Application of the Souriau-Saturnini equations in Schwarzschild spacetime, and gravitational wave detections

Loïc Marsot

Feb. 10, 2019

To the memory of Christian Duval

Quick review of Souriau's universal model of a test particle (see Pierre Saturnini's talk)

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Up to dipole terms we obtain the Mathisson-Papapetrou-Dixon equations [Souriau '74],

$$\dot{P}^{\mu} = -\frac{1}{2} R^{\mu}{}_{\rho\alpha\beta} S^{\alpha\beta} \dot{X}^{\rho},$$
$$\dot{S}^{\mu\nu} = P^{\mu} \dot{X}^{\nu} - P^{\nu} \dot{X}^{\mu}.$$

Overview

For a photon, postulate $P_\mu P^\mu = 0$ and the Tulczyjew constraint $S^\mu{}_\nu P^\nu = 0$. The Souriau-Saturnini equations are then [Saturnini '76],

$$\begin{aligned}
\dot{X}^\mu &= P^\mu + \frac{2}{R(S)^\lambda{}_\sigma S_\lambda{}^\sigma} S^\mu{}_\nu R(S)^\nu{}_\rho P^\rho\,,\\
\dot{P}^\mu &= -s\,\frac{\mathrm{Pf}(R(S)^\mu{}_\nu)}{R(S)^\lambda{}_\sigma S_\lambda{}^\sigma}\,P^\mu\,,\\
\dot{S}^{\mu\nu} &= P^\mu \dot{X}^\nu - \dot{X}^\mu P^\nu.
\end{aligned}$$

with $-\frac{1}{2}\mathrm{Tr}(S^2) = s^2$, $s = \pm\hbar$ and $R(S)^\mu{}_\nu := R^\mu{}_{\nu\alpha\beta}S^{\alpha\beta}$.

We want to see the practical differences induced by these equations. Two examples here:

- Schwarzschild metric, gravitational lensing [Duval, Marsot, Schücker, arXiv:1812.03014]
- Gravitational wave detections [Marsot, arXiv:1904.09260]

Schwarzschild metric, isotropic coordinates, and notations

Schwarzschild metric in isotropic coordinates $(\mathbf{x}, t)$, where $\mathbf{x} = (x^1, x^2, x^3)$,

$$\mathrm{g} = -\left(\frac{r+a}{r}\right)^4 \|d\mathbf{x}\|^2 + \left(\frac{r-a}{r+a}\right)^2 dt^2,$$

with $r := \sqrt{\mathbf{x}\cdot\mathbf{x}}$ and $0 < a < r$, with $a = \frac{1}{2}GM$ the Schwarzschild radius.

If $(\rho, \theta, \varphi, t)$ are the Schwarzschild coordinates, the isotropic polar ones (r, θ, φ, t) are related by

$$\rho = r\left(1 + \frac{GM}{2r}\right)^2 \qquad \text{or} \qquad r = \frac{1}{2}\left(\rho - GM + \sqrt{\rho(\rho - 2GM)}\right).$$

Definitions of P and S

The 4-momentum, with $\mathbf{p} = (p_1, p_2, p_3)$,

$$P = \begin{pmatrix} \dfrac{r^2}{(r+a)^2}\,\mathbf{p} \\ \dfrac{r+a}{r-a}\,\|\mathbf{p}\| \end{pmatrix},$$

such that $P^2 = 0$. The spin tensor, with $\mathbf{s} = (s_1, s_2, s_3)$,

$$S = (S^\mu{}_\nu) = \begin{pmatrix} j(\mathbf{s}) & -\dfrac{(\mathbf{s}\times\mathbf{p})}{\|\mathbf{p}\|}\dfrac{r^2(r-a)}{(r+a)^3} \\ -\dfrac{(\mathbf{s}\times\mathbf{p})^T}{\|\mathbf{p}\|}\dfrac{(r+a)^3}{r^2(r-a)} & 0 \end{pmatrix},$$

such that $SP = 0$, with $j(\mathbf{s}) : \mathbf{p} \mapsto \mathbf{s}\times\mathbf{p}$, and the conserved longitudinal spin,

$$-\frac{1}{2}\mathrm{Tr}(S^2) = \left(\frac{\mathbf{s}\cdot\mathbf{p}}{\|\mathbf{p}\|}\right)^2 = s^2.$$

