A Model of Spinning Massless Particle in the Gravitational Field

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Hot Topics in Modern Cosmology
Spontaneus Workshop XIII

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to the memory of Christian Duval

Introduction I

- Particles endowed with mass, spin, charge, magnetic momentum, are usually described using Quantum Mechanics
- The worldlines of point-like particles in General Relativity, are timelike geodesics for massive particles.
- The description of the motion of spinning particles obeys to differential "Universal Equations" (MDP equations). [Mathisson, 1937], [Papapetrou, 1951], [Dixon, 1970]
- However, no equations were provided for massless spinning particles.

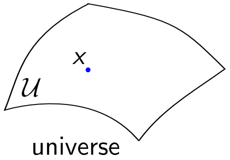
Introduction II

- We present an alternative formulation of the principles of GR [JM Souriau, 1974]
- The MPD universal equations for spinning particles follow in a straightforward manner
- It was used to obtain classical equations of motion for spinning massive charged particles with magnetic momentum in GR with the presence of an ElectroMagnetic field [C Duval, 1972]
- It allows us to build a classical model aimed at describing the motion in gavitational fields of massless spinning particles [PS, 1976]

Outline

- The Principle of General Relativity
- Universal Equations for Spinning Particles
- General Conservation Law
- Motion of Massless and Spinning Particles

The Principle of General Relativity - I



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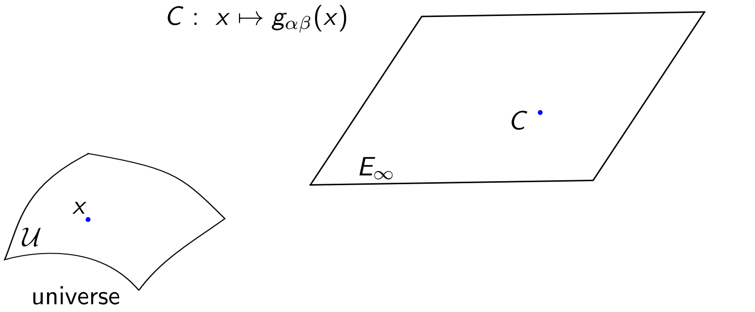

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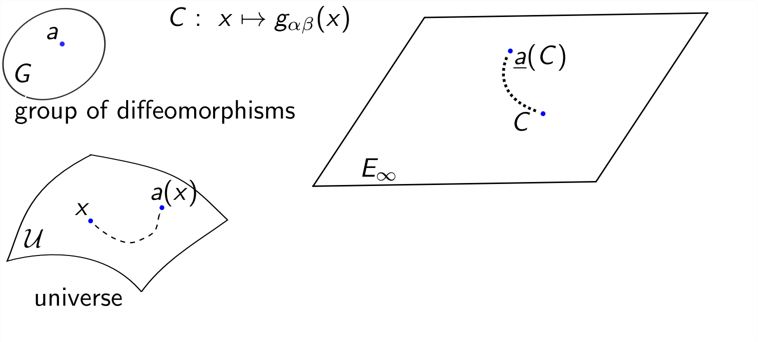

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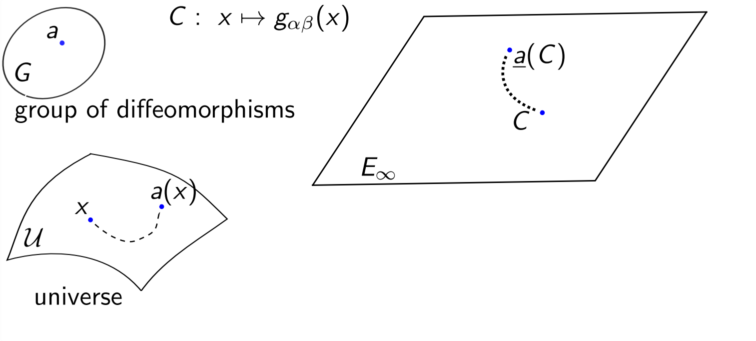

