Notes on QFT and topology in two dimensions: $SL(2,R)$-invariance and the de Sitter universe

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May 5, 2019

Abstract

We study the covering of the 2-dim de Sitter universe.

1 Introduction

Quantum field theory on curved spacetime provides to date the most reliable access to study quantum effects in the presence of gravity. The procedure amounts to finding a metric solving Einstein's equations and then studying Quantum Fields in that background, possibly considering also the back-reaction of the fields on the metric in a semiclassical approach to quantum gravity. It is in this way that the Hawking effect and the spectrum of primordial perturbations are found and characterized.

There is however a caveat : by solving Einstein's equations one gets the local metric structure of a manifold but the global topological properties are not accessible in this way. And the global topological properties may and do play an important role in the interplay between quantization and causality. One very well known example of this status of affaires is the Anti-de Sitter manifold which has closed timelike curves. This problem is "treated" by passing to the universal covering of the manifold which allows for scalar theories of any mass to be considered.

Another well known quantum topological effect is the Aharonov - Bohm phenomenon: the presence of a solenoid makes the configuration space non-simply connected. An example of a similar nature that we consider in this paper is the two-dimensional de Sitter spacetime which is topologically non trivial in the spacelike directions. Why this manifold should be interesting at all? Two-dimensional models of quantum field theory have played and still play an important role as theoretical laboratories where to explore quantum phenomena that have then been recognized also in four dimensional realistic models, the best known being the Schwinger[1] [1] and the Thirring [2, 3] models.

By considering the same models formulated on the two-dimensional de Sitter universe dS_2 one may ask what are the features that survive on the curved manifold. This study may also throw a new light on some of the difficulties encountered in perturbation theory.

We have recently reconsidered the free de Sitter Dirac fields in two dimensions which already showed interesting new features [4]. The two-dimensional de Sitter manifold admits two inequivalent spin structures. Correspondingly, there are two distinct Dirac fields which may be either periodic (Ramond) or anti-periodic (Neveu-Schwarz) w.r.t. spatial rotations of an angle 2π. A requirement of de Sitter covariance (in a certain generalized sense) may be implemented at the quantum level only in the anti-periodic case [4, 5]. As a consequence the Thirring-de Sitter model admits covariant solutions [6] only in the antiperiodic case. The double covering of the de Sitter manifold $\widetilde{dS_2}$ naturally enters in the arena of soluble models of QFT through that door.

The manifold $\widetilde{dS_2}$ is in itself a complete globally hyperbolic manifold. It carries a natural action of $SL(2, R)$, the double covering of $SO_0(1, 2)$, the pseudo-orthogonal group that acts on dS_2. The Lorentzian geometry of $\widetilde{dS_2}$ is locally indistinguishable from that of dS_2 but the global properties are quite different. This fact has profound consequences at the quantum level. We present a few of them in this paper by considering the simplest possible model of quantum field theory, namely a free massive Klein-Gordon field. One of the above mentioned consequences, the most relevent from the physical viewpoint, is that moving from the de Sitter spacetime to its double covering makes the Hawking-Gibbons temperature disappear.

It is known that the thermal effects in QFT have to do with spacetimes possessing a bifurcate Killing horizon. Examples of such spacetimes include Minkowski spacetime, the extended Schwarzschild spacetime and de Sitter spacetime. Kay and Wald have proven a uniqueness theorem for such thermal states but there are counterexamples where the suitable geometrical structure does not imply the existence of the corresponding thermal state as in the Schwarzschild-de Sitter and in the Kerr cases. Our example is in a sense more peculiar: the double covering of the two-dimensional manifold is indistinguishable for a geodesic observer from the uncovered

[1]For example the Schwinger model, which corresponds to two-dimensional quantum electrodynamics, has allowed for the pre-discovery of some of the most important phenomena expected from quantum chromodynamics such as asymptotic freedom and confinement.

manifold. There is no classical experiment that he can do to determine whether he lives in the de Sitter universe or its double covering. Yet at the quantum level things are different and the global geometric structure of the double covering makes the $SL(2,R)$ invariance, locality and analyticity properties incompatible and this forbids the existence of the thermal radiation fro the horizons.

2 The de Sitter universe as a coset space.

Let us consider the two-dimensional de Sitter group $G = SO_0(1,2)$ which is the component connected to the identity of the pseudo-orthogonal Lorentz group acting on the three-dimensional Minkowski spacetime M_3 with metric $(+,-,-)$. The Iwasawa decomposition KNA of a generic element g of G is written as follows:

$$\begin{aligned} g &= k(\zeta)n(\lambda)a(\chi) = \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & \sin\zeta \\ 0 & -\sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} 1+\frac{\lambda^2}{2} & -\frac{\lambda^2}{2} & \lambda \\ \frac{\lambda^2}{2} & 1-\frac{\lambda^2}{2} & \lambda \\ \lambda & -\lambda & 1 \end{pmatrix} \begin{pmatrix} \operatorname{ch}\chi & \operatorname{sh}\chi & 0 \\ \operatorname{sh}\chi & \operatorname{ch}\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{1}$$

The above decomposition gives natural coordinates (λ,ζ) to points $x = x(\lambda,\zeta) = k(\zeta)n(\lambda)$ of the coset space G/A, which is seen to be topologically a cylinder. Here ζ is a real number mod 2π.

The group G acts on G/A by left multiplication: $x(\lambda',\zeta') = gx(\lambda,\zeta)$. The case of a rotation $k(\beta) \in K$ is of course the easiest to account for and amounts simply to a shift of the angle ζ: $\lambda' = \lambda$, $\zeta' = \zeta+\beta$, where both ζ and ζ' are real numbers mod 2π. The two other subgroups give rise to slightly more involved transformation rules which we do not reproduce here. However, by introducing the variables

$$u = \frac{\lambda + \operatorname{tg}\frac{\zeta}{2}}{1-\lambda\operatorname{tg}\frac{\zeta}{2}}, \qquad v = \cot\frac{\zeta}{2}, \tag{2}$$

the action of G admit an interesting simple form:

$$g = k(\alpha)n(\mu)a(\kappa) : u \to u' = \frac{(e^\kappa u+\mu)\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-(e^\kappa u+\mu)\sin\frac{\alpha}{2}}, \quad v \to v' = \frac{(e^\kappa v-\mu)\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+(e^\kappa v-\mu)\sin\frac{\alpha}{2}}. \tag{3}$$

G acts on the variables u and v by homographic transformations.

The simple geometrical interpretation of the above transformation rules may be unveiled by introducing the standard representation of dS_2 as a one-sheeted hyperboloid

$$dS_2 = \left\{ x \in M_3 : \; {x^0}^2 - {x^1}^2 - {x^2}^2 = -1 \right\}. \tag{4}$$

The coset space G/A can indeed be identified with dS_2 as follows: consider a vector whose components are the entries x_{02}, x_{12} and x_{22} of the third column of the matrix $x(\lambda, \zeta) = k(\zeta)n(\lambda)$. Obviously such vector belongs to dS_2 and is invariant by the right action of the subgroup A.

The so-defined map between G/A and dS_2 is a bijection. As a byproduct, the Iwasawa decomposition (1) gives natural global coordinates (λ, ζ) to points[2] $x = x(\lambda, \zeta)$ of the de Sitter hyperboloid:

$$x(\lambda, \zeta) = \begin{cases} x^0 = \lambda, \\ x^1 = \lambda \cos\zeta + \sin\zeta, \\ x^2 = \cos\zeta - \lambda \sin\zeta. \end{cases} \tag{5}$$

The left action of $SO_0(1,2)$ on the coset space G/A by construction coincides with the linear action of $SO_0(1,2)$ in M_3 restricted to the manifold dS_2:

$$x(\lambda', \zeta') = gx(\lambda, \zeta).$$

The base point (origin) $x(0,0) = (0,0,1)$ is invariant by the action of the subgroup A. Of course (5) supposes that we have chosen a certain Lorentz frame in M_3, and this frame will remain fixed in the sequel. Note that the Iwasawa coordinate system (λ, ζ) is not orthogonal:

$$\mathrm{d}s^2 = \left(\mathrm{d}x^{0^2} - \mathrm{d}x^{1^2} - \mathrm{d}x^{2^2}\right)\Big|_{dS_2} = -2\mathrm{d}\lambda \mathrm{d}\zeta - \left(\lambda^2 + 1\right)\mathrm{d}\zeta^2. \tag{6}$$

The variables u and v also have a simple geometric interpretation; they correspond to the two ratios that may be formed by factorizing the equation defining the de Sitter hyperboloid:

$$u = \frac{1 - x^2}{x^1 - x^0}, \quad v = \frac{1 + x^2}{x^1 - x^0}. \tag{7}$$

The above ratios transform "homographically" under the action of the group.

The complexification of the de Sitter manifold can then be equivalently identified either with the coset space G^c/A^c of the corresponding complexified groups or with the complex de Sitter hyperboloid

$$dS_2^c = \{z \in \mathbf{C}^3 : z^{0^2} - z^{1^2} - z^{2^2} = -1\}. \tag{8}$$

Particularly important subsets of dS_2^c are the forward and backward tubes, defined as follows:

$$\mathcal{T}^+ = \{z \in \mathbf{C}^3 : (\mathrm{Im}\, z^0)^2 - (\mathrm{Im}\, z^1)^2 - (\mathrm{Im}\, z^2)^2 > 0, \quad \mathrm{Im}\, z^0 > 0\}, \tag{9}$$

$$\mathcal{T}^- = \{z \in \mathbf{C}^3 : (\mathrm{Im}\, z^0)^2 - (\mathrm{Im}\, z^1)^2 - (\mathrm{Im}\, z^2)^2 > 0, \quad \mathrm{Im}\, z^0 < 0\}. \tag{10}$$

[2]We adopt the same letter x to denote points of the coset space G/A and of the de Sitter hyperboloid dS_2, as they are identified.

In the following we will make also use of the standard global orthogonal coordinate system:

$$x(t,\theta) = \begin{cases} x^0 = \operatorname{sh} t, \\ x^1 = \operatorname{ch} t \sin\theta, \\ x^2 = \operatorname{ch} t \cos\theta. \end{cases} \tag{11}$$

Here θ is a real number mod 2π. The relation between the two above coordinate system is quite simple:

$$\lambda = \operatorname{sh} t, \quad \operatorname{tg}\theta = \operatorname{tg}(\zeta + \arctan\lambda). \tag{12}$$

3 The double covering of the 2-dim de Sitter manifold as a coset space.

The easiest and most obvious way to describe the double covering $\widetilde{dS_2}$ of the two-dimensional de Sitter universe dS_2 consists in unfolding the periodic coordinate θ. More precisely, we may write the covering projection pr : $\widetilde{dS_2} \to dS_2$ as follows:

$$\operatorname{pr}(\tilde{x}(t,\theta)) \to x(t,\theta), \tag{13}$$

where we use the coordinates (t,θ) to parameterize also $\widetilde{dS_2}$; at the lhs θ is a real number mod 4π while at the rhs θ is a real number mod 2π.

In a more elaborate construction the double covering $\widetilde{dS_2}$ arises as a coset space of the double covering $\widetilde{G} = SL(2,R)$ of $SO_0(1,2)$. Let $\widetilde{G}^c = SL(2,\ \mathbf{C})$. An element of $\widetilde{G}^c$ is parametrised by four complex numbers a,b,c,d

$$\tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{14}$$

subject to the condition

$$\det\tilde{g} = ad - bc = 1.$$

For $\tilde{g} \in \widetilde{G}$ the formulae are the same but all entries are real. Let $\widetilde{A}^c$ be the complex subgroup of all 2×2 matrices of the form

$$\tilde{h}(r) = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}, \quad r \in \mathbf{C}, \quad r \neq 0\ . \tag{15}$$

$\widetilde{A}$ is the subgroup of $\widetilde{A}^c$ in which $r > 0$. Note that $\widetilde{A}^c \cap \widetilde{G} = \widetilde{A} \cup -\widetilde{A}$ does not coincide with $\widetilde{A}$. $Z_2 = \{1,\ -1\}$ is the common center of $\widetilde{G}^c$ and $\widetilde{G}$, and is contained in $\widetilde{A}^c$ (but not $\widetilde{A}$).

$\widetilde{G}$ (resp. $\widetilde{G}^c$) operates on the real (resp. complex) 3-dimensional Minkowski space M_3 (resp. $M_3^{(c)}$) by similarity :

$$x \to X = \left(\begin{array}{cc} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{array}\right), \quad X' = \left(\begin{array}{cc} x^{0'} + x^{1'} & x^{2'} \\ x^{2'} & x^{0'} - x^{1'} \end{array}\right) = \tilde{g} X \tilde{g}^T. \tag{16}$$

The real (resp. complex) de Sitter manifold is mapped into itself by the above action. In particular the subgroup $\widetilde{A}$ is seen to be the stability subgroup of the vector $(0,0,1)$ and the quotient $\widetilde{G}^c/\widetilde{A}^c$ can be identified with the complex de Sitter space. The real trace of the latter, i.e. $\widetilde{G}/(\widetilde{A}^c \cap \widetilde{G})$ can be identified to the real de Sitter space dS_2. On the other hand $\widetilde{G}/\widetilde{A}$ can be identified to the two-sheeted covering $\widetilde{dS_2}$ of dS_2. If $g \in \widetilde{G}^c$ and $r \neq 0$,

$$\tilde{g}\tilde{h}(r) = \left(\begin{array}{cc} ra & \frac{b}{r} \\ rc & \frac{d}{r} \end{array}\right) . \tag{17}$$

Let us first consider the case when $g \in \widetilde{G}$ and $r > 0$. Since $ad - bc = 1$, a and c cannot be both equal to 0, we can take $r = (a^2 + c^2)^{-1/2}$ and we get $a'^2 + c'^2 = 1$. Thus every coset $\tilde{g}A$ ($g \in \widetilde{G} = SL(2,R)$) contains exactly one element with this property, i.e. we can represent the double covering $\widetilde{dS_2}$ as the following real algebraic manifold

$$\widetilde{dS_2} = \widetilde{G}/\widetilde{A} \simeq \{(a,\ b,\ c,\ d) \in \mathbb{R}^4\ :\ ad - bc = 1,\ \ a^2 + c^2 = 1\} \tag{18}$$

which is the intersection of two quadrics in $\mathbb{R}^4$ and can be verified to have no singular point.

