fRevolution
Relativistic simulations of $f(R)$ gravity

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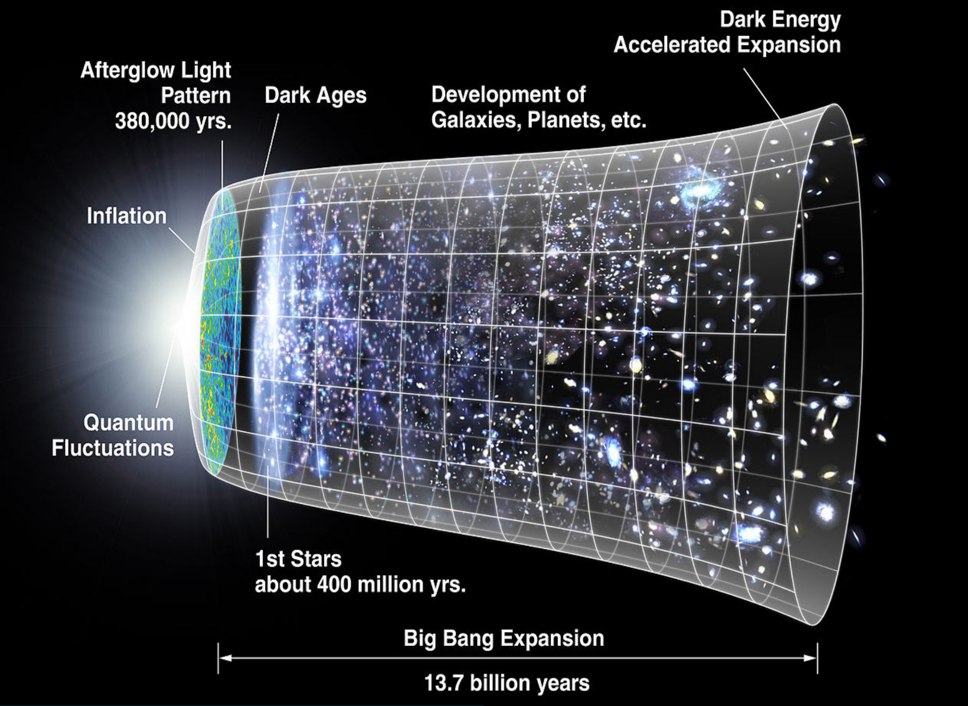
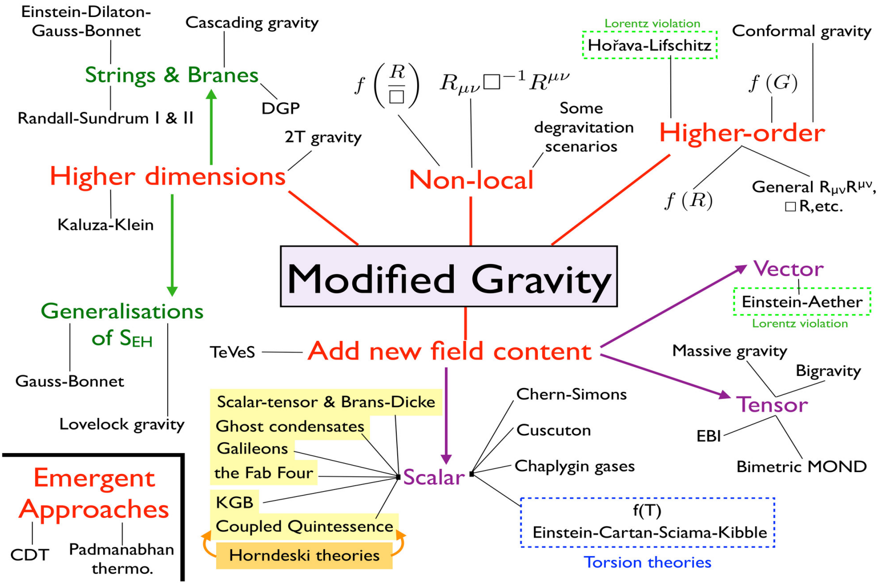

Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
380,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years


Einstein-Dilaton-Gauss-Bonnet
Cascading gravity
Strings & Branes
DGP
Randall-Sundrum I & II
2T gravity
Higher dimensions
Kaluza-Klein
Generalisations of S_{EH}
Gauss-Bonnet
Lovelock gravity
Emergent Approaches
CDT
Padmanabhan thermo.
$f\left(\frac{R}{\Box}\right)$
$R_{\mu\nu}\Box^{-1}R^{\mu\nu}$
Some degravitation scenarios
Non-local
Lorentz violation
Hořava-Lifschitz
Conformal gravity
$f(G)$
Higher-order
$f(R)$
General $R_{\mu\nu}R^{\mu\nu}$, $\Box R$, etc.
Modified Gravity
Vector
Einstein-Aether
Lorentz violation
TeVeS
Add new field content
Massive gravity
Bigravity
Tensor
EBI
Bimetric MOND
Scalar-tensor & Brans-Dicke
Ghost condensates
Galileons
the Fab Four
KGB
Coupled Quintessence
Horndeski theories
Scalar
Chern-Simons
Cuscuton
Chaplygin gases
f(T)
Einstein-Cartan-Sciama-Kibble
Torsion theories

