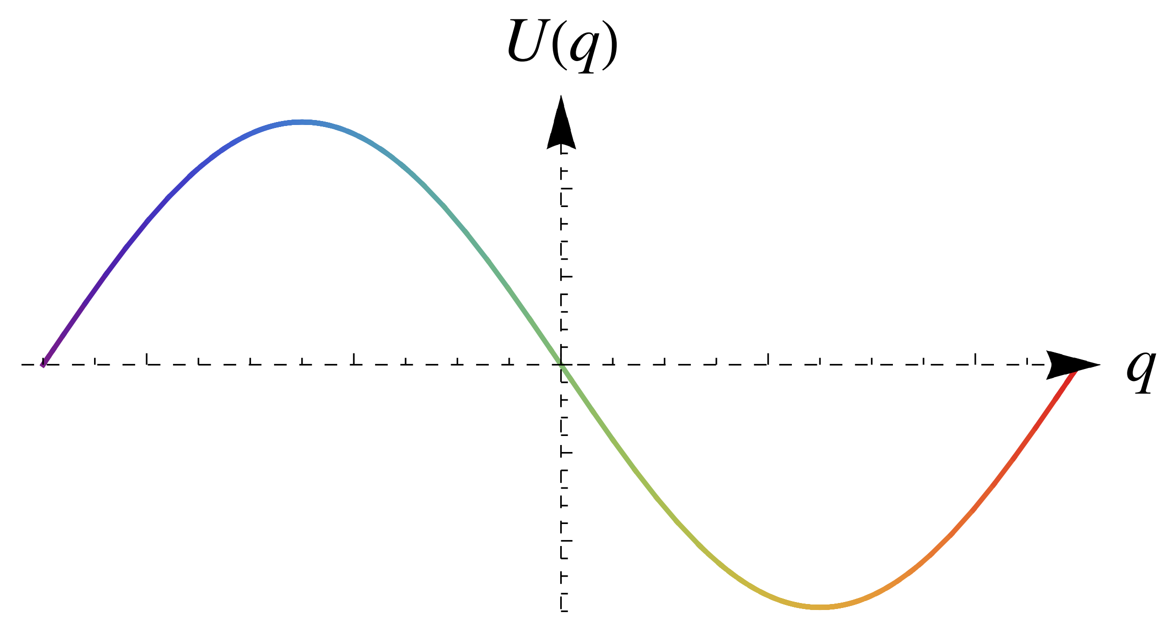
$\longrightarrow$

Vlasov phase space distribution



Cold dark matter: initial conditions $\quad \boldsymbol{X}(t \rightarrow 0, \boldsymbol{q}) = \boldsymbol{q}$





Vlasov equation: Definition

Cold dark matter: phase space sheet

Construct $f(x,u)$ from Lagrangian fields $X(q)$ and $U(q)$

$$f_\mathrm{K}(t,\boldsymbol{x},\boldsymbol{u}) = \frac{m}{\rho_0}\sum_{i=1}^{N}\delta_\mathrm{D}[\boldsymbol{x}-\boldsymbol{x}_i]\,\delta_\mathrm{D}[\boldsymbol{u}-\boldsymbol{u}_i]$$

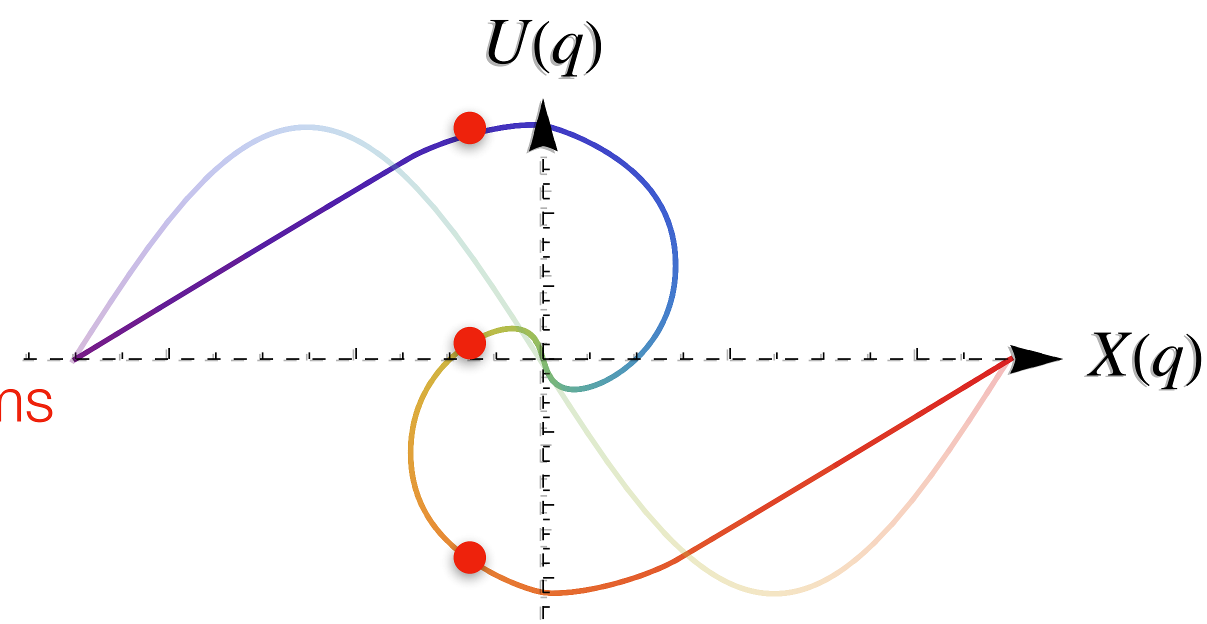
$$\downarrow\; i \to \boldsymbol{q}$$

$$f_\mathrm{c}(t,\boldsymbol{x},\boldsymbol{u}) = \int \mathrm{d}^3q\;\; \delta_\mathrm{D}[\boldsymbol{x}-\boldsymbol{X}(\boldsymbol{q})]\,\delta_\mathrm{D}[\boldsymbol{u}-\boldsymbol{U}(\boldsymbol{q})]$$

$$= \sum_{\substack{\boldsymbol{q}\text{ with}\\ \boldsymbol{x}=\boldsymbol{X}(t,\boldsymbol{q})}} \frac{\delta_\mathrm{D}[\boldsymbol{u}-\boldsymbol{U}(\boldsymbol{q})]}{|\det\partial_{q^i}X^j(\boldsymbol{q})|}$$

sum over streams

Vlasov phase space distribution

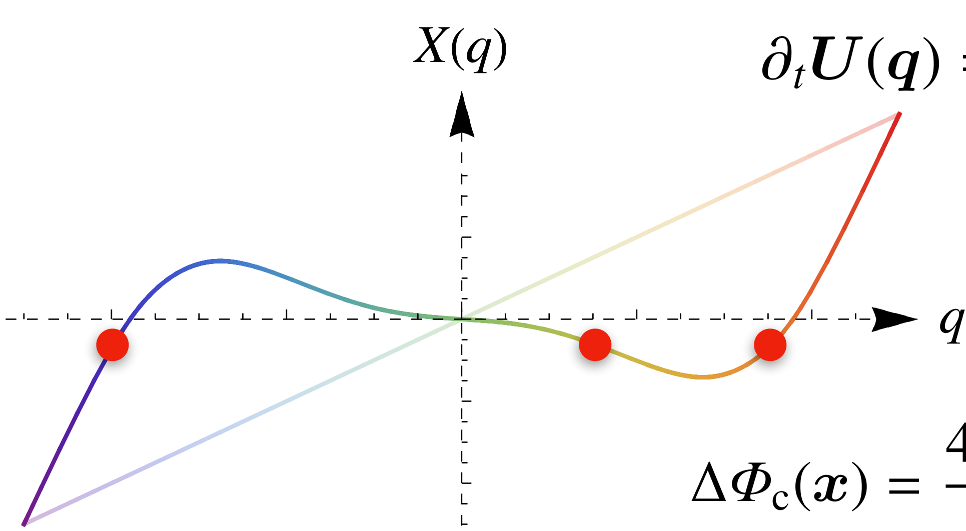


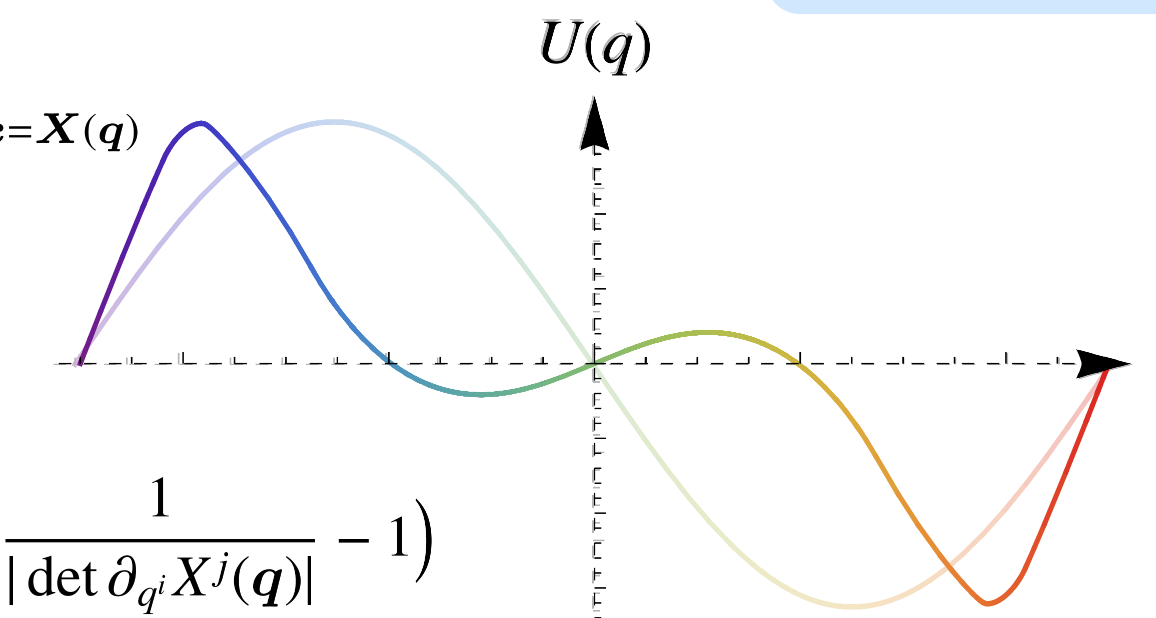
Cold dark matter: dynamics

D.o.f. $6\times\mathcal{R}^3$

$$a^2\partial_t\boldsymbol{X}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})$$

$$\partial_t\boldsymbol{U}(\boldsymbol{q}) = -\boldsymbol{\nabla}_x\Phi_\mathrm{c}(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}$$

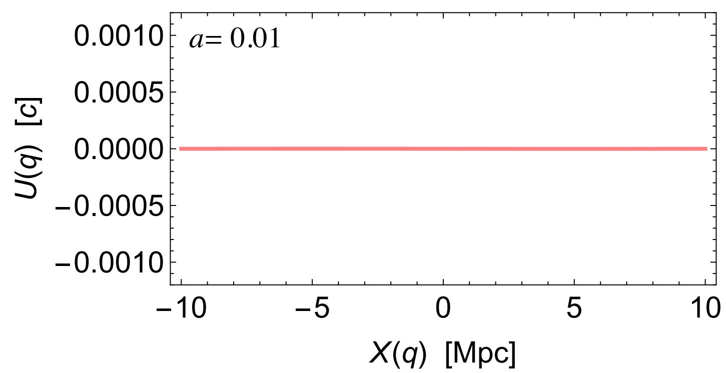




$$\Delta\Phi_\mathrm{c}(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\Bigg(\sum_{\substack{\boldsymbol{q}\text{ with}\\ \boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}}\frac{1}{|\det\partial_{q^i}X^j(\boldsymbol{q})|} - 1\Bigg)$$

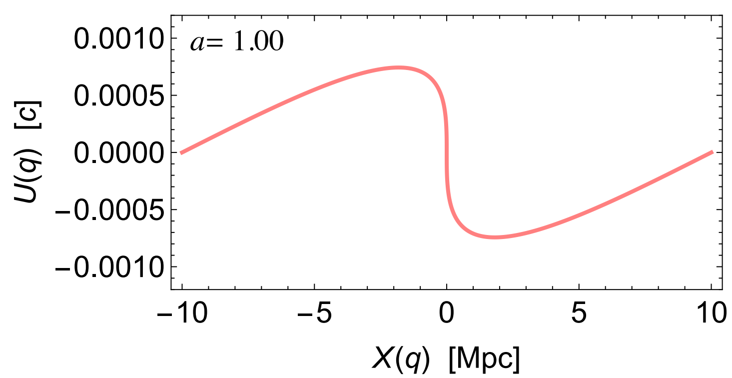
sum over streams is <u>non-local</u> in q-space

Vlasov vs Dust



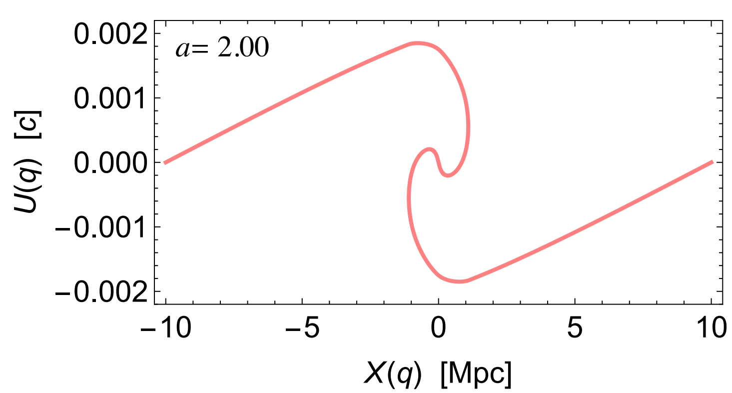
Initial condition

Dust ✓
Vlasov ✓



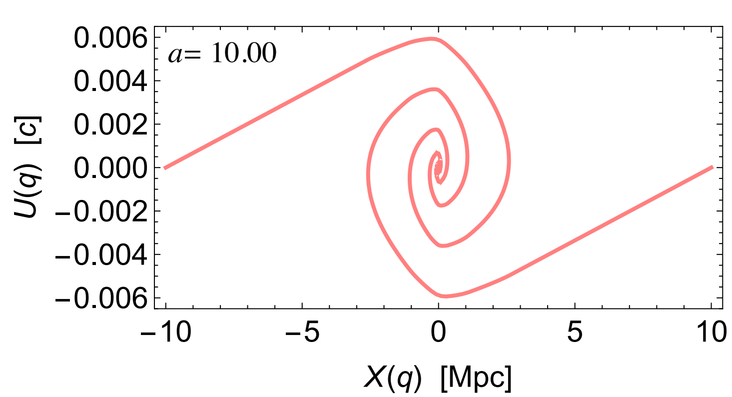
Shell crossing

t ↓



Violent relaxation and virialisation

Dust ✗
Vlasov ✓

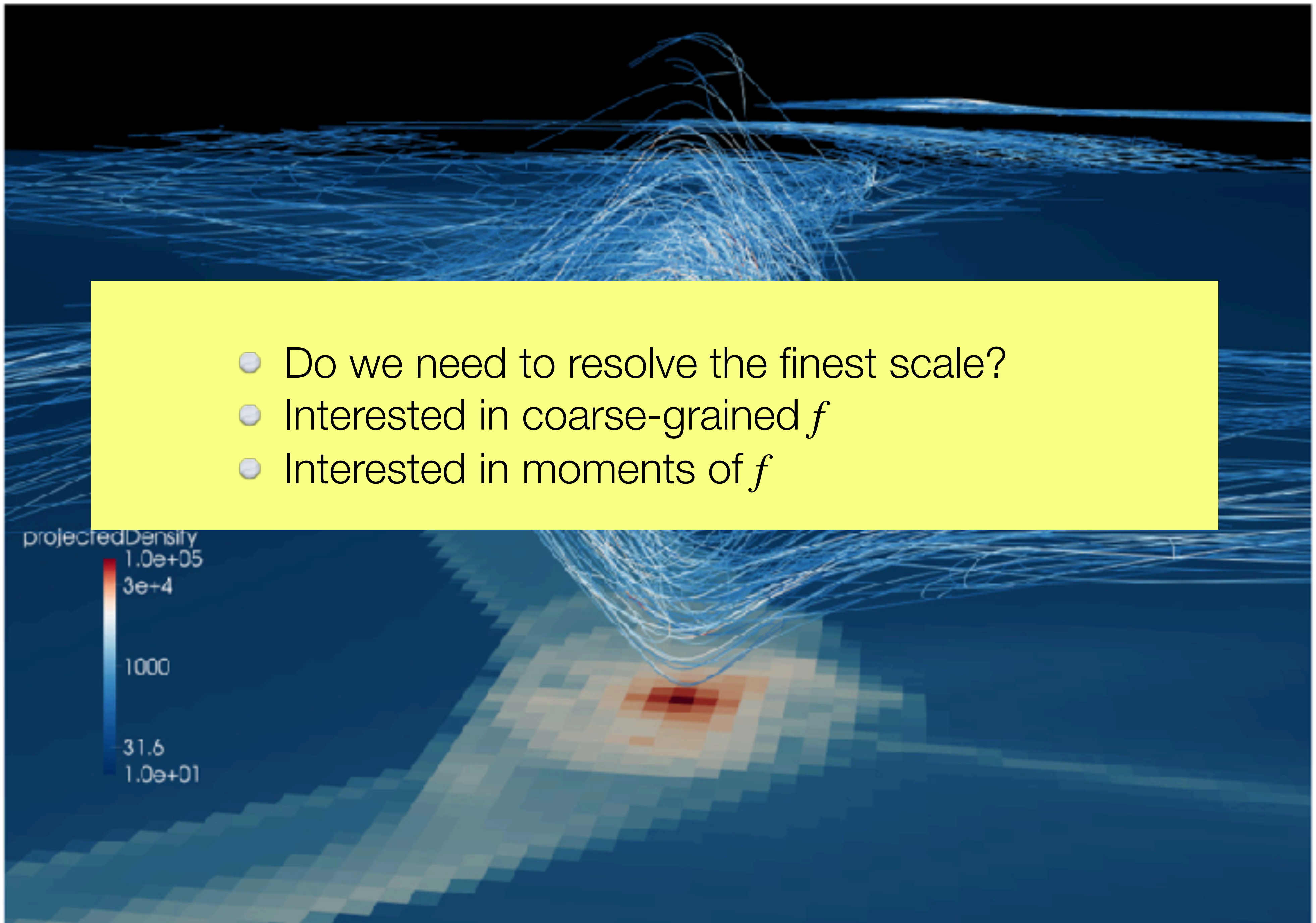


Phase mixing

Vlasov: increasingly complex structure

ColDICE: 3D Vlasov solver with Adaptive Mesh Refinement

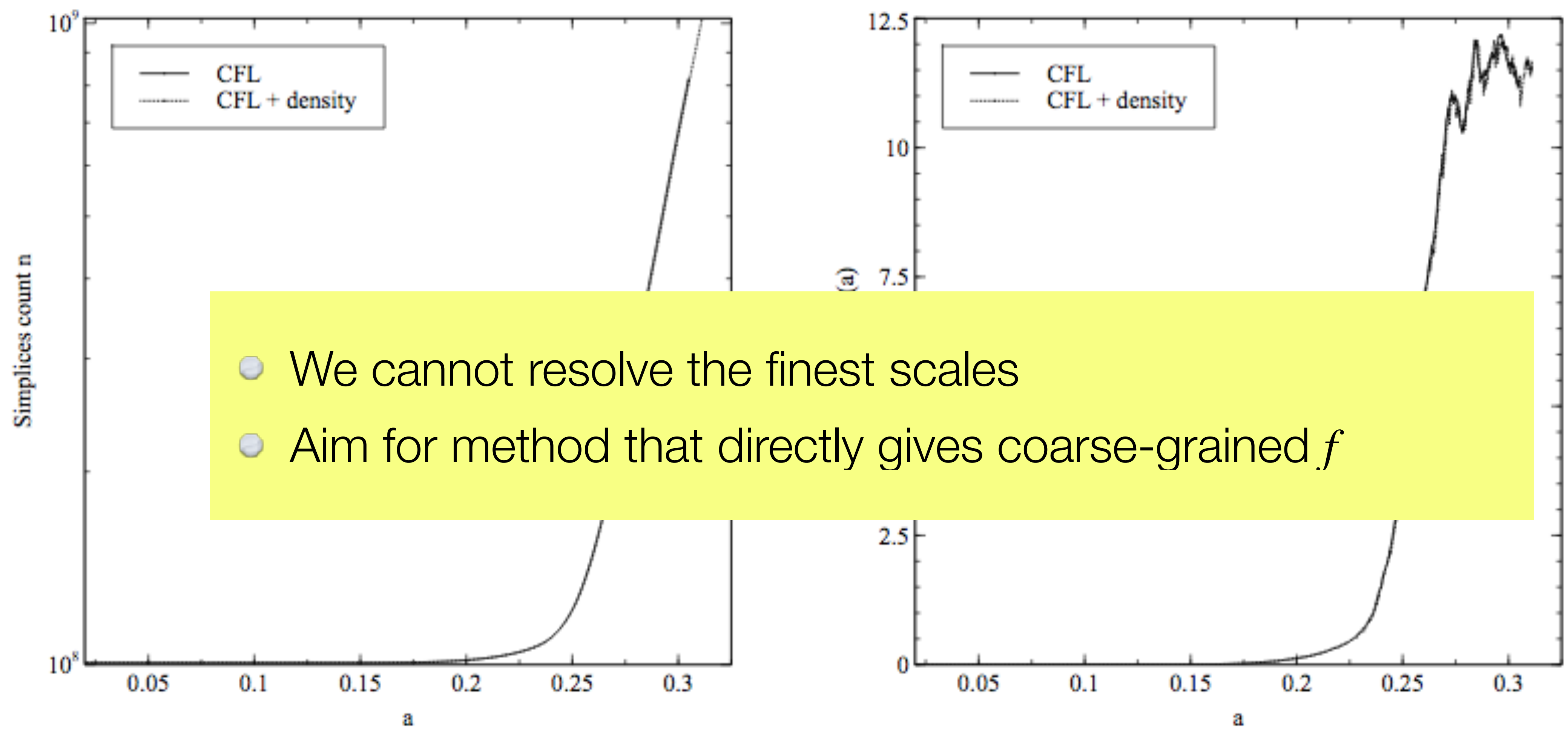
T. Sousbie & S. Colombi, J. Comp. Phys. 321, 644 (2015)