The elements representing the coset can be parametrized by using once more the Iwasawa decomposition of $\widetilde{G}$:

$$\tilde{g} = \tilde{k}(\zeta)\,\tilde{n}(\lambda)\,\tilde{a}(\chi) = \left(\begin{array}{cc} s\cos\frac{\zeta}{2} & \sin\frac{\zeta}{2} \\ -\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{array}\right)\left(\begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array}\right)\left(\begin{array}{cc} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{-\frac{\chi}{2}} \end{array}\right), \tag{19}$$

where λ and χ are real and ζ is a real number mod 4π. The parameters are related to a, b, c and d as follows:

$$\lambda = ab + cd, \quad r = e^{\frac{\chi}{2}} = \sqrt{a^2 + c^2}, \quad \cos\frac{\zeta}{2} = \frac{a}{\sqrt{a^2+c^2}}, \quad \sin\frac{\zeta}{2} = -\frac{c}{\sqrt{a^2+c^2}}. \tag{20}$$

The Iwasawa parametrization of the coset space $\tilde{G}/\tilde{A}$ is then

$$\tilde{x}(\lambda,\zeta) = \tilde{k}(\zeta)\,\tilde{n}(\lambda) = \left(\begin{array}{cc} \cos\frac{\zeta}{2} & \lambda\cos\frac{\zeta}{2} + \sin\frac{\zeta}{2} \\ -\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} - \lambda\sin\frac{\zeta}{2} \end{array}\right). \tag{21}$$

$\widetilde{G}$ acts on $\widetilde{G}/\widetilde{A}$ by left multiplication; the transformation rules of the parameters formally coincide with the previous ones with the only exception that rotations are now defined mod 4π.

The covering map pr $: \widetilde{dS_2} \to dS_2$ can be also represented as follows

$$\mathrm{pr}\,(\tilde{x}(\lambda,\zeta)) \to x(\lambda,\zeta) \tag{22}$$

where at the lhs ζ is a real number mod 4π while at the rhs ζ is a real number mod 2π.

The Maurer-Cartan form provides $\widetilde{G}/\widetilde{A}$ with a natural Lorentzian metric that may be constructed as follows: there is an inner automorphism of $\widetilde{G}$ leaving invariant the elements of the subgroup $\tilde{A}$:

$$\tilde{g} \to \mu(\tilde{g}) = -\gamma^2 \tilde{g} \gamma^2, \quad \gamma^2 = \left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right). \tag{23}$$

It may be used to construct a map from the coset space $\widetilde{G}/\widetilde{A}$ into the group $\widetilde{G}$:

$$\tilde{g}(\tilde{x}) = \tilde{x}\mu(\tilde{x})^{-1} = -\tilde{x}\gamma^2\tilde{x}^{-1}\gamma^2. \tag{24}$$

In turn, this map allows to introduce a left invariant Lorentzian metric on the coset space as follows:

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{Tr}(\mathrm{d}\tilde{g}\,\tilde{g}^{-1})^2 = -2\mathrm{d}\lambda\mathrm{d}\zeta - \left(\lambda^2+1\right)\mathrm{d}\zeta^2. \tag{25}$$

As expected, the metric coincides with the de Sitter metric (6); the only difference is that now the angular variable ζ is defined mod 4π.

The algebraic manifold (18) can be complexified, i.e. we can define

$$\mathcal{V} = \{(a,\ b,\ c,\ d) \in \mathbf{C}^4 \ : \ ad - bc = 1, \quad a^2 + c^2 = 1\} \ . \tag{26}$$

Let us again consider eq. (17) but now with complex $\tilde{g}$ and r. If $\tilde{g}$ is such that $a^2 + c^2 \neq 0$, we can choose $r = \pm(a^2+c^2)^{-1/2}$ and thus the coset $\tilde{g}\tilde{A}^c$ contains two distinct (opposite) elements with $a'^2 + c'^2 = 1$. If p, p' are points of $\mathcal{V}$ such that $p \neq \pm p'$ there is no $r \neq 0$ such that $p' = p\tilde{h}(r)$ hence p and p' belong to different elements of $\widetilde{G}^c/\widetilde{A}^c$. Conversely two opposite points p and $-p$ of the manifold $\mathcal{V}$ belong to the same coset $p\widetilde{A}^c$ i.e. determine a unique element of G^c/A^c. On the other hand if $\tilde{g}$ is such that $a^2+c^2 = 0$, all elements of the coset $\tilde{g}A^c$ have the same property and none belongs to $\mathcal{V}$. The cosets $\widetilde{g}A^c$ such that $a^2 + c^2 = 0$ can be identified to certain points of the complex de Sitter space as follows. Let $g = (a,\ b,\ c,\ d) \in G^c$. Then $x = g(0,\ 0,\ 1)$ is given by

$$\left(\begin{array}{cc} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{array} \right) = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left(\begin{array}{cc} a & c \\ b & d \end{array} \right) = \left(\begin{array}{cc} 2ab & ad+bc \\ ad+bc & 2cd \end{array} \right) . \tag{27}$$

(The fact that the determinant of the lhs is equal to -1 expresses $x \in dS_2^{(c)}$.) a and c cannot be both 0. Supposing $c \neq 0$ we get

$$\frac{a}{c} = \frac{x^0 + x^1}{x^2 - 1} = \frac{x^2 + 1}{x^0 - x^1} \ . \tag{28}$$

The condition $a^2+c^2=0$ is equivalent to $a=\pm ic$ (thus excluding $c=0$) and implies

$$x^0+x^1=\pm i(x^2-1) \Longrightarrow x^0-x^1=\mp i(x^2+1)\ . \tag{29}$$

Conversely one of these conditions implies $a^2+c^2=0$. Therefore

$$a^2+c^2=0 \Longleftrightarrow (x^0+x^1)^2+(x^2-1)^2=0 \Longleftrightarrow (x^0-x^1)^2+(x^2+1)^2=0\ . \tag{30}$$

It follows that $\mathcal{V}$ projects onto $dS_2^{(c)}$ with the exception of the manifold $\mathcal{N}$ defined by the above equations. Note that the points $(\pm i,\ 0,\ 0)$ belong to $\mathcal{N}$. Of course the manifold $\mathcal{N}$ is not invariant under the action of $\widetilde{G}$ or $\widetilde{G}^c$.

Note that $\widetilde{G}^c$ acts transitively on $dS_2^{(c)}$, i.e. given $(x^0,\ x^1,\ x^2)\in \mathbf{C}^3$ satisfying ${x^0}^2-{x^1}^2-{x^2}^2=-1$, there is a $g=(a,\ b,\ c,\ d)$ such that $ad-bc=1$ and eq. (27) holds.

There are other complex manifolds that contain $\widetilde{dS_2}$ as a real form. For example let us represent $\widetilde{dS_2}$ as a cylinder $\mathbb{R}\times S^1$ as follows: a point is associated to a pair $(x^0,\ \theta)$ where $x^0\in\mathbb{R}$ and $\theta\in\mathbb{R}/4\pi\mathbb{Z}$. It projects on the point

$$\begin{pmatrix} x^0 \\ \sqrt{{x^0}^2+1}\cos\theta \\ \sqrt{{x^0}^2+1}\sin\theta \end{pmatrix} \in dS_2\ . \tag{31}$$

The natural complexification of this is $\Sigma\times\mathbf{C}/4\pi\mathbb{Z}$, where Σ is the Riemann surface of $z\mapsto \sqrt{z^2+1}$, a two-sheeted covering of $\mathbf{C}\setminus\{i,\ -i\}$. This complex manifold projects onto $dS_2^{(c)}$ with the exception of the points such that $({x^0}^2+1)=0$, or equivalently $({x^1}^2+{x^2}^2)=0$.

4 More about the covering projection

The group $\widetilde{G}=SL(2,\ \mathbb{R})$ acts on the covering space $\widetilde{dS_2}$ as a group of *spacetime transformations* by left multiplication $\tilde{x}\to\tilde{g}\tilde{x}$ and acts on dS_2 by similarity (**??**). We denote both actions by the shortcut $(\cdot)\to\tilde{g}(\cdot)$. They commute with the covering projection (13):

$$\mathrm{pr}\,(\tilde{g}\tilde{x})=\tilde{g}\,\mathrm{pr}\,(\tilde{x}) \quad \forall\tilde{g}\in\widetilde{G},\ \ \forall\tilde{x}\in\widetilde{dS_2}\ . \tag{32}$$

On the de Sitter manifold dS_2 the antipodal map $x\mapsto -x$ is expressed in the coordinates (t,θ) by $x(t,\theta)\mapsto -x(t,\theta)=x(-t,\theta+\pi)$. Let τ_1 be the operation with the same expression on the covering manifold $\widetilde{dS_2}$, i.e.

$$\tau_1\tilde{x}(t,\theta)=\tilde{x}(-t,\theta+\pi), \quad \tilde{x}\in\widetilde{dS_2} \tag{33}$$

and let τ

$$\tau\tilde{x}(t,\ \theta) = \tilde{x}(t, \theta + 2\pi), \quad \tilde{x} \in \widetilde{dS_2}. \tag{34}$$

Obviously τ_1 is a diffeomorphism of the covering manifold; we have that

$$\tau = \tau_1^2, \quad \mathrm{pr}\,(\tau_1\tilde{x}) = -\mathrm{pr}\,(\tilde{x}), \quad \mathrm{pr}\,(\tau\tilde{x}) = \mathrm{pr}\,(\tilde{x})\,. \tag{35}$$

The following lemma proves that τ_1 commutes with the action of the group $\tilde{G}$.

Lemma 4.1 *Let φ (resp. $\widetilde{\varphi}$) be a continuous map of dS_2 (resp. $\widetilde{dS_2}$) into itself such that $\mathrm{pr}\,\widetilde{\varphi}(\tilde{x}) = \varphi(\mathrm{pr}\,\tilde{x})$ for all $\tilde{x} \in \widetilde{dS_2}$. Suppose that, for every $y \in dS_2$ and every $g \in G$, $\varphi(gy) = g\varphi(y)$. Then, for every $\tilde{x} \in \widetilde{dS_2}$ and every $\tilde{g} \in \widetilde{G}$, $\widetilde{\varphi}(g\tilde{x}) = \tilde{g}\widetilde{\varphi}(\tilde{x})$.*

Proof. Let $\widetilde{x} \in \widetilde{dS_2}$. Let B be the set of all $g \in \widetilde{G}$ such that $\widetilde{\varphi}(g\widetilde{x}) = g\widetilde{\varphi}(\widetilde{x})$. B contains the identity and is obviously closed. Let $g \in B$ and let V be an open neighborhood of $\widetilde{\varphi}(g\widetilde{x}) = g\widetilde{\varphi}(\widetilde{x})$ such that pr is a diffeomorphism of V onto pr V. Let W be an open neighborhood of g in $\widetilde{G}$ such that $h\widetilde{\varphi}(\widetilde{x}) \in V$ and $\widetilde{\varphi}(h\widetilde{x}) \in V$ for all h in W. For any $h \in \widetilde{G}$,

$$\mathrm{pr}\,h\widetilde{\varphi}(\widetilde{x}) = h\mathrm{pr}\,\widetilde{\varphi}(\widetilde{x}) = h\varphi(\mathrm{pr}\,\widetilde{x}) = \varphi(\mathrm{pr}\,h\widetilde{x}) = \mathrm{pr}\,\widetilde{\varphi}(h\widetilde{x})\ . \tag{36}$$

Since pr is a diffeomorphism on V, $h \in W$ implies that $\widetilde{\varphi}(h\widetilde{x}) = h\widetilde{\varphi}(\widetilde{x})$, i.e. $h \in B$, i.e. $W \subset B$. Thus B is open and must coincide with $\widetilde{G}$. If we take $\widetilde{\varphi}(\widetilde{x}) = \tau_1\widetilde{x}$ and $\varphi(x) = -x$ we obtain $g\tau_1 x = \tau_1 g x$ for all $g \in H$.

We can also define τ_2 by

$$\tau_2\tilde{x}(t,\ \theta) = \tilde{x}(-t, \theta - \pi), \tag{37}$$

It satisfies $\tau_2\tau_1 = \tau_1\tau_2 = 1$ and also commutes with the action of H. It follows that $\tau = \tau_1^2 = \tau_2^2$ also commutes with the action of H.

5 Quantum field theory on the 2-dim dS universe vs its double covering

In the spherical coordinate system (both on dS_2 and on $\widetilde{dS_2}$) the de Sitter Klein-Gordon equation takes the form

$$\Box\phi - \lambda(\lambda+1)\phi = \frac{1}{\mathrm{ch}\,t}\ \partial_t(\mathrm{ch}\,t\ \partial_t\phi) - \frac{1}{\mathrm{ch}^2\,t}\ \partial_\theta^2\phi - \lambda(\lambda+1)\phi = 0. \tag{38}$$

The parameter λ and, consequently, the squared mass

$$m_\lambda^2 = -\lambda(\lambda+1) \tag{39}$$

are complex numbers; m_λ^2 is real and positive in the following special cases:

$$\text{either} \qquad \lambda = -\frac{1}{2} + i\rho, \quad \operatorname{Im}\rho = 0, \quad m = \sqrt{\frac{1}{4} + \rho^2} \geq \frac{1}{2}, \tag{40}$$

$$\text{or} \qquad \operatorname{Im}\lambda = 0, \quad -1 < \operatorname{Re}\lambda < 0, \quad 0 < m < \frac{1}{2}. \tag{41}$$

Let us introduce the complex variable $z = i \operatorname{sh} t$, so that $1 - z^2 = \operatorname{ch}^2 t$, and separate the variables by posing

$$\phi = f(z) e^{il\theta} \tag{42}$$

Eq. (38) implies that f has to solve the Legendre differential equation:

$$(1 - z^2) f''(z) - 2z f'(z) + \lambda(\lambda + 1) f(z) - \frac{l^2}{(1 - z^2)} f(z) = 0. \tag{43}$$

All the difference between dS_2 and its covering $\widetilde{dS_2}$ is that in the first case l is an integer number while in the second case $2l$ is integer. Enlarging the set of possible values of l in this way will cause many unexpected (and dramatic!) new features. We will describe some of them below.