$f(R)$ Gravity

Action

$$\mathcal{S}_{\rm grav} = \int d^4x \sqrt{-g}\,(R + 2\Lambda) \quad \to \quad \int d^4x \sqrt{-g}\,[R + f(R)]$$

Field Equations

$$R_{\mu\nu} - \frac{R}{2}\,g_{\mu\nu} - \Lambda\,g_{\mu\nu} = T_{\mu\nu}$$

$$\downarrow$$

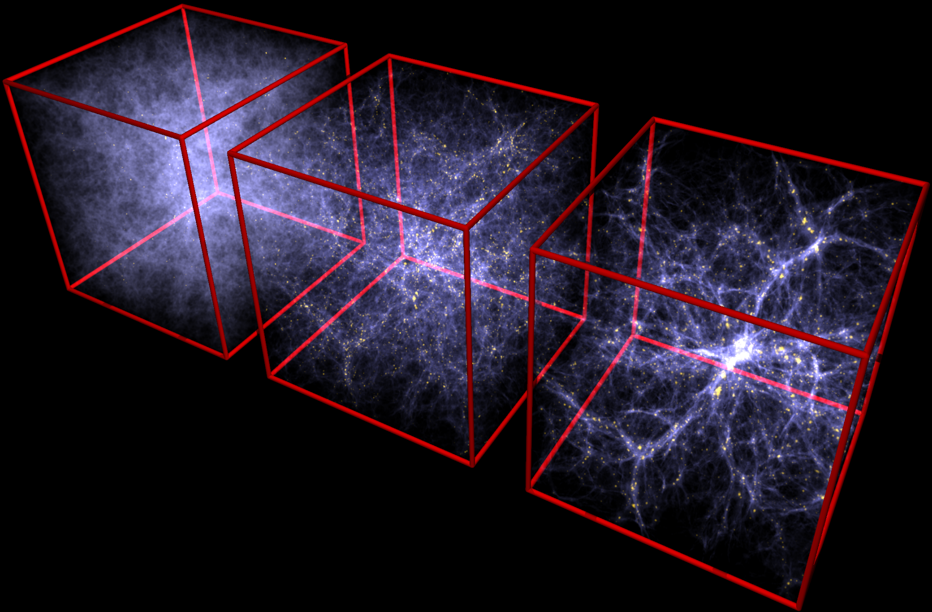
$$(1 + f_R)\,R_{\mu\nu} - \frac{R + f}{2}\,g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = T_{\mu\nu}$$

- Higher (4th) order theory in $g_{\mu\nu}$, but no Ostrogradski instability
- Massless spin-2 graviton + massive scalar $\sim f_R$ (scalaron)
- Additional dynamics: scalar waves, screening mechanisms, cosmic acceleration without explicit Λ, …

