# Quantum expectation values on black hole space-times

Elizabeth Winstanley

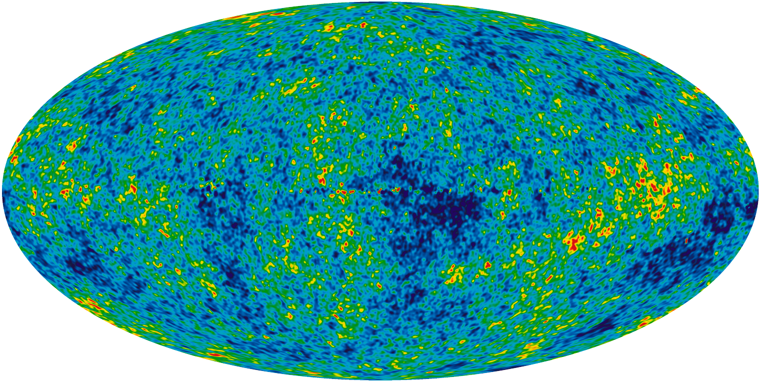
Consortium for Fundamental Physics
School of Mathematics and Statistics
The University of Sheffield

CFP

The University Of Sheffield.

# QFT in curved space-time

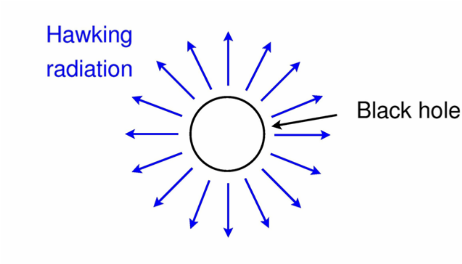
- Fixed classical background geometry
- Quantum field propagating on this background

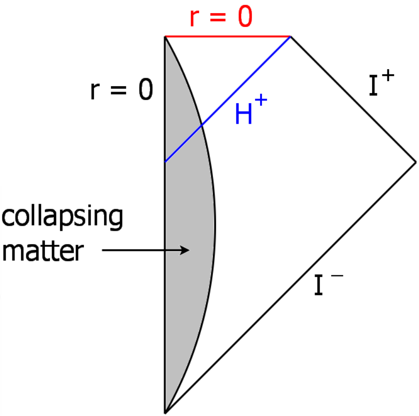


# Hawking radiation

- Black hole formed by gravitational collapse
- Thermal flux at $\mathscr{I}^+$

$$T = \frac{\kappa}{2\pi}$$



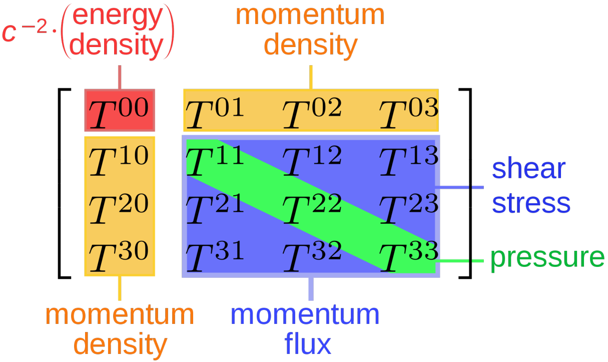




[ Hawking *CMP* **43** 199 (1975) ]

# Stress-energy tensor expectation value

## Semi-classical Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \langle \hat{T}_{\mu\nu} \rangle$$



# Massless, conformally coupled scalar field $\Phi$

## Klein-Gordon equation

$$\left[ \Box - \frac{1}{6} R \right] \Phi = 0$$

# Massless, conformally coupled scalar field $\Phi$

## Klein-Gordon equation

$$\left[ \Box - \frac{1}{6} R \right] \Phi = 0$$

- $\Box$ curved space-time Laplacian $\nabla_\mu \nabla^\mu$

# Massless, conformally coupled scalar field $\Phi$

## Klein-Gordon equation

$$\left[\Box - \frac{1}{6}R\right]\Phi = 0$$

- $\Box$ curved space-time Laplacian $\nabla_{\mu}\nabla^{\mu}$
- $R$ Ricci scalar curvature

# Massless, conformally coupled scalar field $\Phi$

## Klein-Gordon equation

$$\left[ \Box - \frac{1}{6} R \right] \Phi = 0$$

- $\Box$ curved space-time Laplacian $\nabla_\mu \nabla^\mu$
- $R$ Ricci scalar curvature

## Classical stress-energy tensor

$$T_{\mu\nu} = \frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}\Phi^{;\alpha}\Phi_{;\alpha} - \frac{1}{3}\Phi\Phi_{;\mu\nu} + \frac{1}{3}g_{\mu\nu}\Phi\Box\Phi + \frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\Phi^2$$