Two linearly independent[3] solutions of the above equation are the Ferrers functions (also called "Legendre functions on the cut") $\mathbf{P}_\nu^\mu(z)$ and $\mathbf{Q}_\nu^\mu(z)$, where

$$\nu = \lambda, \ \mu = -l.$$

$\mathbf{P}_\nu^\mu(z)$ and $\mathbf{Q}_\nu^\mu(z)$ are holomorphic in the cut-plane

$$\Delta_2 = \mathbf{C} \setminus (-\infty - 1] \cup [1, \infty). \tag{47}$$

[3] The following formulae are useful to compute the various Wronskians $\mathcal{W}\{w_1, w_2\} = w_1(z) w_2'(z) - w_2(z) w_1'(z)$ needed:

$$\mathbf{P}_\nu^\mu(0) = \frac{2^{\mu+1} \sin\left(\frac{1}{2}\pi(\mu + \nu)\right) \Gamma\left(\frac{1}{2}(\mu + \nu + 2)\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(-\mu + \nu + 1)\right)}, \tag{44}$$

$$\mathbf{P'}_\nu^\mu(0) = \frac{2^{\mu} \cos\left(\frac{1}{2}\pi(\mu + \nu)\right) \Gamma\left(\frac{1}{2}(\mu + \nu + 1)\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(-\mu + \nu + 2)\right)}. \tag{45}$$

where $\mathbf{P}'(z) = \frac{d\mathbf{P}}{dz}$. We get

$$\begin{aligned}
\mathcal{W}\{\mathbf{P}_\nu^\mu(i \operatorname{sh} t), \mathbf{P}_\nu^\mu(-i \operatorname{sh} t)\} &= \frac{2}{\Gamma(-\mu - \nu)\Gamma(-\mu + \nu + 1)} (\operatorname{ch} t)^{-2} \\
\mathcal{W}\{\mathbf{P}_\nu^\mu(i \operatorname{sh} t), \mathbf{P}_\nu^{-\mu}(i \operatorname{sh} t)\} &= -\frac{2\sin(\pi\mu)}{\pi} (\operatorname{ch} t)^{-2} \\
\mathcal{W}\{\mathbf{P}_\nu^\mu(-i \operatorname{sh} t), \mathbf{P}_\nu^{-\mu}(-i \operatorname{sh} t)\} &= \frac{2\sin(\pi\mu)}{\pi} (\operatorname{ch} t)^{-2} \\
\mathcal{W}\{\mathbf{P}_\nu^\mu(i \operatorname{sh} t), \mathbf{P}_\nu^{-\mu}(-i \operatorname{sh} t)\} &= -\frac{2\sin(\pi\nu)}{\pi} (\operatorname{ch} t)^{-2}
\end{aligned} \tag{46}$$

and satisfy the reality conditions

$$\overline{\mathbf{P}^{\mu}_{\nu}(z)} = \mathbf{P}^{\overline{\mu}}_{\overline{\nu}}(\bar{z}), \quad \overline{\mathbf{Q}^{\mu}_{\nu}(z)} = \mathbf{Q}^{\overline{\mu}}_{\overline{\nu}}(\bar{z}) \tag{48}$$

for all $z \in \Delta_2$. $\mathbf{P}^{\mu}_{\nu}(z)$ respects the symmetry (39) of the mass squared: for all $z \in \Delta_2$ it satisfies the identity [47]

$$\mathbf{P}^{-l}_{\lambda}(z) = \mathbf{P}^{-l}_{-\lambda-1}(z). \tag{49}$$

If $\lambda - l$ and $\lambda + l - 1$ are not non-negative integers, $\mathbf{P}^{-l}_{\lambda}(z)$ and $\mathbf{P}^{-l}_{\lambda}(-z)$ also constitute two linearly independent solutions of Eq. (43). In this case the general solution has the form

$$\phi_l(t,\theta) = [a_l\, \mathbf{P}^{-l}_{\lambda}(i \operatorname{sh} t) + b_l\, \mathbf{P}^{-l}_{\lambda}(-i \operatorname{sh} t)]\ e^{il\theta}. \tag{50}$$

In the following we will restrict our attention to values of λ such that

$$\operatorname{Re}\lambda + \frac{1}{2} \geq 0 \tag{51}$$

i.e. we do not consider here tachyon fields.

6 Canonical commutation relations.

Let us focus on the modes

$$\begin{aligned} \phi_l(t,\theta) &= [a_l \mathbf{P}^{-l}_{\lambda}(i \operatorname{sh} t) + b_l \mathbf{P}^{-l}_{\lambda}(-i \operatorname{sh} t)] e^{il\theta} \\ \phi^*_l(t,\theta) &= [a^*_l \mathbf{P}^{-l}_{\lambda}(-i \operatorname{sh} t) + b^*_l \mathbf{P}^{-l}_{\lambda}(i \operatorname{sh} t)] e^{-il\theta} \end{aligned} \tag{52}$$

where either $\lambda = -1/2 + i\nu$ or λ real. The KG product is defined as usual:

$$(f,g)_{KG} = i \int_{\Sigma} (f^* \partial_\mu g - g\, \partial_\mu f^*) d\Sigma^\mu(x) = i \int_{\Sigma} (f^* \partial_t g - g\, \partial_t f^*) d\theta \tag{53}$$

On dS_2 the integral is over the interval $\Sigma = [0, 2\pi]$ and l is integer. When we consider fields on the covering manifold $\widetilde{dS_2}$ the integral is over the interval $\Sigma = [0, 4\pi]$ and $2l$ is integer.

The first condition imposed by the canonical quantization procedure is the orthogonality $(\phi^*_l, \phi_{l'})_{KG} = 0$ of the modes; it gives rise to the following conditions on the coefficients:

$$a_l b_{-l} - b_l a_{-l} = 0 \quad \text{for } l \in \mathbb{Z} \text{ (i.e. on both } dS_2 \text{ and } \widetilde{dS_2}), \tag{54}$$

$$a_l a_{-l} - b_l b_{-l} = c_l \sin(\pi\lambda), \quad a_l b_{-l} - b_l a_{-l} = c_l \sin(\pi l) \quad \text{for } l \in \tfrac{1}{2} + \mathbb{Z} \text{ (only on } \widetilde{dS_2}). \tag{55}$$

The constants c_l are unrestricted by the above condition. Both conditions are summarized as follows:

$$a_{-l} = c_l(a_l \sin(\pi\lambda) + b_l \sin(\pi l)), \quad b_{-l} = c_l(b_l \sin(\pi\lambda) + a_l \sin(\pi l)). \tag{56}$$

The normalization condition is given by

$$(\phi_l, \phi_{l'})_{KG} = \frac{2k\pi}{\gamma_l} \left(|a_l|^2 - |b_l|^2\right) \delta_{ll'} = \frac{1}{N_l} \delta_{ll'} \tag{57}$$

where $k = 1$ for dS_2 and $k = 2$ for $\widetilde{dS_2}$ and

$$\gamma_l = \frac{1}{2} \Gamma(l - \lambda) \Gamma(1 + \lambda + l) \tag{58}$$

so that

$$N_l = \frac{\gamma_l}{2k\pi(|a_l|^2 - |b_l|^2)} = 1. \tag{59}$$

As a function of l the product γ_l is always positive for $\lambda = -\frac{1}{2} + i\nu$. When $-1 < \lambda < 0$ it takes negative values for negative half integer l's.

The commutator finally takes the following form:

$$\begin{aligned} C(t,\theta,t',\theta') &= \sum_{kl \in \mathbb{Z}} N_l [\phi_l(t,\theta)\phi_l^*(t',\theta') - \phi_l(t',\theta')\phi_l^*(t,\theta)] = \\ &= \sum_{kl \in \mathbb{Z}} N_l(|a_l|^2 - |b_l|^2)[\mathbf{P}_\lambda^{-l}(i \operatorname{sh} t)\mathbf{P}_\lambda^{-l}(-i \operatorname{sh} t') - \mathbf{P}_\lambda^{-l}(-i \operatorname{sh} t)\mathbf{P}_\lambda^{-l}(i \operatorname{sh} t')] \cos(l\theta - l\theta') \\ &+ \sum_{kl \in \mathbb{Z}} iN_l(|a_l|^2 + |b_l|^2)[\mathbf{P}_\lambda^{-l}(i \operatorname{sh} t)\mathbf{P}_\lambda^{-l}(-i \operatorname{sh} t') + \mathbf{P}_\lambda^{-l}(-i \operatorname{sh} t)\mathbf{P}_\lambda^{-l}(i \operatorname{sh} t')] \sin(l\theta - l\theta') \\ &+ \sum_{kl \in \mathbb{Z}} [2iN_l a_l b_l^* \mathbf{P}_\lambda^{-l}(i \operatorname{sh} t)\mathbf{P}_\lambda^{-l}(i \operatorname{sh} t') + 2iN_l a_l^* b_l \mathbf{P}_\lambda^{-l}(-i \operatorname{sh} t)\mathbf{P}_\lambda^{-l}(-i \operatorname{sh} t')] \sin(l\theta - l\theta'). \end{aligned} \tag{60}$$

where, again, $k = 1$ for dS_2 and $k = 2$ for $\widetilde{dS_2}$. We left in this expression explicitly indicated N_l as a function of a_l and b_l, as in Eq. (59), even though the normalization condition imposes $N_l = 1$. This allows to verify the locality property of the above expression more easily. Let us indeed verify that the equal time commutator

$$C(0,\theta,0,\theta') = 2i \sum_{kl \in \mathbb{Z}} \frac{\gamma_l(|a_l + b_l|^2)}{2k\pi\,(|a_l|^2 - |b_l|^2)} [\mathbf{P}_\lambda^{-l}(0)]^2 \sin(l\theta - l\theta') \tag{61}$$

vanishes. The terms contributing to $C(0,\theta,0,\theta')$ are the ones antisymmetric in the exchange of θ and θ' (the second and third line in Eq. (60)). By using Eq. (56) we have that

$$\frac{|a_l + b_l|^2}{|a_l|^2 - |b_l|^2} = \cot\left(\frac{1}{2}\pi(l + \lambda)\right) \operatorname{tg}\left(\frac{1}{2}\pi(\lambda - l)\right) \frac{|a_{-l} + b_{-l}|^2}{|a_{-l}|^2 - |b_{-l}|^2} \tag{62}$$

In deriving the above identity we took in to account the hypothesis $\lambda = -1/2 + i\nu$ or λ real, which implies that $\sin \pi\lambda$ is a real number.

On the other hand formula (44) gives

$$\frac{\gamma_l \mathbf{P}_\lambda^{-l}(0)^2}{\gamma_{-l}\mathbf{P}_\lambda^{l}(0)^2} = \mathrm{tg}\left(\frac{1}{2}\pi(\lambda+l)\right)\cot\left(\frac{1}{2}\pi(\lambda-l)\right). \qquad (63)$$

Therefore the coefficients of $\sin[l(\theta-\theta')]$ and of $\sin[-l(\theta-\theta')]$ are equal and the equal time commutator $C(0,\theta,0,\theta')$ vanishes.

Let us verify now the CCR's:

$$\begin{aligned}
\partial_{t'}C(t,\theta,t,\theta')|_{t=t'=0} &= -2i\sum_{kl\in\mathbb{Z}}\frac{\gamma_l}{2k\pi}[\mathbf{P}_\lambda^{-l}(0)\mathbf{P'}_\lambda^{-l}(0)]\cos(l\theta-l\theta')+\\
+2i\sum_{kl\in\mathbb{Z}}\sum\frac{\gamma_l}{2k\pi(|a_l|^2-|b_l|^2)}&(a_l b_l^*-a_l^* b_l)[\mathbf{P}_\lambda^{-l}(0)\mathbf{P'}_\lambda^{-l}(0)]\sin(l\theta-l\theta') =\\
= i\sum_{kl\in\mathbb{Z}}\frac{1}{2k\pi}\cos(l\theta-l\theta') &= i\delta(\theta-\theta')
\end{aligned} \qquad (64)$$

where we used again Eq. (56) As a byproduct we deduce that the second and third line in Eq. (60) vanish identically and the covariant commutator may be re-expressed as follows:

$$\begin{aligned}
C(t,\theta,t',\theta') &= \sum_{kl\in\mathbb{Z}}\frac{\gamma_l}{2k\pi}[\mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}\,t)\mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}\,t')-\mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}\,t)\mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}\,t')]\cos(l\theta-l\theta')\\
&= \sum_{kl\in\mathbb{Z}}\frac{\gamma_l}{2k\pi}[\mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}\,t)\mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}\,t')-\mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}\,t)\mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}\,t')]\exp(il\theta-il\theta').
\end{aligned} \qquad (65)$$

The second step follows from the symmetry of the generic term of the series at the right hand side of Eq. (65) under the change $l\to -l$.

7 SL(2,R)-invariance of the commutator

While the $SL(2,R)$ invariance of the commutator is (somehow) a priori guaranteed by the vanishing of the equal time commutator and by the CCR's (64), it is instructive for what follows to give a direct proof based on the recurrence relations satisfied by the Legendre functions on the cut. This will prepare the general proof of the following Section 8, where the question of finding the more general invariant two-point function will be addressed.