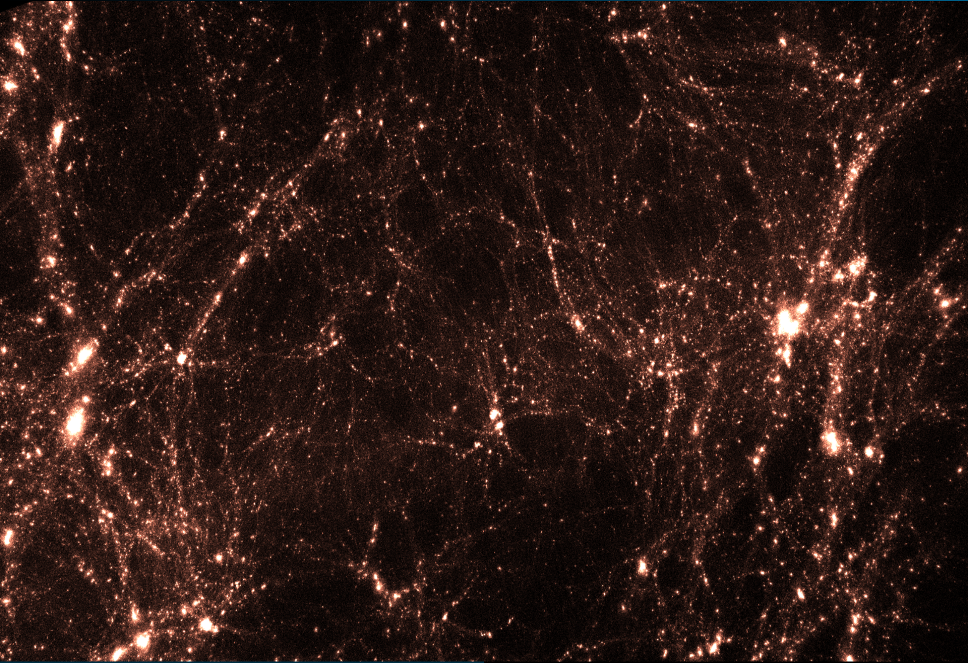
Viability Conditions

$$\begin{cases} 1 + f_R > 0 & \text{no ghosts} \\ f_{RR} < 0 & \text{no tachyons} \end{cases} \qquad \begin{cases} f \to 0 \\ f_R \to 0 \end{cases} \text{for } |R| \gg |R_c|$$

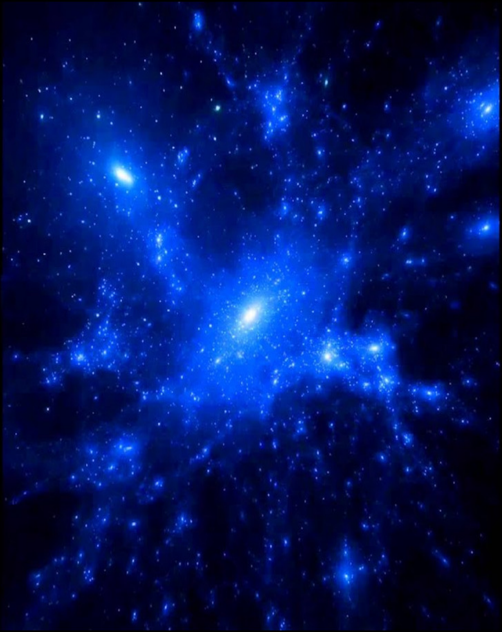
Cosmological Simulations



Cosmological Simulations



Cosmological Simulations



Cosmological Simulations

- Metric (Poisson Gauge)

$$ds^2 = a^2(\tau)[-(1+2\Phi-2\chi)d\tau^2 - 2B_i\,dx^i d\tau + (1-2\Phi)\delta_{ij}dx^i dx^j + h_{ij}dx^i dx^j]$$

- Scalar potential Φ; gravitational slip $\chi \equiv \Phi - \Psi$; frame dragging B_i; tensor perturbations h_{ij}
- Background

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 (T^0_0 - \bar{T}^0_0)$$

- **Weak field expansion**: metric perturbations remain small at all times on this background
- Homogenous modes are computed consistently so that the observables do not depend on the precise choice of $\bar{T}^\mu_\nu$ – can capture **backreaction!**
- Density perturbations are kept fully general. Because $\Delta\Phi \sim \delta\varrho$, keep terms up to second order in Φ if they contain 2 spatial derivatives

- CDM particles can have **arbitrary positions and velocities**
- Geodesic motion computed interpolating the metric fields

Action

$$\mathcal{S}_\mathrm{p} = -m \int |\mathrm{d}s| \simeq -m \int \mathrm{d}\tau \, a\sqrt{1 - v^2} \left(1 + \frac{\Psi + v^2\Phi + B_i v_i - \frac{1}{2} h_{ij} v^i v^j}{1 - v^2}\right)$$

Conjugate Momentum

$$q_i \equiv \frac{\partial \mathcal{L}_\mathrm{p}}{\partial v^i} = \frac{ma}{\sqrt{1 - v^2}} \left[v_i \left(1 - 2\Phi - \frac{\Psi + v^2\Phi + B_j v^j - \frac{1}{2} h_{jk} v^j v^k}{1 - v^2}\right) - B_i + h_{ij} v^j \right]$$

Geodesic Equations

$$\begin{cases} v^i = \dfrac{\mathrm{d}x^i}{\mathrm{d}\tau} = \dfrac{q_i}{\sqrt{q^2 + m^2 a^2}} \left[1 + \Psi + \left(2 - \dfrac{q^2}{q^2 + m^2 a^2}\right) \Phi + \dfrac{q^j q^k h_{jk}}{2(q^2 + m^2 a^2)} \right] + B_i - \dfrac{h_{ij} q^j}{\sqrt{q^2 + m^2 a^2}} \\ \dfrac{\mathrm{d}q_i}{\mathrm{d}\tau} = \dfrac{\partial \mathcal{L}_\mathrm{p}}{\partial x^i} = \sqrt{q^2 + m^2 a^2} \left(\Psi_{,i} + \dfrac{q^2}{q^2 + m^2 a^2} \Phi_{,i} + \dfrac{q^j B_{j,i}}{\sqrt{q^2 + m^2 a^2}} - \dfrac{q^j q^k h_{jk,i}}{2(q^2 + m^2 a^2)} \right) \end{cases}$$