# Massless, conformally coupled scalar field $\Phi$

## Klein-Gordon equation

$$\left[\Box - \frac{1}{6}R\right]\Phi = 0$$

- $\Box$ curved space-time Laplacian $\nabla_\mu \nabla^\mu$
- $R$ Ricci scalar curvature

## Classical stress-energy tensor

$$T_{\mu\nu} = \frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}\Phi^{;\alpha}\Phi_{;\alpha} - \frac{1}{3}\Phi\Phi_{;\mu\nu} + \frac{1}{3}g_{\mu\nu}\Phi\Box\Phi + \frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\Phi^2$$

# Vacuum polarization

$$\langle \hat{\Phi}^2 \rangle$$

# Vacuum polarization

$$\langle \hat{\Phi}^2 \rangle$$

## Technical motivation

- Simplest nontrivial expectation value
- Simpler to compute than SET

# Vacuum polarization

$$\langle \hat{\Phi}^2 \rangle$$

## Technical motivation

- Simplest nontrivial expectation value
- Simpler to compute than SET

## Physical motivation

- Has some physical features in common with SET
- Related to local temperature

$$T_{\text{local}} \propto \sqrt{\langle \hat{\Phi}^2 \rangle}$$

[ Buchholz & Schlemmer *CQG* **24** F25 (2007) ]

# Renormalizing the vacuum polarization

DeWitt *Phys. Rept.* **19** 295 (1975)
Christensen *PRD* **14** 2490 (1976)
Wald *CMP* **54** 1 (1977)
Christensen *PRD* **17** 946 (1978)
Decanini & Folacci *PRD* **78** 044025 (2008)

# Divergence of the VP

## $\hat{\Phi}^2$

- Involves products of field operators at the same space-time point
- Expectation values are divergent

# Divergence of the VP

## $\hat{\Phi}^2$

- Involves products of field operators at the same space-time point
- Expectation values are divergent

## Expectation value

$$\langle \hat{\Phi}^2(x) \rangle = \lim_{x' \to x} \left[ -\mathrm{i} G_{\mathrm{F}}(x, x') \right]$$

# Divergence of the VP

## $\hat{\Phi}^2$

- Involves products of field operators at the same space-time point
- Expectation values are divergent

## Expectation value

$$\langle \hat{\Phi}^2(x) \rangle = \lim_{x' \to x} \left[ -\mathrm{i} G_{\mathrm{F}}(x, x') \right]$$

Feynman Green's function $G_{\mathrm{F}}(x, x')$

$$\left[ \Box - \frac{1}{6} R \right] G_{\mathrm{F}}(x, x') = -\left(-g\right)^{-\frac{1}{2}} \delta(x - x')$$

# Overall strategy

# Overall strategy

## Regularization by point-splitting

$$-\mathrm{i}G_{\mathrm{F}}(x, x')$$

- Finite for $x' \neq x$
- Divergences as $x' \to x$ are purely geometric and independent of the quantum state

# Overall strategy

## Regularization by point-splitting

$$-\mathrm{i}G_{\mathrm{F}}(x,x')$$

- Finite for $x' \neq x$
- Divergences as $x' \to x$ are purely geometric and independent of the quantum state

## Renormalized expectation value

- Subtract off appropriate divergent terms $G_{\mathrm{S}}(x,x')$
- Take the limit $x' \to x$

$$\langle\hat{\Phi}^2(x)\rangle_{\mathrm{ren}} = \lim_{x'\to x}\left\{-\mathrm{i}\left[G_{\mathrm{F}}(x,x') - G_{\mathrm{S}}(x,x')\right]\right\}$$

# Hadamard renormalization

## Hadamard parametrix

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{U(x,x')}{8\pi^2\sigma(x,x')} + \mathcal{O}\left(\sigma\log\sigma\right)$$

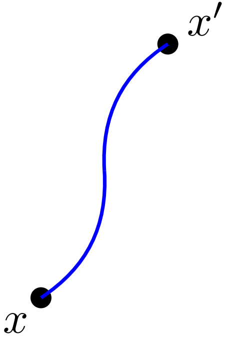
[ Decanini & Folacci *PRD* **78** 044025 (2008) ]

# Hadamard renormalization

## Hadamard parametrix

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{U(x,x')}{8\pi^2\sigma(x,x')} + \mathcal{O}\left(\sigma\log\sigma\right)$$

- $2\sigma(x,x')$ square of the geodesic distance between $x$ and $x'$



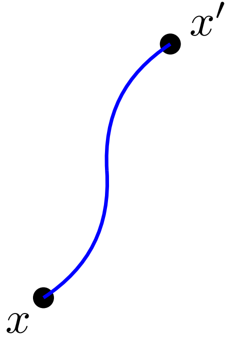

[ Decanini & Folacci *PRD* **78** 044025 (2008) ]