To this aim, let us start by considering the first term at the RHS of Eq. (65) :

$$W_0(\tilde{x},\tilde{x}') = \frac{1}{4\pi}\sum_{2l\in\mathbb{Z}}\gamma_l\left[\mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}\,t)\mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}\,t')\right]\exp(il(\theta-\theta')). \qquad (66)$$

This kernel is non local[4] but it turns out be $SL(2,R)$-invariant. The proof of this statement

[4] Note that the partial sum over the integers provides a local kernel on the de Sitter manifold which is precisely the Bunch-Davies two point function. We stress again that locality on the de Sitter manifold and on its covering are two distinct notions.

amounts to checking that the following infinitesimal condition holds:

$$\delta W = \sin\theta\,\partial_t W + \mathrm{th}\,t\,\cos\theta\,\partial_\theta W + \sin\theta'\,\partial_{t'} W + \mathrm{th}\,t'\,\cos\theta'\,\partial_{\theta'} W = 0 \tag{67}$$

namely

$$\begin{aligned}\delta W_0 \;=\; & \sum_l \gamma_l \left[i\sin\theta\,\mathrm{ch}\,t\,\mathbf{P'}^{-l}_\lambda(z) + il\,\mathrm{th}\,t\,\cos\theta\,\mathbf{P}^{-l}_\lambda(z)\right]\mathbf{P}^{-l}_\lambda(-z')\,e^{il(\theta-\theta')} \\ & -\sum_l \gamma_l \mathbf{P}^{-l}_\lambda(z)\left[i\sin\theta'\,\mathrm{ch}\,t'\,\mathbf{P'}^{-l}_\lambda(-z') + il\,\mathrm{th}\,t'\,\cos\theta'\,\mathbf{P}^{-l}_\lambda(-z')\right]e^{il(\theta-\theta')} = 0. \end{aligned} \tag{68}$$

Singling out the Fourier coefficient of $\exp(il\theta)$, the above condition translates into the following requirement:

$$\begin{aligned} & \gamma_{l-1}\mathbf{P}^{1-l}_\lambda(-z')e^{i\theta'}\;\left[\mathrm{ch}\,t\,\mathbf{P'}^{1-l}_\lambda(z) + i(l-1)\,\mathrm{th}\,t\;\mathbf{P}^{1-l}_\lambda(z)\right] + \\ + \;& \gamma_{l+1}\;\mathbf{P}^{-1-l}_\lambda(-z')e^{-i\theta'}\left[-\,\mathrm{ch}\,t\,\mathbf{P'}^{-1-l}_\lambda(z) + i(l+1)\,\mathrm{th}\,t\;\mathbf{P}^{-1-l}_\lambda(z)\right] + \\ - \;& 2\gamma_l\mathbf{P}^{-l}_\lambda(z)\left[i\sin\theta'\,\mathrm{ch}\,t'\,\mathbf{P'}^{-l}_\lambda(-z') + il\,\mathrm{th}\,t'\,\cos\theta'\,\mathbf{P}^{-l}_\lambda(-z')\right] = 0. \end{aligned} \tag{69}$$

This expression may be simplified by using the following crucial identities:

$$\begin{aligned} & \mathrm{ch}\,t\,\mathbf{P'}^{1-l}_\lambda(z) + i(l-1)\,\mathrm{th}\,t\;\mathbf{P}^{1-l}_\lambda(z) \\ = \;& (\lambda-l+1)(\lambda+l)\left[-\,\mathrm{ch}\,t\,\mathbf{P'}^{-1-l}_\lambda(z) + i(l+1)\,\mathrm{th}\,t\;\mathbf{P}^{-1-l}_\lambda(z)\right] && (70) \\ = \;& (\lambda-l+1)(\lambda+l)\mathbf{P}^{-l}_\lambda(z). && (71) \end{aligned}$$

It takes a little work to verify that the above formulae are nothing but a rewriting of known relations among the Legendre functions. To prove Eq. (70) one first removes the derivative $\mathbf{P'} = \frac{d\mathbf{P}}{dz}$ by using Eq. 3.8.19 from Bateman's book [33] and get

$$\begin{aligned} & i(\lambda-l+1)\,\mathrm{sh}\,t\;\mathbf{P}^{1-l}_\lambda(z) + (\lambda-l+1)\mathbf{P}^{1-l}_{\lambda-1}(z) \\ & -(\lambda-l+1)(\lambda+l)\left[-i\,\mathrm{sh}\,t\,(\lambda+l+1)\mathbf{P}^{-1-l}_\lambda(z) + (\lambda-l-1)\mathbf{P}^{-1-l}_{\lambda-1}(z)\right] = 0. \end{aligned} \tag{72}$$

Eqs. 3.8.11 and 3.8.15 from Bateman's book allow to show that (72) is equivalent to

$$\mathbf{P}^{1-l}_\lambda(z) - 2l\,i\,\mathrm{th}\,t\,\mathbf{P}^{-l}_\lambda(z) + (\lambda-l)\,(\lambda+l+1)\mathbf{P}^{-1-l}_\lambda(z) = 0 \tag{73}$$

which in turn coincides with Bateman's Eq. 3.8.11. To prove the second equality (71) one invokes Bateman's Eqs. 3.8.17 and 3.8.19.

Now we are ready to show the $SL(2,R)$-invariance of the kernel (66). Let us insert Eqs. (70) and (71) in Eq. (69) and divide by γ_{l-1}; we get the following equivalent expression

$$[e^{i\theta'}\;\mathbf{P}^{1-l}_\lambda(-z') - e^{-i\theta'}(l-\lambda)(\lambda+l+1)\;\mathbf{P}^{-1-l}_\lambda(-z')]+$$

$$+2\left[i\sin\theta'\,\mathrm{ch}\,t'\,\mathbf{P}'^{-l}_{\lambda}(-z') + il\,\mathrm{th}\,t'\,\cos\theta'\,\mathbf{P}^{-l}_{\lambda}(-z')\right] = 0. \tag{74}$$

Here the variable z has disappeared and the condition (69) is now tractable. By singling out the coefficients of $\cos\theta'$ and $\sin\theta'$ we are led to examine the validity of the following identities:

$$\mathbf{P}^{1-l}_{\lambda}(-z') - (l-\lambda)(\lambda+l+1)\,\mathbf{P}^{-1-l}_{\lambda}(-z') + 2il\,\mathrm{th}\,t'\,\mathbf{P}^{-l}_{\lambda}(-z') = 0, \tag{75}$$

$$\mathbf{P}^{1-l}_{\lambda}(-z') + (l-\lambda)(\lambda+l+1)\,\mathbf{P}^{-1-l}_{\lambda}(-z') + 2\,\mathrm{ch}\,t'\,\mathbf{P}'^{-l}_{\lambda}(-z') = 0. \tag{76}$$

Eq. (75) it is once more a known relationship among Legendre functions on the cut, namely Eq. 3.8.11 of Bateman's book. As regards the second identity, it can be proven by observing that the difference of the above two equations coincides with the relation given in Eq. (70). The $SL(2,R)$-invariance of the kernel (66) is proven.

An immediate corollary is that the kernels obtained by taking the even and the odd parts of $W_0(\tilde{x},\tilde{x}')$, namely

$$W_{0,even}(\tilde{x},\tilde{x}') = \frac{1}{4\pi}\sum_{l\in\mathbb{Z}}\gamma_l\left[\mathbf{P}^{-l}_{\lambda}(z)\mathbf{P}^{-l}_{\lambda}(-z')\right]\exp(il(\theta-\theta')) \tag{77}$$

and

$$W_{0,odd}(\tilde{x},\tilde{x}') = \frac{1}{4\pi}\sum_{\frac{1}{2}+l\in\mathbb{Z}}\gamma_l\left[\mathbf{P}^{-l}_{\lambda}(z)\mathbf{P}^{-l}_{\lambda}(-z')\right]\exp(il(\theta-\theta')) \tag{78}$$

are separately invariant; this follows from Eqs. (75) and (76).

The second corollary is the invariance of the commutator. Let us consider indeed the map τ_1 given in Eq. (116). Since it commutes with the action of $SL(2,R)$ on the covering manifold $\widetilde{dS_2}$ we immediately get that also the kernels

$$W_1(\tilde{x},\tilde{x}') = \sum_{2l\in\mathbb{Z}}\gamma_l\left[\mathbf{P}^{-l}_{\lambda}(i\,\mathrm{sh}\,t)\mathbf{P}^{-l}_{\lambda}(-i\,\mathrm{sh}\,t')\right]\exp(-il(\theta-\theta')). \tag{79}$$

is $SL(2,R)$ invariant. The invariance of the commutator follows.

8 Invariance under $SL(2,R)$ and other properties of general two-point functions

Once given the commutator, the crucial step to get a physical model is to represent the field ϕ as an operator-valued distribution in a Hilbert space $\mathcal{H}$. This can be done by finding a positive-semidefinite bivariate distribution $W(\tilde{x},\tilde{y})$ solving the KG equation and the functional equation

$$C(\tilde{x},\tilde{x}') = W(\tilde{x},\tilde{x}') - W(\tilde{x}',\tilde{x}). \tag{80}$$

Actually, C and W are not functions but distributions so the above equation must be understood in the sense of distributions. Given a solution W, the Gelfand-Naimark-Segal (GNS) procedure provides the Fock space of the theory and a representation of the field as a local operator-valued distribution (the word local here refers to *local commutativity*).

There are of course infinitely many inequivalent solution of Eq. (80). Here we will characterize the most general $SL(2,R)$–invariant solution.

To this aim let us first consider a general two-point function, i.e. a distribution W on $\widetilde{dS_2} \times \widetilde{dS_2}$. There are several conditions that we may (or may not) want to impose on such a function.

1. Local commutativity (locality): Let

$$\mathcal{R} = \{(\tilde{x},\ \tilde{x}') \in \widetilde{dS_2} \times \widetilde{dS_2}\ :\ \tilde{x} \text{ and } \tilde{x}' \text{ are spacelike separated}\}. \tag{81}$$

   $W(\tilde{x},\tilde{x}')$ has the property of local commutativity (or locality) if

$$W(\tilde{x},\tilde{x}') - W(\tilde{x}',\tilde{x}) = 0 \quad \forall(\tilde{x},\tilde{x}') \in \mathcal{R}\ . \tag{82}$$

2. Symmetry or anti-symmetry:

$$W(\tilde{x},\tilde{x}') = \pm W(\tilde{x}',\tilde{x}) \quad \forall(\tilde{x},\tilde{x}') \in \widetilde{dS_2} \times \widetilde{dS_2}. \tag{83}$$

3. Invariance under the group $SL(2,R)$:

$$W(\tilde{g}\tilde{x},\tilde{g}\tilde{x}') = W(\tilde{x},\tilde{x}') \quad \forall(\tilde{x},\tilde{x}') \in \widetilde{dS_2} \times \widetilde{dS_2}.\ \ \forall \tilde{g} \in SL(2,R). \tag{84}$$

4. Hermiticity:

$$W(\tilde{x},\tilde{x}') = \overline{W(\tilde{x}',\tilde{x})} \quad \forall(\tilde{x},\tilde{x}') \in \widetilde{dS_2} \times \widetilde{dS_2}. \tag{85}$$

5. Positive definiteness:

$$\int\int W(\tilde{x},\tilde{x}')\bar{f}(\tilde{x})f(\tilde{x}')d\tilde{x}d\tilde{x}' \geq 0 \qquad \forall f \in \mathcal{C}_0^\infty(\widetilde{dS_2}). \tag{86}$$

6. Klein-Gordon equation in $\tilde{x}$ and $\tilde{x}'$ with a "mass" λ.

7. Canonical Commutation Relations (80).

8. Analyticity. By this we mean that there is an open tuboid[5] $\mathcal{U}_+$ in the complexified version of $\widetilde{dS_2} \times \widetilde{dS_2}$ such that, in a neighborhood of any real point $(x,\ x') \in \widetilde{dS_2} \times \widetilde{dS_2}$ we have, in the sense of distributions,

$$F(x,\ x') = \lim_{(w,\ w')\in\mathcal{U}_+\ ,\ (w,\ w')\to(x,\ x')} F_+(w,\ ,w')\ , \tag{87}$$

where F_+ is holomorphic with locally polynomial behavior in $\mathcal{U}_+$. $F_-(w,\ w') \stackrel{\text{def}}{=} F_+(w',\ w)$ is analytic in

$$\mathcal{U}_- = \{(w,\ w')\ :\ (w',\ w) \in \mathcal{U}_+\} \tag{88}$$

and we suppose

$$\mathcal{U}_- = \overline{\mathcal{U}_+}\ . \tag{89}$$

9. Local analyticity:

By this we mean that there is a complex open connected neighborhood $\mathcal{N}$ of $\mathcal{R}$ such that, in $\mathcal{R}$, both $F(\tilde{x},\ \tilde{x}')$ and $F(\tilde{x}',\ \tilde{x})$ are restrictions of the same function holomorphic in $\mathcal{N}$. If a two-point function F has the two properties of locality and analyticity as defined above, then it also has the property of local analyticity by the edge-of-the-wedge theorem.

If W is any two-point function, it can be written as $W = W_r + iW_i$, where

$$W_r(\tilde{x},\tilde{x}') = \frac{1}{2}W(\tilde{x},\tilde{x}') + \frac{1}{2}\overline{W(\tilde{x}',\tilde{x})}, \quad W_i(\tilde{x},\tilde{x}') = \frac{1}{2i}W(\tilde{x},\tilde{x}') - \frac{1}{2i}\overline{W(\tilde{x}',\tilde{x})}\ . \tag{90}$$

W_r and W_i are hermitic, and if W satisfies any one of the conditions 2, 3, or 6, so do W_r and W_i.

If W is any two-point function, it can be written as $W = W_{even} + W_{odd}$ where

$$W_{even}(x,\ x') = \frac{1}{2}W(x,\ x') + \frac{1}{2}W(x,\ \tau x'), \quad W_{odd}(x,\ x') = \frac{1}{2}W(x,\ x') - \frac{1}{2}W(x,\ \tau x')\ . \tag{91}$$

[5] See a general discussion of tuboids in [45].

8.1 Fourier expansion

We will now restrict our attention to those two-point functions which are rotation-invariant in our fixed frame, i.e. such that

$$W((t,\theta),(t',\theta')) = W((t,\theta+a),(t',\theta'+a)) \quad \forall a \in \mathbb{R} \tag{92}$$

(recall that τ is such a rotation for $a = 2\pi$); they can be expanded in a Fourier series as follows:

$$W(\tilde{x},\tilde{x}') = \sum_{l\in\mathcal{L}} u_l(z,z')e^{il(\theta-\theta')}\ , \tag{93}$$

As before $z = i\,\mathrm{sh}(t)$ and $z' = i\,\mathrm{sh}(t')$. The set $\mathcal{L}$ can be $\mathbb{Z}$, $\frac{1}{2}\mathbb{Z}$, or $\frac{1}{2}+\mathbb{Z}$. If $\mathcal{L} = \frac{1}{2}\mathbb{Z}$ then W_{even} (resp. W_{odd}), as defined in (91), is the sum over $\mathbb{Z}$ (resp. $\frac{1}{2}+\mathbb{Z}$).

Let us now address the question of finding the most general $SL(2,R)$–invariant hermitic two-point function satisfying the Klein-Gordon equation in each variable *for a positive squared mass* (i.e for the principal and the complementary series). Since such a function has to be rotation-invariant we may write

$$\begin{aligned} W(\tilde{x},\tilde{x}') &= \sum_{2l\in\mathbb{Z}} \gamma_l[A_l\mathbf{P}_\lambda^{-l}(z)\mathbf{P}_\lambda^{-l}(-z') + B_l\mathbf{P}_\lambda^{-l}(-z)\mathbf{P}_\lambda^{-l}(z')]e^{il(\theta-\theta')} \\ &+ \sum_{2l\in\mathbb{Z}} \gamma_l[e^{i\pi l}C_l\mathbf{P}_\lambda^{-l}(z)\mathbf{P}_\lambda^{-l}(z') + e^{-i\pi l}C_l^*\mathbf{P}_\lambda^{-l}(-z)\mathbf{P}_\lambda^{-l}(-z')]e^{il(\theta-\theta')}. \end{aligned} \tag{94}$$

Here we neither impose the locality property nor the positive definiteness. We have put γ_l in evidence for future convenience, by taking inspiration from the previous section. For the chosen mass parameters λ and $z \in i\mathbb{R}$ it happens that $\overline{\mathbf{P}_\lambda^{-l}(z)} = \mathbf{P}_\lambda^{-l}(-z)$. With this restriction, F is hermitic iff $A_l = A_l^*$, $B_l = B_l^*$.