The equations in Schwarzschild spacetime (1/2)

Let us introduce the shorthand,

$$D := r^2(\mathbf{s}\cdot\mathbf{p}) - 3(\mathbf{p}\cdot\mathbf{x})(\mathbf{s}\cdot\mathbf{x}).$$

$$\frac{d\mathbf{x}}{dt} = \frac{r^2(r-a)}{(r+a)^3 D}\left\{r^2(\mathbf{s}\cdot\mathbf{p})\,\frac{\mathbf{p}}{\|\mathbf{p}\|} - 3\|\mathbf{p}\|(\mathbf{s}\cdot\mathbf{x})\,\mathbf{x} + 3[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]\,\frac{\mathbf{x}\times\mathbf{p}}{\|\mathbf{p}\|}\right\},$$

$$\frac{d\mathbf{p}}{dt} = \frac{2\,a}{(r+a)^4 D}\Bigg\{r^2(r-a)\left[(\mathbf{s}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{x}) - \frac{3r}{(r+a)^3}\,(\mathbf{s}\cdot\mathbf{x})[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]\right]\frac{\mathbf{p}}{\|\mathbf{p}\|}$$
$$- r\,\|\mathbf{p}\|\left[D + r\,(r-a)\,(\mathbf{s}\cdot\mathbf{p})\right]\mathbf{x} + 3\,(r-a)[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]\,(\mathbf{p}\cdot\mathbf{x})\,\frac{\mathbf{x}\times\mathbf{p}}{\|\mathbf{p}\|}\Bigg\},$$

The equations in Schwarzschild spacetime (2/2)

$$\frac{d\mathbf{s}}{dt} = \frac{1}{\|\mathbf{p}\|\,(r+a)^4 D}\Big\{3(r-a)(r+a)^3\Big[\Big(-r^2\|\mathbf{p}\|^2 + (\mathbf{x}\cdot\mathbf{p})^2\Big)\,\mathbf{s}\times\mathbf{p}$$

$$+\Big(2\|\mathbf{p}\|^2(\mathbf{x}\cdot\mathbf{s}) - (\mathbf{x}\cdot\mathbf{p})(\mathbf{s}\cdot\mathbf{p})\Big)\,\mathbf{x}\times\mathbf{p}\Big] + 2arD\,((\mathbf{x}\cdot\mathbf{s})\mathbf{p} - (\mathbf{x}\cdot\mathbf{p})\mathbf{s})$$

$$+2a(r-a)\big(-r^2(\mathbf{s}\cdot\mathbf{p})^2\mathbf{x} - 3[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]^2\mathbf{x}$$

$$+r^2(\mathbf{x}\cdot\mathbf{s})(\mathbf{s}\cdot\mathbf{p})\mathbf{p} + 3[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}](\mathbf{x}\cdot\mathbf{s})\mathbf{x}\times\mathbf{p}\big)\Big\}.$$

Conserved quantities

The energy $\mathcal{E}$, the angular momentum $\mathcal{L}$ and logitudinal spin s,

$$\mathcal{E} = \frac{r-a}{r+a}\|\mathbf{p}\| + \frac{2ar}{(r+a)^4\|\mathbf{p}\|}[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]\,,$$

$$\boldsymbol{\mathcal{L}} = \left(\frac{r+a}{r}\right)^2\mathbf{x}\times\mathbf{p} + \frac{r-a}{r+a}\mathbf{s} + \frac{2a}{r^2(r+a)}(\mathbf{s}\cdot\mathbf{x})\,\mathbf{x}\,,$$

$$s = \frac{\mathbf{s}\cdot\mathbf{p}}{\|\mathbf{p}\|}\,,$$

are conserved. Possible to eliminate $\mathbf{s}$ in the equations of motion with the relations,