- Do we need to resolve the finest scale?
- Interested in coarse-grained f
- Interested in moments of f

Vlasov: increasingly complex structure

Slide from T. Sousbie's talk https://www.cirm-math.fr/ProgWeebly/Renc1683/SousbieTalk.pdf



- We cannot resolve the finest scales
- Aim for method that directly gives coarse-grained f

- Very fast growth: $n \propto a^{\alpha}$ with $\alpha \simeq 12 \pm 1/2$
- Need to stop refining at some point to reach $a = 1$ ...

2b) Mathematical formulations and 1D example

of coarse grained CDM and Schrödinger Method

Schrödinger-Poisson system

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\,\psi \qquad\qquad \Delta\Phi = 4\pi G\rho(|\psi|^2 - 1)$$

coarse grained CDM and Husimi phase space density

Degrees of freedom: 2×d

$$\mathbb{R}^d \to \mathbb{R}^{2\times d}: \quad \boldsymbol{X}(\boldsymbol{q}), \boldsymbol{U}(\boldsymbol{q})$$

Dynamics: non-local

$$a^2\partial_t \boldsymbol{X}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})$$ Hamiltonian equations

$$\partial_t \boldsymbol{U}(\boldsymbol{q}) = -\nabla_x \Phi_c(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}$$

$$\Delta\Phi_c(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\Big(\sum_{\boldsymbol{q}\text{ with } \boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})} \frac{1}{|\det \partial_{q^i} X^j(\boldsymbol{q})|} - 1\Big)$$

sum over streams

Phase space distr.: non-local

$$f_c(\boldsymbol{x},\boldsymbol{u}) = \int \mathrm{d}^d q\, \delta_D[\boldsymbol{x}-\boldsymbol{X}(\boldsymbol{q})]\,\delta_D[\boldsymbol{u}-\boldsymbol{U}(\boldsymbol{q})]$$

Gaussian smoothing with width σ_x and $\sigma_u = \tilde{\hbar}/(2\sigma_x)$

$$\bar{f}_c(\boldsymbol{x},\boldsymbol{u}) = \int \mathrm{d}^d q\, \frac{e^{-\frac{[\boldsymbol{x}-\boldsymbol{X}(\boldsymbol{q})]^2}{2\sigma_x^2}}}{(2\pi)^{d/2}\sigma_x{}^d}\, \frac{e^{-\frac{[\boldsymbol{u}-\boldsymbol{U}(\boldsymbol{q})]^2}{2\sigma_u^2}}}{(2\pi)^{d/2}\sigma_u{}^d}$$

2

$$\mathbb{R}^d \to \mathbb{R}^2: \quad \Re\{\psi(\boldsymbol{x})\}, \Im\{\psi(\boldsymbol{x})\}$$

local

Hamiltonian equations   Arriola, Soler, (JSP, 103, 2001),

$$i\tilde{\hbar}\partial_t\psi(\boldsymbol{x}) = -\frac{\tilde{\hbar}^2}{2a^2}\Delta\psi(\boldsymbol{x}) + \Phi_\psi(\boldsymbol{x})\psi(\boldsymbol{x})$$

$$\Delta\Phi_\psi(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\left(|\psi(\boldsymbol{x})|^2 - 1\right)$$

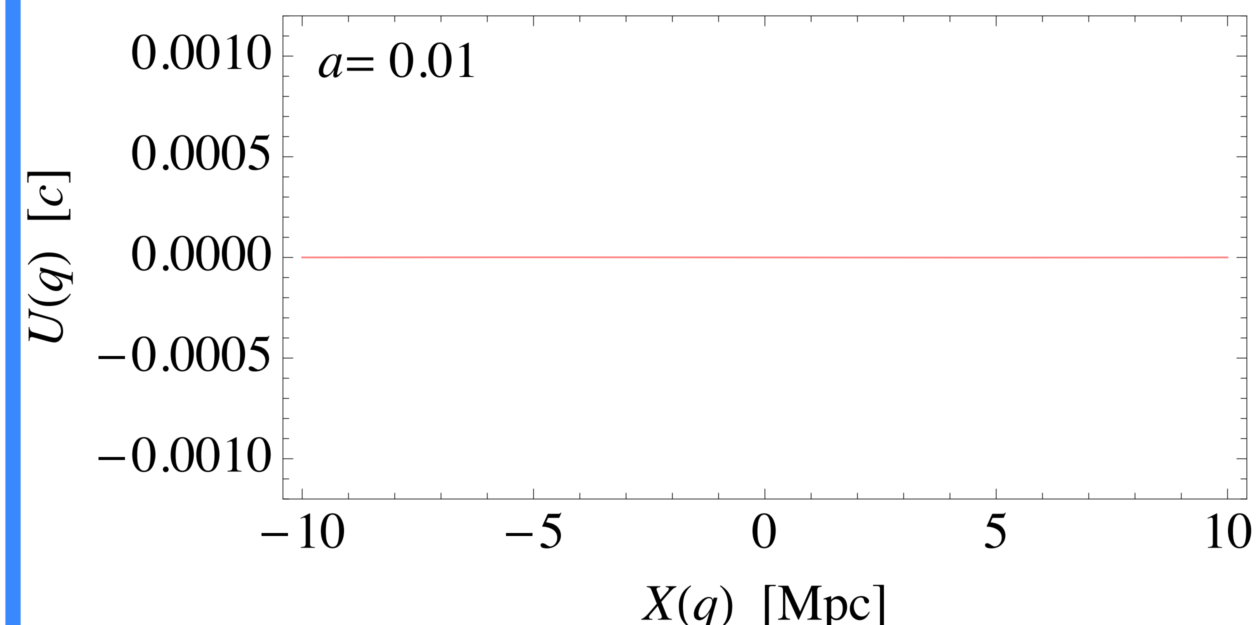
new parameter: $\tilde{\hbar} = \hbar/m$

quasi-local

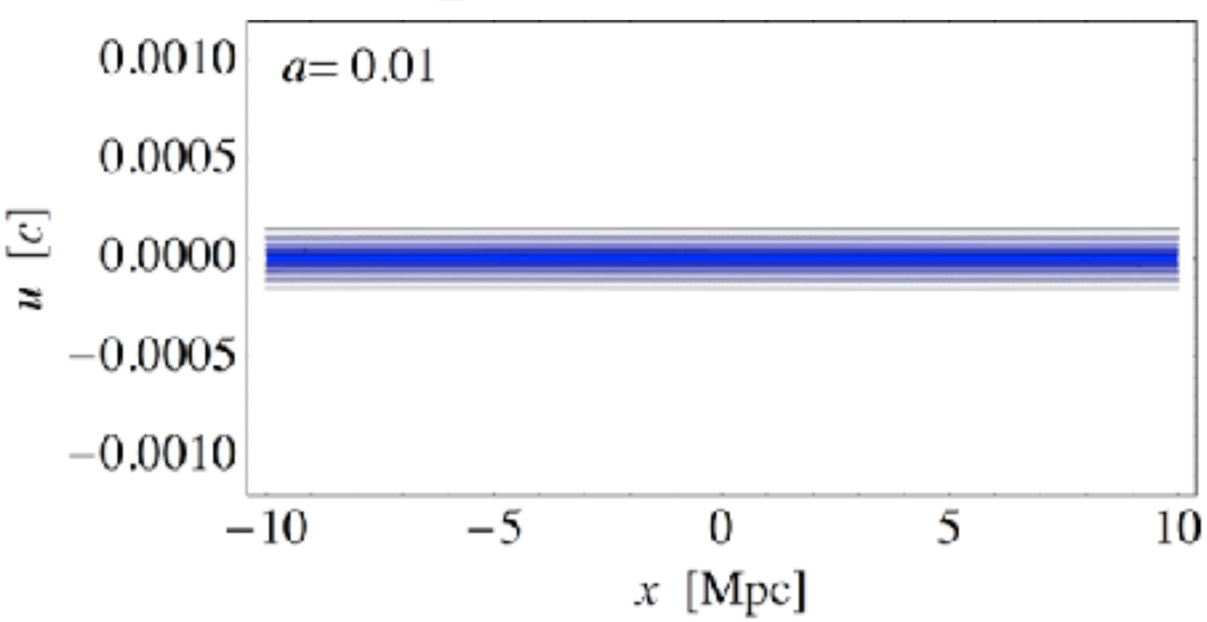
In eulerian space Gaussian filter has effective range of few σ_x

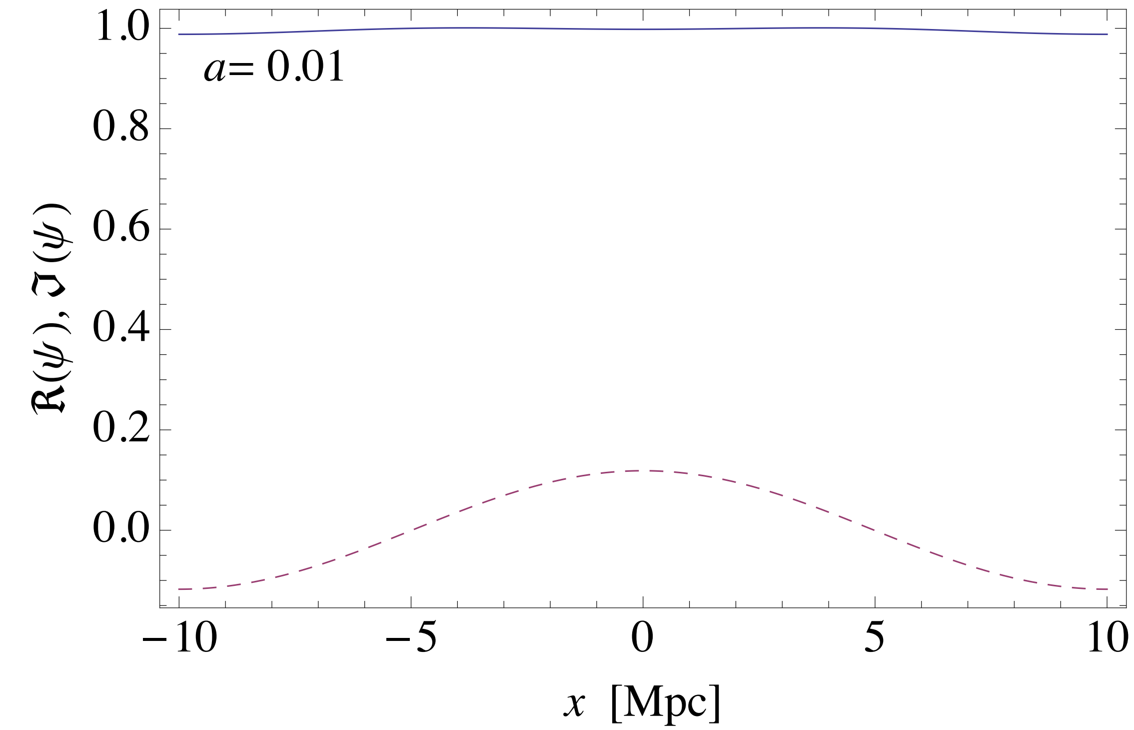
$$f_H(\boldsymbol{x},\boldsymbol{u}) = \left|\int \mathrm{d}^d x'\, \frac{e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{4\sigma_x^2} - \frac{i}{\tilde{\hbar}}\boldsymbol{u}\cdot\boldsymbol{x}'}}{(2\pi\tilde{\hbar})^{d/2}(2\pi\sigma_x{}^2)^{d/4}}\psi(\boldsymbol{x}')\right|^2$$

1D pancake collapse: coarse grained CDM and ScM

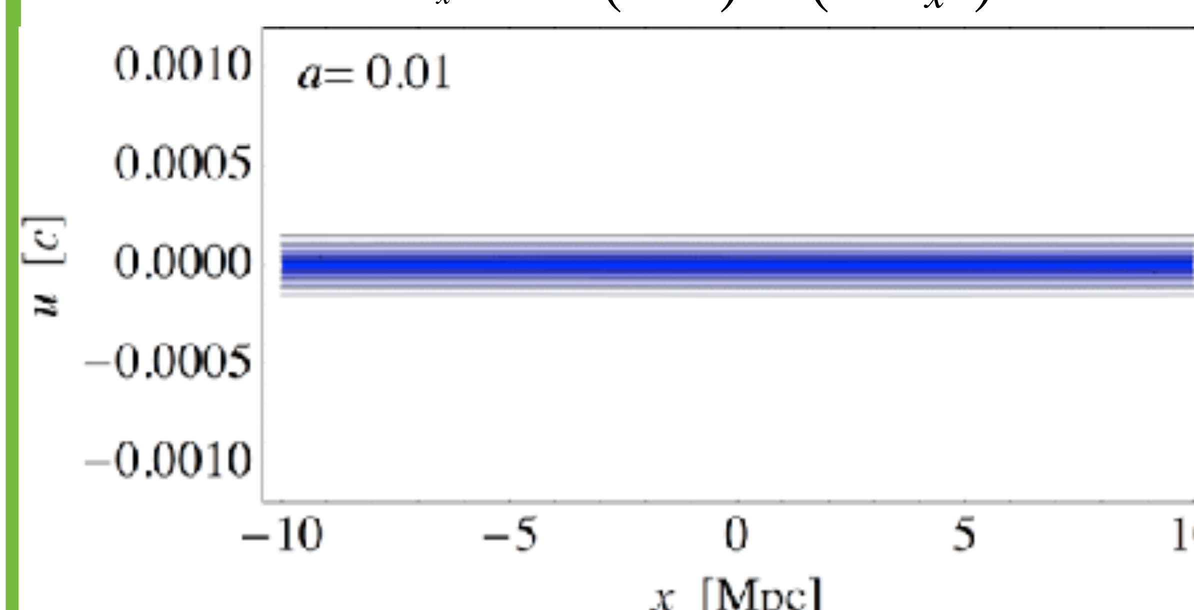


$$\bar{f}_c(x,u) = \int_{-L}^{L} \frac{dq}{2\pi\sigma_x\sigma_u} e^{-\frac{(u-U(q))^2}{2\sigma_u^2}} e^{-\frac{(x-X(q))^2}{2\sigma_x^2}}$$

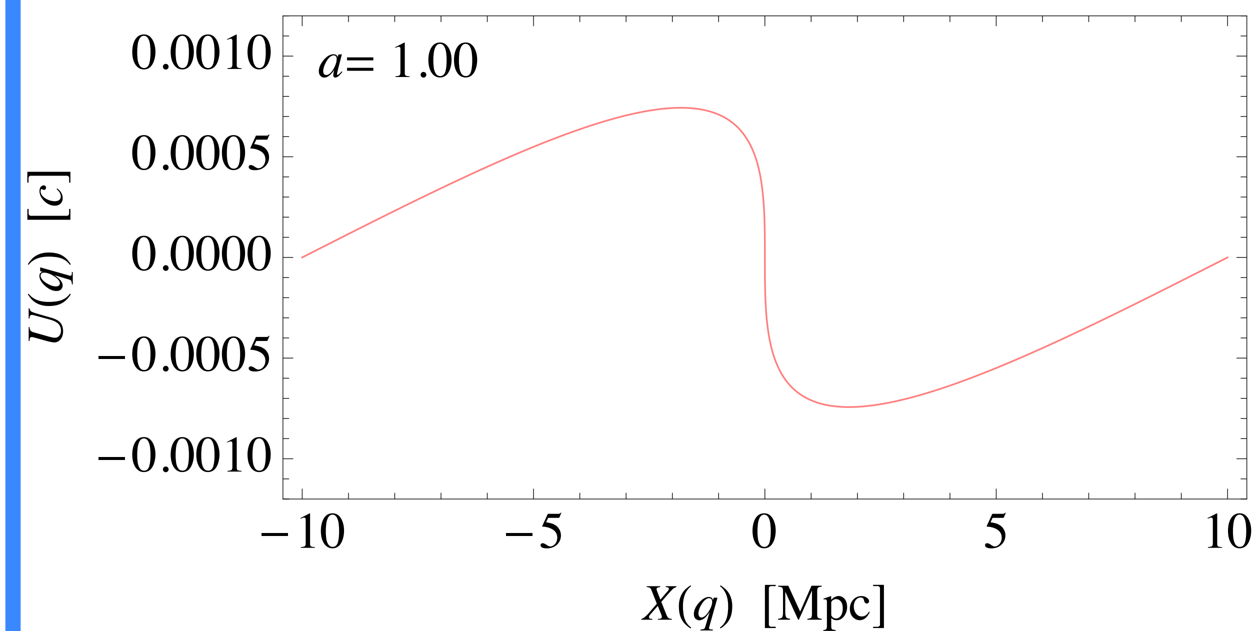




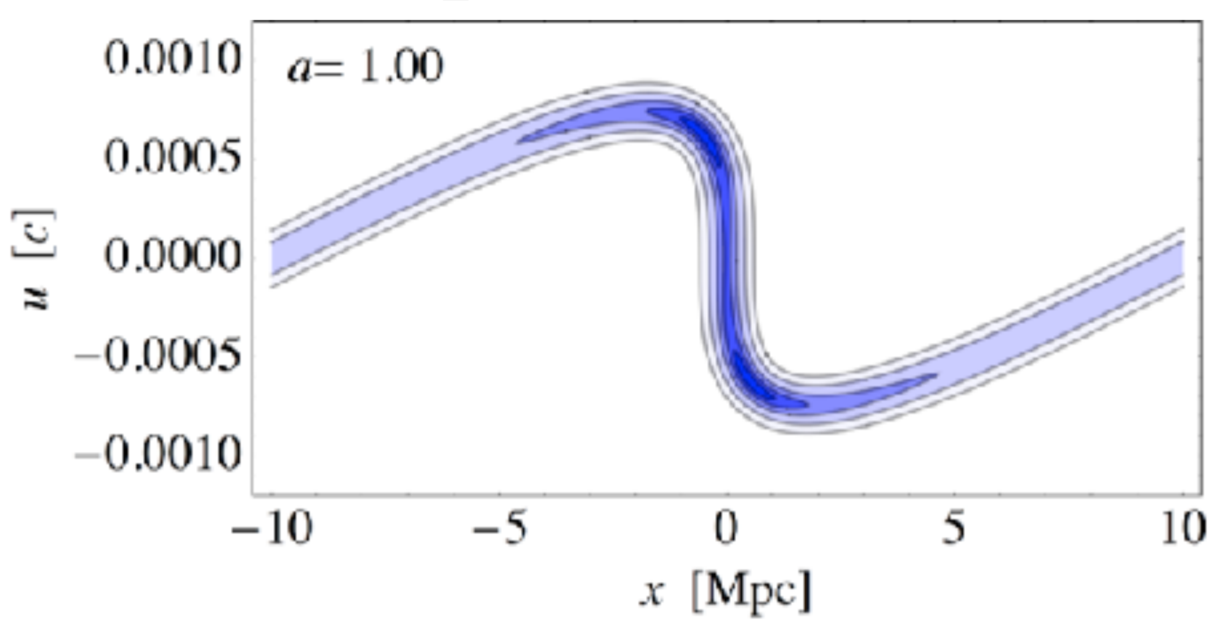
$$f_H(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} dx' \frac{e^{-\frac{(x-x')^2}{4\sigma_x^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x^2)^{1/4}} \psi(x') \right|^2$$