The above two-point function is $SL(2,R)$-invariant if and only if the condition (67) holds. This amounts to

$$\begin{aligned} \delta_A + \delta_B + \delta_C + \delta_{C^*} = &\textstyle\sum_{2l\in\mathbb{Z}} i\gamma_l \sin\theta\,\mathrm{ch}\,t \Big[(A_l\mathbf{P}_\lambda^{-l}(-z') + C_le^{i\pi l}\mathbf{P}_\lambda^{-l}(z'))\mathbf{P'}_\lambda^{-l}(z) \\ &- (B_l\mathbf{P}_\lambda^{-l}(z') + C_l^*e^{-i\pi l}\mathbf{P}_\lambda^{-l}(-z'))\mathbf{P'}_\lambda^{-l}(-z)]\Big]\, e^{il(\theta-\theta')} \\ &+ \textstyle\sum_{2l\in\mathbb{Z}} il\gamma_l\,\mathrm{th}\,t\,\cos\theta\,\Big[(A_l\mathbf{P}_\lambda^{-l}(-z') + C_le^{i\pi l}\mathbf{P}_\lambda^{-l}(z'))\mathbf{P}_\lambda^{-l}(z) \\ &+(B_l\mathbf{P}_\lambda^{-l}(z') + C_l^*e^{-i\pi l}\mathbf{P}_\lambda^{-l}(-z'))\mathbf{P}_\lambda^{-l}(-z)\Big]\, e^{il(\theta-\theta')} \\ &+ \textstyle\sum_{2l\in\mathbb{Z}} i\gamma_l \sin\theta'\,\mathrm{ch}\,t' \Big[(B_l\mathbf{P}_\lambda^{-l}(-z) + C_le^{i\pi l}\mathbf{P}_\lambda^{-l}(z))\mathbf{P'}_\lambda^{-l}(z') \\ &-(A_l\mathbf{P}_\lambda^{-l}(z) + C_l^*e^{-i\pi l}\mathbf{P}_\lambda^{-l}(-z))\mathbf{P'}_\lambda^{-l}(-z')\Big]\, e^{il(\theta-\theta')} \\ &- \textstyle\sum_{2l\in\mathbb{Z}} il\gamma_l\,\mathrm{th}\,t'\,\cos\theta'\,\Big[[(B_l\mathbf{P}_\lambda^{-l}(-z) + C_le^{i\pi l}\mathbf{P}_\lambda^{-l}(z))\mathbf{P}_\lambda^{-l}(z') \\ &+(A_l\mathbf{P}_\lambda^{-l}(z) + C_l^*e^{-i\pi l}\mathbf{P}_\lambda^{-l}(-z))\mathbf{P}_\lambda^{-l}(-z')]\Big]\, e^{il(\theta-\theta')} = 0 \end{aligned} \tag{95}$$

where δ_A includes all the terms containing A and so on. Singling out the Fourier coefficient of $\exp il\theta$ we get

$$\begin{aligned}
\delta_A(l) &= \frac{1}{2}e^{-il\theta'}\gamma_{l-1}e^{i\theta'}\ A_{l-1}\mathbf{P}_\lambda^{1-l}(-z')\left[\operatorname{ch}t\,\mathbf{P'}_\lambda^{1-l}(z)+i(l-1)\operatorname{th}t\ \mathbf{P}_\lambda^{1-l}(z)\right]+\\
&+ \frac{1}{2}e^{-il\theta'}\gamma_{l+1}e^{-i\theta'}\ A_{l+1}\mathbf{P}_\lambda^{-1-l}(-z')\left[-\operatorname{ch}t\,\mathbf{P'}_\lambda^{-1-l}(z)+i(l+1)\operatorname{th}t\ \mathbf{P}_\lambda^{-1-l}(z)\right]+\\
&- e^{-il\theta'}\gamma_l A_l\mathbf{P}_\lambda^{-l}(z)\left[i\sin\theta'\operatorname{ch}t'\,\mathbf{P'}_\lambda^{-l}(-z')+il\cos\theta'\operatorname{th}t'\,\mathbf{P}_\lambda^{-l}(-z')\right] \qquad (96)
\end{aligned}$$

By itaking into account the crucial identities (70) and (71) and also Eq. (58) this expression takes the following simpler form:

$$\begin{aligned}
\delta_A(l) &= -\frac{1}{2}e^{-il\theta'}\gamma_l e^{i\theta'}\ A_{l-1}\mathbf{P}_\lambda^{1-l}(-z')\mathbf{P}_\lambda^{-l}(z)\\
&+ \frac{1}{2}e^{-il\theta'}\gamma_l(l-\lambda)(\lambda+l+1)e^{-i\theta'}\ A_{l+1}\mathbf{P}_\lambda^{-1-l}(-z')\mathbf{P}_\lambda^{-l}(z)+\\
&+ e^{-il\theta'}\gamma_l A_l\mathbf{P}_\lambda^{-l}(z)\left[i\sin\theta'(il\operatorname{th}t'\ \mathbf{P}_\lambda^{-l}(-z')+\mathbf{P}_\lambda^{1-l}(-z'))-il\cos\theta'\operatorname{th}t'\,\mathbf{P}_\lambda^{-l}(-z')\right]. \qquad (97)
\end{aligned}$$

By using the operator τ_1 and τ_2 we also immediately get that

$$\begin{aligned}
\delta_B(l) &= \frac{1}{2}e^{-il\theta'}\gamma_l e^{i\theta'}\ B_{l-1}\mathbf{P}_\lambda^{1-l}(z')\mathbf{P}_\lambda^{-l}(-z)\\
&- \frac{1}{2}e^{-il\theta'}\gamma_l(l-\lambda)(\lambda+l+1)e^{-i\theta'}\ B_{l+1}\mathbf{P}_\lambda^{-1-l}(z')\mathbf{P}_\lambda^{-l}(-z)+\\
&- e^{-il\theta'}\gamma_l B_l\mathbf{P}_\lambda^{-l}(-z)\left[i\sin\theta'(il\operatorname{th}t'\ \mathbf{P}_\lambda^{-l}(z')+\mathbf{P}_\lambda^{1-l}(z'))-il\cos\theta'\operatorname{th}t'\,\mathbf{P}_\lambda^{-l}(z')\right],
\end{aligned}$$

$$\begin{aligned}
\delta_C(l) &= \frac{1}{2}e^{-il\theta'}e^{i\pi l}\gamma_l e^{i\theta'}C_{l-1}\mathbf{P}_\lambda^{1-l}(z')\mathbf{P}_\lambda^{-l}(z)\\
&- \frac{1}{2}e^{-il\theta'}e^{i\pi l}\gamma_l(l-\lambda)(\lambda+l+1)e^{-i\theta'}C_{l+1}\mathbf{P}_\lambda^{-1-l}(z')\mathbf{P}_\lambda^{-l}(z)+\\
&- e^{-il\theta'}e^{i\pi l}\gamma_l C_l\mathbf{P}_\lambda^{-l}(z)\left[i\sin\theta'(il\operatorname{th}t'\ \mathbf{P}_\lambda^{-l}(z')+\mathbf{P}_\lambda^{1-l}(z'))-il\cos\theta'\operatorname{th}t'\,\mathbf{P}_\lambda^{-l}(-z')\right],
\end{aligned}$$

$$\begin{aligned}
\delta_{C^*}(l) &= -\frac{1}{2}e^{-il\theta'}e^{-i\pi l}\gamma_l e^{i\theta'}\ C^*_{l-1}\mathbf{P}_\lambda^{1-l}(-z')\mathbf{P}_\lambda^{-l}(-z)\\
&+ \frac{1}{2}e^{-il\theta'}e^{-i\pi l}\gamma_l(l-\lambda)(\lambda+l+1)e^{-i\theta'}\ C^*_{l+1}\mathbf{P}_\lambda^{-1-l}(-z')\mathbf{P}_\lambda^{-l}(-z)+\\
&+ e^{-il\theta'}e^{-i\pi l}\gamma_l C^*_l\mathbf{P}_\lambda^{-l}(-z)\left[i\sin\theta'(il\operatorname{th}t'\ \mathbf{P}_\lambda^{-l}(-z')+\mathbf{P}_\lambda^{1-l}(-z'))-il\cos\theta'\operatorname{th}t'\,\mathbf{P}_\lambda^{-l}(-z')\right].
\end{aligned}$$

Let us begin by supposing that ($B = C = 0$). In this case we must have $\delta_A(l) = 0$. Singling out as before the coefficients of $\cos\theta'$ and $\sin\theta'$ in the above expression we get

$$\left(-\frac{A_{l-1}}{2}\mathbf{P}_\lambda^{1-l}(-z')+(l-\lambda)(\lambda+l+1)\frac{A_{l+1}}{2}\mathbf{P}_\lambda^{-1-l}(-z')-ilA_l\operatorname{th}t'\mathbf{P}_\lambda^{-l}(-z')\right)\mathbf{P}_\lambda^{-l}(z)=0.$$

$$\left(\left(A_l - \frac{A_{l-1}}{2}\right)\mathbf{P}_\lambda^{1-l}(-z') - (l-\lambda)(\lambda+l+1)\frac{A_{l+1}}{2}\mathbf{P}_\lambda^{-1-l}(-z') + ilA_l \operatorname{th} t'\,\mathbf{P}_\lambda^{-l}(-z')\right)\mathbf{P}_\lambda^{-l}(z) = 0.$$

Taking the sum of the above two equations we get

$$(A_l - A_{l-1})\,\mathbf{P}_\lambda^{1-l}(-z')\mathbf{P}_\lambda^{-l}(z) = 0. \tag{98}$$

This shows that there are only two possible values for A_l: $A_l = A_0$ for $l \in \mathbb{Z}$ and $A_l = A_{\frac{1}{2}}$ for $l \in \frac{1}{2} + \mathbb{Z}$. The two equations now reduce to

$$\mathbf{P}_\lambda^{1-l}(-z') - (l-\lambda)(\lambda+l+1)\mathbf{P}_\lambda^{-1-l}(-z') + 2il \operatorname{th} t'\mathbf{P}_\lambda^{-l}(-z') = 0 \tag{99}$$

and this is a known relation between contiguous Legendre functions (Bateman Eq. 3.8.11). In the general case, proceeding in the same way, we get the condition (95) implies the following one:

$$\begin{aligned} &(A_l - A_{l-1})\,\mathbf{P}_\lambda^{1-l}(-z')\mathbf{P}_\lambda^{-l}(z) - (B_l - B_{l-1})\,\mathbf{P}_\lambda^{1-l}(z')\mathbf{P}_\lambda^{-l}(-z) + \\ - \;& e^{i\pi l}\,(C_l - C_{l-1})\,\mathbf{P}_\lambda^{1-l}(z')\mathbf{P}_\lambda^{-l}(z) + e^{-i\pi l}\left(C_l^* - C_{l-1}^*\right)\mathbf{P}_\lambda^{1-l}(-z')\mathbf{P}_\lambda^{-l}(-z) = 0 \end{aligned} \tag{100}$$

Therefore also in the general case the are only two possible values for A_l, B_l and C_l:

$$A_l = A_0, \quad B_l = B_0, \quad C_l = C_0 \quad for \quad l \in \mathbb{Z} \tag{101}$$

$$A_l = A_{\frac{1}{2}}, \quad B_l = B_{\frac{1}{2}}, \quad C_l = C_{\frac{1}{2}} \quad for \quad l \in \frac{1}{2} + \mathbb{Z}. \tag{102}$$

The verification that these conditions indeed guarantee that $\delta_A(l) + \delta_B(l) + \delta_C(l) + \delta_{C^*}(l) = 0$ proceeds as in the previous case.

8.2 Canonicity

If we impose that an invariant two-point function satisfies the canonical commutation relation (80), a lenghty but simple calculation shows that

$$A_0 - B_0 = 1, \quad A_{\frac{1}{2}} = \frac{1}{2} - e^{i\pi l}C_{\frac{1}{2}}, \quad B_{\frac{1}{2}} = -\frac{1}{2} - e^{i\pi l}C_{\frac{1}{2}}, \quad C_{\frac{1}{2}} = -C_{\frac{1}{2}}^* \tag{103}$$

while C_0 is unrestricted.

8.3 Positivity

Let us again consider a hermitic 2-point function

$$W(\tilde{x},\tilde{x}') = \sum_{l\in\mathcal{L}} u_l(z,z')e^{il(\theta-\theta')} \ ,$$
$$u_l(z,z') = \gamma_l[A_l\mathbf{P}_\lambda^{-l}(z)\mathbf{P}_\lambda^{-l}(-z') + B_l\mathbf{P}_\lambda^{-l}(-z)\mathbf{P}_\lambda^{-l}(z')$$
$$+ e^{i\pi l}C_l\mathbf{P}_\lambda^{-l}(z)\mathbf{P}_\lambda^{-l}(z') + e^{-i\pi l}C_l^*\mathbf{P}_\lambda^{-l}(-z)\mathbf{P}_\lambda^{-l}(-z')] \ , \tag{104}$$

where again $z = i\,\mathrm{sh}(t)$, $z' = i\,\mathrm{sh}(t')$, $A_l = A_l^*$, $B_l = B_l^*$. W is said to be of positive type if, for every test-function f,

$$\int_{\widetilde{dS_2}\times\widetilde{dS_2}} \overline{f(\tilde{x})}W(\tilde{x},\ \tilde{x}')f(\tilde{x}')\,d\tilde{x}\,d\tilde{x}' \geq 0 \ . \tag{105}$$

A necessary and sufficient condition for this is that, for every l and every test function f on $\mathbb{R}$,

$$\int_{\mathbb{R}\times\mathbb{R}} \overline{f(t)}u_l(t,\ t')f(t')\,dt\,dt' \geq 0 \ . \tag{106}$$

If f is a test-function on $\mathbb{R}$, let

$$f_1 = \int_{\mathbb{R}} f(t)\check{\mathbf{P}}_\lambda^{-l}(i\,\mathrm{sh}(t))\,dt, \quad f_2 = \int_{\mathbb{R}} f(t)\mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}(t))\,dt, \tag{107}$$

hence

$$\overline{f_1} = \int_{\mathbb{R}} \overline{f(t)}\mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}(t))\,dt, \quad \overline{f_2} = \int_{\mathbb{R}} \overline{f(t)}\check{\mathbf{P}}_\lambda^{-l}(i\,\mathrm{sh}(t))\,dt. \tag{108}$$

Then

$$\int_{\mathbb{R}\times\mathbb{R}} \overline{f(t)}u_l(t,\ t')f(t')\,dt\,dt' = \gamma_l \begin{pmatrix} \overline{f_1} \\ \overline{f_2} \end{pmatrix} \begin{pmatrix} A_l & e^{i\pi l}C_l \\ e^{-i\pi l}C_l^* & B_l \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} . \tag{109}$$

Therefore W is of positive type if

$$\gamma_l A_l \geq 0, \quad \gamma_l B_l \geq 0, \quad (\gamma_l)^2(A_lB_l - C_lC_l^*) \geq 0 \quad \forall l \in \mathcal{L} \ . \tag{110}$$

Let us now suppose that W is invariant, i.e. the conditions (101) and (102) are satisfied. If $l \in \mathbb{Z}$, with our choices of λ, $\gamma_l > 0$ for all $l \in \mathbb{Z}$, hence W_{even} is of positive type iff

$$A_0 > 0, \quad B_0 > 0, \quad A_0B_0 - C_0C_0^* > 0. \tag{111}$$

If $l \in \frac{1}{2} + \mathbb{Z}$ and $\lambda = -\frac{1}{2} + i\rho$, $\rho \neq 0$, then γ_l is always > 0 and W_{odd} is of positive type iff

$$A_{\frac{1}{2}} > 0, \quad B_{\frac{1}{2}} > 0, \quad A_{\frac{1}{2}}B_{\frac{1}{2}} - C_{\frac{1}{2}}C_{\frac{1}{2}}^* > 0. \tag{112}$$

If $l \in \frac{1}{2} + \mathbb{Z}$ and $-1 < \lambda < 0$, then γ_l has the sign of l and W_{odd} is never of positive type.