- From the particle ensemble once can construc the stress-energy tensor and interpolate on the grid points
- It then acts as a source term for the metric perturbations (see later)

$$T^0_0 = \frac{1}{a^4}\sum_n \delta^{(3)}(\mathbf{x}-\mathbf{x}_n)\left[\sqrt{q^2+m^2a^2}\left(1+3\Phi+\frac{q^2}{q^2+m^2a^2}\Phi+q^iB_i-\frac{q^iq^jh_{ij}}{2\sqrt{q^2+m^2a^2)}}\right)\right]$$

$$T^i_j = \frac{\delta^{ik}}{a^4}\sum_n \delta^{(3)}(\mathbf{x}-\mathbf{x}_n)\left[\frac{q_jq_k}{\sqrt{q^2+m^2a^2}}\left(1+4\Phi+\frac{m^2a^2}{q^2+m^2a^2}\Phi+\frac{h_{\ell m}q^\ell q^m}{2(q^2+m^2a^2)}\right)+q_jB_k-\frac{q_jq^\ell h_{k\ell}}{\sqrt{q^2+m^2a^2}}\right]$$

$$T^i_0 = -\frac{\delta^{ij}}{a^4}\sum_n \delta^{(3)}(\mathbf{x}-\mathbf{x}_n)\left[q_j\left(1+5\Phi+\Psi+\frac{q^\ell B_\ell}{\sqrt{q^2+m^2a^2}}\right)+B_j\sqrt{q^2+m^2a^2}-q^kh_{jk}\right]$$

$$T^0_i = \frac{1}{a^4}\sum_n \delta^{(3)}(\mathbf{x}-\mathbf{x}_n)q_i(1+3\Phi-\Psi)$$

Background Evolution

$$3\ddot{\bar{f}}_R + 6\mathcal{H}\dot{\bar{f}}_R + \left(2\bar{f} - \bar{f}_R\bar{R}\right) a^2 = -a^2 \left(\bar{R} + 8\pi G a^2 \bar{T}^{\mu}_{\mu}\right)$$

or

$$3\bar{f}_{RR}\ddot{\bar{R}} + 3\bar{f}_{RRR}\dot{\bar{R}}^2 + 6\mathcal{H}\bar{f}_{RR}\dot{\bar{R}} + (2\bar{f} - \bar{f}_R\bar{R})a^2 = -a^2 \left(\bar{R} + 8\pi G\bar{T}^{\mu}_{\mu}\right)$$

- Metric perturbations are the same as in GR
- Curvature

$$R = \bar{R} + \delta R \qquad (\delta R/\bar{R} \text{ arbitrary})$$

- In general, we will have

$$\bar{R} \neq -8\pi G\bar{T} \qquad \delta R \neq -8\pi G\,\delta T$$

- Scalaron

$$f_R = \bar{f}_R + \delta f_R$$

- Important to subtract the correct background to test for backreaction

$$\langle\delta R\rangle \neq \langle\delta T\rangle = 0$$
$$\langle\Phi\rangle \to \langle\Phi_{\rm GR}\rangle + \frac{1}{2}\langle\delta f_R\rangle \qquad \text{and} \qquad \langle\delta f_R\rangle \sim \bar{f}_{RR}\,\langle\delta R\rangle \neq 0$$

Newtonian Limit

- Newtonian limit of $f(R)$ gravity very well understood: Oyaizu *0807.2449*, MG-Gadget [*Puchwein, Baldi, Springel, 1305.2418*], ISIS [*Llinares, Mota, Winther, 1307.6748*], …
- But: interesting effects if one relaxes quasi-static approximation (*Arbuzova, Dolgov, LR: 1306.5694, 1507.02152*; *Hagala, Llinares & Mota 1607.02600*, *Sawicki & Bellini 1503.06831*)
- Relativistic effects? (backreaction, slip, …)

Poisson Equation

$$\Delta\Phi_N \approx 4\pi G a^2 \delta\varrho + \frac{1}{2}\Delta f_R \quad \Rightarrow \quad \Phi_{f(R)} \approx \Phi_{\mathrm{GR}} + \frac{\delta f_R}{2}$$

Scalaron Evolution

$$\Delta f_R \approx \frac{a^3}{3}\left[\delta R(f_R) - 8\pi G \delta\varrho\right]$$

Gravitational Slip

$$\chi \approx \chi_{\mathrm{GR}} + \delta f_R$$

Geodesic Motion

$$\ddot{\mathbf{x}} = -\nabla\Psi_{\mathrm{GR}} \approx -\nabla\Phi_{\mathrm{GR}} + \frac{1}{2}\nabla f_R$$