# Hadamard renormalization

## Hadamard parametrix

$$-iG_S(x,x') = \frac{U(x,x')}{8\pi^2\sigma(x,x')} + \mathcal{O}(\sigma \log \sigma)$$

- $2\sigma(x,x')$ square of the geodesic distance between $x$ and $x'$
- $U(x,x')$ symmetric biscalar regular as $x' \to x$




[ Decanini & Folacci *PRD* **78** 044025 (2008) ]

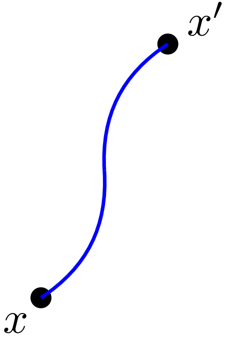
# Hadamard renormalization

## Hadamard parametrix

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{U(x,x')}{8\pi^2\sigma(x,x')} + \mathcal{O}\left(\sigma\log\sigma\right)$$

- $2\sigma(x,x')$ square of the geodesic distance between $x$ and $x'$
- $U(x,x')$ symmetric biscalar regular as $x' \to x$

$$U(x,x') = 1 + \frac{1}{12}R_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu} + \dots$$




[ Decanini & Folacci *PRD* **78** 044025 (2008) ]

# Hadamard renormalization

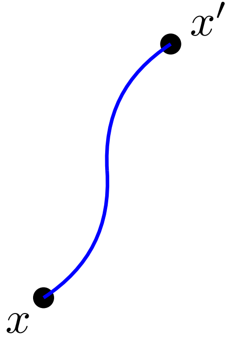
## Hadamard parametrix

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{U(x,x')}{8\pi^2\sigma(x,x')} + \mathcal{O}\left(\sigma \log \sigma\right)$$

- $2\sigma(x,x')$ square of the geodesic distance between $x$ and $x'$
- $U(x,x')$ symmetric biscalar regular as $x' \to x$

$$U(x,x') = 1 + \frac{1}{12} R_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu} + \ldots$$

- Taylor series expansion for $\sigma$, $\sigma^{;\mu}$ in terms of $\Delta x^{\mu}$




[ Decanini & Folacci *PRD* **78** 044025 (2008) ]

# Static, spherically symmetric black hole

## Metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

# Static, spherically symmetric black hole

## Metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

## Positive frequency field modes

$$\phi_{\omega\ell m} = \mathrm{e}^{-\mathrm{i}\omega t}\psi_{\omega\ell}(r)Y_{\ell m}(\theta,\varphi) \qquad \omega > 0$$

# Static, spherically symmetric black hole

## Metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

## Positive frequency field modes

$$\phi_{\omega\ell m} = e^{-i\omega t}\psi_{\omega\ell}(r)Y_{\ell m}(\theta,\varphi) \qquad \omega > 0$$

- $\omega$ frequency

# Static, spherically symmetric black hole

## Metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

## Positive frequency field modes

$$\phi_{\omega\ell m} = \mathrm{e}^{-\mathrm{i}\omega t}\psi_{\omega\ell}(r)Y_{\ell m}(\theta,\varphi) \qquad \omega > 0$$

- $\omega$ frequency
- $Y_{\ell m}(\theta,\varphi)$ spherical harmonic

# Static, spherically symmetric black hole

## Metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

## Positive frequency field modes

$$\phi_{\omega\ell m} = e^{-i\omega t}\psi_{\omega\ell}(r)Y_{\ell m}(\theta,\varphi) \qquad \omega > 0$$

- $\omega$ frequency
- $Y_{\ell m}(\theta,\varphi)$ spherical harmonic
- $\psi_{\omega\ell}(r)$ radial function

# Canonical quantization

$$\hat{\Phi} = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \hat{a}_{\omega\ell m} \phi_{\omega\ell m} + \hat{a}^{\dagger}_{\omega\ell m} \phi^{*}_{\omega\ell m} \right]$$

# Canonical quantization

$$\hat{\Phi} = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \hat{a}_{\omega\ell m}\phi_{\omega\ell m} + \hat{a}^{\dagger}_{\omega\ell m}\phi^{*}_{\omega\ell m} \right]$$

**Boulware vacuum $\hat{a}_{\omega\ell m}|B\rangle = 0$**

State which is as empty as possible far from the black hole

# Canonical quantization

$$\hat{\Phi} = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \hat{a}_{\omega\ell m} \phi_{\omega\ell m} + \hat{a}^{\dagger}_{\omega\ell m} \phi^{*}_{\omega\ell m} \right]$$