8.4 Convergence and (lack of) analyticity

We again consider a series of the form (104). In the preceding subsections such a series was regarded as the Fourier expansion of some two-point function (or rather distribution). In this subsection we suppose the series given and ask about its convergence. No generality is lost by the restriction to a hermitic series. However we will consider only a particular example from which the general case can be understood. Let

$$F_0(x,\ x') = \sum_{l\in\mathcal{L}} c_l(z,\ z')e^{il(\theta-\theta')}\ ,$$
$$c_l(z,\ z') = \gamma_l \mathbf{P}_\lambda^{-l}(z)\check{\mathbf{P}}_\lambda^{-l}(z')\ . \tag{113}$$

The dependence of c_l on λ has been omitted for simplicity. $\mathcal{L} = \frac{1}{2}\mathbb{Z}$. This series is the simplest example of the $SL(2,\ \mathbb{R})$ invariant series discussed in Subsect. 8.1. Recall that

$$x = (t,\ \theta),\quad z = i\,\mathrm{sh}(t),\quad x' = (t',\ \theta'),\quad z' = i\,\mathrm{sh}(t')\ . \tag{114}$$

We also set

$$x^0 = \mathrm{sh}(t) = \mathrm{tg}(s),\quad u = s+\theta,\quad v = s-\theta,\quad x'^0 = \mathrm{sh}(t') = \mathrm{tg}(s'),\quad u' = s'+\theta',\quad v' = s'-\theta',$$
$$z = ix^0 = i\,\mathrm{sh}(t) = -\,\mathrm{ch}(\mathrm{Re}\,t)\,\sin(\mathrm{Im}\,t) + i\,\mathrm{sh}(\mathrm{Re}\,t)\,\cos(\mathrm{Im}\,t)\ ,$$
$$z' = ix'^0 = i\,\mathrm{sh}(t') = -\,\mathrm{ch}(\mathrm{Re}\,t')\,\sin(\mathrm{Im}\,t') + i\,\mathrm{sh}(\mathrm{Re}\,t')\,\cos(\mathrm{Im}\,t')\ . \tag{115}$$

$c_l(z,\ z')$ is holomorphic in z (resp. z') in the cut-plane Δ_2 (see (47)). Values such that $\mathrm{Re}\,z > 0,\ \mathrm{Re}\,z' < 0$ correspond to $x \in \mathcal{T}_-\ ,\ x' \in \mathcal{T}_+$ while $\mathrm{Re}\,z < 0,\ \mathrm{Re}\,z' > 0$ correspond to $x \in \mathcal{T}_+\ ,\ x' \in \mathcal{T}_-$. The convergence of the series can be studied separately for $l \in \mathbb{Z}$ and $l \in \frac{1}{2} + \mathbb{Z}$.

8.4.1 Case of integer l

We first check that:

If $l \in \mathbb{Z}$, then $c_l(z,\ z') = c_{-l}(z,\ z')$.

It is sufficient to assume that $l > 0$. Then [47, 3.4 (17) p. 144]

$$\mathbf{P}_\lambda^{l}(z) = \frac{\cos(l\pi)\Gamma(\lambda+l+1)}{\Gamma(\lambda-l+1)}\mathbf{P}_\lambda^{-l}(z)\ , \tag{116}$$

$$c_{-l}(z,\ z',\ \lambda) = \frac{1}{2}\Gamma(-l-\lambda)\Gamma(-l+\lambda+1)\left[\frac{\Gamma(\lambda+l+1)}{\Gamma(\lambda-l+1)}\right]^2 \mathbf{P}_\lambda^{-l}(z)\mathbf{P}_\lambda^{-l}(-z')$$
$$= \frac{-\pi}{2\sin\pi(l+\lambda)}\frac{\Gamma(\lambda+l+1)}{\Gamma(\lambda-l+1)}\mathbf{P}_\lambda^{-l}(z)\mathbf{P}_\lambda^{-l}(-z')\ , \tag{117}$$

and using

$$\frac{1}{\Gamma(\lambda-l+1)}=\frac{1}{\pi}\sin\pi(l-\lambda)\Gamma(l-\lambda), \tag{118}$$

we obtain $c_{-l}(z,\ z')=c_l(z,\ z')$. We wish to investigate the convergence of the series (113) in the case when $l\in\mathbb{Z}$, $\operatorname{Re}z>0$ and $\operatorname{Re}z'<0$. By the preceding observation, it suffices to examine the half series $l\geq 0$. We use [47, 3.4 (6) p.143] with $\mu=-l$, $\nu=\lambda$, i.e.

$$\mathbf{P}_\lambda^{-l}(z)=\frac{1}{\Gamma(1+l)}\left(\frac{1-z}{1+z}\right)^{\frac{l}{2}}F\left(-\lambda,\ 1+\lambda\ ;\ 1+l\ ;\ \frac{1-z}{2}\right)\ . \tag{119}$$

Thus

$$\begin{aligned}c_l(z,\ z')=&\frac{\gamma_l(\lambda)}{\Gamma(1+l)^2}\left(\frac{1-z}{1+z}\right)^{\frac{l}{2}}\left(\frac{1+z'}{1-z'}\right)^{\frac{l}{2}}\times\\ &F\left(-\lambda,\ 1+\lambda\ ;\ 1+l\ ;\ \frac{1-z}{2}\right)F\left(-\lambda,\ 1+\lambda\ ;\ 1+l\ ;\ \frac{1+z'}{2}\right)\ .\end{aligned} \tag{120}$$

Simple geometry shows that

$$\pm\operatorname{Re}z>0\Longleftrightarrow\left|\frac{1\mp z}{1\pm z}\right|<1. \tag{121}$$

Using the discussion in Appendix A (eqs (181-187)), we make estimates of all the factors occuring in $c_l(z,\ z')$ which will be valid even if l is not an integer, provided $l\geq l_0$ for some $l_0>0$.

We first let N be the smallest integer $\geq|\operatorname{Re}\lambda|+1$. Then for $l>N$

$$\begin{aligned}\left|\frac{\Gamma(l-\lambda)}{\Gamma(l+1)}\right|&\leq\frac{\Gamma(l+N)}{\Gamma(l+1)}\leq(l+N)^{N-1}\leq(2l)^{N-1},\\ \left|\frac{\Gamma(l+1+\lambda)}{\Gamma(l+1)}\right|&\leq\frac{\Gamma(l+1+N)}{\Gamma(l+1)}\leq(2l)^N\ .\end{aligned} \tag{122}$$

We now set $z=i\operatorname{tg}(s)$ with $\operatorname{Im}s<0$ and $z'=i\operatorname{tg}(s')$ with $\operatorname{Im}s'>0$. Then

$$\frac{1-z}{1+z}=e^{-2is},\quad\left|\frac{1-z}{1+z}\right|<1,\quad\operatorname{Re}z>0,\quad\frac{1-z'}{1+z'}=e^{-2is'},\quad\left|\frac{1+z'}{1-z'}\right|<1,\quad\operatorname{Re}z'<0\ . \tag{123}$$

To discuss the first hypergeometric function appearing in (120) we temporarily denote $w=(1-z)/2$ which satisfies

$$\operatorname{Re}w<\frac{1}{2},\quad w=\frac{e^{-is}}{2\cos(s)},\quad|w|\leq\frac{1}{2|\operatorname{Im}s|}\ . \tag{124}$$

According to (181-187)

$$\begin{aligned}|F\left(-\lambda,\ 1+\lambda\ ;\ 1+l\ ;\ w\right)-1|&\leq\frac{1}{l+1}|w|M(w)|1+\operatorname{Re}\lambda|\operatorname{ch}(\pi\operatorname{Im}\lambda),\\ M(w)&=\sup_{0\leq u\leq 1}|(1-uw)^{\lambda-1}|\ .\end{aligned} \tag{125}$$

We have, for $0 \le u \le 1$, $\mathrm{Re}\, uw < \frac{1}{2}$, $|\,\mathrm{Arg}(1-uw)| < \pi$,

$$|(1-uw)^{\lambda-1}| \le |1-uw|^{\mathrm{Re}\,\lambda-1} e^{\pi|\,\mathrm{Im}\,\lambda|}\ . \tag{126}$$

with our choices of λ, $\mathrm{Re}\,\lambda - 1 < -1$. Since $\mathrm{Re}\, uw < \frac{1}{2}$, $|1-uw| > \frac{1}{2}$, so that

$$M(w) \le 2^{1-\mathrm{Re}\,\lambda} e^{\pi|\,\mathrm{Im}\,\lambda|}, \tag{127}$$

$$\left| F\left(-\lambda,\ 1+\lambda\ ;\ 1+l\ ;\ \frac{1-z}{2}\right) - 1 \right| \le \frac{1}{(l+1)|\,\mathrm{Im}\, s|} 2^{-\,\mathrm{Re}\,\lambda} e^{2\pi|\,\mathrm{Im}\,\lambda|}\ . \tag{128}$$

An analogous bound, with s' instead of s, holds for the second hypergeometric function occuring in (120). Gathering all this shows that there are positive constants E and Q depending only on λ such that

$$|c_l(z,\ z')| \le E\left(1 + \frac{1}{|\,\mathrm{Im}\, s|}\right)\left(1 + \frac{1}{|\,\mathrm{Im}\, s'|}\right)(l+1)^Q e^{l(\mathrm{Im}\, s - \mathrm{Im}\, s')}\ . \tag{129}$$

Recall again that here $\mathrm{Im}\, s < 0$ and $\mathrm{Im}\, s' > 0$, and that the bound (129) only requires $l \ge l_0$ for some $l_0 > 0$, and the genericity of λ.

Returning to the case of integer l, we see that the two series

$$\sum_{l\in\mathbb{Z},\ l\ge 0} c_l(z,\ z') e^{il(\theta-\theta')} \quad \text{and} \quad \sum_{l\in\mathbb{Z},\ l>0} c_{-l}(z,\ z') e^{il(\theta'-\theta)} = \sum_{l\in\mathbb{Z},\ l>0} c_l(z,\ z') e^{il(\theta'-\theta)} \tag{130}$$

converge absolutely and uniformly on any compact subset of the tubes

$$\{(s,\ s',\ \theta,\ \theta')\ : \mathrm{Im}\, s < 0,\quad \mathrm{Im}\, s' > 0,\quad \mathrm{Im}(s' - s - \theta' + \theta) > 0\} \tag{131}$$

and

$$\{s,\ s',\ \theta,\ \theta')\ : \mathrm{Im}\, s < 0,\quad \mathrm{Im}\, s' > 0,\quad \mathrm{Im}(s' - s + \theta' - \theta) > 0\} \tag{132}$$

respectively, and that the limits are holomorphic functions having boundary values in the sense of tempered distributions at the real values of $(s,\ s',\ \theta,\ \theta')$. Hence

$$\sum_{l\in\mathbb{Z}} c_l(z,\ z',\ \lambda) e^{il(\theta-\theta')} \tag{133}$$

converges to a function holomorphic in the tube

$$\mathrm{T}_{-,+} = \{(s,\ s',\ \theta,\ \theta')\ : \mathrm{Im}\, s < 0,\quad \mathrm{Im}\, s' > 0,\quad \mathrm{Im}(s' - s) - |\,\mathrm{Im}(\theta' - \theta)| > 0\} \tag{134}$$

which has a tempered boundary value at the real values of $(s,\ s',\ \theta,\ \theta')$. Denoting $u = s + \theta$, $v = s - \theta$, $u' = s' + \theta'$, $v = s' - \theta'$, the tube (134) contains the tube

$$\mathcal{T}_{-,+} = \{(u,\ v,\ u',\ v')\ : \mathrm{Im}\, u < 0,\quad \mathrm{Im}\, v < 0,\quad \mathrm{Im}\, u' > 0,\quad \mathrm{Im}\, v' > 0\}. \tag{135}$$

8.4.2 Case of half-odd-integers

We still have

$$c_l(z,\ z') = \gamma_l(\lambda)\mathbf{P}_\lambda^{-l}(z)\check{\mathbf{P}}_\lambda^{-l}(z')\ , \tag{136}$$

$$\gamma_l(\lambda) = \frac{1}{2}\Gamma(l-\lambda)\Gamma(l+\lambda+1)\ , \tag{137}$$

now assuming that $l \in \frac{1}{2} + \mathbb{Z}$.

8.4.3 Positive l

Here $l = n + \frac{1}{2}$, with integer $n \geq 0$. Eqs. (119) and (120) remain valid. We set again $z = i\,\mathrm{tg}(s)$, $z' = i\,\mathrm{tg}(s')$. With $\mathrm{Im}\, s < 0$ and $\mathrm{Im}\, s' > 0$, the estimate (129) still holds and therefore the series

$$\sum_{l\in\frac{1}{2}+\mathbb{Z},\ l>0} c_l(z,\ z')e^{il(\theta-\theta')} \tag{138}$$

converges uniformly on every compact of the tube (131) (hence also of $\mathrm{T}_{-,+}$ or $\mathcal{T}_{-,+}$ (see (134, 135))) to a holomorphic function that has a boundary value in the sense of tempered distributions at real values of s, s', θ, θ' (or u, v, u', v').