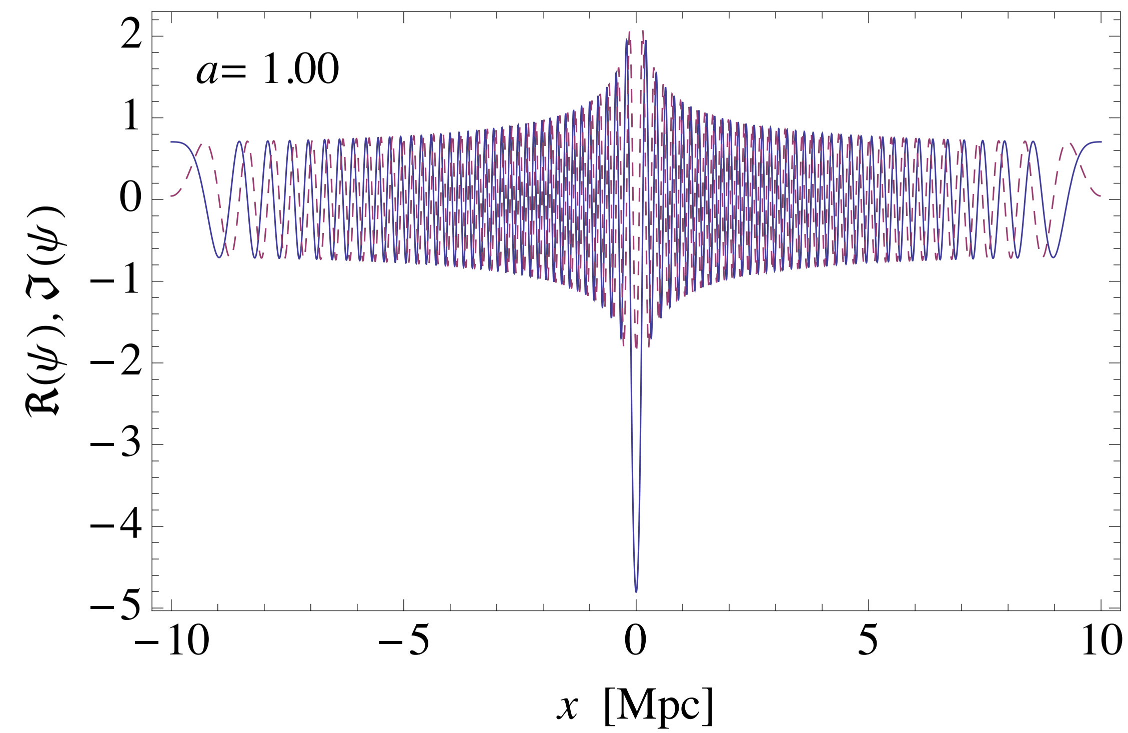


1D pancake collapse: coarse grained CDM and ScM

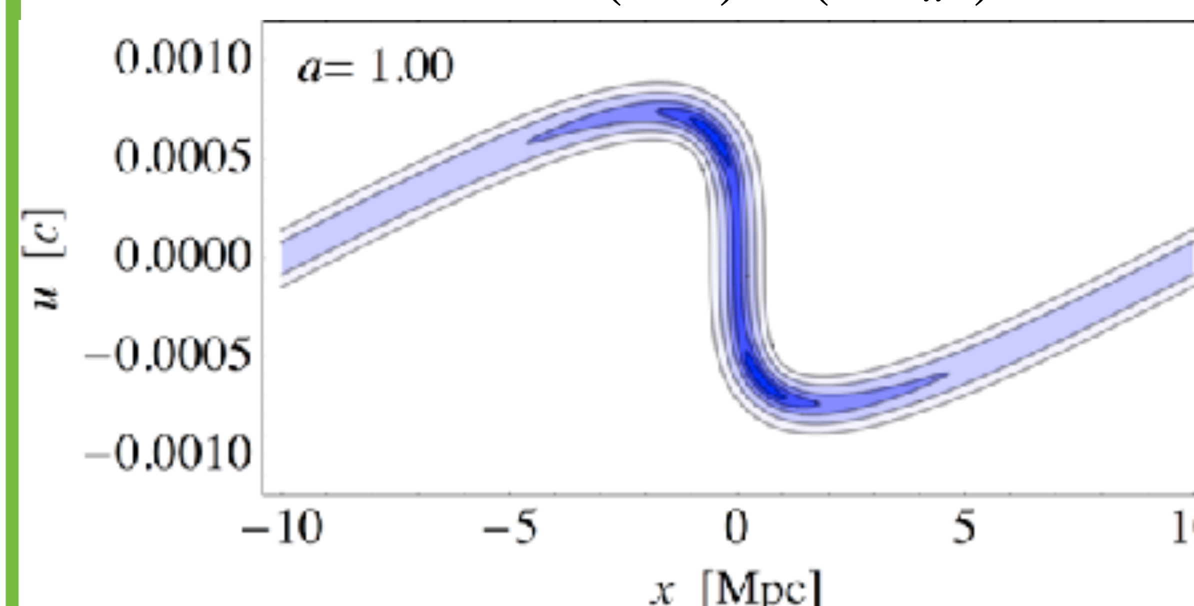


$$\bar{f}_{\mathrm{c}}(x,u) = \int_{-L}^{L} \frac{\mathrm{d}q}{2\pi\sigma_x\sigma_u}\, e^{-\frac{(u-U(q))^2}{2\sigma_u{}^2}}\, e^{-\frac{(x-X(q))^2}{2\sigma_x{}^2}}$$

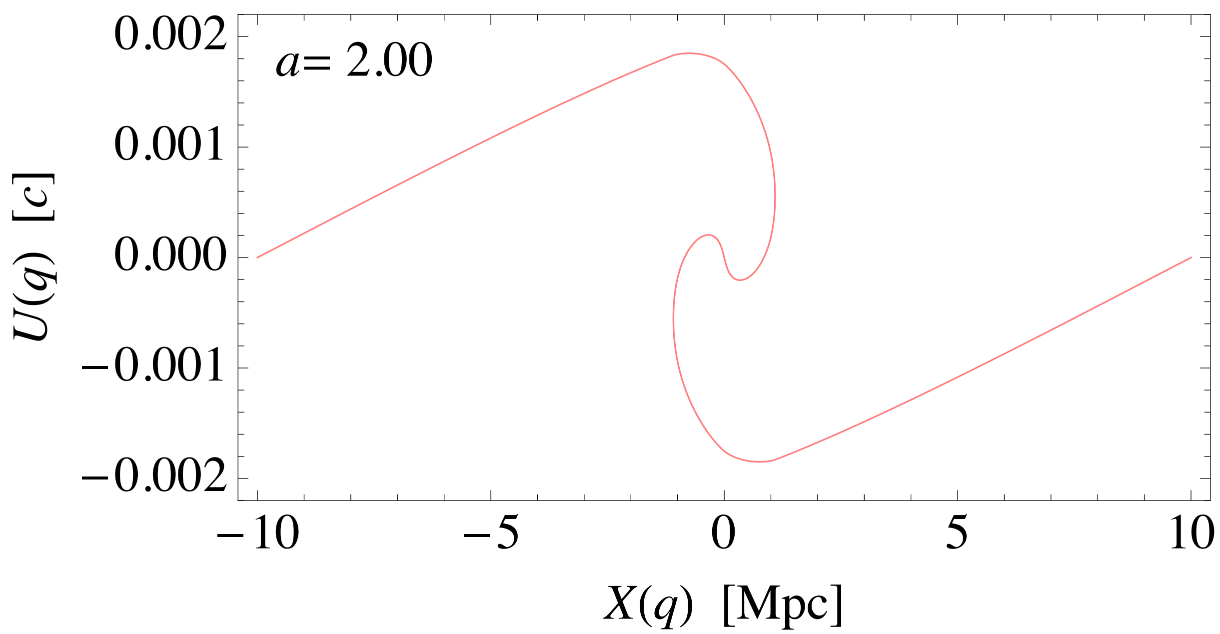




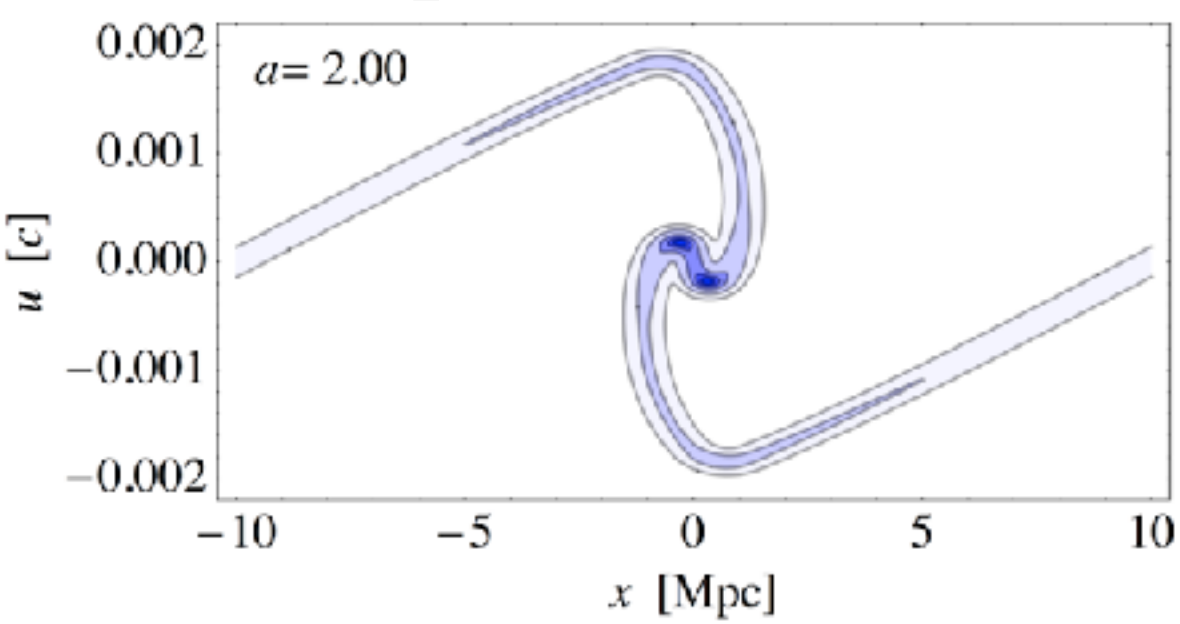
$$f_{\mathrm{H}}(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} \mathrm{d}x' \; \frac{e^{-\frac{(x-x')^2}{4\sigma_x{}^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x{}^2)^{1/4}}\, \psi(x') \right|^2$$

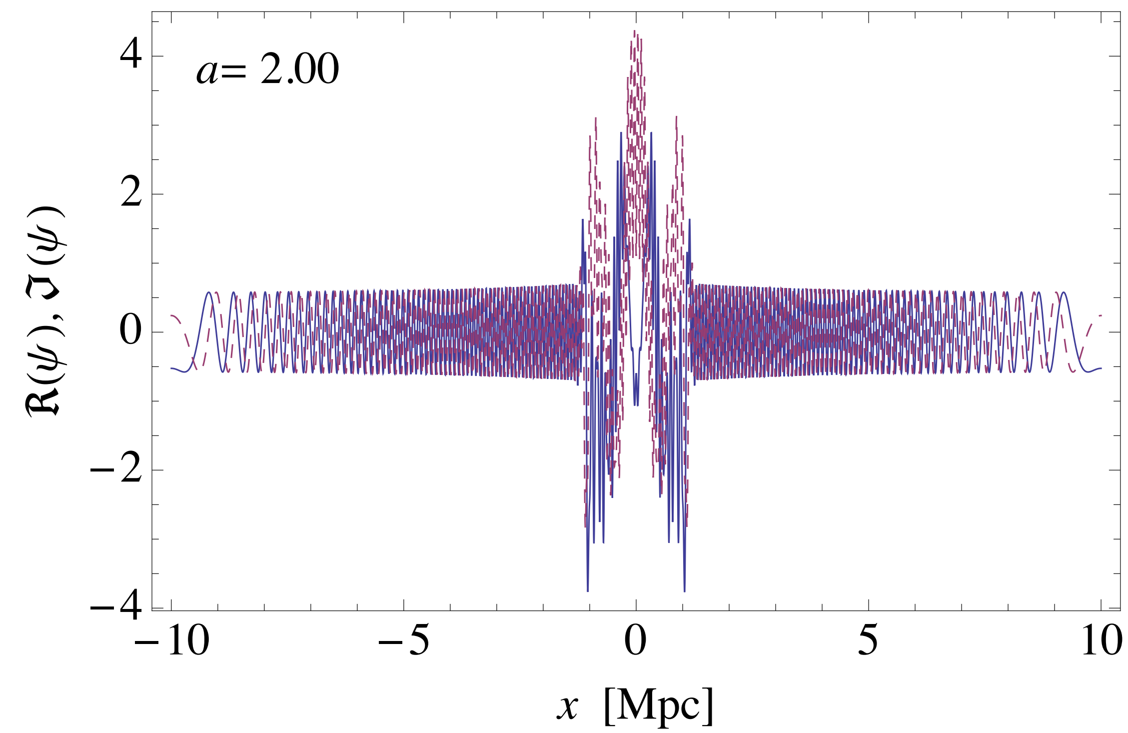


1D pancake collapse: coarse grained CDM and ScM

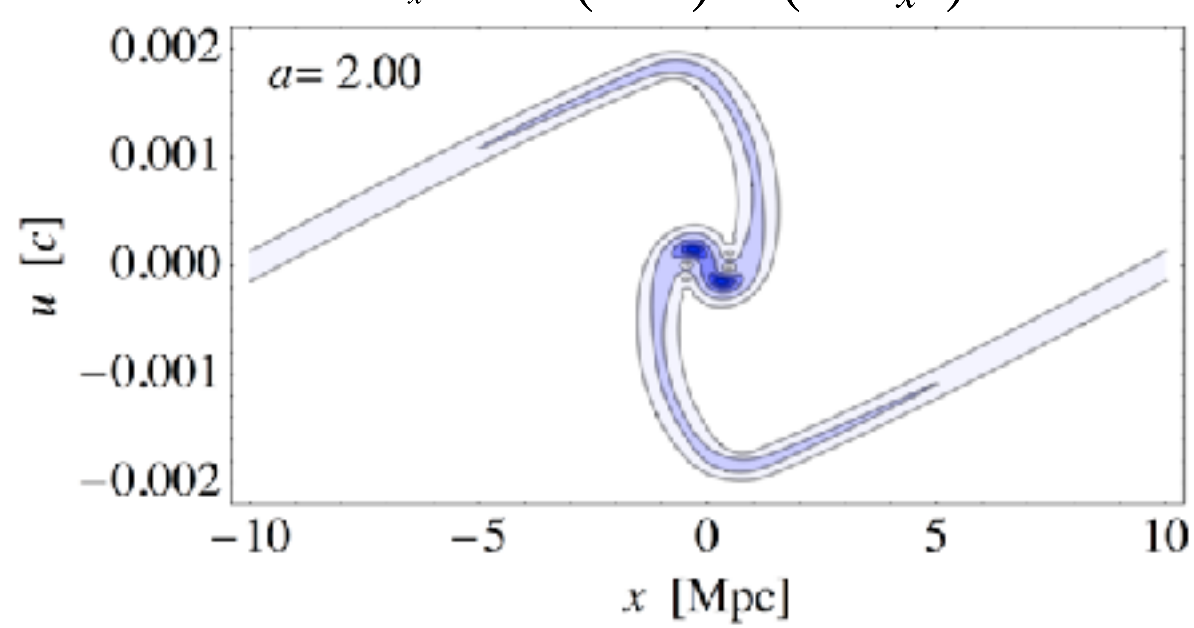


$$\bar{f}_{c}(x,u) = \int_{-L}^{L} \frac{dq}{2\pi\sigma_x\sigma_u} e^{-\frac{(u-U(q))^2}{2\sigma_u^2}} e^{-\frac{(x-X(q))^2}{2\sigma_x^2}}$$