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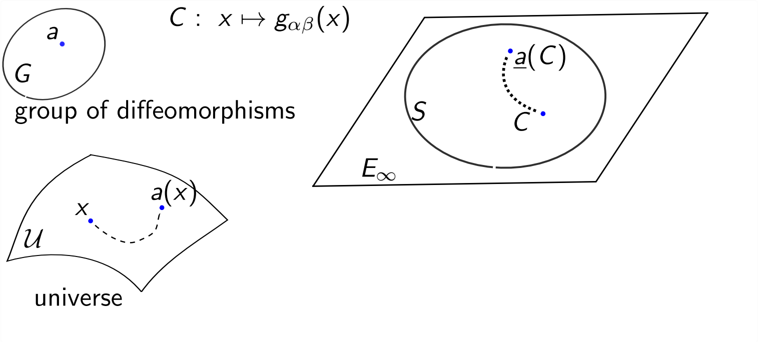

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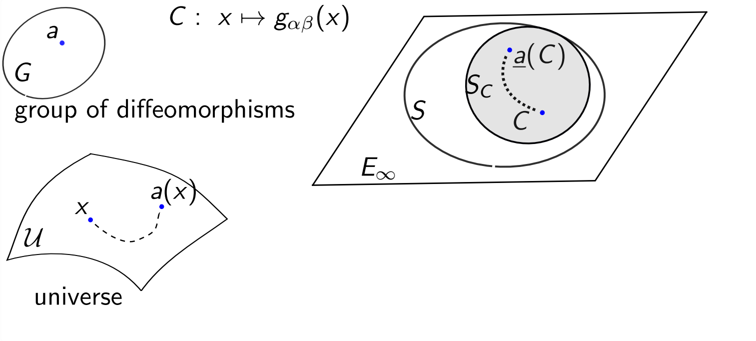

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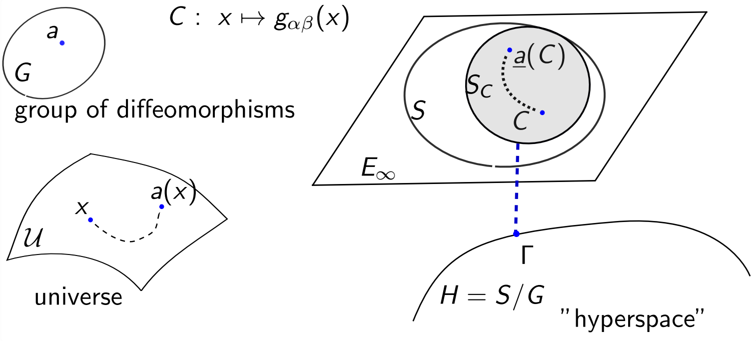

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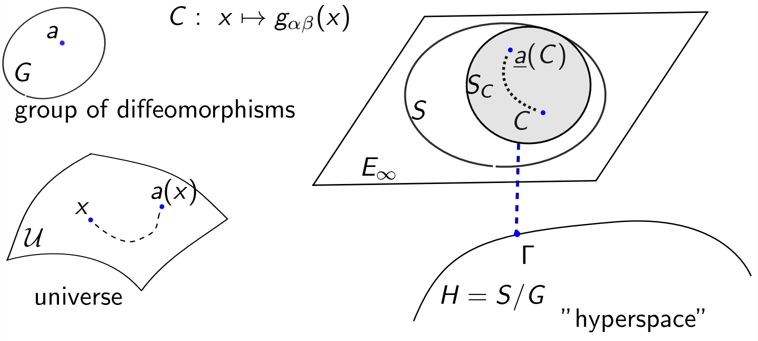
$C \in S$: subset of metrics with signature (+ - - -)
S_C: the orbit of C under G
Manifold $H = S/G$: "hyperspace"; $\Gamma \in H$ is the class of C

The Principle of General Relativity - II



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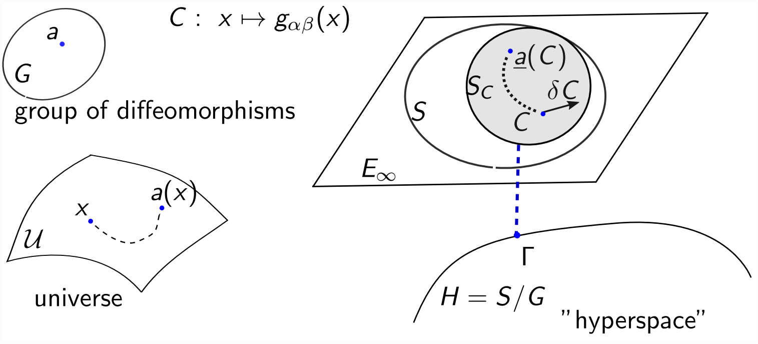