8.4.4 Negative l

Taking again $l = \frac{1}{2} + n$ with integer $n \geq 0$ we consider

$$c_{-l}(z,\ z') = \gamma(-l)\mathbf{P}_\lambda^{l}(z)\check{\mathbf{P}}_\lambda^{l}(z') \tag{139}$$

and use the formula (obtainable from [47, 3.3.2 (17) p. 141])

$$\begin{aligned}
&\mathbf{P}_\lambda^l(z) = \frac{\Gamma(l+\lambda+1)\Gamma(l-\lambda)}{\pi\Gamma(1+l)}\times \\
&\left[-\sin(\lambda\pi)\left(\frac{1-z}{1+z}\right)^{\frac{l}{2}} F\left(-\lambda,\ 1+\lambda\ ;\ 1+l\ ;\ \frac{1-z}{2}\right)\right. \\
&\left.+\sin(l\pi)\left(\frac{1+z}{1-z}\right)^{\frac{l}{2}} F\left(-\lambda,\ 1+\lambda\ ;\ 1+l\ ;\ \frac{1+z}{2}\right)\right]\ .
\end{aligned} \tag{140}$$

Note that

$$\begin{aligned}
\frac{1}{2}\Gamma(-l-\lambda)\Gamma(\lambda-l+1)\frac{\Gamma(\lambda+l+1)^2\Gamma(l-\lambda)^2}{\pi^2\Gamma(l+1)^2} &= -\frac{\Gamma(l-\lambda)\Gamma(l+\lambda+1)}{2\cos^2(\pi\lambda)\Gamma(l+1)^2} \\
&= \frac{-\gamma_l}{\cos^2(\pi\lambda)\Gamma(l+1)^2}\ .
\end{aligned} \tag{141}$$

We set again $z = i\,\mathrm{tg}(s)$, $z' = i\,\mathrm{tg}(s')$, $u = s+\theta$, $v = s-\theta$, $u' = s'+\theta'$, $v' = s'-\theta'$. We can rewrite

$$c_{-l}(z,\ z') = \sum_{\varepsilon,\varepsilon'=\pm} c_{-l,\varepsilon,\varepsilon'}(z,\ z') \tag{142}$$

where $\varepsilon = \mp$ (resp. $\varepsilon' = \pm$) denotes the choice of the first or second term in the bracket of (140). Thus

$$c_{-l,-,-}(z,\ z') = \frac{\Gamma(l-\lambda)\Gamma(l+\lambda+1)\sin(\pi l)\sin(\pi\lambda)}{2\Gamma(l+1)^2\cos(\pi\lambda)^2}\left(\frac{1-z}{1+z}\right)^{\frac{l}{2}}\left(\frac{1-z'}{1+z'}\right)^{\frac{l}{2}}\times$$
$$F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1-z}{2}\right)F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1-z'}{2}\right)\ , \tag{143}$$
$$c_{-l,+,+}(z,\ z',\ \lambda) = \frac{\Gamma(l-\lambda)\Gamma(l+\lambda+1)\sin(\pi l)\sin(\pi\lambda)}{2\Gamma(l+1)^2\cos(\pi\lambda)^2}\left(\frac{1+z}{1-z}\right)^{\frac{l}{2}}\left(\frac{1+z'}{1-z'}\right)^{\frac{l}{2}}\times$$
$$F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1+z}{2}\right)F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1+z'}{2}\right)\ , \tag{144}$$
$$c_{-l,-,+}(z,\ z',\ \lambda) = \frac{-\Gamma(l-\lambda)\Gamma(l+\lambda+1)\sin^2(\pi\lambda)}{2\Gamma(l+1)^2\cos(\pi\lambda)^2}\left(\frac{1-z}{1+z}\right)^{\frac{l}{2}}\left(\frac{1+z'}{1-z'}\right)^{\frac{l}{2}}\times$$
$$F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1-z}{2}\right)F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1+z'}{2}\right)\ , \tag{145}$$
$$c_{-l,+,-}(z,\ z',\ \lambda) = \frac{-\Gamma(l-\lambda)\Gamma(l+\lambda+1)\sin^2(\pi l)}{2\Gamma(l+1)^2\cos(\pi\lambda)^2}\left(\frac{1+z}{1-z}\right)^{\frac{l}{2}}\left(\frac{1-z'}{1+z'}\right)^{\frac{l}{2}}\times$$
$$F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1+z}{2}\right)F\left(-\lambda,\ \lambda+1\ ;\ 1+l\ ;\ \frac{1-z'}{2}\right)\ . \tag{146}$$

For a given choice of ε and ε' the estimates (122-129) are readily adapted so that the series

$$\sum_{l=\frac{1}{2}+n,\ n\geq 0} c_{-l,\varepsilon,\varepsilon'}(z,\ z',\ \lambda)e^{-il(\theta-\theta')} \tag{147}$$

converges absolutely to a holomorphic function of $(s,\ s',\ \theta,\ \theta')$ in the tube

$$\mathrm{T}_{\varepsilon,\varepsilon'} = \{(s,\ s',\ \theta,\ \theta')\ :\ \varepsilon\,\mathrm{Im}\,s > 0,\ \ \varepsilon'\,\mathrm{Im}\,s' > 0,\ \ \mathrm{Im}(\varepsilon' s' + \varepsilon s) - |\,\mathrm{Im}(\theta'-\theta)| > 0\} \tag{148}$$

as well as in

$$\mathcal{T}_{\varepsilon,\varepsilon'} = \{(u,\ v,\ u',\ v')\ :\ \varepsilon\,\mathrm{Im}\,u > 0,\ \ \varepsilon\,\mathrm{Im}\,v > 0,\ \ \varepsilon'\,\mathrm{Im}\,u' > 0,\ \ \varepsilon'\,\mathrm{Im}\,v' > 0\}. \tag{149}$$

This function has a boundary value at real values of these variables in the sense of tempered distributions.

8.4.5 Conclusion

The series (113) converges to a distribution F_0 which is a finite sum of boundary values of functions holomorphic in several non-intersecting open tuboids. Thus F_0 is not the boundary value of a function holomorphic in a single open tuboid. It is possible to verify that this is also true for the invariant functions of the type given by (101) and (102). It might be asked if F_0 (or one of its siblings) could not still have a tuboid of analyticity beyond what follows from the above proofs of convergence. However in Sect. 10 a general lemma will show that analyticity is incompatible with the simultaneous requirements of locality (1), invariance (3) and Klein-Gordon equation (6). The proof of convergence given in this subsection also works for more general (non-invariant) series of the form (94) provided the $A_l,\ \ldots C_l^*$ are polynomially bounded in l.

9 Study of local "vacuum" states invariant under $SL(2, R)$

One immediate solution of the KG equation constructed in terms of the system of modes ϕ_l (52) is provided by their "vacuum"; the corresponding two-point function is given by

$$\begin{aligned} W(x, x') &= \sum_l \phi_l(x)\phi_l^*(x') = \\ &= \sum_l [a_l \mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}\,t) + b_l \mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}\,t)][a_l^* \mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}\,t') + b_l^* \mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}\,t')]e^{il\theta - il\theta'} \end{aligned} \tag{150}$$

The standard theory of Bogoliubov canonical transformations provides infinitely many other, possibly inequivalent, vacua by specifying the corresponding two-point functions in terms of two operators a and b as follows:

$$W_{a,b}(x, x') = \sum [a_{ij}\phi_j(x) + b_{ij}\,\phi_j^*(x)][a_{il}^*\phi_l^*(x') + b_{il}^*\,\phi_l(x')]. \tag{151}$$

Since the commutator must not depend on the choice of a and b, Eq. (80) tells us the conditions $\sum(a_{ij}a_{il}^* - b_{ij}^* b_{il}) = \delta_{jl}$ and $\sum(a_{ij}b_{il}^* - a_{il}b_{ij}^*) = 0$. The states given by Eq. (151) are "vacuum states" or else "pure states" i.e. they provide through the GNS construction irreducible representations of the field algebra.

Let us now single out among the states (150) those who are $SL(2, \mathbb{R})$–invariant. This is an easy corollary of the theorem of the previous section. Eqs. (101) and (102) imply the following relations:

$$|a_l|^2 = c_1(\epsilon)\gamma_l, \quad |b_l|^2 = c_2(\epsilon)\gamma_l, \quad a_l b_l^* = c_3(\epsilon)\gamma_l e^{il\pi} \tag{152}$$

where $\epsilon = 0$ for $l \in \mathbb{Z}$ and $\epsilon = 1$ for $l \in \frac{1}{2} + \mathbb{Z}$ (i.e. there are six independent constants). The above equations, together with the normalization condition (59), can be solved as follows:

$$a_l = \sqrt{\frac{\gamma_l}{2\pi k}}\,\mathrm{ch}\,\alpha_\epsilon, \qquad b_l = \sqrt{\frac{\gamma_l}{2\pi k}}\,\mathrm{sh}\,\alpha_\epsilon e^{i\phi_\epsilon - il\pi}, \qquad \epsilon = 0, 1. \tag{153}$$

Here we took a_l real without loss of generality.

9.1 Local Commutativity: pure de Sitter

Let us examine whether the above equations are compatible with the requirements imposed by local commutativity. In the pure de Sitter case (as opposed to its covering) l is integer and the CCR's amount to the condition (54) which imposes no further restriction and any choice of α_0 and ϕ_0 gives rise to a de Sitter invariant state which has the right commutator (relatively to the de Sitter manifold). These states are well-known: they are the so-called alpha vacua [36, 37, 38, 39].

Among them, there is a particularly important state corresponding to the choice $\alpha_0 = 0$: this is the so-called Bunch-Davies vacuum [40, 41, 42, 43, 44, 39]

$$W_{BD}(x,x') = W^{(0)}_{\alpha_0=0}(x,x') = \sum_{l\in\mathbb{Z}} \frac{\gamma_l}{2\pi} \mathbf{P}^{-l}_{\lambda}(i\,\mathrm{sh}\,t)\mathbf{P}^{-l}_{\lambda}(-i\,\mathrm{sh}\,t')e^{il\theta-il\theta'} = \\ = \frac{\Gamma(-\lambda)\Gamma(\lambda+1)}{4\pi} P_\lambda(\zeta), \tag{154}$$

where $P_\lambda(\zeta)$ is the associated Legendre function of the first kind [33] and the de Sitter invariant variable ζ is the scalar product $\zeta = x(t - i\epsilon, \theta) \cdot x'(t' + i\epsilon, \theta)$ in the ambient space sense. Actually, $W_{BD}(x,x')$ admits an extension to the complex de Sitter manifold and satisfies there the *maximal analyticity property* [43, 44]: it is holomorphic for all $\zeta \in \mathbf{C} \setminus (-\infty,\ -1]$ i.e. everywhere except on the locality cut. This crucial property singles the Bunch-Davies vacuum out of all the other invariant vacua and has a very well known thermal interpretation [42, 43]: the restriction of the Bunch-Davies state a wedge-like region is a thermal state at temperature $T = 1/2\pi$. A similar property is expected in interacting theories based on an analogue of the Bisognano-Wichmann theorem [45].

9.2 Covering

In the antiperiodic case the CCR's

$$\text{for } l \in \tfrac{1}{2} + \mathbb{Z} \quad \begin{cases} a_l a_{-l} - b_l b_{-l} = c_l \sin(\pi\lambda) \\ a_l b_{-l} - b_l a_{-l} = c_l \sin(\pi l) \end{cases} \tag{155}$$

imply the following relation between the constants α and ϕ and the mass parameter λ of the field:

$$e^{i\phi}\sin(\pi l)(-i\,\mathrm{sh}(2\alpha)\sin(\pi\lambda) - i\,\mathrm{ch}(2\alpha)\sin\phi + \cos\phi) = 0. \tag{156}$$

For $\lambda = -1/2 + i\nu$ there is only one possible solution given by

$$\coth 2\alpha = \mathrm{ch}\,\pi\nu, \quad \phi = \frac{\pi}{2}. \tag{157}$$

We denote the corresponding two-point function $W^{(\frac{1}{2})}_{\nu}(x,x')$. Note that the value $\alpha = 0$, that would correspond to the above-mentioned maximal analyticity property, is excluded: it would be attained only for an infinite value of the mass. On the other hand Equation (156) has no solution at all when λ is real: there is no invariant vacuum of the complementary series.

In conclusion, for $\lambda = -\frac{1}{2} + i\nu$ the most general invariant vacuum state is the superposition of an arbitrary alpha vacuuum (the even part) plus a fixed odd part $W^{(\frac{1}{2})}_{\nu}(x,x')$ as follows

$$W(x,x') = W^{(0)}_{\alpha_0,\phi_0}(x,x') + W^{(\frac{1}{2})}_{\nu}(x,x'). \tag{158}$$

For $\lambda = -\frac{1}{2} + \nu$ there is no $SL(2,R)$ local invariant vacuum state.

10 Incompatibility of analyticity with some other requirements

In this section the following lemma will be proved:

Lemma 10.1 *A two-point function F on $\widetilde{dS_2}\times\widetilde{dS_2}$ that simultaneously satisfies (3) Invariance under $SL(2,\ \mathbb{R})$, (6) Klein-Gordon equation, and (9) Local analyticity (see Sect. 8), vanishes on $\mathcal{R}$ (i.e. $F(x,\ x')=0$ whenever x and x' are space-like separated).*

An example of a non-zero two-point function satisfying these requirements is the canonical commutator (65). As an obvious corollary of this lemma,

Lemma 10.2 *A two-point function F on $\widetilde{dS_2}\times\widetilde{dS_2}$ that simultaneously satisfies (1) Locality, (3) Invariance under $SL(2,\ \mathbb{R})$, (6) Klein-Gordon equation, and (8) Analyticity (see Sect. 8), is equal to 0.*

Proof. Suppose F satisfies the conditions (3), (6), and (9). As in Sect. 8, Let $\mathcal{R}$ denote the (open, connected) set of space-like separated points in $\widetilde{dS_2}\times\widetilde{dS_2}$. and $\mathcal{N}$ the complex connected neighborhood of $\mathcal{R}$ in which $F(\tilde{x},\ \tilde{x}')$ and $F(\tilde{x}',\ \tilde{x})$ have a common analytic continuation. We denote F_+ this analytic continuation. For any pair $k=(\widetilde{w},\ \widetilde{w}')$ we denote, by abuse of notation, $\widetilde{w}\cdot\widetilde{w}'$ or also $\psi(k)$ the scalar product of the projections of $\widetilde{w}$ and $\widetilde{w}'$ into the complex Minkowski space $M_3^{(c)}$ (i.e. with an abuse of notation, $\psi(k)=-1-\frac{1}{2}(x-x')^2$). In particular $\mathcal{N}$ contains the subset

$$E_{\varepsilon,\ \eta}=\{(t,\ \theta+iy),\ (t',\ \theta'))\ :\ t=t'=\theta'=0,\ \ \varepsilon<\theta<4\pi-\varepsilon,\quad |y|<\eta\}\ , \tag{159}$$

where $\varepsilon>0$ and $\eta>0$ must be chosen small enough. Note that for points of the form (159),

$$\zeta=-\cos(\theta+iy)=-\cos(\theta)\,\mathrm{ch}(y)+i\sin(\theta)\,\mathrm{sh}(y)\ . \tag{160}$$

Let $k_0=(\widetilde{w}_0,\ \widetilde{w}_0')\in\mathcal{N}$ be such that $\zeta_0=\psi(k_0)\neq\pm1$. There exists open neighborhoods $U_1\subset\subset U_2\subset\subset\mathcal{N}$ of k_0, and an open neighborhood W_0 of the identity in the group $SL(2,\ \mathbf{C})$ such that, for all $g\in W_0$ and $k\in U_1$, $gk\in U_2$ and $F(gk)=F(k)$ (since this holds for real g). Moreover we suppose U_2 small enough that the restriction to U_2 of the projection $\mathrm{pr}\times\mathrm{pr}$ is an isomorphism, and also that $k\mapsto\sqrt{1-\psi(k)^2}$ can be defined as a holomorphic function on U_2 (in particular $\psi(k)\neq\pm1$ for all $k\in U_2$).