Trace Equation – Scalaron Evolution

$$\Delta f_R \approx \frac{a^3}{3}\left[\delta R(f_R) - 8\pi G\delta\varrho\right]$$

$$\downarrow$$

$$(1+2\Phi)\Delta f_R - 2\mathcal{H}\delta\dot{f}_R = \frac{a^2}{3}\left[(1-f_R)\delta R(f_R) + 8\pi G\delta T + 2\delta f - \bar{R}\,\delta f_R\right]$$

- Non-linear relation $R \leftrightarrow f_R$, so spectral methods (FFT) not applicable
- Relaxation solver (Newton-Raphson)

$$Y[f_R, T_{\mu\nu}] = 0 \qquad f_R^{(i+1)} = f_R^{(i)} - \left.\frac{Y}{\partial Y/\partial f_R}\right|_{f_R = f_R^{(i)}}$$

- Where needed change of variable prevents f_R from assuming forbidden values. E.g Hu-Sawicki: $f_R \leftrightarrow R$ well-defined only for $f_R < 0$

$$f_R = \bar{f}_R e^u \qquad u \equiv \ln(f_R/\bar{f}_R)$$

- By far the most computationally expensive equation
  $\Rightarrow$ **MULTIGRID** acceleration: solve on coarser grid first, then extend to finer grids

00 Equation – Gravitational Potential Φ

$$\Delta\Phi \simeq -4\pi G a^2 \delta T^0_0 + \frac{1}{2}\Delta f_R$$

$$\downarrow$$

$$\left(\Delta - \frac{3\mathcal{H}}{d\tau} - 3\mathcal{H}^2\right)\Phi_t =$$

$$= -4\pi G a^2(1-4\Phi)\delta T^0_0 - 3\mathcal{H}^2\chi - \frac{3\mathcal{H}}{d\tau}\Phi - \frac{3}{2}\delta^{ij}\Phi_{,i}\Phi_{,j} +$$

$$+ \frac{1-2\Phi-f_R}{2}\Delta f_R - \frac{3\mathcal{H}}{2}\delta\dot{f}_R - \frac{3\mathcal{H}^2}{2}\delta f_R - \frac{1}{2}\delta^{ij}\Phi_{,i}\delta f_{R,j} + \frac{R\delta f_R + \bar{f}_R\delta R - \delta f}{4}a^2$$

- Already solved for the scalaron $\rightarrow$ can put δf_R in the source term and use spectral methods (FFT)
- Minor increase in computational time compared to GR

Traceless ij Equation

$$\dot{B}_{(i,j)} + 2\mathcal{H}B_{(i,j)} + \chi_{,ij} - \frac{1}{3}\delta_{ij}\Delta\chi =$$

$$= 8\pi G a^2 \Pi_{ij} - \left(\delta_i^k \delta_j^\ell - \frac{1}{3}\delta^{k\ell}\delta_{ij}\right)\left[2\Phi_{,k}\Phi_{,\ell} + 2(2\Phi - \chi)\Phi_{,k\ell} - \delta f_{R,k\ell}\right]$$

$$\equiv S_{ij} - \frac{1}{3}\delta_{ij}S + \delta f_{R,ij} - \frac{1}{3}\delta_{ij}\Delta\delta f_R \,.$$

where

$$\Pi_{ij} \equiv \left(\delta_{ik}\delta_j^\ell - \frac{1}{3}\delta_k^\ell \delta_{ij}\right) T_\ell^k$$

- Spin-0 component (Fourier space)

$$\widetilde{\chi} = \frac{\left(k^2\delta^{ij} - 3k^i k^j\right)}{2k^4}\widetilde{S}_{ij}\left(\Phi, \chi\right) + \delta\widetilde{f}_R \,,$$

- Spin-1 component: parabolic equation for $\mathcal{B}_i \equiv a^2 B_i$ (“same” as in GR)

$$\dot{\widetilde{\mathcal{B}}}_i = -\frac{2ia^2}{k^4}\delta_{i\ell}\left(k^2\delta^{j\ell} - k^j k^\ell\right) k^m \widetilde{S}_{jm}$$

0i Equation

$$-\frac{1}{2}\Delta B_i - B_i\Delta\Phi + \delta^{jk}B_j(\delta f_R - \Phi)_{,ik} - \mathcal{H}(2\Phi - 2\chi + \delta f_R)_{,i} - \\ - 2\dot{\Phi}_{,i} + \delta\dot{f}_{R,i} - \dot{f}_R\Phi_{,i} - 2(\Phi - \chi)\delta\dot{f}_{R,i} = 8\pi G a^2 T^0_i\,,$$