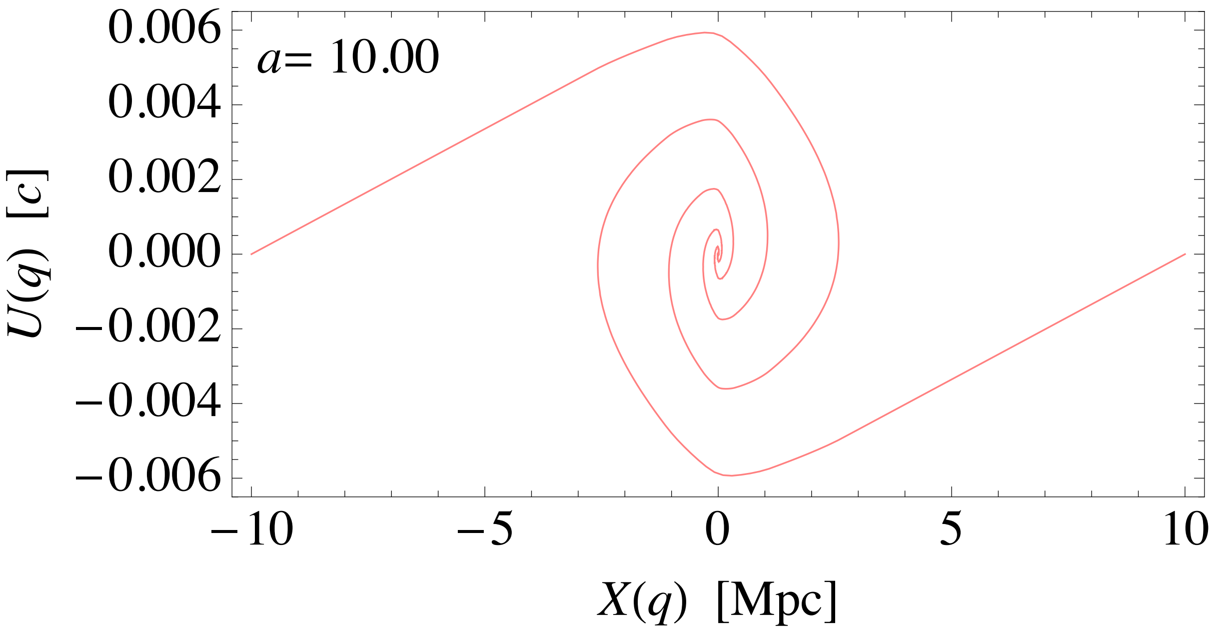




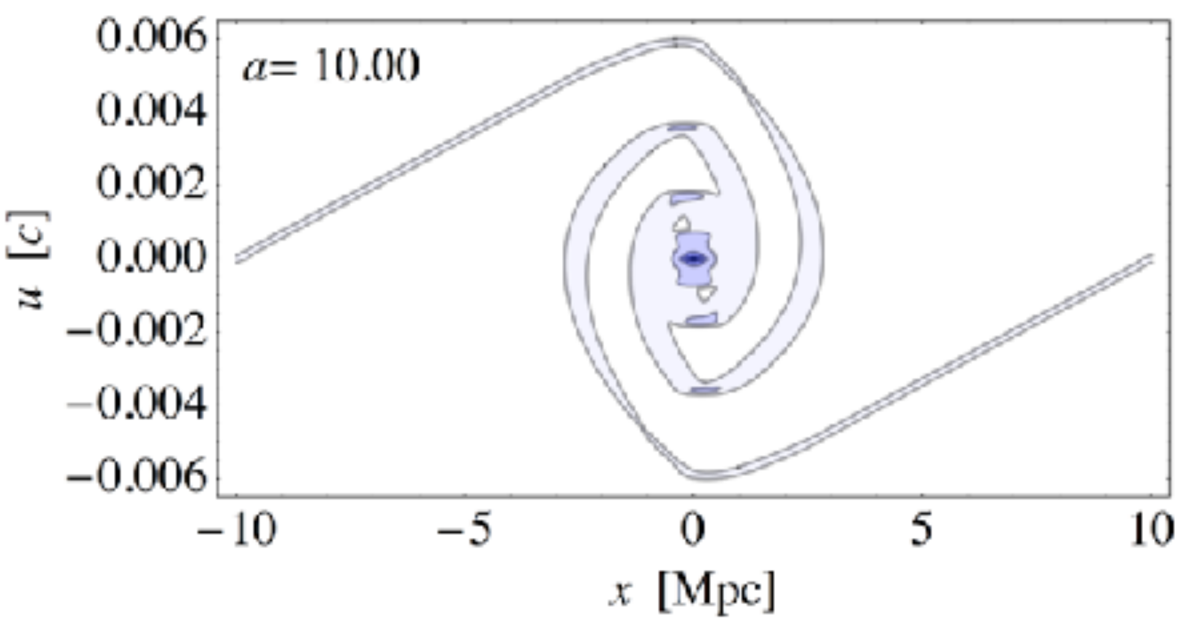
$$f_{H}(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} dx' \; \frac{e^{-\frac{(x-x')^2}{4\sigma_x^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x^2)^{1/4}} \psi(x') \right|^2$$

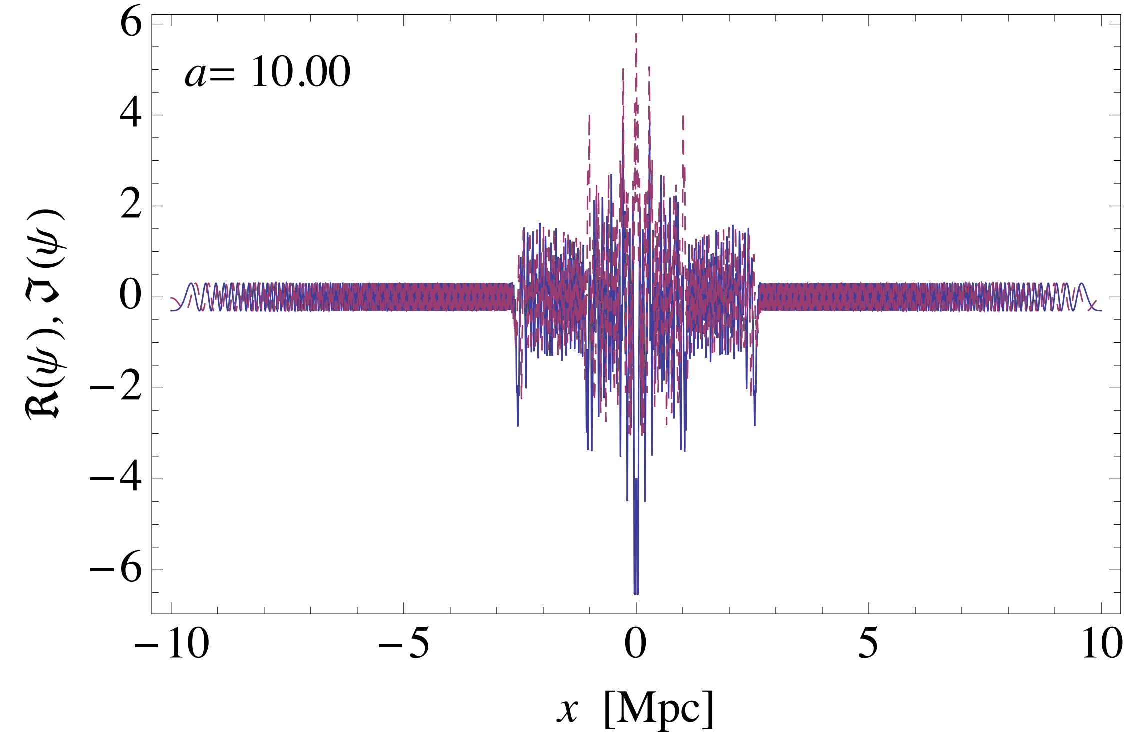


1D pancake collapse: coarse grained CDM and ScM

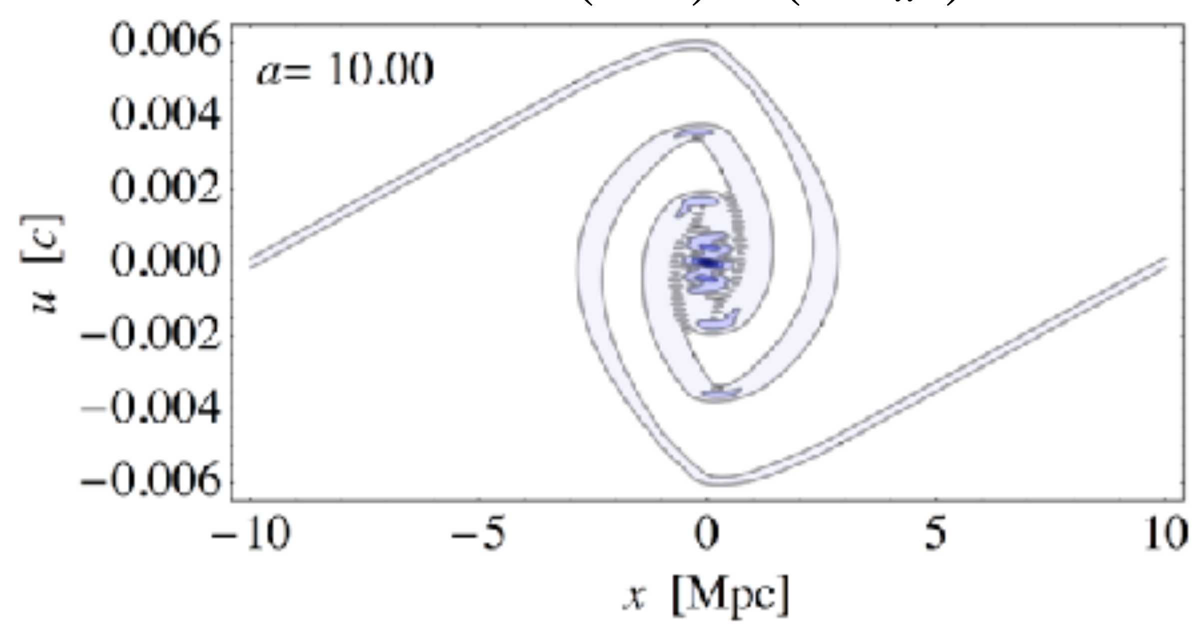


$$\bar{f}_{\mathrm{c}}(x,u) = \int_{-L}^{L} \frac{\mathrm{d}q}{2\pi\sigma_x\sigma_u}\, e^{-\frac{(u-U(q))^2}{2\sigma_u^2}}\, e^{-\frac{(x-X(q))^2}{2\sigma_x^2}}$$





$$f_{\mathrm{H}}(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} \mathrm{d}x' \; \frac{e^{-\frac{(x-x')^2}{4\sigma_x^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x^2)^{1/4}}\, \psi(x') \right|^2$$



Convergence of Schrödinger to dust

$$f_{\mathrm{d}}(t,\boldsymbol{x},\boldsymbol{u}) = n_{\mathrm{d}}(t,\boldsymbol{x})\,\delta_{\mathrm{D}}\big(\boldsymbol{u} - \boldsymbol{\nabla}\phi_{\mathrm{d}}(t,\boldsymbol{x})\big)$$

↓ Vlasov equation $\quad \boldsymbol{u}_{\mathrm{d}} \equiv \boldsymbol{\nabla}\phi_{\mathrm{d}}$

$$\partial_t n_{\mathrm{d}} = -\frac{1}{a^2}\boldsymbol{\nabla}\cdot(n_{\mathrm{d}}\boldsymbol{u}_{\mathrm{d}})\,,$$

$$\partial_t \boldsymbol{u}_{\mathrm{d}} = -\frac{1}{a^2}(\boldsymbol{u}_{\mathrm{d}}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{d}} - \boldsymbol{\nabla}\Phi_{\mathrm{d}}\,,$$

$$\boldsymbol{\nabla}\times\boldsymbol{u}_{\mathrm{d}} = 0$$

$$\Delta\Phi_{\mathrm{d}} = \frac{4\pi G\rho_0}{a}\big(n_{\mathrm{d}} - 1\big)$$

$$\psi(\boldsymbol{x}) =: \sqrt{n_\psi(\boldsymbol{x})}\,\exp\big(i\phi(\boldsymbol{x})/\tilde{\hbar}\big)$$

$\boldsymbol{u}_\psi \equiv \boldsymbol{\nabla}\phi$ ↓ Schrödinger-Poisson and $n_\psi \neq 0$

$$\partial_t n_\psi = -\frac{1}{a^2}\boldsymbol{\nabla}_x\cdot(n_\psi\boldsymbol{u})$$

$$\partial_t \boldsymbol{u}_\psi = -\frac{1}{a^2}(\boldsymbol{u}_\psi\cdot\boldsymbol{\nabla})\,\boldsymbol{u}_\psi - \boldsymbol{\nabla}\Phi_\psi + \underbrace{\frac{\tilde{\hbar}^2}{2a^2}\boldsymbol{\nabla}\left(\frac{\Delta\sqrt{n_\psi}}{\sqrt{n_\psi}}\right)}_{\text{"Quantum pressure"}}$$

$$\boldsymbol{\nabla}\times\boldsymbol{u}_\psi = 0$$

$$\Delta\Phi_\psi = \frac{4\pi G\rho_0}{a}\big(n_\psi - 1\big)$$

Madelung 1927
cf. Philippe Brax's talk

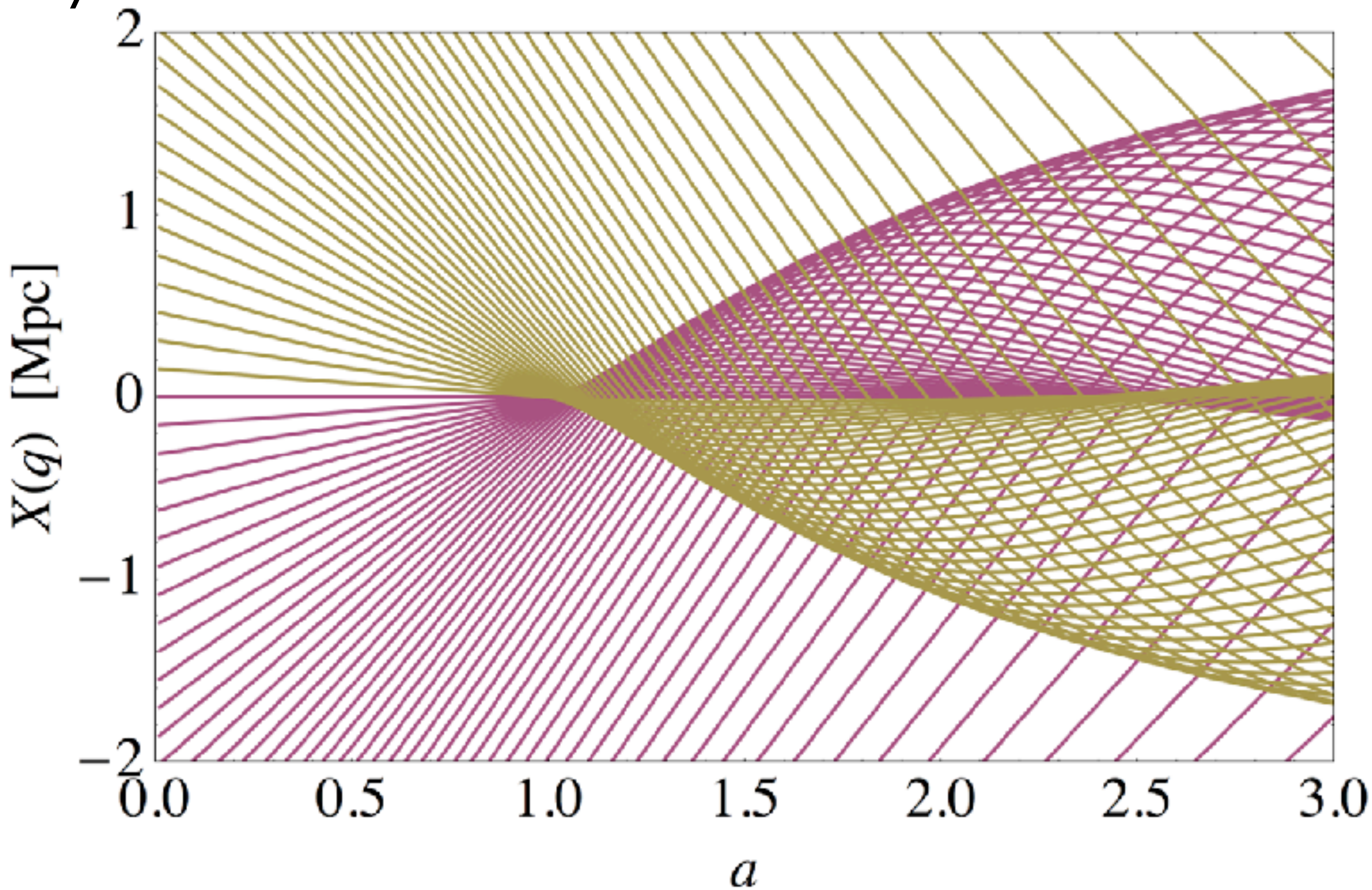
$$\left|\frac{\tilde{\hbar}^2}{2a^2}\frac{\Delta\sqrt{n_\psi}}{\sqrt{n_\psi}}\right| \ll \left|\Phi_\psi\right| \quad \Longrightarrow$$ Schrödinger-Poisson becomes dust

- Tells you how to set up single stream initial conditions $\psi_{\mathrm{ini}}(\boldsymbol{x}) = \sqrt{n_{\mathrm{d}}^{\mathrm{ini}}}\exp(i\phi_{\mathrm{d}}^{\mathrm{ini}}/\tilde{\hbar})$
- Breakdown of this condition and $n_\psi = 0$ are generic when CDM undergoes shell crossing and multi-streaming. Why does the ScM not break down at this point?

shell-crossing : 1D pancake

CDM trajectories

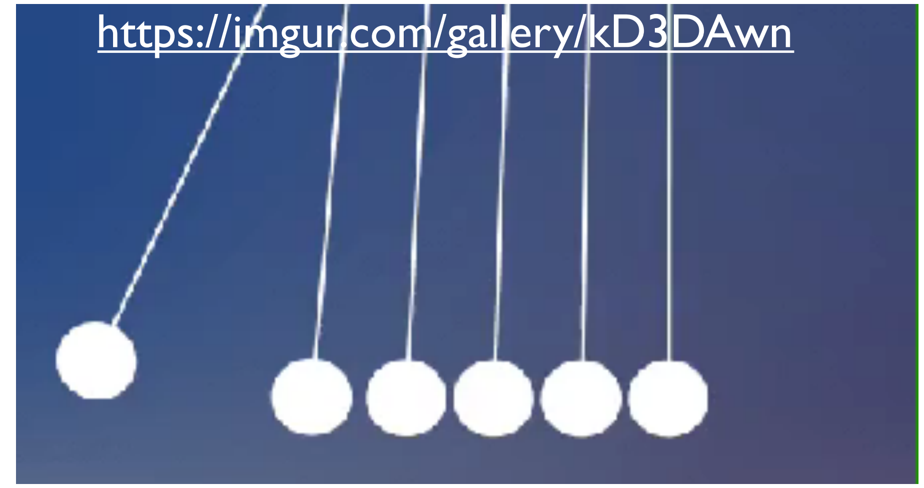
- Integral lines of u_d (until first shell crossing)
- Fundamental dynamical variable
- Cross freely

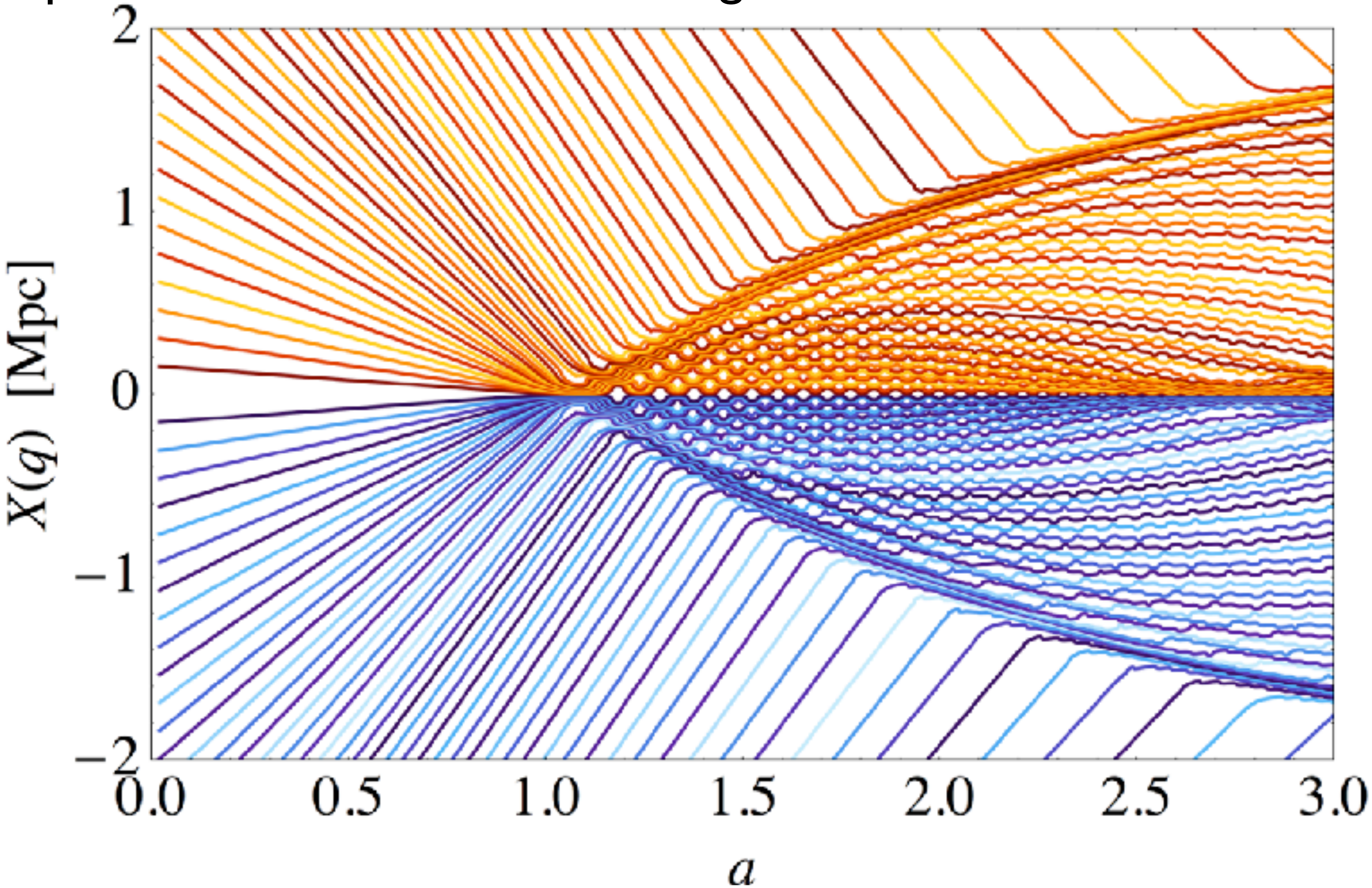


shell-crossing without shell-crossing: 1D pancake

ScM trajectories

- Integral lines of $\nabla\phi$: Bohmian trajectories
- Derived concept in ScM (and not needed)
- Quantum pressure emulates shell crossing