We will prove[6] that there is an open neighborhood $V_0\subset\subset U_1$ of k_0, and a function f_0 holomorphic on $\psi(V_0)$ such that $f_0(\psi(k))=F_+(k)$ for all $k\in V_0$. To do this we adopt the simplifying notation whereby if $\widetilde{t}\in\widetilde{dS_2}^{(c)}$ then t denotes $\mathrm{pr}\,\widetilde{t}$ and conversely if $t\in dS_2^{(c)}$

[6]These arguments are special cases of more general well-known facts. See e.g. [48], [49].

then $\widetilde{t}$ denotes $\mathrm{pr}^{-1}t$. For any $k = (\widetilde{w},\ \widetilde{w'}) \in U_2$ we contruct a complex Lorentz frame $(e_0(k),\ e_1(k), e_2(k))$ as follows:

$$\begin{aligned} &e_0(k) = iw\ , \\ &e_1(k) = \alpha w + \beta w', \quad e_0(k)\cdot e_1(k) = 0, \quad e_1(k)^2 = -1\ . \end{aligned} \tag{161}$$

Denoting $\zeta = w \cdot w' = \psi(k)$, this implies

$$\alpha = \beta\zeta, \quad \beta = \frac{1}{\sqrt{1-\zeta^2}}\ , \tag{162}$$

$$e_2(k) = e_0(k) \wedge e_1(k)\ . \tag{163}$$

(Here $\sqrt{1-\zeta^2}$ denotes the determination of $\sqrt{1-\psi(k)^2}$ mentioned above). This implies

$$w = -ie_0(k), \quad w' = \sqrt{1-\zeta^2}\, e_1(k) + i\zeta\, e_0(k)\ . \tag{164}$$

For any ζ sufficiently close to ζ_0 let $h(\zeta) = (\widetilde{v(\zeta)},\ \widetilde{v'(\zeta)})$ be defined by

$$v(\zeta) = -ie_0(k_0), \quad v'(\zeta) = \sqrt{1-\zeta^2}\, e_1(k_0) + i\zeta\, e_0(k_0)\ . \tag{165}$$

It is clear that $v(\zeta)^2 = v'(\zeta)^2 = -1$, $v(\zeta)\cdot v'(\zeta) = \zeta$, and $e_j(h(\zeta)) = e_j(k_0)$ for $j = 0,\ 1,\ 2)$. If $k = (\widetilde{w},\ \widetilde{w'})$ is close to k_0 and $\psi(k) = \zeta$, there exists an element $g(k)$ of $SL(2,\ \mathbf{C})$ close to the identity, such that $k = g(k)h(\zeta)$. This element projects onto the unique Lorentz transformation such that $e_j(k) = \Lambda(k)e_j(h(\zeta)) = \Lambda(k)e_j(k_0)$ for $j = 0,\ 1,\ 2)$. We have therefore $F(k) = F(h(\zeta))$, i.e. a holomorphic function of ζ. We have now shown that every $k_0 \in \mathcal{N}$ such that $\psi(k_0) \neq \pm 1$ has an open neighborhood V_0 such that $F_+(k) = f_0(\psi(k))$ for all $k \in V_0$, where f_0 is holomorphic in $\psi(V_0)$.

If k_1 is another point of $\mathcal{N}$ such that $\psi(k_1) \neq \pm 1$, and V_1, f_1 are the analogous objects, and if V_0 and V_1 overlap, it is clear that f_1 is an analytic continuation of f_0. Thus for any compact arc contained in $\mathcal{N}$ from k_0 to k_2, f_0 can be analytically continued along the image under ψ of that arc, provided this image avoids the points ± 1.

Since F satisfies the Klein-Gordon equation in x and in x', it follows, by a well-known calculation, that the $f = f_0$ obtained by the above procedure at a point k_0 (with $\psi(k_0) \neq \pm 1$) must be a solution of the Legendre equation :

$$(1-\zeta^2)f''(\zeta) - 2\zeta f'(\zeta) + \lambda(\lambda+1)f(\zeta) = 0\ . \tag{166}$$

By the general theory of such equations, f may be analytically continued along any arc in the complex plane which avoids the points ± 1. Two linearly independent solutions of the equation are

$$\mathbf{P}_\lambda(\zeta) = \mathbf{P}^0_\lambda(\zeta) \quad \text{and} \quad \check{\mathbf{P}}_\lambda(\zeta) = \check{\mathbf{P}}^0_\lambda(\zeta) = \mathbf{P}^0_\lambda(-\zeta)\ , \tag{167}$$

$$\mathbf{P}^0_\lambda(\zeta) = F\left(-\lambda,\ \lambda+1\ ;\ 1\ ;\ \frac{1-\zeta}{2}\right)\ . \tag{168}$$

Note that $\mathbf{P}_\lambda$ is holomorphic in the cut-plane with a cut along $(-\infty,\ -1]$ and is singular at -1 : according to [47, p. 164] it has a logarithmic singularity at -1. Hence $\check{\mathbf{P}}_\lambda$ is holomorphic in the cut-plane with a cut along $[1,\ \infty)$ and has a logarithmic singularity at 1. By [47, (10) p. 140], for real $t > 1$,

$$\check{\mathbf{P}}_\lambda(t+i0) - \check{\mathbf{P}}_\lambda(t-i0) = -2i\sin(\lambda\pi)\mathbf{P}_\lambda(t)\ , \quad t > 1\ . \tag{169}$$

We must have

$$f(\zeta) = a\mathbf{P}_\lambda(\zeta) + b\check{\mathbf{P}}_\lambda(\zeta)\ , \tag{170}$$

where a and b are constants. We specialize k_0 as

$$k_0 = ((t_0 = 0,\ \theta_0 = 2\varepsilon),\ (t'_0 = 0,\ \theta'_0 = 0))\ , \tag{171}$$

where $0 < \varepsilon$ is as in (159), and we suppose $\varepsilon < \pi/8$. We denote

$$G(\theta + iy) = F_+((t = 0,\ \theta + iy),\ (t' = 0,\ \theta' = 0)). \tag{172}$$

As the restriction of F_+ to the set $E_{\varepsilon,\ \eta}$, G is holomorphic in the rectangle $\varepsilon < \theta < 4\pi - \varepsilon$, $|y| < \eta$. We consider an arc (actually a straight line) γ_y lying in $E_{\varepsilon,\ \eta}$, given by

$$\theta \mapsto \gamma_y(\theta) = ((t = 0,\ \theta + iy),\ (t' = 0,\ \theta' = 0)). \tag{173}$$

Here y is real with $|y| \le \tau$ and θ varies in the interval $[2\varepsilon,\ 2\pi]$. We also require $0 < \tau < \eta$ to be small enough that the starting point $\gamma_y(2\varepsilon)$ be always contained in the neighborhood V_0 of k_0 where the function $f = f_0$ is initially defined. Let y be fixed with $0 < y < \tau$. As θ varies in $[2\varepsilon,\ 2\pi]$, $\zeta = \psi(\gamma_y(\theta)) = -\cos(\theta + iy)$ runs along an arc of an ellipse with foci at ± 1, starting in the upper half-plane, crossing the real axis at $t = \mathrm{ch}(y)$, and returning through the lower half-plane to $-t - i0$. Along this arc, f can be analytically continued; starting as $f(\zeta) = a\mathbf{P}_\lambda(\zeta) + b\check{\mathbf{P}}_\lambda(\zeta)$ it becomes, after crossing the real axis at t, equal to $[a - 2ib\sin(\lambda\pi)]\mathbf{P}_\lambda(\zeta) + b\check{\mathbf{P}}_\lambda(\zeta)$ (as a consequence of (169)). Thus at the end point,

$$f(-t - i0) = G(2\pi + iy) = [a - 2ib\sin(\lambda\pi)]\mathbf{P}_\lambda(-t - i0) + b\check{\mathbf{P}}_\lambda(-t - i0)\ . \tag{174}$$

Since $|G(2\pi + iy)|$ is bounded uniformly in y, while $|\mathbf{P}_\lambda(-t - i0)| \to \infty$ as $t \to 1$, we must have

$$a - 2ib\sin(\lambda\pi) = 0. \tag{175}$$

Repeating the argument with $y < 0$ (the arc of ellipse starts in the lower half-plane and finishes in the upper half-plane) we now get

$$a + 2ib\sin(\lambda\pi) = 0. \tag{176}$$

This implies $f = 0$, and therefore that F_+ vanishes in an open complex neighborhood of k_0. Hence $F_+ = 0$, hence F vanishes on $\mathcal{R}$.

There are examples of 2-point functions that satisfy any three out of the four conditions (1), (3), (6), and (8). For instance let

$$\begin{aligned} F_2(x,\ x') = {} & u_0(t,\ t',\ \lambda) \\ & + \sum_{l\in\mathcal{L},\ l>0} u_l(t,\ t',\ \lambda)\left[e^{il(\theta-\theta')} + e^{-il(\theta-\theta')}\right] , \end{aligned} \tag{177}$$

$$u_l(t,\ t',\ \lambda) = \gamma_l \mathbf{P}_\lambda^{-l}(i\,\mathrm{sh}(t))\mathbf{P}_\lambda^{-l}(-i\,\mathrm{sh}(t')) \ , \quad l \geq 0 \ , \tag{178}$$

Then F_2 satisfies (1), (6) as well as canonicity and positivity. It also satisfies (8) and is the boundary value of a function holomorphic in the tuboid $\mathrm{T}_{-,+}$ (see (134)) (or $\mathcal{T}_{-,+}$ (see (135))). But F_2 is not invariant under $SL(2,\ \mathbb{R})$.

11 Concluding remarks

One important fact to be stressed again is that the above states do not share the global analyticity property characteristic of the Bunch-Davis maximally analytic vacuum[32, 44, 43, 45]. Correspondingly, restriction of the above states to the wedge between the horizons does not give rise to a thermal state as it happens in the pure de Sitter case (only for the BD vacuum)[42, 43, 44, 45]. The de Sitter temperature has disappeared.

A Appendix. Estimations for hypergeometric functions

We reproduce here for completeness some estimates from [47, 2.3.2 pp 76-77]. Recall (see e.g. [46, 5.6(ii)])

$$|\Gamma(x+iy)| \leq |\Gamma(x)|, \tag{179}$$

$$|\Gamma(x+iy)| \geq \frac{1}{\sqrt{\mathrm{ch}\, y}}|\Gamma(x)|, \quad x \geq \frac{1}{2} \ . \tag{180}$$

Define

$$\rho_{n+1}(a,\ b,\ ;\ c\ ;\ z) = F(a,\ b,\ ;\ c\ ;\ z) - 1 - \frac{ab}{c}z - ... - \frac{(a)_n(b)_n}{(c)_n n!}z^n$$
$$= \frac{\Gamma(c)\Gamma(a+n)z^{n+1}}{\Gamma(b)\Gamma(c-b)\Gamma(a)n!}\times$$
$$\int_0^1 ds \int_0^1 dt\, t^{b+n}(1-t)^{c-b-1}(1-s)^n(1-stz)^{-a-n-1}\ . \tag{181}$$

Let $a = \alpha + i\alpha'$, $b = \beta + i\beta'$, $c = \gamma + i\gamma'$. Then

$$|\rho_{n+1}| \leq \mu(n)\, |z|^{n+1}|c|^{-\beta}\gamma^{-\beta-n-1}\ , \tag{182}$$

where it is assumed that $|\arg(1-z)| < \pi$, $\gamma > \beta$, $n + \beta > 0$, $|\arg c| < \pi - \varepsilon$, $\gamma > 0$ sufficiently large, n sufficiently large. $\mu(n)$ depends on n, a, b, z.

Example: $n = 0$

$$\rho_1(a,\ b,\ ;\ c\ ;\ z) = \frac{\Gamma(c)z}{\Gamma(b)\Gamma(c-b)}\, I\ ,$$
$$I = \int_0^1 ds \int_0^1 dt\, t^b(1-t)^{c-b-1}(1-stz)^{-a-1}\ . \tag{183}$$

We assume $\beta + 1 > 0$, $\gamma - \beta > 0$, $\gamma > 0$, and that $|1 - z| > \varepsilon$, $|\arg(1-z)| < \pi - \varepsilon$ for some $\varepsilon > 0$. In this case the modulus $|I|$ of the integral is bounded by

$$|I| \leq M(z) \int_0^1 t^\beta(1-t)^{\gamma-\beta-1}\, dt \tag{184}$$

with

$$M(z) = \sup_{0 \leq u \leq 1} |(1-uz)^{-a-1}|. \tag{185}$$

$$|I| \leq M(z)\frac{\Gamma(\beta+1)\Gamma(\gamma-\beta)}{\Gamma(\gamma+1)}. \tag{186}$$

Hence using (179)

$$|\rho_1(a,\ b,\ ;\ c\ ;\ z)| \leq \frac{|z|M(z)|\beta|\sqrt{\mathrm{ch}(\pi\beta')\,\mathrm{ch}\,\pi(\gamma'-\beta')}}{\gamma}\ . \tag{187}$$

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