- Spin-1 component: elliptic constraint for B_i ("same" as GR)

$$\delta^{ij}(\widetilde{B}_i)_t = 2\,k^{-4}\left(k^2\delta^{ij} - k^i k^j\right)\text{ Fourier}\left\{8\pi G a^2 T^0_i + B_i\Delta\Phi + \delta^{k\ell}B_k(\Phi - \delta f_R)_{,i\ell}\right\} \\ \equiv k^{-4}\left(k^2\delta^{ij} - k^i k^j\right)\widetilde{S}^0_i\,.$$

- Spin-0 component: parabolic equation for scalaron

$$\delta\dot{\widetilde{f}}_R - \mathcal{H}\delta\widetilde{f}_R = -8\pi G a^2 i k^{-2} k^i \widetilde{T}^0_i + \cdots$$

$\rightarrow$ **UNSTABLE!** Deviations at small scales $\sim \ell_{\text{cell}}$ propagate to larger scales

Models

Hu-Sawicki [*Hu & Sawicki 2007*]

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{1 + c_2 (R/m^2)^n}$$

$$f \approx -m^2 \frac{c_1}{c_2} + m^2 \frac{c_1}{c_2^2} \left(\frac{m^2}{R}\right)^n \qquad f_R \approx -\frac{n\, c_1}{c_2^2} \left(\frac{m^2}{R}\right)^{n+1}$$

- Fix $m^2 = H_0^2 \Omega_m$, then c_1/c_2 to produce the observed cosmological constant
- Specify $\bar{f}_{R0}$

$R^{1+\delta}$

$$F(R) = \alpha R^{1+\delta}$$

$$f_{R0} \approx \alpha(1+\delta) R_0^\delta - 1 \quad \Rightarrow \quad \alpha \approx \frac{1 + f_{R0}}{(1+\delta) R_0^\delta}$$

$R + R^n$

$$F(R) = R + \beta R^n \quad \to \quad f(R) = R \left(\frac{R}{\alpha R_0}\right)^{n-1}$$

Point Mass Tests: Setup

- We consider the static field produced by a point mass located in the centre of a cubic box (with periodic boundary conditions) with resolution ℓ_{cell}
- Compare with the analytical solutions obtained linearising the trace equation:

$$\Delta f_R \simeq \frac{\delta f_R}{3\bar{f}_{RR}} - \frac{8\pi G}{3}\delta\varrho$$

with a density field

$$\delta\varrho = \begin{cases} 10^{-4}(N^3-1)\bar{\varrho} & \text{point mass cell} \\ -10^{-4}\bar{\varrho} & \text{elsewhere} \end{cases}$$

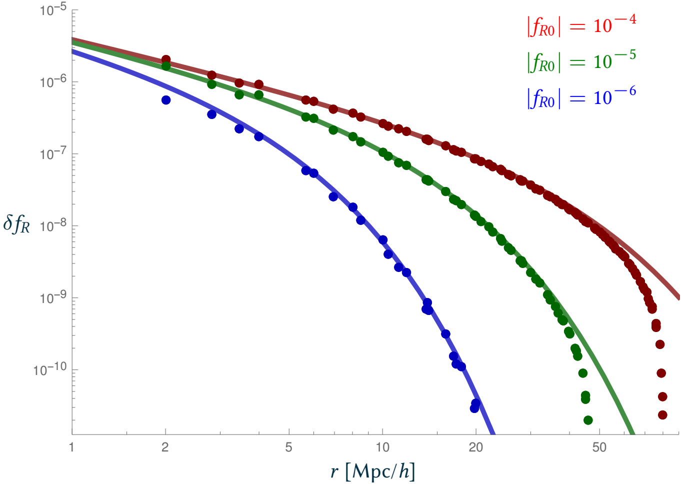
- The formal solution for a point mass $\varrho = m\,\delta^{(3)}(\mathbf{r})$ in an asymptotically flat Universe is a Yukawa-like profile

$$\delta f_R = \frac{2Gm}{3}\frac{e^{-r/\sigma}}{r} \qquad \text{with} \qquad \begin{cases} \sigma^2 = 3\bar{f}_{RR} \\ m \to 10^{-4}(N^3-1)\bar{\varrho}\,\ell_{\mathrm{cell}}^3 \end{cases}$$

- For $|f_{R0}| = 10^{-6}$, $\delta f_R > |f_{R0}|$ at small distances, and hence the linear approximation fails. Thus we replace the point mass with a gaussian density profile