## Boulware vacuum $\hat{a}_{\omega\ell m} |B\rangle = 0$

State which is as empty as possible far from the black hole

## Feynman Green's function

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \langle B | \mathcal{T} \left[ \hat{\Phi}(x) \hat{\Phi}(x') \right] | B \rangle$$

# Canonical quantization

$$\hat{\Phi} = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \hat{a}_{\omega\ell m} \phi_{\omega\ell m} + \hat{a}^{\dagger}_{\omega\ell m} \phi^{*}_{\omega\ell m} \right]$$

## Boulware vacuum $\hat{a}_{\omega\ell m}|B\rangle = 0$

State which is as empty as possible far from the black hole

## Feynman Green's function

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \langle B|\mathcal{T}\left[\hat{\Phi}(x)\hat{\Phi}(x')\right]|B\rangle$$
$$= \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} (2\ell+1)\, \mathrm{e}^{\mathrm{i}\omega\Delta t} \psi_{\omega\ell}(r)\psi^{*}_{\omega\ell}(r') P_{\ell}(\cos\gamma)$$

# Canonical quantization

$$\hat{\Phi} = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \hat{a}_{\omega\ell m} \phi_{\omega\ell m} + \hat{a}^{\dagger}_{\omega\ell m} \phi^{*}_{\omega\ell m} \right]$$

## Boulware vacuum $\hat{a}_{\omega\ell m}|B\rangle = 0$

State which is as empty as possible far from the black hole

## Feynman Green's function

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \langle B| \mathcal{T} \left[ \hat{\Phi}(x)\hat{\Phi}(x') \right] |B\rangle$$

$$= \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} (2\ell+1)\, \mathrm{e}^{\mathrm{i}\omega\Delta t} \psi_{\omega\ell}(r)\psi^{*}_{\omega\ell}(r') P_{\ell}(\cos\gamma)$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\Delta\varphi$$

# Renormalized VP

$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{x' \to x} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}}(x,x') - G_{\mathrm{S}}(x,x') \right] \right\}$$

# Renormalized VP

$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{x' \to x} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}}(x, x') - G_{\mathrm{S}}(x, x') \right] \right\}$$

## Feynman Green's function $G_{\mathrm{F}}(x, x')$

- Mode sum over separable solutions of the scalar field equation
- Modes can only be found numerically

# Renormalized VP

$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{x' \to x} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}}(x,x') - G_{\mathrm{S}}(x,x') \right] \right\}$$

## Feynman Green's function $G_{\mathrm{F}}(x,x')$

- Mode sum over separable solutions of the scalar field equation
- Modes can only be found numerically

## Hadamard parametrix $G_{\mathrm{S}}(x,x')$

- Purely geometric
- Taylor series expansions for $x'$ close to $x$

# Renormalized VP

$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{x' \to x} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}}(x,x') - G_{\mathrm{S}}(x,x') \right] \right\}$$

## Feynman Green's function $G_{\mathrm{F}}(x,x')$

- Mode sum over separable solutions of the scalar field equation
- Modes can only be found numerically

## Hadamard parametrix $G_{\mathrm{S}}(x,x')$

- Purely geometric
- Taylor series expansions for $x'$ close to $x$

How can we subtract $G_{\mathrm{S}}(x,x')$ from $G_{\mathrm{F}}(x,x')$ so that limit can be taken and answer computed numerically?

# WKB-based method

Candelas & Howard *PRD* **29** 1618 (1984)
Howard & Candelas *PRL* **53** 403 (1984)
Howard *PRD* **30** 2532 (1984)
Anderson, Hiscock & Samuel *PRD* **51** 4337 (1995)
EW & Young *PRD* **77** 024008 (2008)
Flachi & Tanaka *PRD* **78** 064011 (2008)
Breen & Ottewill *PRD* **82** 084019 (2010)
Breen & Ottewill *PRD* **85** 084029 (2012)

# Euclideanization

- Wick rotation $t \rightarrow -\mathrm{i}\tau$

# Euclideanization

- Wick rotation $t \to -i\tau$
- Euclidean time coordinate periodic $\tau \to \tau + 2\pi/\kappa$

# Euclideanization

- Wick rotation $t \to -\mathrm{i}\tau$
- Euclidean time coordinate periodic $\tau \to \tau + 2\pi/\kappa$
- Thermal state at temperature $\kappa/2\pi$

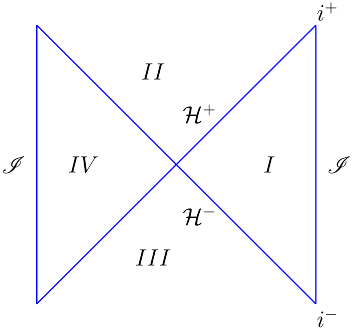
# Euclideanization

- Wick rotation $t \to -i\tau$
- Euclidean time coordinate periodic $\tau \to \tau + 2\pi/\kappa$
- Thermal state at temperature $\kappa/2\pi$