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$V : x \mapsto \delta x = V$: C^∞ vector field with compact support $\Omega_V \subset \mathcal{U}$

Lie derivative $L_V C = x \mapsto [L_V g]_{\alpha\beta} = \nabla_\alpha V_\beta + \nabla_\beta V_\alpha$ where $V_\mu = g_{\mu\nu} V^\nu$

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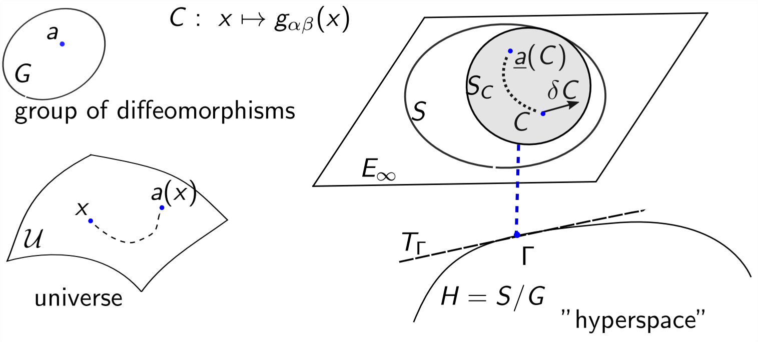

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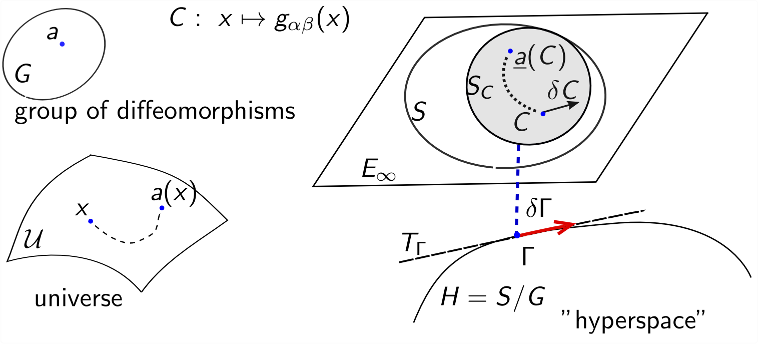

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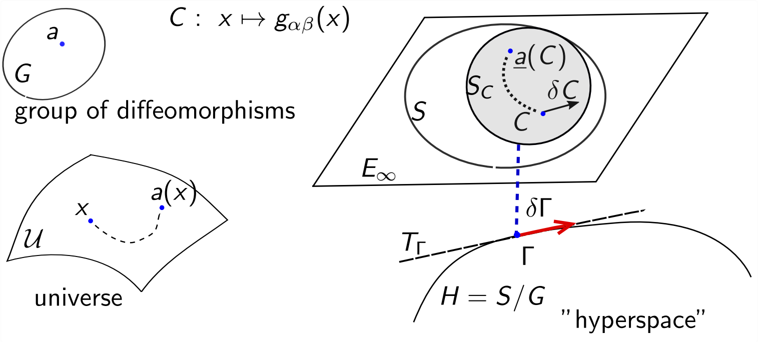

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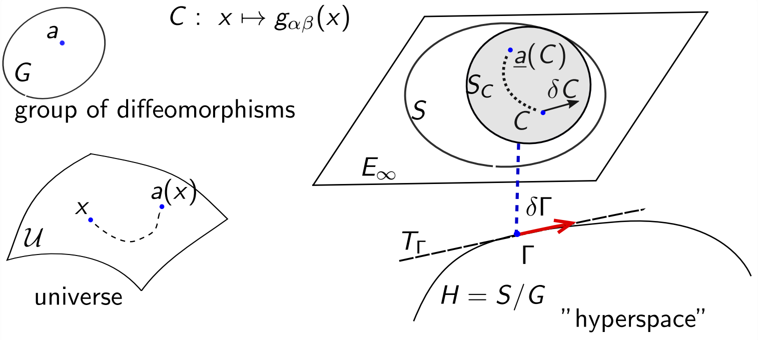

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T_Γ is a vector space as the quotient of two vector spaces