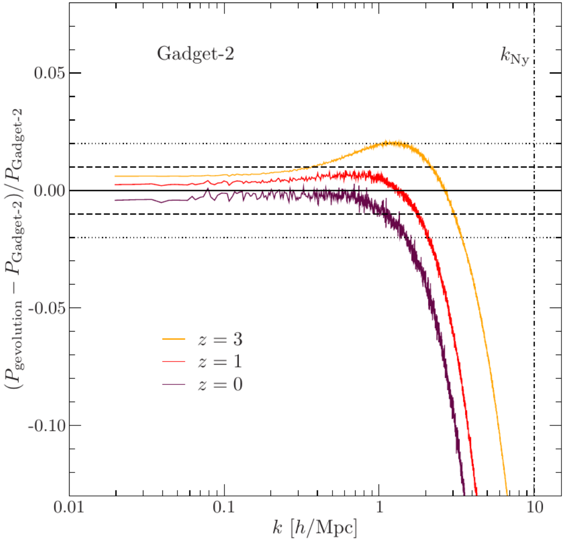
$$\varrho \propto \exp\left(-\frac{r^2}{\ell_{\mathrm{cell}}^2}\right) ,$$

Point Mass Tests: Results (Hu-Sawicki)



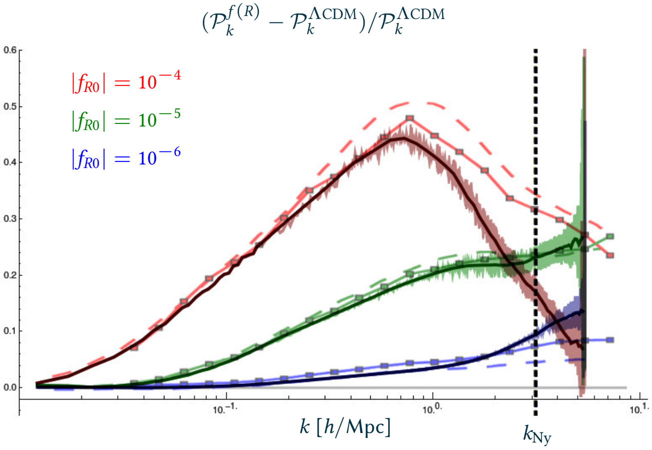
- Deviations from analytical solution expected at large radii (boundary effects – periodic vs. asymptotically flat boundary conditions) and small radii for $|f_{R0}| = 10^{-6}$ (finite resolution)
- Excellent agreement at intermediate radii

Results: gevolution vs. gadget-2



- Excellent agreement at $k \ll k_{\mathrm{Ny}}$, but too little power at small scales $\sim k_{\mathrm{Ny}}$

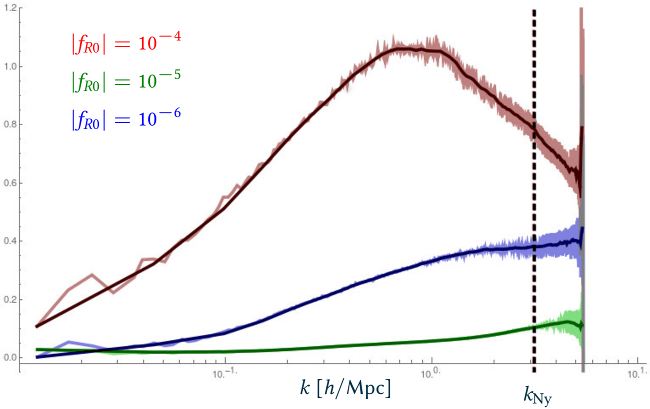
Matter Power Spectrum – Hu-Sawicki
Comparison with MG-gadget



- Increase of power up to about 50% at scales of about $1h/\text{Mpc}$
- No evidence of backreaction :(
- Excellent agreement at all scales except for $|f_{R0}| = 10^{-4}$ at small scales $\sim k_{\text{Ny}}$: true effect or consequence of lower resolution? Either way: why?

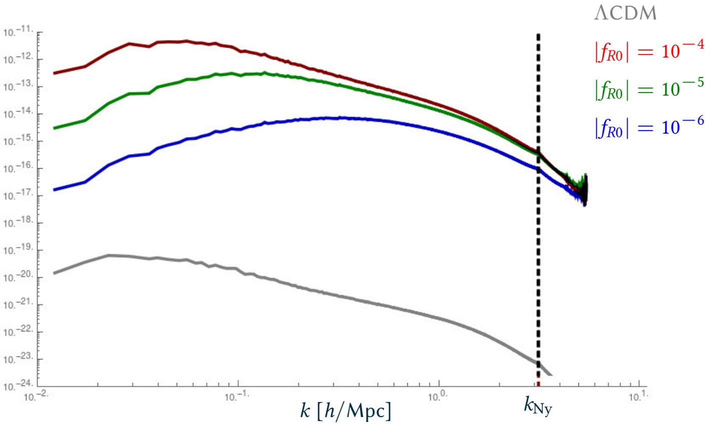
Vector Modes B_i – Hu-Sawicki

$(\mathcal{P}_k^{f(R)} - \mathcal{P}_k^{\Lambda\mathrm{CDM}})/\mathcal{P}_k^{\Lambda\mathrm{CDM}}$