## HHI state $|H\rangle$

- Black hole in thermal equilibrium at the Hawking temperature
- Regular on and outside event horizon

[ Hartle & Hawking *PRD* **13** 2188 (1976)
Israel *PLA* **57** 107 (1976) ]



# WKB-based method $\Delta r = 0$, $\gamma = 0$

# WKB-based method $\Delta r = 0$, $\gamma = 0$

Euclidean Green's function

$$G_{\mathrm{E}}(x, x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{\mathrm{i}n\kappa\Delta\tau} (2\ell + 1)\, p_{n\ell}(r) q_{n\ell}(r)$$

# WKB-based method $\Delta r = 0, \gamma = 0$

Euclidean Green's function

$$G_{\mathrm{E}}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} (2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r)$$

## Renormalized VP

$$\langle \hat{\Phi}^2(x) \rangle_{\mathrm{ren}} = \lim_{\Delta\tau \to 0} \{ G_{\mathrm{E}} - G_{\mathrm{S}} \}$$

# WKB-based method $\Delta r = 0, \gamma = 0$

Euclidean Green's function

$$G_{\rm E}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} {\rm e}^{in\kappa\Delta\tau} (2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r)$$

## Renormalized VP

$$\begin{aligned} \langle \hat{\Phi}^2(x) \rangle_{\rm ren} &= \lim_{\Delta\tau\to 0} \{ G_{\rm E} - G_{\rm S} \} \\ &= \lim_{\Delta\tau\to 0} \{ G_{\rm E} - \text{WKB expansion} \} \\ &\quad + \lim_{\Delta\tau\to 0} \{ \text{WKB expansion} - G_{\rm S} \} \end{aligned}$$

# WKB-based method $\Delta r = 0$, $\gamma = 0$

## Euclidean Green's function

$$G_{\mathrm{E}}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} (2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r)$$

### Renormalized VP

$$\begin{aligned}\langle \hat{\Phi}^2(x) \rangle_{\mathrm{ren}} &= \lim_{\Delta\tau\to 0} \{ G_{\mathrm{E}} - G_{\mathrm{S}} \} \\ &= \lim_{\Delta\tau\to 0} \{ G_{\mathrm{E}} - \text{WKB expansion} \} \\ &\quad + \lim_{\Delta\tau\to 0} \{ \text{WKB expansion} - G_{\mathrm{S}} \}\end{aligned}$$

- WKB expansion known in closed form

# WKB-based method $\Delta r = 0, \gamma = 0$

Euclidean Green's function

$$G_{\mathrm{E}}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} \left(2\ell+1\right) p_{n\ell}(r) q_{n\ell}(r)$$

## Renormalized VP

$$\begin{aligned} \langle \hat{\Phi}^2(x) \rangle_{\mathrm{ren}} &= \lim_{\Delta\tau\to 0} \left\{ G_{\mathrm{E}} - G_{\mathrm{S}} \right\} \\ &= \lim_{\Delta\tau\to 0} \left\{ G_{\mathrm{E}} - \text{WKB expansion} \right\} \\ &\quad + \lim_{\Delta\tau\to 0} \left\{ \text{WKB expansion} - G_{\mathrm{S}} \right\} \end{aligned}$$

- WKB expansion known in closed form
- Numerical sum over modes

# WKB-based method $\Delta r = 0$, $\gamma = 0$

Euclidean Green's function

$$G_{\mathrm{E}}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} (2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r)$$

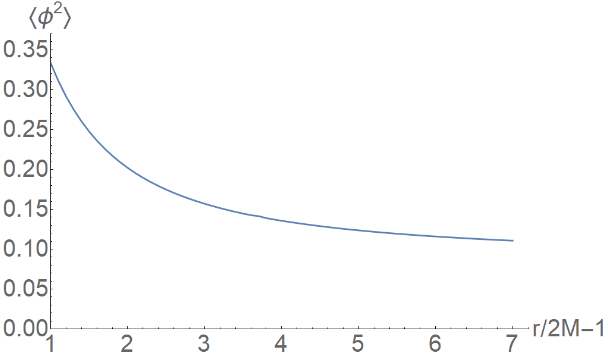
## Renormalized VP

$$\begin{aligned} \langle \hat{\Phi}^2(x) \rangle_{\text{ren}} &= \lim_{\Delta\tau\to 0} \{ G_{\mathrm{E}} - G_{\mathrm{S}} \} \\ &= \lim_{\Delta\tau\to 0} \{ G_{\mathrm{E}} - \text{WKB expansion} \} \\ &\quad + \lim_{\Delta\tau\to 0} \{ \text{WKB expansion} - G_{\mathrm{S}} \} \end{aligned}$$