No further assumption needed about the manifold structure of H

The Principle of General Relativity - III

Matter distribution on the universe $\mathcal{U}$

Definition of the cotangent space $\quad \mu \in T^\star_\Gamma$:

$$\mu(\delta\Gamma) = M(\delta C) \iff \begin{cases} \forall (x \mapsto V) \text{ compactly supported} \\ \delta C : \ \delta g_{\alpha\beta} = \nabla_\alpha V_\beta + \nabla_\beta V_\alpha \\ \mu(\delta\Gamma) = 0 \end{cases}$$

$$\langle \mathcal{T} | \delta g \rangle = M(\delta C) = \int_{\mathcal{U}} \frac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta} \, \mathrm{vol}, \quad \forall \delta C \in E_\infty$$

$$\text{where } \ \mathrm{vol} = \sqrt{|\det(g)|} \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3 \, \mathrm{d}x^4$$

$T^{\alpha\beta}$: the stress-energy tensor and $x \mapsto \delta C = \delta g_{\alpha\beta}$: test function

Matter distribution on $\mathcal{U}$: $\quad \left(x \mapsto T^{\alpha\beta}\right) \ \leftrightarrow \ \mu \in T^\star_\Gamma \ \leftrightarrow \ \mathcal{T}$: $\quad$ tensor distribution

The Principle of General Relativity - IV

Souriau's general covariance condition

$$
\textbf{(S)} \qquad \left\{ \begin{array}{l} \forall (x \mapsto V) \text{ with compact support } \Omega_V \subset \mathcal{U} : \\ \langle \mathcal{T} | L_V g \rangle = \displaystyle\int_{\mathcal{U}} \frac{1}{2} T^{\alpha\beta} \left(\nabla_\alpha V_\beta + \nabla_\beta V_\alpha \right) \mathrm{vol} = 0 \end{array} \right.
$$

T is a symmetric tensor: $\qquad T^{\alpha\beta} = T^{\beta\alpha} \quad \Rightarrow$

$$
\int_{\mathcal{U}} T^{\alpha\beta} \nabla_\alpha V_\beta \, \mathrm{vol} = 0 \quad \Leftrightarrow \quad \int_{\mathcal{U}} \nabla_\alpha \left(T^{\alpha\beta} V_\beta \right) \mathrm{vol} - \int_{\mathcal{U}} \left(\nabla_\alpha T^{\alpha\beta} \right) V_\beta \, \mathrm{vol} = 0 \quad \forall V
$$

$$
\boxed{\textbf{(S)} \quad \Leftrightarrow \quad \nabla_\alpha T^{\alpha\beta} = 0}
$$

Universal Equations for Spinning Particles - I

The distribution $\mathcal{T}$ may be discontinue: it is then supported by a submanifold $\mathcal{M} \subset \mathcal{U}$, e.g., 3, 2 or 1-dimensional for condensed states of matter

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The worldline Λ of a point-like particle is a one-dimensional submanifold of $\mathcal{U}$.

$$\langle \mathcal{T}_\Lambda | \delta g \rangle = \int_\Lambda \frac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta} \, \mathrm{d}\tau \quad \text{where } \Lambda \text{ is parametrized by } \tau \in \mathbb{R}$$

- $\delta g_{\alpha\beta}$ are test functions for the distribution $\mathcal{T}$
- $\frac{1}{2} T^{\alpha\beta} \, \mathrm{d}\tau$ is the tensor density on the curve Λ

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Expanding the distribution $\mathcal{T}$ to first order yields the general form

$$\langle \mathcal{T}_\Lambda | \delta g \rangle = \frac{1}{2} \int_\Lambda \left[\Theta^{\alpha\beta} \delta g_{\alpha\beta} + \Psi^{\alpha\beta\gamma} \nabla_\alpha \delta g_{\beta\gamma} \right] \mathrm{d}\tau$$

where $\frac{1}{2}\Theta \, \mathrm{d}\tau$ and $\frac{1}{2}\Psi \, \mathrm{d}\tau$ are tensor densities on Λ that define $\mathcal{T}_\Lambda$

Universal Equations for Spinning Particles - II

Souriau's general covariance condition on Λ

$$\textbf{(S)} \quad \langle \mathcal{T}_\Lambda | L_V g \rangle = 0, \quad \forall (x \mapsto V) \text{ with compact support } \Omega_V \subset \mathcal{U}$$

$$\Downarrow$$

$$\exists P \in T_x,\ S \in T_x \otimes T_x \mid S \text{ is skew symmetric: } S^{\alpha\beta} + S^{\beta\alpha} = 0$$

$$\langle \mathcal{T}_\Lambda | \delta g \rangle = \frac{1}{2} \int_\Lambda \left[P^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \delta g_{\mu\nu} + S^{\mu\nu} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \nabla_\mu \delta g_{\nu\rho} \right] \mathrm{d}\tau$$