$$\mathbf{x}\times\mathbf{p}\cdot\mathbf{s} = \frac{r+a}{r-a}\left(\mathbf{x}\times\mathbf{p}\cdot\boldsymbol{\mathcal{L}} - \left(\frac{r+a}{r}\right)^2\left(r^2\|\mathbf{p}\|^2 - (\mathbf{x}\cdot\mathbf{p})^2\right)\right),$$

$$\mathbf{s}\cdot\mathbf{x} = \boldsymbol{\mathcal{L}}\cdot\mathbf{x}\,,$$

$$\mathbf{s}\cdot\mathbf{p} = s\|\mathbf{p}\|.$$

From 9 to 5 equations of motion

The conservation of energy gives us

$$\|\mathbf{p}\| = \frac{r-a}{r+a}\,\frac{(r+a)^2\mathcal{E} - \frac{2ar}{(r^2-a^2)\|\mathbf{p}\|}\,(\mathbf{x}\times\mathbf{p}\cdot\boldsymbol{\mathcal{L}})}{(r-a)^2 \;-\; \frac{2a}{r\|\mathbf{p}\|^2}\|\mathbf{x}\times\mathbf{p}\|^2}.$$

The equations become,

$$\frac{d\mathbf{x}}{dt} = \frac{r^2(r-a)}{(r+a)^3 D}\left\{r^2 s\mathbf{p} - 3\|\mathbf{p}\|(\boldsymbol{\mathcal{L}}\cdot\mathbf{x})\,\mathbf{x} + 3[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]\,\frac{\mathbf{x}\times\mathbf{p}}{\|\mathbf{p}\|}\right\},$$

$$\frac{d}{dt}\left(\frac{\mathbf{p}}{\|\mathbf{p}\|}\right) = \frac{2a}{(r+a)^4 D}\left\{3(r-a)\frac{(\mathbf{x}\cdot\mathbf{p})}{\|\mathbf{p}\|^2}\,[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]\;\mathbf{x}\times\mathbf{p} + \right.$$

$$\left. + \left(3r(\boldsymbol{\mathcal{L}}\cdot\mathbf{x})(\mathbf{x}\cdot\mathbf{p}) - (2r-a)s\|\mathbf{p}\|r^2\right)\left(\mathbf{x} - \frac{(\mathbf{x}\cdot\mathbf{p})\,\mathbf{p}}{\|\mathbf{p}\|^2}\right)\right\}.$$

with $D := r^2 s\|\mathbf{p}\| - 3(\mathbf{p}\cdot\mathbf{x})(\boldsymbol{\mathcal{L}}\cdot\mathbf{x})$, and $[\mathbf{x}\times\mathbf{p}\cdot\mathbf{s}]$ as above.

Radial propagation

If the initial momentum is parallel to the initial position, the equations of motion become much simpler,

$$\frac{d\mathbf{x}}{dt} = \frac{r^2(r-a)}{(r+a)^3}\frac{\mathbf{p}}{\|\mathbf{p}\|},$$

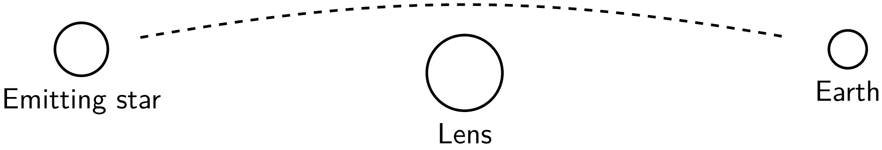
$$\frac{d\mathbf{p}}{dt} = -\frac{2\,a\,r^2}{(r+a)^4}\,\mathbf{p},$$

$$\frac{d\mathbf{s}^\perp}{dt} = -\frac{2\,a\,r^2}{(r+a)^4}\,\mathbf{s}^\perp$$

The trajectory described is exactly the null geodesic one. In radial propagation, the trajectory and redshift are the same whether we consider the spin of the photon or not. While s is conserved, $\mathbf{s}^\perp$ is only parallel transported.

Quick reminder: gravitational lensing without spin

The dashed line is the null geodesic trajectory followed by spinless light. It is bent around the lens, which can be our Sun, a galaxy, or other.