- WKB expansion known in closed form
- Numerical sum over modes
- Analytic part plus numerical integrals

# VP on Schwarzschild

$$ds^2 = f(r)\,d\tau^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\varphi^2 \qquad f(r) = 1 - \frac{2M}{r}$$




[ Candelas & Howard *PRD* **29** 1618 (1984) ]

- Regular on and outside the horizon
- As $r \to \infty$ approaches value for a thermal state on $\mathbb{M}$

# Extended coordinates method

Taylor & Breen *PRD* **94** 125024 (2016)
Taylor & Breen *PRD* **96** 105020 (2017)
Morley, Taylor & EW *CQG* **35** 235010 (2018)
Breen & Taylor *PRD* **98** 105006 (2018)

# An alternative approach on Euclidean spacetime

$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{x' \to x} \{ G_{\text{E}} - G_{\text{S}} \}$$

$$G_{\text{E}}(x, x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \text{e}^{\text{i}n\kappa\Delta\tau} \, (2\ell + 1) \, p_{n\ell}(r) q_{n\ell}(r) P_\ell \, (\cos\gamma)$$

Taylor & Breen *PRD* **94** 125024 (2016)

Taylor & Breen *PRD* **96** 105020 (2017)

# An alternative approach on Euclidean spacetime

$$\langle \hat{\Phi}^2(x) \rangle_{\text{ren}} = \lim_{x' \to x} \{ G_{\text{E}} - G_{\text{S}} \}$$

$$G_{\text{E}}(x, x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} \left(2\ell + 1\right) p_{n\ell}(r) q_{n\ell}(r) P_\ell \left(\cos\gamma\right)$$

$$G_{\text{S}}(x, x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} \left(2\ell + 1\right) \Gamma_{n\ell}(r) P_\ell \left(\cos\gamma\right)$$

Taylor & Breen *PRD* **94** 125024 (2016)

Taylor & Breen *PRD* **96** 105020 (2017)

# An alternative approach on Euclidean spacetime

$$\langle \hat{\Phi}^2(x) \rangle_{\rm ren} = \lim_{x' \to x} \left\{ G_{\rm E} - G_{\rm S} \right\}$$

$$G_{\rm E}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} {\rm e}^{in\kappa\Delta\tau} \left( 2\ell + 1 \right) p_{n\ell}(r) q_{n\ell}(r) P_\ell \left( \cos\gamma \right)$$

$$G_{\rm S}(x,x') = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} {\rm e}^{in\kappa\Delta\tau} \left( 2\ell + 1 \right) \Gamma_{n\ell}(r) P_\ell \left( \cos\gamma \right)$$

$$\langle \hat{\Phi}^2(x) \rangle_{\rm ren} = \lim_{\Delta\tau \to 0, \gamma \to 0} \left\{ \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} {\rm e}^{in\kappa\Delta\tau} \left( 2\ell + 1 \right) P_\ell \left( \cos\gamma \right) \times \left[ p_{n\ell}(r) q_{n\ell}(r) - \Gamma_{n\ell}(r) \right] \right\}$$

Taylor & Breen *PRD* **94** 125024 (2016)

Taylor & Breen *PRD* **96** 105020 (2017)

# Mode-sum representation of $G_S$

## Extended coordinates

$$w^2 = \frac{2}{\kappa^2}\left[1 - \cos\left(\kappa\Delta\tau\right)\right] \qquad s^2 = f(r)w^2 + 2r^2\left(1 - \cos\gamma\right)$$

Taylor & Breen *PRD* **94** 125024 (2016)

Taylor & Breen *PRD* **96** 105020 (2017)

# Mode-sum representation of $G_S$

## Extended coordinates

$$w^2 = \frac{2}{\kappa^2}\left[1 - \cos\left(\kappa\Delta\tau\right)\right] \qquad s^2 = f(r)w^2 + 2r^2\left(1 - \cos\gamma\right)$$

## Hadamard parametrix

$$G_S(x, x') = \frac{\kappa}{8\pi^2}\sum_{i=0}^{2}\sum_{j=-i}^{i}\mathcal{D}_{ij}(r)\frac{w^{2i+2j}}{s^{2j+2}} + \ldots$$

Taylor & Breen *PRD* **94** 125024 (2016)

Taylor & Breen *PRD* **96** 105020 (2017)

# Mode-sum representation of $G_S$

## Extended coordinates

$$w^2 = \frac{2}{\kappa^2}\left[1 - \cos\left(\kappa\Delta\tau\right)\right] \qquad s^2 = f(r)w^2 + 2r^2\left(1 - \cos\gamma\right)$$