Convergence of ScM to coarse grained Vlasov

Husimi - Vlasov correspondence

“Theorem”:

$$\frac{\partial}{\partial t}\left(f_H - \bar{f}\right) \rightarrow \hbar^2$$

Bertrand et al (JPP 23, 1980)
Takahashi 1989
Widrow-Kaiser 1993

Convergence of ScM to coarse grained Vlasov

Coarse grained f automatically satisfies (if f solves Vlasov):

$$\partial_t \bar{f} = -\frac{\boldsymbol{u}}{a^2}\nabla_x \bar{f} - \frac{{\sigma_u}^2}{a^2}\nabla_x\nabla_u \bar{f} + \nabla_x \bar{\Phi}\exp({\sigma_x}^2 \overleftarrow{\nabla}_x \overrightarrow{\nabla}_x)\nabla_u \bar{f}$$

$$\bar{f}(\boldsymbol{x},\boldsymbol{u}) = \int \frac{\mathrm{d}^3x'\,\mathrm{d}^3u'}{(2\pi\sigma_x\sigma_u)^3} e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{2{\sigma_x}^2}-\frac{(\boldsymbol{u}-\boldsymbol{u}')^2}{2{\sigma_u}^2}} f(\boldsymbol{x}',\boldsymbol{u}') = e^{\frac{{\sigma_x}^2}{2}\Delta_x + \frac{{\sigma_u}^2}{2}\Delta_u}\{f\} \qquad \sigma_u = \frac{\tilde{\hbar}}{2\sigma_x}$$

Husimi equation (automatically satisfied ψ solves Schrödinger-Poisson)

$$\partial_t f_{\mathrm{H}} = -\frac{\boldsymbol{u}}{a^2}\nabla_x f_{\mathrm{H}} - \frac{{\sigma_u}^2}{a^2}\nabla_x\nabla_u f_{\mathrm{H}} + \Phi_{\mathrm{H}}\exp({\sigma_x}^2 \overleftarrow{\nabla}_x \overrightarrow{\nabla}_x)\frac{2}{\tilde{\hbar}}\sin\left(\frac{\tilde{\hbar}}{2}\overleftarrow{\nabla}_x \overrightarrow{\nabla}_u\right) f_{\mathrm{H}}$$

$$S_{\mathrm{V}} + S_{\mathrm{cgV}} + S_{\tilde{\hbar}} + \mathcal{O}(\tilde{\hbar}^2{\sigma_x}^2, \tilde{\hbar}^4)$$

$$S_{\mathrm{V}} \equiv -\frac{\boldsymbol{u}}{a^2}\cdot\nabla_x f_{\mathrm{H}} + \nabla_x \Phi_{\mathrm{H}}\cdot\nabla_u f_{\mathrm{H}}$$ Vlasov source

$$S_{\mathrm{cgV}} \equiv -\frac{{\sigma_u}^2}{a^2}\nabla_x\cdot\nabla_u f_{\mathrm{H}} + {\sigma_x}^2(\partial_{x_i}\partial_{x_j}\Phi_{\mathrm{H}})(\partial_{x_i}\partial_{u_j} f_{\mathrm{H}}),$$

coarse graining source

$$S_{\tilde{\hbar}} \equiv -\frac{\tilde{\hbar}^2}{24}(\partial_{x_i}\partial_{x_j}\partial_{x_k}\Phi_{\mathrm{H}})(\partial_{u_i}\partial_{u_j}\partial_{u_k} f_{\mathrm{H}})$$

quantum artifact source

Necessary to approximate coarse grained Vlasov:

$$|S_{\tilde{\hbar}}| \ll |S_{\mathrm{cgV}}|$$

coarse grained CDM and ScM moments

Phase space sheet has to be tracked | **Completely avoids phase space**

Dynamics: non-local

$$a^2 \partial_t \boldsymbol{X}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})$$

$$\partial_t \boldsymbol{U}(\boldsymbol{q}) = -\boldsymbol{\nabla}_x \Phi_c(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}$$

$$\Delta \Phi_c(\boldsymbol{x}) = \frac{4\pi G \rho_0}{a} \Big(\sum_{\substack{\boldsymbol{q} \text{ with} \\ \boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}} \frac{1}{|\det \partial_{q^i} X^j(\boldsymbol{q})|} - 1 \Big)$$

local

$$i\tilde{\hbar}\partial_t \psi(\boldsymbol{x}) = -\frac{\tilde{\hbar}^2}{2a^2}\Delta\psi(\boldsymbol{x}) + \Phi_\psi(\boldsymbol{x})\psi(\boldsymbol{x})$$

$$\Delta \Phi_\psi(\boldsymbol{x}) = \frac{4\pi G \rho_0}{a}\left(|\psi(\boldsymbol{x})|^2 - 1\right)$$

Moments: non-local

$$G_c(\boldsymbol{x}, \boldsymbol{J}) = \sum_{\substack{\boldsymbol{q} \text{ with} \\ \boldsymbol{x}=\boldsymbol{X}(t,\boldsymbol{q})}} \frac{e^{i\boldsymbol{J}\cdot\boldsymbol{U}(\boldsymbol{q})}}{|\det \partial_{q^i} X^j(\boldsymbol{q})|}$$

sum over streams

$$\bar{M}^{c(n)}_{i_1,\ldots,i_n}(\boldsymbol{x}) = e^{\frac{\sigma_x^2}{2}\Delta} \left\{ \frac{(-i)^n \partial^n}{\partial J_{i_1} \ldots \partial J_{i_n}} e^{-\frac{1}{2}\sigma_u^2 \boldsymbol{J}^2} G_c(\boldsymbol{x}, \boldsymbol{J}) \right\}\Big|_{\boldsymbol{J}=0}$$

quasi-local

W for Wigner

$$G_W(\boldsymbol{x}, \boldsymbol{J}) = \psi\left(\boldsymbol{x} + \frac{\tilde{\hbar}}{2}\boldsymbol{J}\right)\bar{\psi}\left(\boldsymbol{x} - \frac{\tilde{\hbar}}{2}\boldsymbol{J}\right)$$

$$M^{H(n)}_{i_1,\ldots,i_n}(\boldsymbol{x}) = e^{\frac{\sigma_x^2}{2}\Delta} \left\{ \frac{(-i)^n \partial^n}{\partial J_{i_1} \ldots \partial J_{i_n}} e^{-\frac{1}{2}\sigma_u^2 \boldsymbol{J}^2} G_W(\boldsymbol{x}, \boldsymbol{J}) \right\}\Big|_{\boldsymbol{J}=0}$$

n derivatives of ψ

2c) Quantitative **Comparison**
of
2D cosmological
simulations

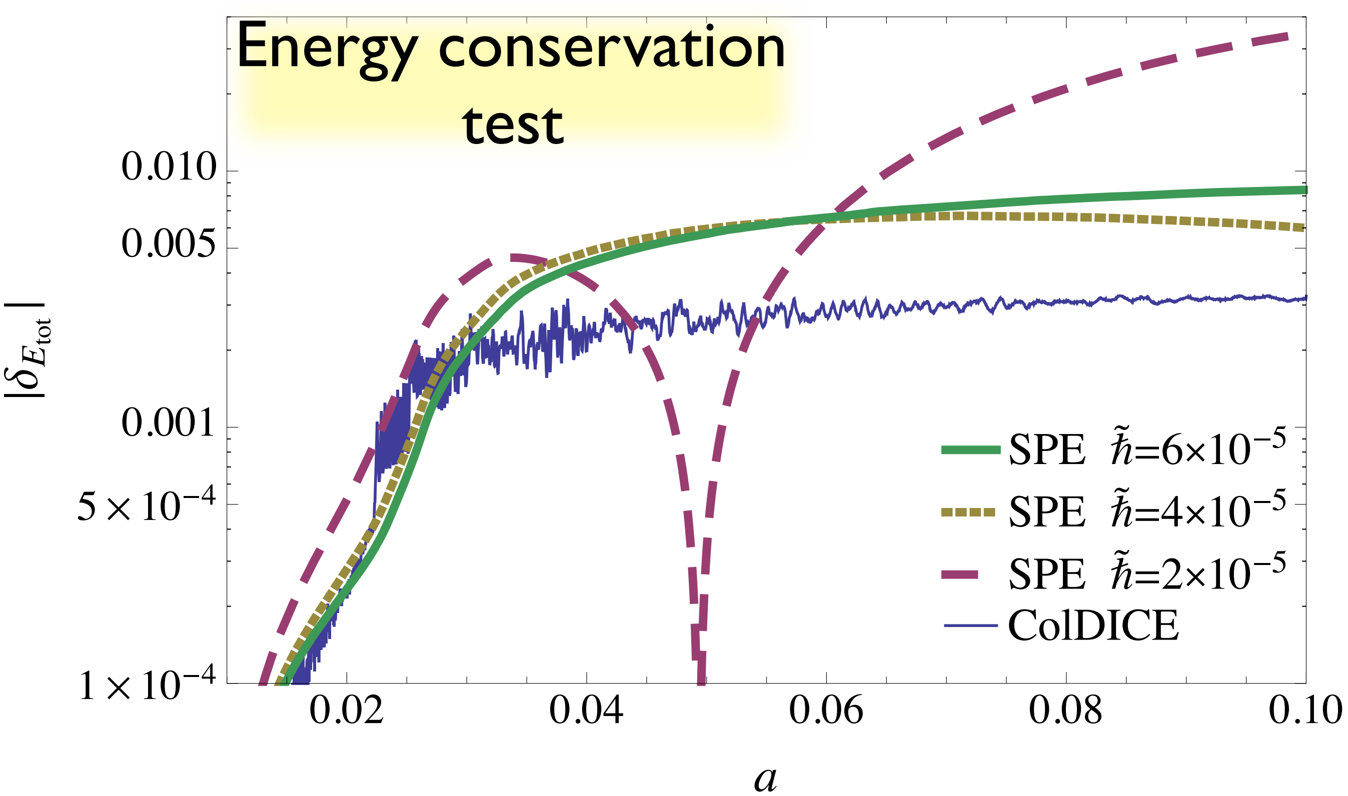
for ColDICE and ScM

Sine collapse

Numerical convergence

- the larger $\tilde{\hbar}$, the better
- similar to `ColDICE`

$$\delta_{E_{\rm tot}} \equiv \frac{E_{\rm tot}(a)}{E(a_{\rm ini})} - 1 \overset{?}{=} 0$$



$$K(t) = \frac{\tilde{\hbar}^2}{2a^2}\int d^3x\ |\nabla_x\psi|^2 \qquad E(t) := K(t) + W(t)$$

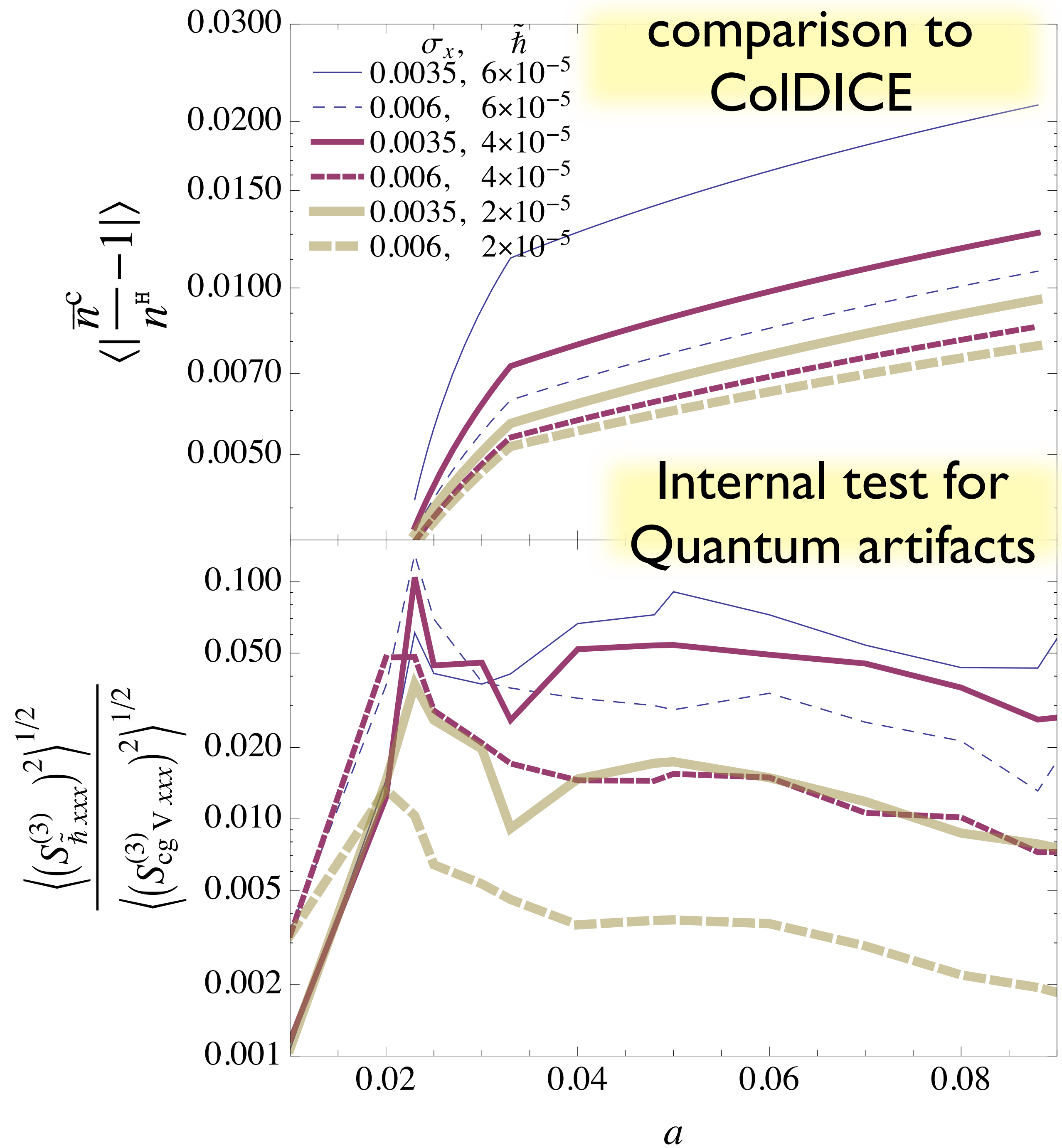
$$W(t) = \frac{1}{2}\int d^3x\ \Phi_\psi\,|\psi|^2 \qquad E_{\rm tot} := E(t) + E_{\rm exp}(t)$$

$$E_{\rm exp} = \int_{a_{\rm ini}}^{a} \frac{2K(a') + W(a')}{a'}da'$$

Convergence to Vlasov

- the smaller $\tilde{\hbar}$, the better

$$\langle \bar{n}^{\rm C}/n^{\rm H} - 1\rangle \lesssim 1\%$$



3a) Discussion:

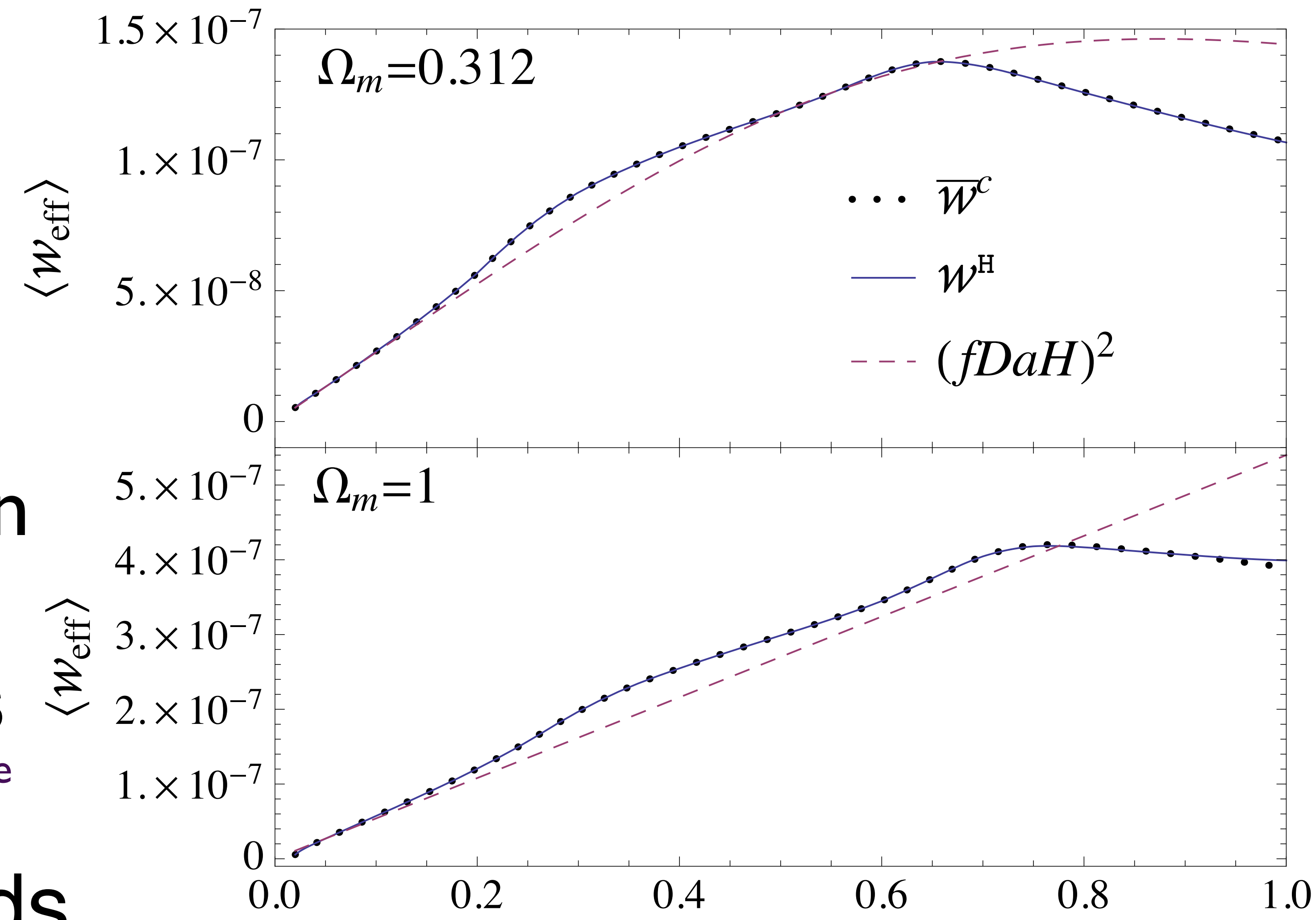
DM equation of state and cosmological backreaction estimate from Vlasov and ScM