The full trajectory of spinless light is contained in a plane.

Schwarzschild numerical solution (ChD, LM, TS)

Starting at perihelion together with a spinless photon,

$$\mathbf{x}_0 = \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{p}_0 = \begin{pmatrix} 0 \\ p_0 \\ 0 \end{pmatrix}, \qquad \mathbf{s}_0 = \begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix},$$

with $r_0 = 3 \cdot 10^5$ m, $\lambda_0 = 600$ nm, and Schwarzschild radius $a = 3 \cdot 10^3$ m.

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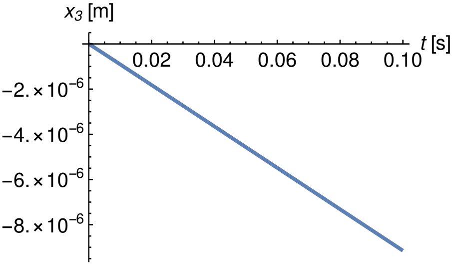

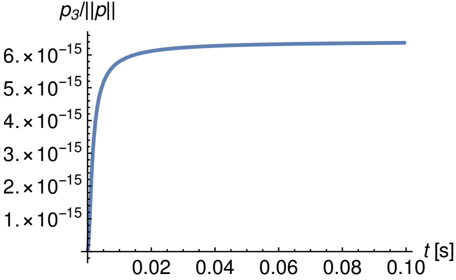

Schwarzschild perturbative solution

Same initial conditions at perihelion. Two small parameters,

$$\alpha = \frac{a}{r_0} \qquad \& \qquad \epsilon = \frac{\hbar}{r_0\, p_0} = \frac{\lambda_0}{2\pi\, r_0},$$

Find a deviation of order ϵ to the geodesic.

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Find a deviation of order ϵ to the geodesic. At lowest order in ϵ and α,

$$x_3 = -\epsilon\, \chi\, t \qquad \& \qquad p_3 = 2\, \epsilon\, \alpha\, \chi\, p_0 \frac{\sqrt{r_0^2 + t^2} - r_0}{\sqrt{r_0^2 + t^2}}\,,$$

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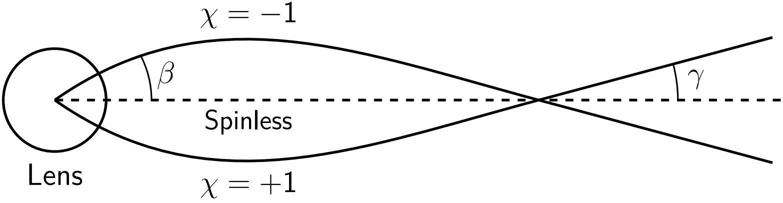
corresponding to deviation angles out of the geodesic planes,

$$\beta \sim -(1 - 4\alpha) \frac{\chi\, \lambda_0}{2\pi\, r_0} \qquad \& \qquad \gamma \sim \chi \frac{a\, \lambda_0}{\pi\, r_0^2}.$$

For the sun, and $\lambda_0 = 600\text{nm}$, $2|\beta| \sim 5 \cdot 10^{-11\,\prime\prime}$ and $2|\gamma| \sim 5 \cdot 10^{-16\,\prime\prime}$.

Not following its momentum?

x_3 and p_3 have different signs, intuitively expect something like:



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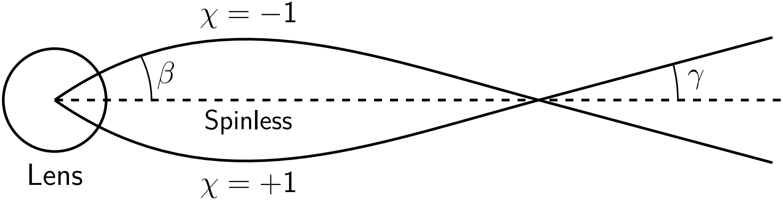

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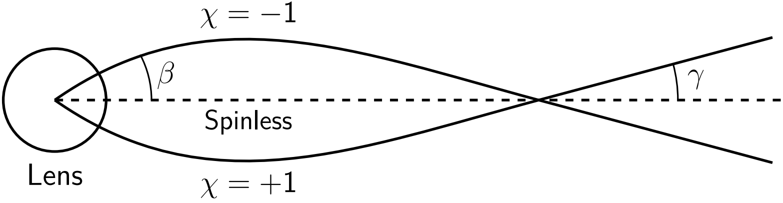

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This problem is solved when considering the cosmological constant. The equations become well defined “far away” from the star, and the photon follows its momentum. But not the most satisfactory explanation.