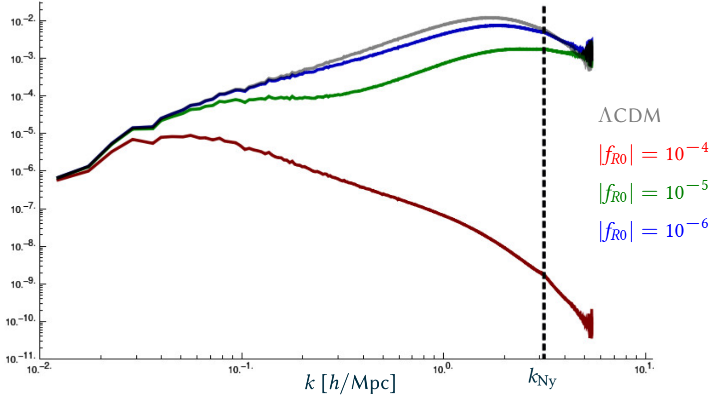
- Increase of power up to about 100% at scales of about $1h/\mathrm{Mpc}$

Gravitational slip χ – Hu-Sawicki



- χ huge compared to ΛCDM (while still $\ll \Phi$). Detectable? (e.g. *Pizzuti et al. 1901.01961*)

Curvature δR – Hu-Sawicki

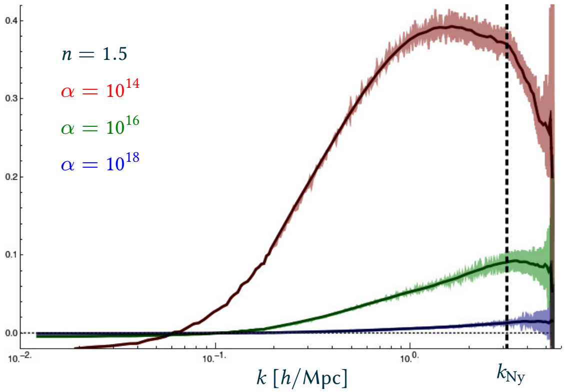


- δR can be completely different than $8\pi G\delta\varrho$, even for relatively small deviations in $\delta\varrho$ (see previous slides)
- Assuming $\delta R \sim \delta\varrho$ for estimates and semi-analytical results can lead to BIG errors

Matter Power Spectrum – $R + R^n$

$$f(R) = R\left(\frac{R}{\alpha R_0}\right)^{n-1}$$

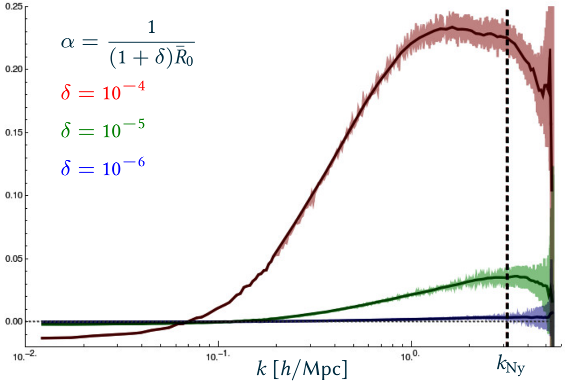
$$(\mathcal{P}_k^{f(R)} - \mathcal{P}_k^{\Lambda\mathrm{CDM}})/\mathcal{P}_k^{\Lambda\mathrm{CDM}}$$



Matter Power Spectrum – $R^{1+\delta}$

$$F(R) = \alpha R^{1+\delta}$$

$$(\mathcal{P}_k^{f(R)} - \mathcal{P}_k^{\Lambda\text{CDM}})/\mathcal{P}_k^{\Lambda\text{CDM}}$$



Conclusions and Future Work

- $f(R)$ gravity is a popular class of modified gravity theories that has not yet been ruled out
- Cosmological simulations might help us constrain these models
- Until now, $f(R)$ simulations have used a modified Newtonian approach – is it enough?
- `fRevolution`: extend `gevolution` to $f(R)$ gravity. Relativistic effects taken into account; good agreement with existing codes at scales $\gg$ Nyquist, even in the non-linear regime. GR recovered in the appropriate limits.
- Future work:
  - Tests of quasi-static approximation (but: large time resolution needed, computationally challenging)
  - Degeneracy between modified gravity and relativistic neutrinos
  - Inclusion of ray-tracing module: weak-lensing, RSD, etc.
  - Halo and void statistics
  - Extensions to other scalar-tensor theories?