Effective pressure

$$T^i{}_i = \rho_0 M^{(2)}_{ii}/a^5 = 2P_{\text{eff}}$$

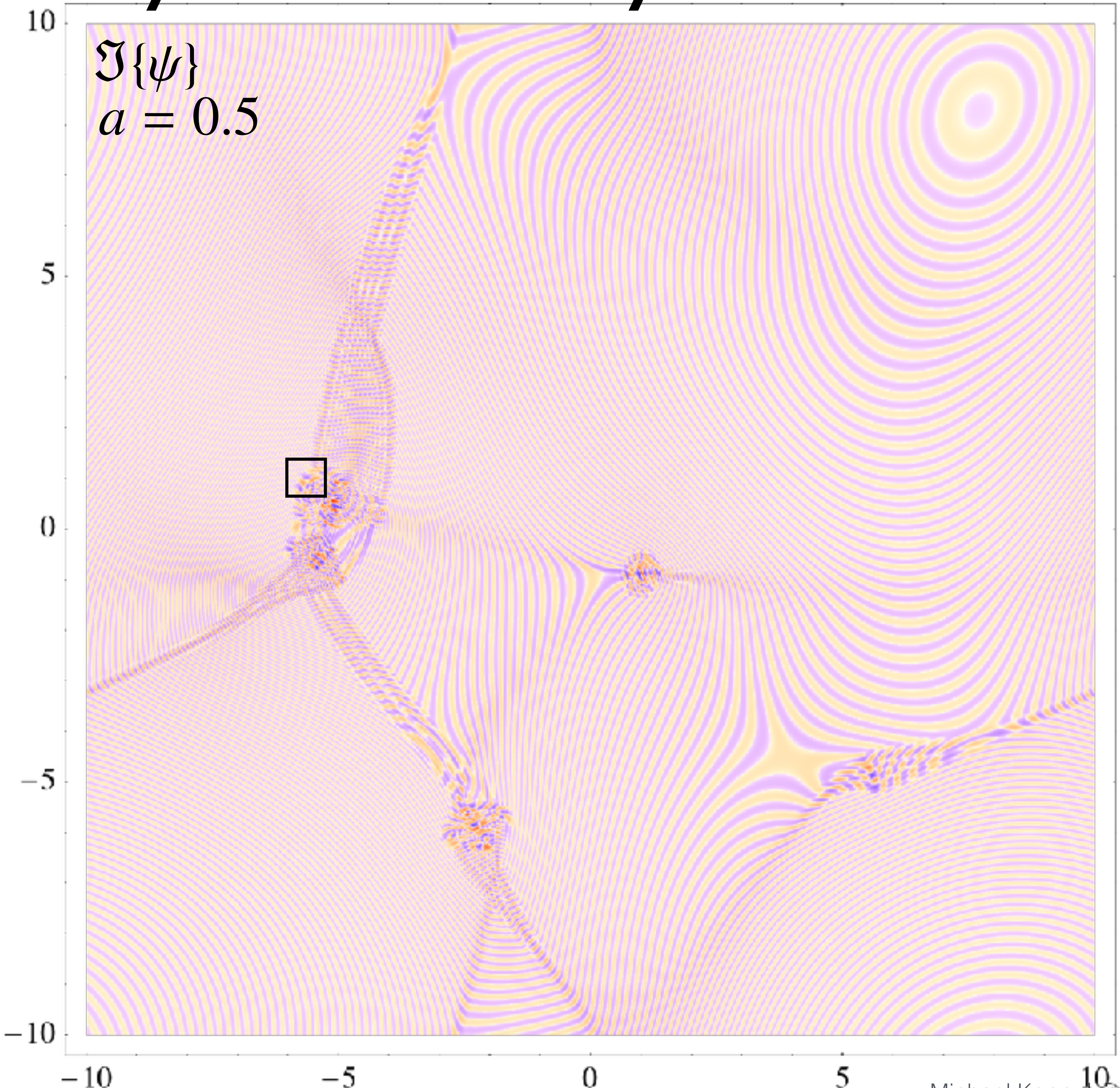
$$w_{\text{eff}} \equiv P_{\text{eff}} a^3/\rho_0$$

- Excellent agreement between ColDICE and ScM
- ScM can be used as basis for EFTofLSS, Baumann, Nicolis, Senatore et al (JCAP, 2012) and nonperturbative methods

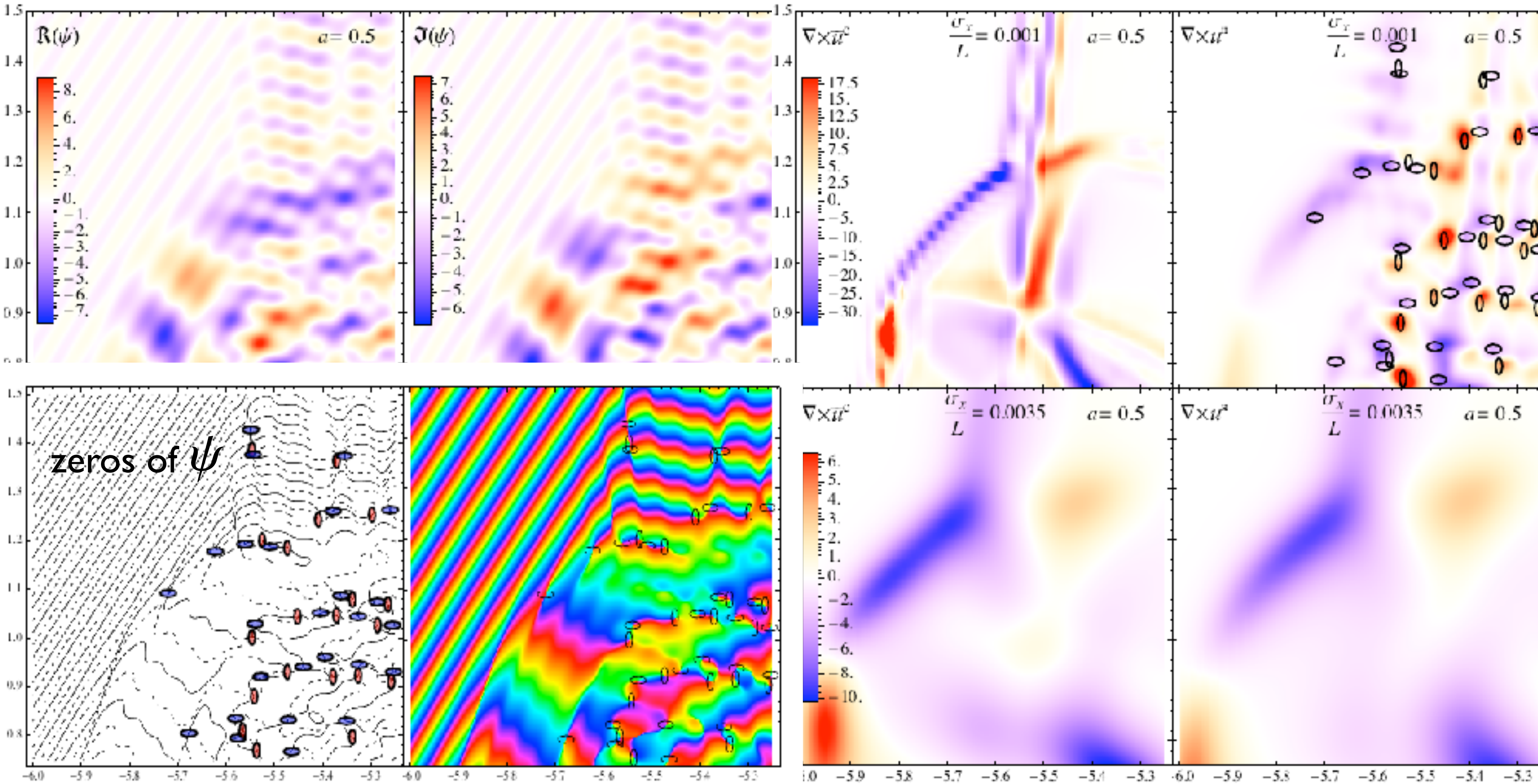


3b) Discussion:
Microscopic and macroscopic vorticity
in Vlasov and ScM

vorticity without vorticity



vorticity without vorticity



ψ single valued $\Longrightarrow \dfrac{1}{2\pi\tilde{\hbar}}\oint_C \nabla\phi\cdot d\boldsymbol{l} = m \in \mathbb{Z} \overset{\text{Stokes}}{\Longrightarrow} \nabla\times(\nabla\phi) = \hat{\boldsymbol{z}}\,2\pi\,\tilde{\hbar}\,m\,\delta_{\mathrm{D}}(\boldsymbol{x}_{\mathrm{vort}})$

$$\nabla\times\boldsymbol{u}_{\mathrm{H}} = \hat{\boldsymbol{z}}\,2\frac{\sigma_u}{\sigma_x}\sum_i^{N_{\mathrm{vort}}} m_i e^{\frac{(\boldsymbol{x}-\boldsymbol{x}_i)^2}{2\sigma_x{}^2}} + \sigma_x^2\left(\nabla\frac{n^{\mathrm{H}}_{,i}}{n^{\mathrm{H}}}\times\nabla\bar{\phi}_{,i}\right) + \mathcal{O}(\sigma_x{}^4)$$

Vorticity in CDM:
Pueblas, Scoccimarro
(PRD 2009)

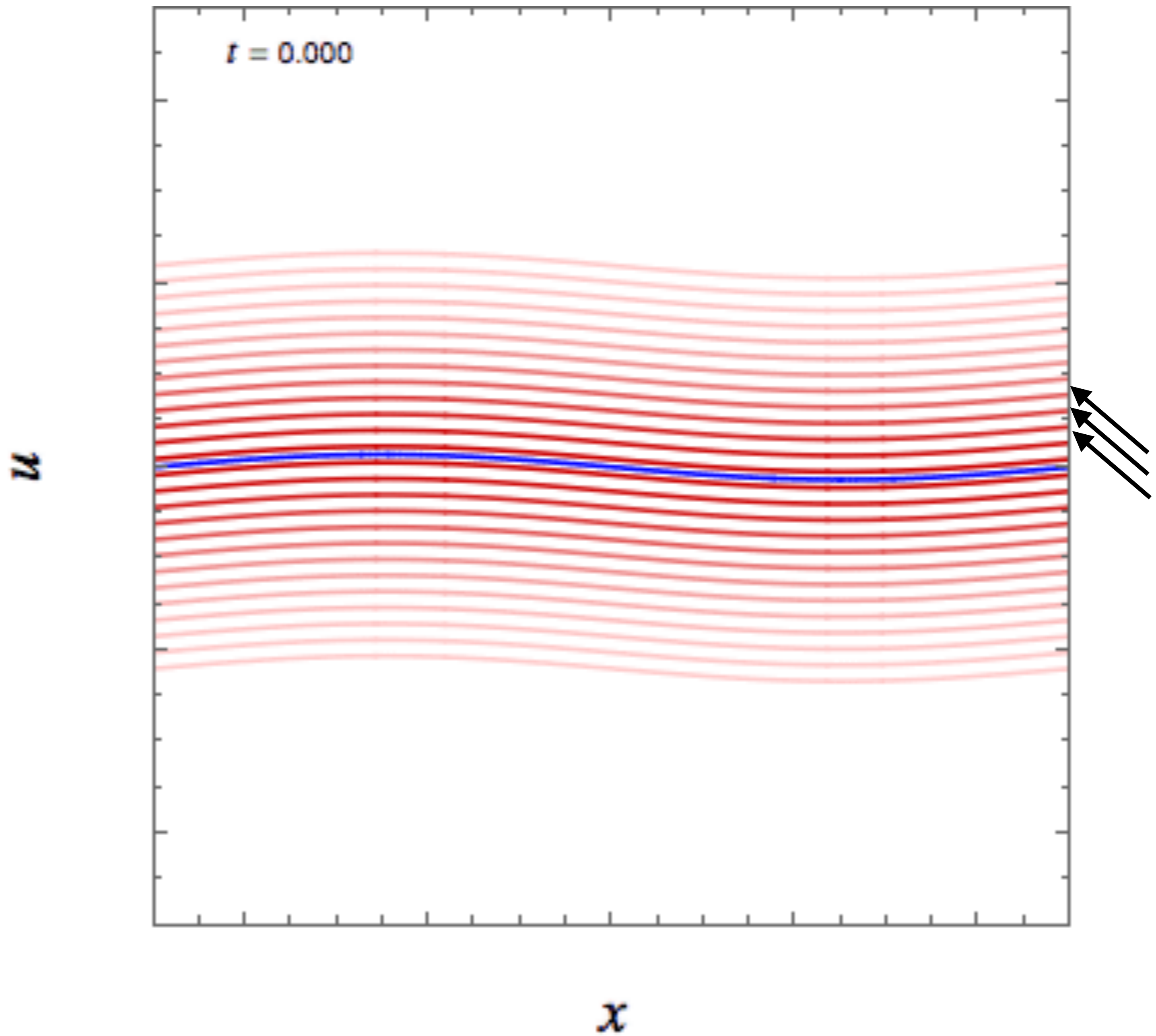
4) Comparison
of
1D dynamics and methods
for coarse grained Vlasov and ScM
with warm initial conditions

Relevant literature:

ScM:
Nguyen, Izrar, Bertrand et al 1981 Physics Letters
Mocz et al 1801.03507

Vlasov:
Banerjee et al, arXiv:1801.03906
Kates-Harbeck et al arXiv:1506.07207

Vlasov equation: warm vs cold initial conditions



$$\boldsymbol{X}(\boldsymbol{q}) \to \boldsymbol{X}_a(\boldsymbol{q})$$
$$\boldsymbol{U}(\boldsymbol{q}) \to \boldsymbol{U}_a(\boldsymbol{q})$$

a labels the sheets used to sample the initial velocity dispersion

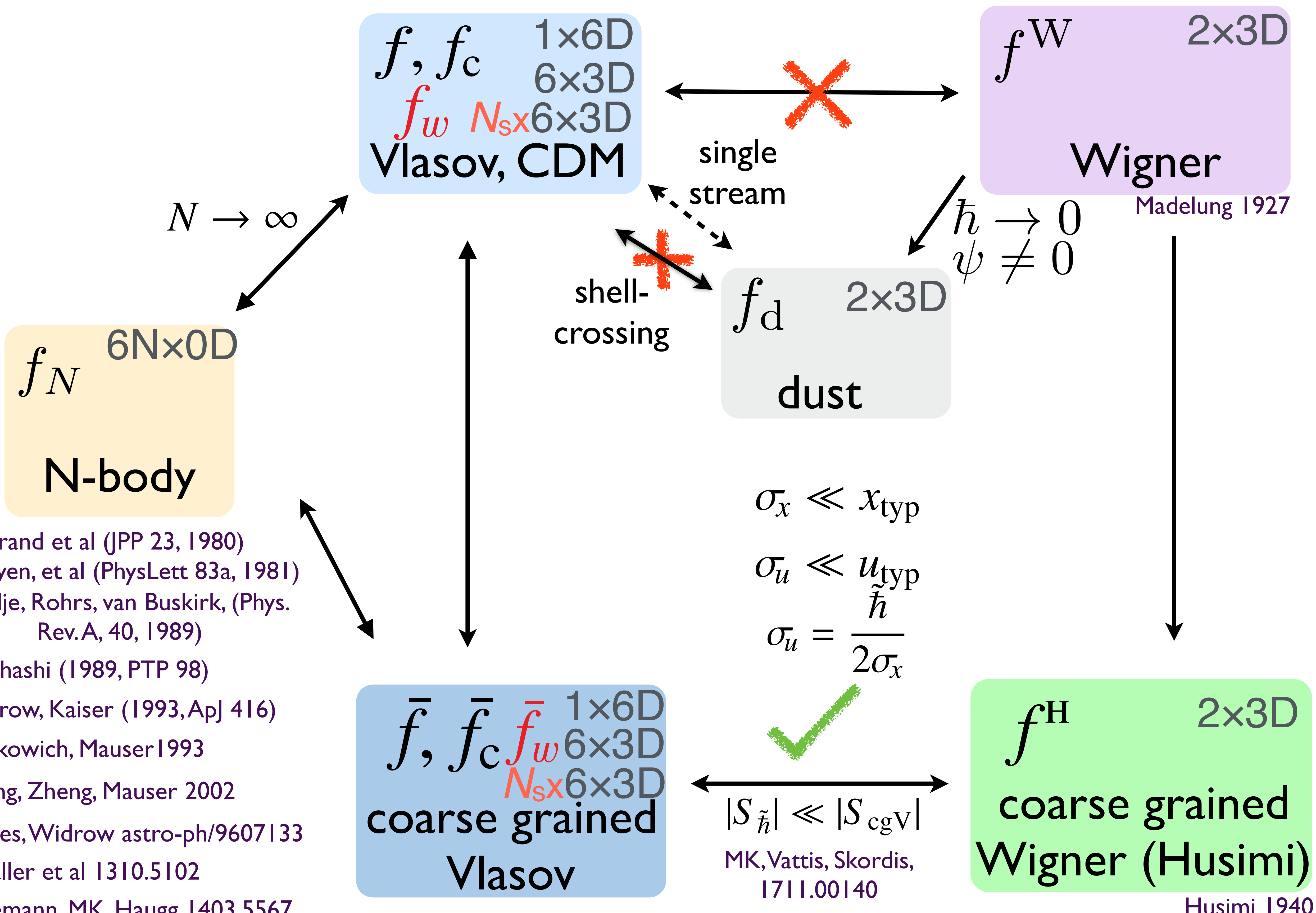
Definition: Warm Dark Matter (WDM)

For instance *massive neutrinos* or DM with initial velocity dispersion

$$\Omega_\nu = 0.02 \frac{m_\nu}{\mathrm{eV}} \qquad \sqrt{\langle C_\nu^{(2)} \rangle} \simeq 200(1+z)\left(\frac{1\mathrm{eV}}{m_\nu}\right)\mathrm{km\,s^{-1}} \gg \sqrt{\langle (C^{(1)})^2_\nu \rangle}$$

$$\lim_{t\to 0} f_w(t, \boldsymbol{x}, \boldsymbol{u}) = \sum_{b=1}^{N_s} w_b f_c^{(b)}(t, \boldsymbol{x}, \boldsymbol{u})$$

Weighted sum over N_s displaced cold sheets



- Bertrand et al (JPP 23, 1980)
- Nguyen, et al (PhysLett 83a, 1981)
- Skodje, Rohrs, van Buskirk, (Phys. Rev. A, 40, 1989)
- Takahashi (1989, PTP 98)
- Widrow, Kaiser (1993, ApJ 416)
- Markowich, Mauser 1993
- Zhang, Zheng, Mauser 2002
- Davies, Widrow astro-ph/9607133
- Schaller et al 1310.5102
- Uhlemann, MK, Haugg 1403.5567
- Garny, Konstandin 1710.04846
- Mocz et al 1801.03507