Observations and (potential) problems with this approach

- Angles out of the geodesic plane,

$$\beta \sim -(1-4\alpha)\frac{\chi\,\lambda_0}{2\pi\, r_0} \qquad \& \qquad \gamma \sim \chi\frac{a\,\lambda_0}{\pi\, r_0^2},$$

  featuring a rainbow effect.

- Experimental upper bound with radio waves deviated by the sun: $\gamma_{\rm exp} < 10^{-3}\,''$ vs $\gamma_{\rm th} \sim 10^{-11}\,''$ [Harwit, et. al. '74]

- Other approach found the same expression for γ, but in different direction [Gosselin, Bérard, Mohrbach '07]

- Transverse spin is 0 at perihelion then grows linearly with time (better with Λ). Due to classical equations ?

Spin effects in gravitational wave detections?

Gravitational wave (GW) detection involves a laser beam, hence photons with their spin, travelling through an inhomogeneous gravitational field.

Recall the metric describing a GW in spacetime to linear order in the GW amplitude $\epsilon \sim 10^{-20}$,

$$g = dt^2 - \Big(1 - \epsilon \cos(\omega(t-z))\Big) dx_1^2 - \Big(1 + \epsilon \cos(\omega(t-z))\Big) dx_2^2 - dx_3^2.$$

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...experimentally we only know an upper bound to the mass of the photon,

$$m_{\gamma_{\mathrm{exp}}} < 10^{-54}\mathrm{kg}.$$

Justifications for the massive photon trick

The MPD equations with $P^2 = {m_\gamma}^2$ and $SP = 0$ are,

$$\dot{X} = P - \frac{2\, SR(S)P}{4\, P^2 - R(S)(S)},$$
$$\dot{P} = -\frac{1}{2} R(S)\dot{X},$$
$$\dot{S} = P\overline{\dot{X}} - \dot{X}\overline{P}.$$

We recover the Souriau-Saturnini equations in the limit $m_\gamma \to 0$.

Photon such as $\mathbf{p} = (0, p_2, 0)$, we have,

$${p_2}^2 \gg {m_\gamma}^2 \gg R(S)(S)$$

The mass drops out in the direction of propagation we are interested in.

Equation of motion for the spinning photon

In the direction of propagation defined by **p** we have,

$$\frac{dx_2}{dt} = \underbrace{1 - \frac{\epsilon}{2}\cos(\omega(t-z))}_{\text{null geodesic}} \underbrace{-\frac{\epsilon}{2}\frac{\lambda_\gamma^2}{\lambda_{\mathrm{GW}}^2}\frac{s_1^2 - s_3(s_2+s_3))}{\hbar^2}\cos(\omega(t-z))}_{\text{Spin-GW interaction}}.$$

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With values taken from LIGO/Virgo, $\lambda_\gamma = 1064$ nm,

$$\frac{\epsilon}{2}\frac{\lambda_\gamma^2}{\lambda_{\mathrm{GW}}^2} \sim 10^{-46}$$

The time of flight depends on the polarization state of the photon, but the amplitude of the change is lower than second order geodesic terms in ϵ.

Conclusions

- Schwarzschild spacetime:
  - No contribution of the spin to radial propagation
  - Additional deviation angle for gravitational lensing, featuring birefringence and a rainbow effect
  - Theoretical prediction 8 orders of magnitude below experimental upper bound in 1974
- Gravitational wave background:
  - Additional time delay in time of flight in LIGO/Virgo detectors
  - Predictions 25 orders of magnitude below current experimental values
- A better interpretation of the classical transverse spin is needed