P and S satisfying the

Mathisson-Papapetrou-Dixon "Universal Equations"

$$\textbf{(MPD)} \quad \begin{cases} \dfrac{\hat{\mathrm{d}} P^\mu}{\mathrm{d}\tau} = -\dfrac{1}{2} R^\mu_{\rho,\alpha\beta} S^{\alpha\beta} \dfrac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \\ \dfrac{\hat{\mathrm{d}} S^{\mu\nu}}{\mathrm{d}\tau} = P^\mu \dfrac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - P^\nu \dfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \end{cases}$$

where the hat ($\hat{}$) denotes the covariant derivative on Λ with respect of τ

General Conservation Law

$$\textbf{(S)} \quad \Rightarrow \quad \langle \mathcal{T} | L_V g \rangle = 0 \mid \forall (x \mapsto V) \text{ with compact support } \Omega_V \subset \mathcal{U}$$

$$\langle \mathcal{T} | L_V g \rangle = \int_{\mathcal{U}} \nabla_\alpha \left(T^{\alpha\beta} V_\beta \right) \mathrm{vol} - \int_{\mathcal{U}} \left(\nabla_\alpha T^{\alpha\beta} \right) V_\beta \, \mathrm{vol}$$

If $(x \mapsto W)$ is **not** compactly supported, we still have $\nabla_\alpha T^{\alpha\beta} = 0$, but

$$\langle \mathcal{T} | L_W g \rangle = \int_{\mathcal{U}} \nabla_\alpha \left(T^{\alpha\beta} W_\beta \right) \mathrm{vol} \neq 0$$

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For a 1-dimensional manifold Λ, and first order distribution:

$$\langle \mathcal{T} | L_W g \rangle = \int_{\Lambda} \mathrm{d} \left(P^\alpha W_\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\alpha W_\beta \right) \neq 0$$

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But if Z is a Killing vector of the metrics $(\mathcal{U}, g)$, i.e., leaves g invariant

$$L_Z g = 0 \quad \Rightarrow \quad \langle \mathcal{T} | L_Z g \rangle = 0 \quad \Rightarrow \quad P^\alpha Z_\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\alpha Z_\beta = \text{ const.}$$

Noetherian-like first integral, independent from any model of particle

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Motion of Massless and Spinning Particles

(MPD) equations are **not** deterministic: an equation for $\frac{\mathrm{d}x}{\mathrm{d}\tau}$ is missing

Equations of State for P and S, compatible with **(MPD)**:

$$\begin{cases} P_\mu P^\mu = 0 \; ; \quad P \text{ future-pointing} \\ S^{\alpha\beta} P_\alpha = 0 \; ; \; S \neq 0 \; \forall x \in \Lambda \end{cases}$$

$$\Longrightarrow \qquad \star(S)^{\alpha\beta} P_\beta = \chi s P^\alpha \Rightarrow \; S^{\mu\nu} S_{\nu\mu} = -2s^2 = \text{const.}$$

$\chi = \pm 1$: helicity, and $s \geq 0$: (scalar) spin ($s = \hbar$ for a photon)
$\star$ is the Hodge star, $R(S)_{\mu\nu} = R_{\alpha\beta,\mu\nu} S^{\alpha\beta}$, $\mathrm{pf}(R(S) = \star(R(S)) \, .R(S)$

definitions

Motion of Massless and Spinning Particles

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Complete system of equations:

$$\boxed{\begin{aligned} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} &= P^\mu + \frac{2}{R_{\lambda\mu,\nu\rho} S^{\lambda\mu} S^{\nu\rho}} S^{\mu\alpha} R_{\alpha\beta,\lambda\rho} S^{\beta\lambda} P^\rho \\ \frac{\hat{\mathrm{d}} P^\mu}{\mathrm{d}\tau} &= -\chi s \frac{\mathrm{pf}\,(R(S))}{R_{\lambda\mu,\nu\rho} S^{\lambda\mu} S^{\nu\rho}} P^\mu \\ \frac{\hat{\mathrm{d}} S^{\mu\nu}}{\mathrm{d}\tau} &= P^\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} P^\nu \end{aligned}}$$

Motion of Spinless Particles

In the present framework, equations for spinless particles are obtained by limiting the distribution $\mathcal{T}_\Lambda$ to the monopole term, i.e., by setting the dipolar term to zero.