Setting up warm initial conditions for f and ψ

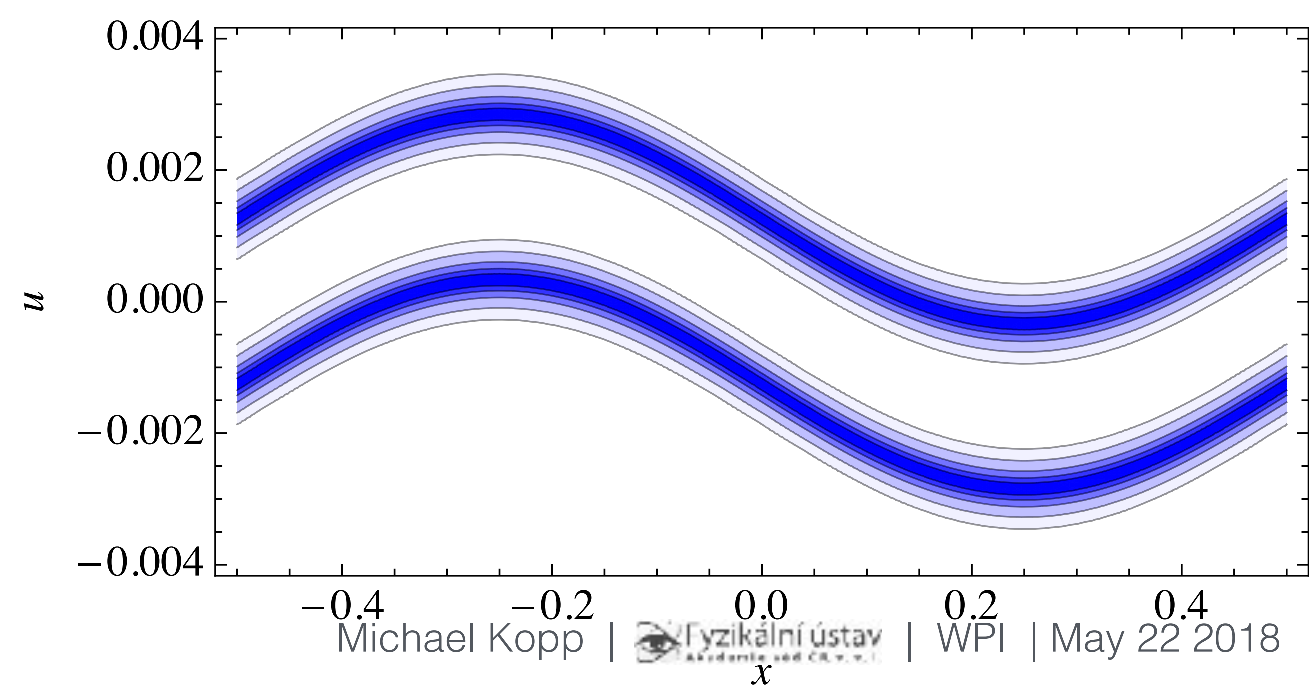
Example: 2 sheets with uniform density

$$\delta_u = U_2^{\text{ini}}(q) - U_1^{\text{ini}}(q) = \tilde{\hbar}\frac{2\pi}{L}N_u$$

$$f^{\text{ini}}(x,u) = \sum_{b=1}^{N_s=2} \delta_{\text{D}}(U_b^{\text{ini}}(q) - u)$$
$$= \delta_{\text{D}}(U_1^{\text{ini}}(q) - u) + \delta_{\text{D}}(U_2^{\text{ini}}(q) - u)$$

$$\bar{f}^{\text{ini}}(x,u) = \text{norm}\left(\exp\left[\frac{(U_1^{\text{ini}}(q) - u)^2}{2\sigma_u^2}\right] + \exp\left[\frac{(U_2^{\text{ini}}(q) - u)^2}{2\sigma_u^2}\right]\right)$$

$\Rightarrow$

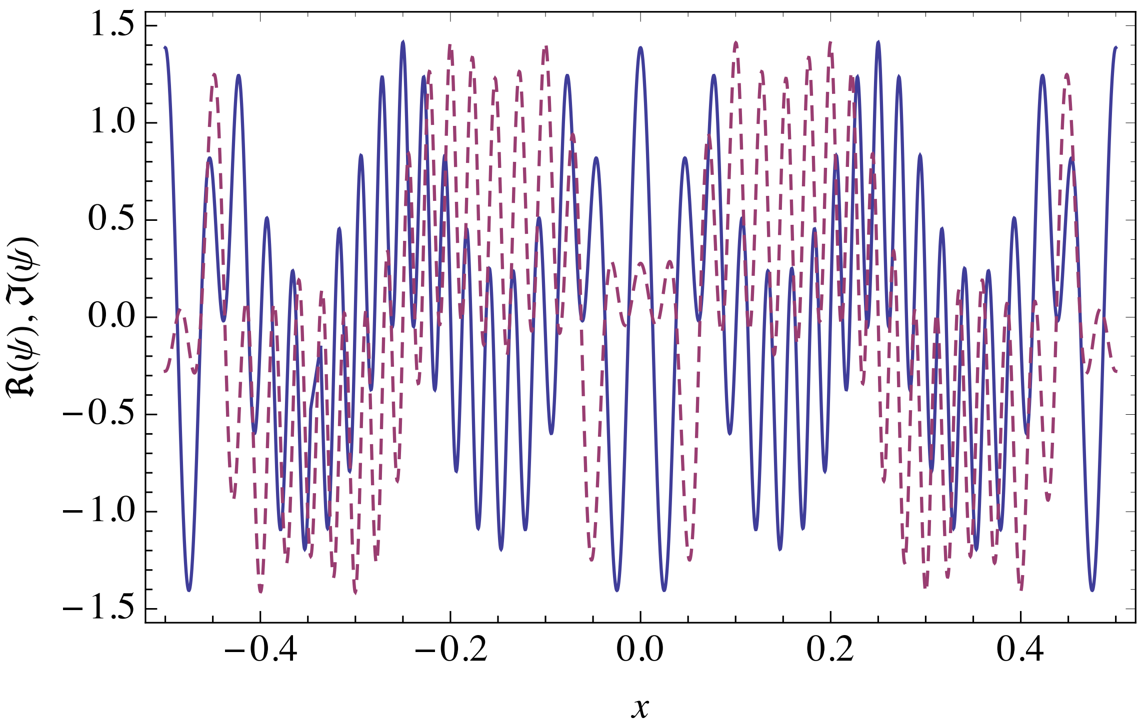


Setting up warm initial conditions for f and ψ

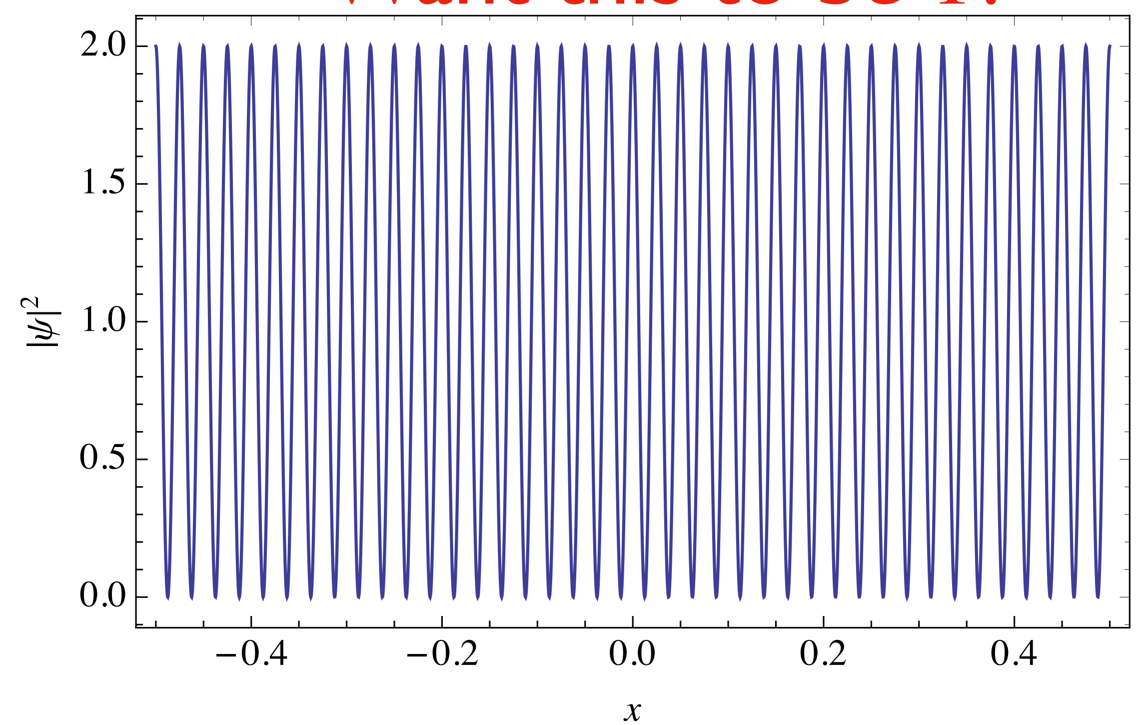
Example: 2 sheets with uniform density

$$\psi^{\rm ini}(x) = \sum_{b=1}^{N_s=2} \sqrt{\tfrac{1}{2}}\, e^{i\,\phi_b^{\rm ini}(x)/\tilde{\hbar}} = e^{i\,\phi_{\rm c}^{\rm ini}(x)/\tilde{\hbar}}\,\sqrt{2}\cos\left(\frac{\delta_u}{2\tilde{\hbar}}x\right) \qquad \delta_u = U_2^{\rm ini}(q) - U_1^{\rm ini}(q) = \tilde{\hbar}\frac{2\pi}{L}N_u$$

Want this to be 1!



$\Rightarrow$



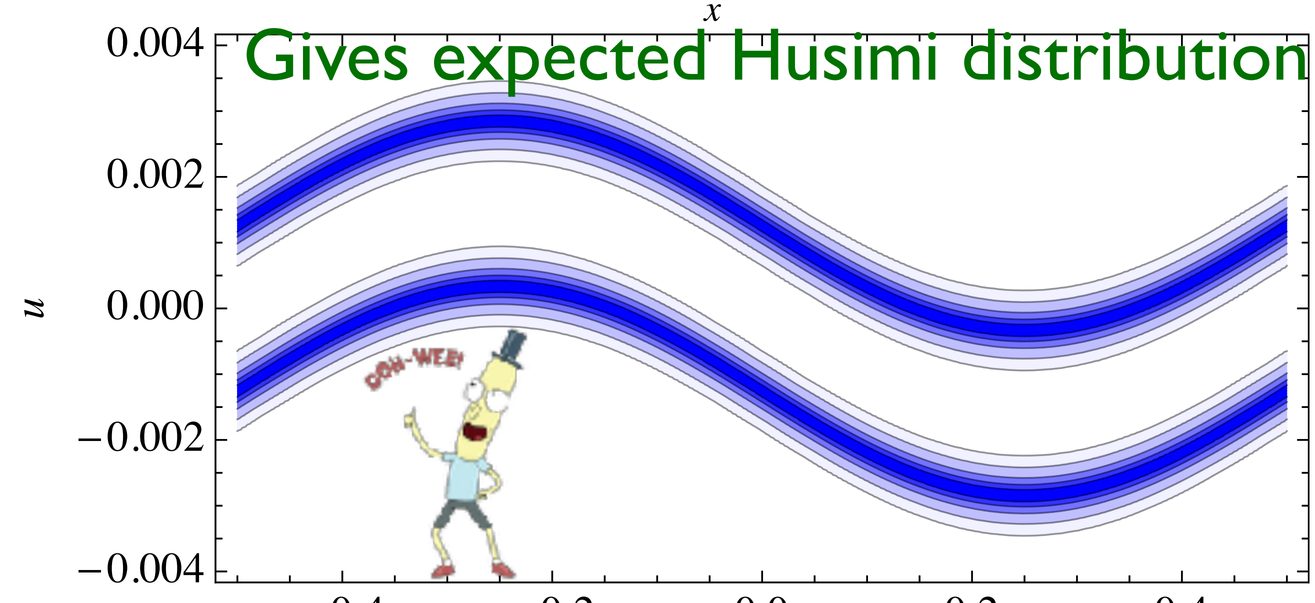
Need to choose at least

$$\sigma_x = \frac{2\pi\tilde{\hbar}}{\delta_u} = \frac{L}{N_u} = 0.025$$

$$\Downarrow$$

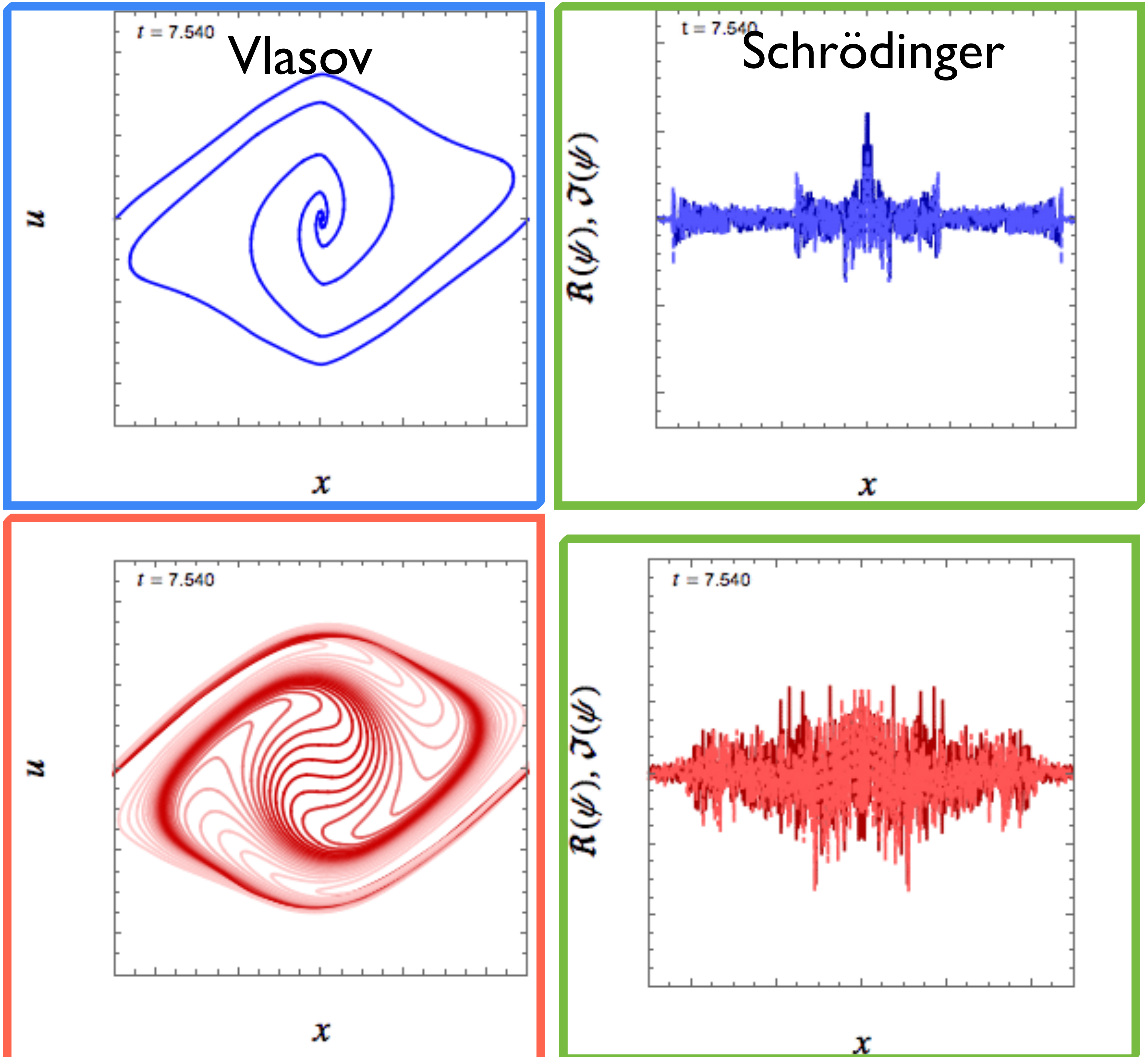
$$\sigma_u \equiv \frac{\tilde{\hbar}}{2\sigma_x} = \frac{\delta_u}{4\pi}$$ sheets automatically well separated

$\Rightarrow$



coarse grained WDM and ScM moments

Phase space sheet has to be tracked	Completely avoids phase space
Dynamics: non-local	local
$$a^2\partial_t \boldsymbol{X}_b(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})$$ $$\partial_t \boldsymbol{U}_b(\boldsymbol{q}) = -\boldsymbol{\nabla}_x \Phi_c(\boldsymbol{x})\vert_{\boldsymbol{x}=\boldsymbol{X}_b(\boldsymbol{q})}$$ $$\Delta\Phi_w(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\left(\sum_{b=1}^{N_s} w_b \sum_{\substack{\boldsymbol{q}\text{ with}\\ \boldsymbol{x}=\boldsymbol{X}_b(\boldsymbol{q})}} \frac{1}{\vert\det \partial_{q^i} X_b^j(\boldsymbol{q})\vert} - 1\right)$$	$$i\tilde{\hbar}\partial_t\psi(\boldsymbol{x}) = -\frac{\tilde{\hbar}^2}{2a^2}\Delta\psi(\boldsymbol{x}) + \Phi_\psi(\boldsymbol{x})\psi(\boldsymbol{x})$$ $$\Delta\Phi_\psi(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\left(\vert\psi(\boldsymbol{x})\vert^2 - 1\right)$$
Moments: non-local w for warm	quasi-local W for Wigner
$$G_w(\boldsymbol{x},\boldsymbol{J}) = \sum_{b=1}^{N_s} w_b \sum_{\substack{\boldsymbol{q}\text{ with}\\ \boldsymbol{x}=\boldsymbol{X}_b(t,\boldsymbol{q})}} \frac{e^{i\boldsymbol{J}\cdot\boldsymbol{U}_b(\boldsymbol{q})}}{\vert\det \partial_{q^i} X_b^j(\boldsymbol{q})\vert}$$ sum over streams	$$G_{\mathrm{W}}(\boldsymbol{x},\boldsymbol{J}) = \psi\left(\boldsymbol{x} + \tfrac{\tilde{\hbar}}{2}\boldsymbol{J}\right)\bar{\psi}\left(\boldsymbol{x} - \tfrac{\tilde{\hbar}}{2}\boldsymbol{J}\right)$$
$$\bar{M}^{w(n)}_{i_1,\ldots,i_n}(\boldsymbol{x}) = e^{\frac{\sigma_x^2}{2}\Delta}\left\{\frac{(-i)^n\partial^n}{\partial J_{i_1}\ldots\partial J_{i_n}} e^{-\frac{1}{2}\sigma_u^2\boldsymbol{J}^2} G_{\hat{w}}(\boldsymbol{x},\boldsymbol{J})\right\}\Bigg\vert_{\boldsymbol{J}=0}$$	$$M^{\mathrm{H}(n)}_{i_1,\ldots,i_n}(\boldsymbol{x}) = e^{\frac{\sigma_x^2}{2}\Delta}\left\{\frac{(-i)^n\partial^n}{\partial J_{i_1}\ldots\partial J_{i_n}} e^{-\frac{1}{2}\sigma_u^2\boldsymbol{J}^2} G_{\mathrm{W}}(\boldsymbol{x},\boldsymbol{J})\right\}\Bigg\vert_{\boldsymbol{J}=0}$$ *n* derivatives of ψ