## Hadamard parametrix

$$G_S(x,x') = \frac{\kappa}{8\pi^2}\sum_{i=0}^{2}\sum_{j=-i}^{i}\mathcal{D}_{ij}(r)\frac{w^{2i+2j}}{s^{2j+2}} + \dots$$

$$\frac{w^{2i+2j}}{s^{2j+2}} = \sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau}\left(2\ell+1\right)P_\ell\left(\cos\gamma\right)\Psi_{n\ell}(i,j|r)$$

Taylor & Breen *PRD* **94** 125024 (2016) Taylor & Breen *PRD* **96** 105020 (2017)

# Mode-sum representation of $G_S$

## Extended coordinates

$$w^2 = \frac{2}{\kappa^2}\left[1 - \cos\left(\kappa\Delta\tau\right)\right] \qquad s^2 = f(r)w^2 + 2r^2\left(1 - \cos\gamma\right)$$

## Hadamard parametrix

$$G_S(x,x') = \frac{\kappa}{8\pi^2}\sum_{i=0}^{2}\sum_{j=-i}^{i}\mathcal{D}_{ij}(r)\frac{w^{2i+2j}}{s^{2j+2}} + \ldots$$

$$\frac{w^{2i+2j}}{s^{2j+2}} = \sum_{n=-\infty}^{\infty}\sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau}\left(2\ell+1\right)P_\ell\left(\cos\gamma\right)\Psi_{n\ell}(i,j|r)$$

$$\Psi_{n\ell}(i,j|r) = \frac{\kappa}{2\pi}\int_{\Delta\tau=0}^{\kappa/2\pi}\int_{\gamma=0}^{\pi}\frac{w^{2i+2j}}{s^{2j+2}}e^{-in\kappa\Delta\tau}P_\ell(\cos\gamma)\sin\gamma\, d\gamma\, d\Delta\tau$$

Taylor & Breen *PRD* **94** 125024 (2016)

Taylor & Breen *PRD* **96** 105020 (2017)

# VP calculated using extended coordinates

# VP calculated using extended coordinates

$$G_{\rm E} = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} e^{in\kappa\Delta\tau} (2\ell+1)\, p_{n\ell}(r) q_{n\ell}(r) P_\ell (\cos\gamma)$$

# VP calculated using extended coordinates

$$G_{\mathrm{E}} = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{\mathrm{i}n\kappa\Delta\tau} \left(2\ell + 1\right) p_{n\ell}(r) q_{n\ell}(r) P_{\ell}\left(\cos\gamma\right)$$

$$G_{\mathrm{S}} = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i=0}^{2} \sum_{j=-i}^{i} \mathrm{e}^{\mathrm{i}n\kappa\Delta\tau} \left(2\ell + 1\right) P_{\ell}\left(\cos\gamma\right) \mathcal{D}_{ij}(r) \Psi_{n\ell}(i, j|r)$$

# VP calculated using extended coordinates

$$G_{\mathrm{E}} = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \mathrm{e}^{in\kappa\Delta\tau} \left(2\ell+1\right) p_{n\ell}(r) q_{n\ell}(r) P_{\ell}\left(\cos\gamma\right)$$

$$G_{\mathrm{S}} = \frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i=0}^{2} \sum_{j=-i}^{i} \mathrm{e}^{in\kappa\Delta\tau} \left(2\ell+1\right) P_{\ell}\left(\cos\gamma\right) \mathcal{D}_{ij}(r) \Psi_{n\ell}(i,j|r)$$

## Renormalized VP $\langle\hat{\Phi}^2\rangle_{\mathrm{ren}}$

$$\frac{\kappa}{8\pi^2} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} (2\ell+1) \left[ p_{n\lambda}(r) q_{n\lambda}(r) - \sum_{i=0}^{2} \sum_{j=-i}^{i} \mathcal{D}_{ij}(r) \Psi_{n\ell}(i,j|r) \right]$$

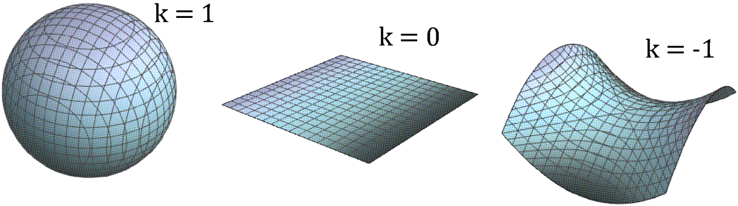
# Topological black holes in adS

## Euclidean metric

$$ds^2 = f(r)\,d\tau^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\mathcal{F}_k(\theta)^2\,d\varphi^2$$