This means that the equations of state for these particles are:

$$\begin{cases} P_\mu P^\mu = \mathrm{const.} \geq 0 \\ S = 0 \end{cases}$$

Then the **(MPD)** equations become

$$\begin{cases} \dfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = P^\mu \\ \dfrac{\hat{\mathrm{d}}P^\nu}{\mathrm{d}\tau} = 0 \end{cases}$$

which is the equation of a time-like or null geodesic: the usual "Principle of Geodesics" is recovered.

Summary

Souriau's formulation of the principles of General Relativity yields

- ✓ the MPD universal equations for the motion of spinning particles;
- ✓ the conservation law associated with a Killing vector;
- ✓ by setting the equation of state

$$\overline{P}.P = 0, \quad SP = 0 \;\Rightarrow\; \{\; \star(S)P = \chi s P \;;\; \mathrm{Tr}\,(S^2) = -s^2 \;\}$$

it also yields the complete system of equations for a massless spinning particle:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = P + \frac{2}{R(S)(S)} S.R(S)P$$

$$\frac{\hat{\mathrm{d}}P}{\mathrm{d}\tau} = -\chi s \frac{\mathrm{pf}\,(R(S))}{R(S)(S)} P$$

$$\frac{\hat{\mathrm{d}}S}{\mathrm{d}\tau} = P.\overline{\frac{\mathrm{d}x}{\mathrm{d}\tau}} - \frac{\mathrm{d}x}{\mathrm{d}\tau}.\overline{P}$$

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Thank you for your attention

BACKUP

Some definitions

- **The Hodge star**

  (e_1, e_2, e_3, e_4) orthonormal oriented basis of T_x

  $\star(e_\lambda \wedge e_\mu) = \varepsilon_{\lambda\mu\nu\rho}\, e_\nu \wedge e_\rho$ $\star()$ linear map

  $\varepsilon_{\lambda\mu\nu\rho}$ Levi-Civita tensor $(\varepsilon_{1234} = 1)$

- **Tensors & linear map**

  $S^{\lambda\mu} \in T_x \otimes T_x$: skew symmetric contravariant tensor: $S^{\mu\nu} + S^{\nu\mu} = 0$

  ${S^\lambda}_\mu = g_{\mu\nu} S^{\lambda\nu}$: skew symmetric linear map $S: \ T_x \to T_x$:

  $g(SV, W) + g(V, SW) = 0 \quad \forall V, W \in T_x$

  $S_{\mu\nu} = g_{\lambda\mu} {S^\lambda}_\nu$: skew symmetric covariant tensor: $S_{\mu\nu} + S_{\nu\mu} = 0$

- **Pfaffian**

  F skew linear map: $F.(\star F) = (\star F).F = \mathrm{pf}(F).\mathbb{I}$

  $\star F$ skew linear map: $\mathrm{pf}(\star F) = -\mathrm{pf}(F)$

  $\det F = -\mathrm{pf}(F)^2$

- **Coordinate-free notation**

  $R(S)$: skew symmetric linear map; $R(S)^\mu_\nu = R^\mu_{\nu,\alpha\beta} S^{\alpha\beta}$

  $S.R(S)$: linear map

  $R(S)(S) = R_{\alpha\beta,\mu\nu} S^{\alpha\beta} S^{\mu\nu} \in \mathbb{R}$

back

Coordinate-free equations

P is a (contravariant-)**vector** whose components are P^μ
$\overline{P}$ is the corresponding (co-)**vector** whose components are $\overline{P}_\lambda = g_{\mu\lambda} P^\mu$

$$\boxed{\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= P + \frac{2}{R(S)(S)} S.R(S)P \\ \frac{\hat{\mathrm{d}}P}{\mathrm{d}\tau} &= -\chi s \frac{\mathrm{pf}\,(R(S))}{R(S)(S)} P \\ \frac{\hat{\mathrm{d}}S}{\mathrm{d}\tau} &= P.\overline{\frac{\mathrm{d}x}{\mathrm{d}\tau}} - \frac{\mathrm{d}x}{\mathrm{d}\tau}.\overline{P} \end{aligned}}$$