Cold

- $N_{grid}=2^{14}$
- $N\ \ =2^{12}$

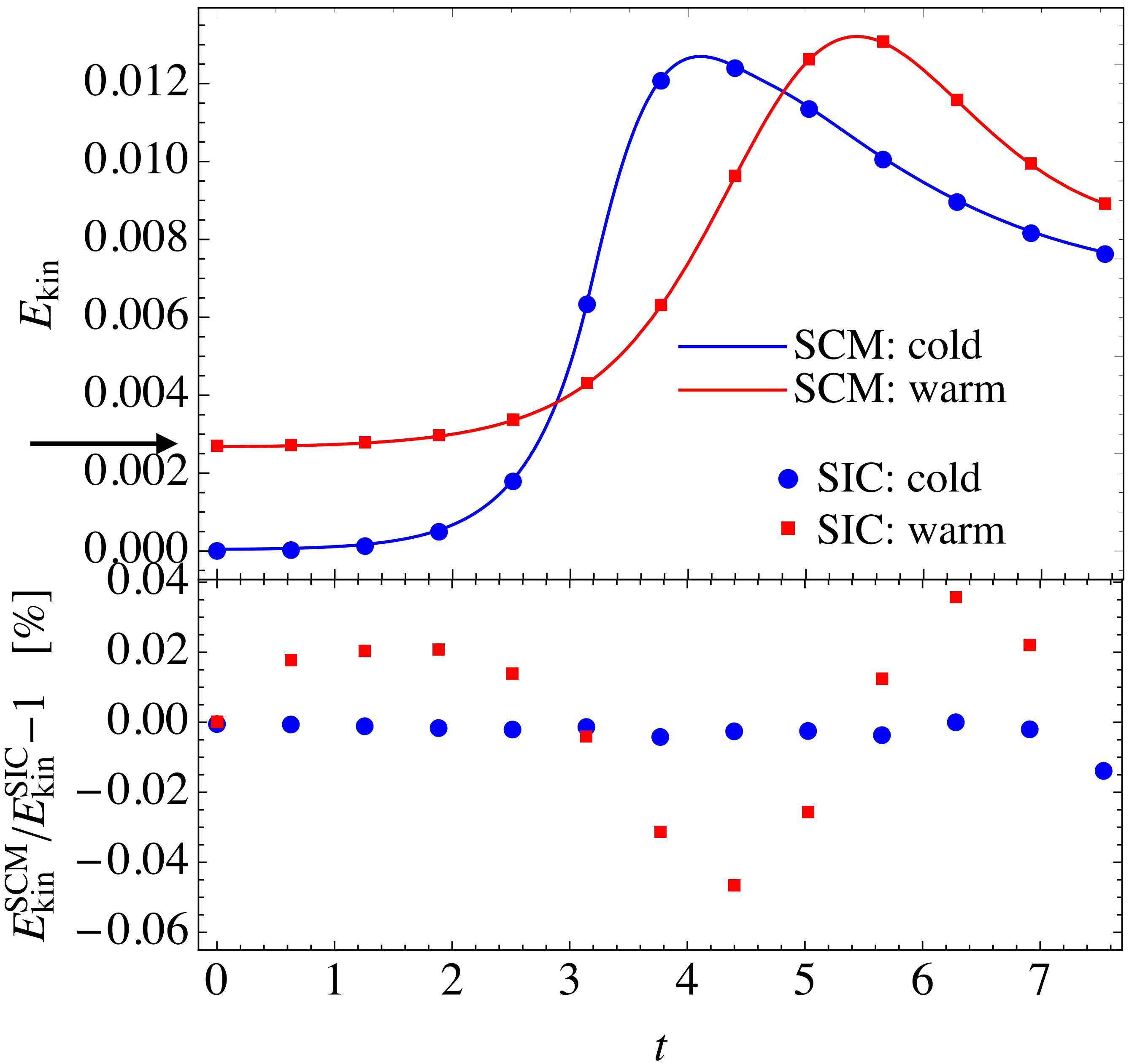
Warm

- $N_{grid}=2^{14}$
- $N\ N_s=3328$

coarse grained WDM and ScM agreement

On the arXiv in 2 month or so

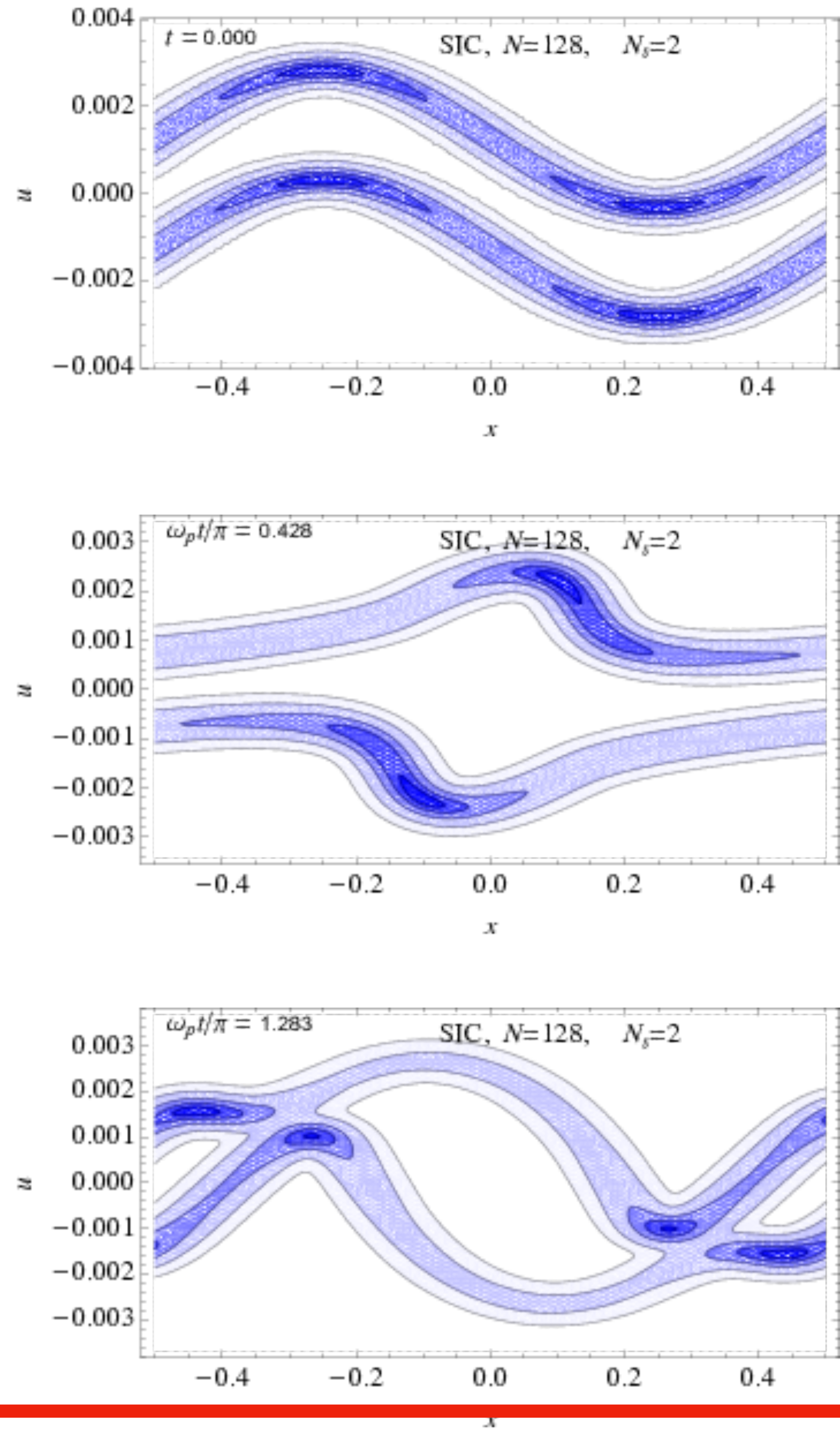
Non-zero initial velocity dispersion →



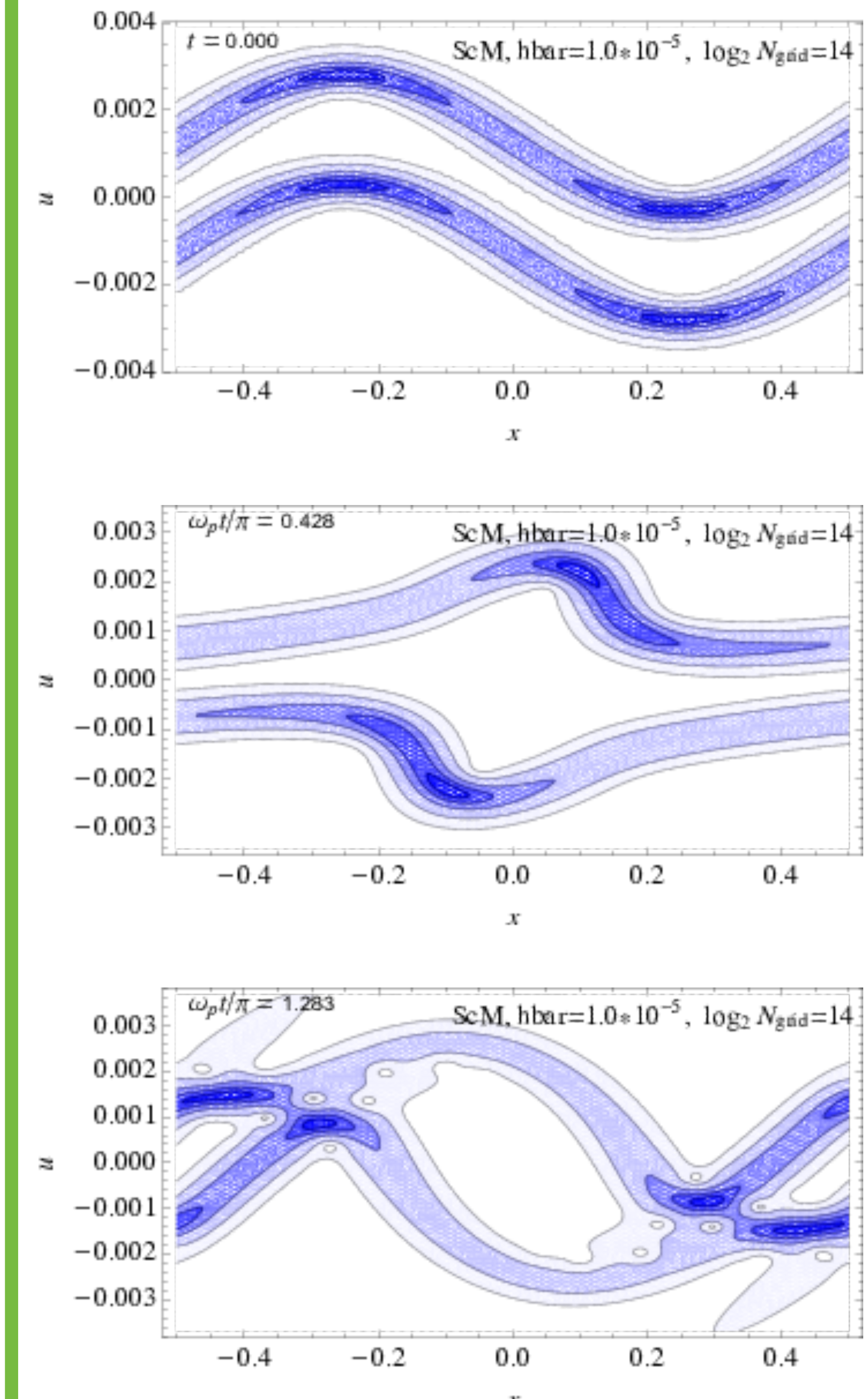
Agreement better than $5 \cdot 10^{-4}$

Repulsive force (plasmas): *two-stream instability*

Vlasov

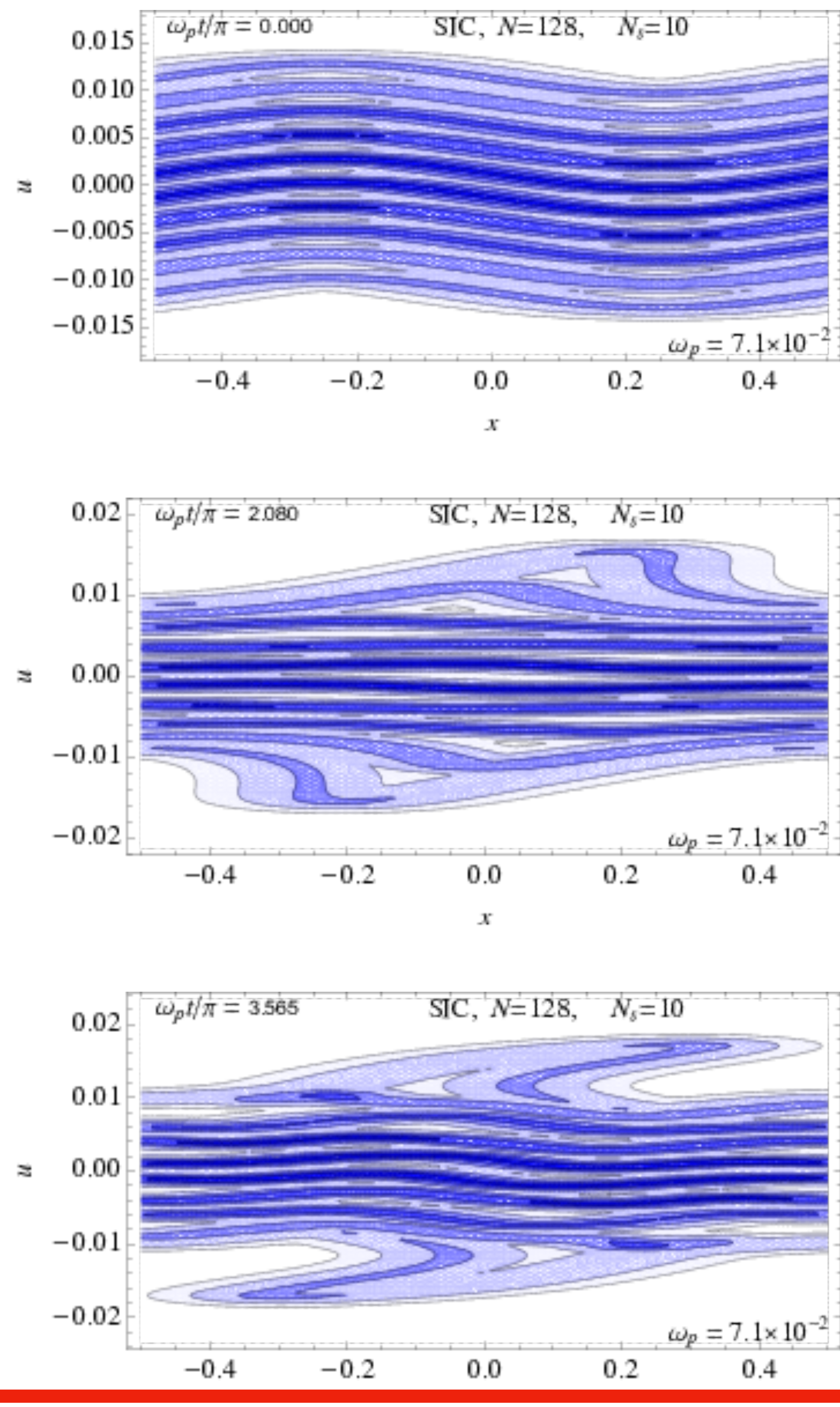


Schrödinger method

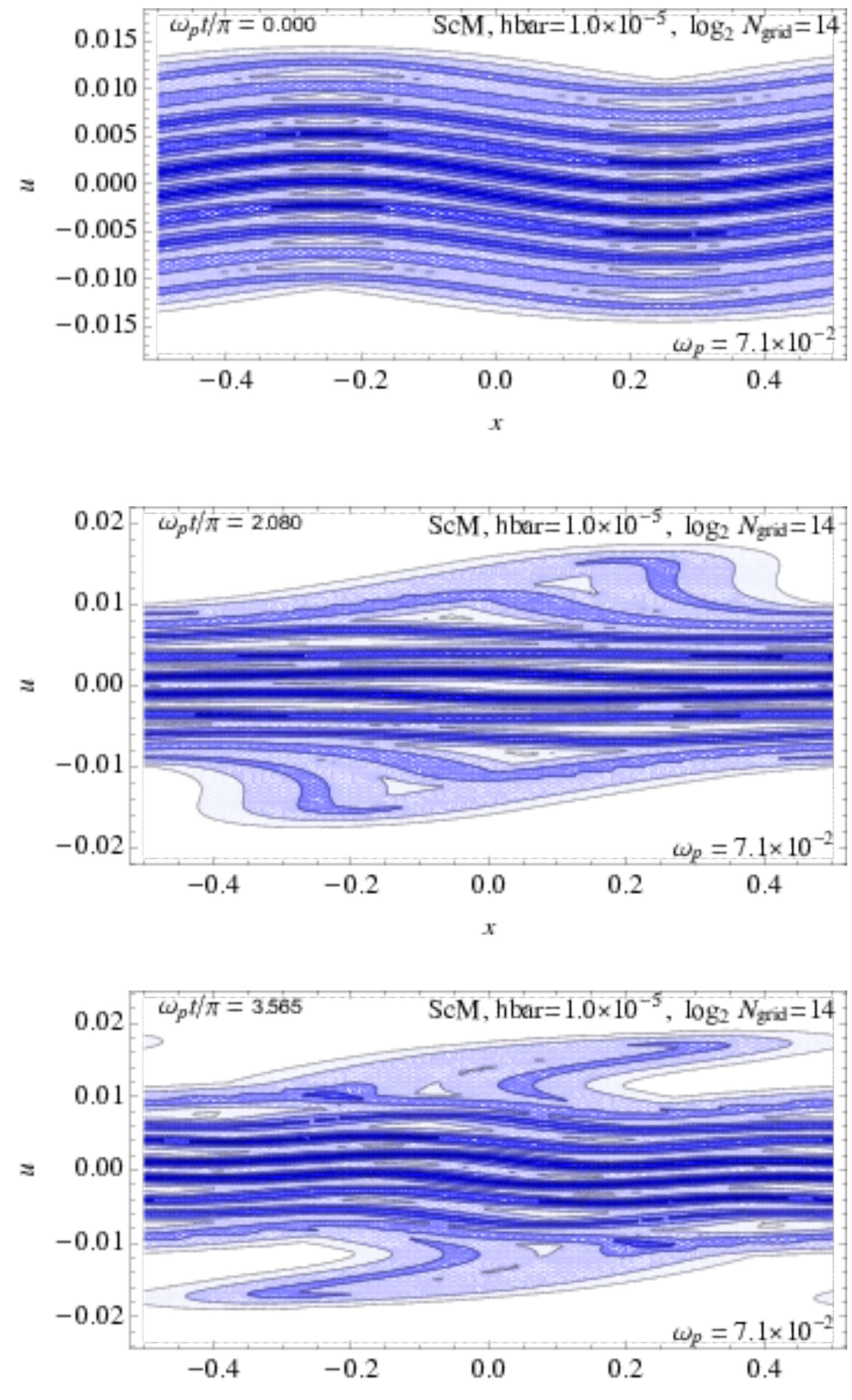


Repulsive force (plasmas): *Landau damping*

Vlasov



Schrödinger method



5) Summary

MK, Vattis, Skordis, 1711.00140

1. Convergence of ScM to coarse grained Vlasov for $\tilde{\hbar} \to 0$
2. Excellent agreement with ColDICE — Sousbie, Colombi 1509.07720
3. Many advantages: — Uhlemann, MK, Haugg 1403.5567
   a) only 2 degrees of freedom, UV complete
   b) phase space can be avoided
   c) quasi-local in eulerian space
   d) *f* sampled uniformly, but minimal resolution $\tilde{\hbar}$

Ongoing and Future

1. 3D implementation with AMR, using GAMER — Schive et al (ApJS, 186, 2010) Schive Chiueh, Broadhurst (Nature 2014)
2. Warm initial conditions, modeling neutrinos
3. Non-perturbative field theory methods applied to ScM:
   Solve evolution equation for $\langle \psi(t, \boldsymbol{x}_1)\bar{\psi}(t, \boldsymbol{y}_1)...\psi(t, \boldsymbol{x}_n)\bar{\psi}(t, \boldsymbol{y}_n)\rangle$