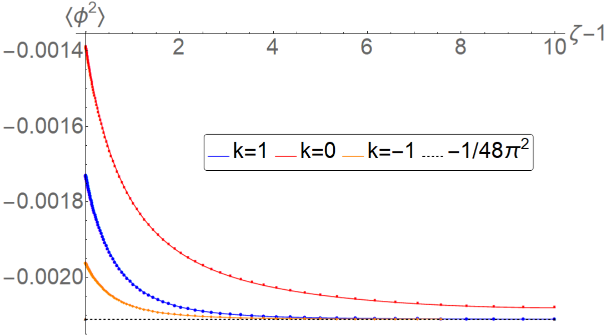
$$f(r) = k - \frac{2M}{r} - \frac{\Lambda r^2}{3} \qquad \Lambda < 0$$

$$\mathcal{F}_k(\theta) = \begin{cases} \sin\theta & k = 1 \\ \theta & k = 0 \\ \sinh\theta & k = -1 \end{cases}$$



# VP on topological black holes

$$f(r) = k - \frac{2M}{r} - \frac{\Lambda r^2}{3} \qquad k = 1, 0, -1$$




As $r \to \infty$, approach values for vacuum state on pure adS

[ Morley, Taylor & EW *CQG* **35** 235010 (2018) ]

# Pragmatic mode sum method

Levi & Ori *PRD* **91** 104028 (2015)
Levi & Ori *PRD* **94** 044054 (2016)
Levi & Ori *PRL* **117** 231101 (2016)
Levi, Eilon, Ori & van de Meent *PRL* **118** 141102 (2017)
Levi *PRD* **95** 025007 (2017)
Lanir, Levi, Ori & Sela *PRD* **97** 024033 (2018)
Lanir, Levi & Ori *PRD* **98** 084017 (2018)
Lanir, Ori, Zilberman, Sela, Maline & Levi *PRD* **99** 061502 (2019)

# Lorentzian space-time

## Static, spherically symmetric metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

## Feynman Green's function in the Boulware state

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} (2\ell+1)\,\mathrm{e}^{\mathrm{i}\omega\Delta t}\psi_{\omega\ell}(r)\psi^{*}_{\omega\ell}(r')P_{\ell}(\cos\gamma)$$

# Lorentzian space-time

## Static, spherically symmetric metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

## Feynman Green's function in the Boulware state

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} (2\ell+1)\,\mathrm{e}^{\mathrm{i}\omega\Delta t}\psi_{\omega\ell}(r)\psi_{\omega\ell}^{*}(r')P_{\ell}(\cos\gamma)$$

Time-like point-splitting $r = r'$, $\gamma = 0$

$$-\mathrm{i}G_{\mathrm{F}}(x,x') = \int_{\omega=0}^{\infty} d\omega \sum_{\ell=0}^{\infty} (2\ell+1)\,\mathrm{e}^{\mathrm{i}\omega\Delta t}\left|\psi_{\omega\ell}(r)\right|^2$$

# Hadamard parametrix

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{1}{4\pi^2 f \Delta t^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots$$

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Hadamard parametrix

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{1}{4\pi^2 f \Delta t^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots$$

Integral representation of singular term

$$\frac{1}{\Delta t^2} = -\int_{\omega=0}^{\infty} \omega \mathrm{e}^{\mathrm{i}\omega\Delta t}\, d\omega$$

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Hadamard parametrix

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = \frac{1}{4\pi^2 f \Delta t^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots$$

Integral representation of singular term

$$\frac{1}{\Delta t^2} = -\int_{\omega=0}^{\infty} \omega e^{\mathrm{i}\omega\Delta t}\, d\omega$$

$$-\mathrm{i}G_{\mathrm{S}}(x,x') = -\frac{1}{4\pi^2 f}\int_{\omega=0}^{\infty} \omega e^{\mathrm{i}\omega\Delta t}\, d\omega + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} + \ldots$$

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Renormalized VP

$$\langle \hat{\Phi}^2 \rangle_{\text{ren}} = \lim_{\Delta t \to 0} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}} - G_{\mathrm{S}} \right] \right\}$$

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Renormalized VP

$$\langle \hat{\Phi}^2 \rangle_{\mathrm{ren}} = \lim_{\Delta t \to 0} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}} - G_{\mathrm{S}} \right] \right\}$$

$$= \lim_{\Delta t \to 0} \left\{ \int_{\omega=0}^{\infty} \omega e^{\mathrm{i}\omega \Delta t} \, d\omega \left[ \sum_{\ell=0}^{\infty} (2\ell+1) \left| \psi_{\omega\ell}(r) \right|^2 + \frac{\omega}{4\pi^2 f} \right] \right\}$$

$$+ \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r}$$

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Renormalized VP

$$\begin{aligned}
\langle \hat{\Phi}^2 \rangle_{\text{ren}} &= \lim_{\Delta t \to 0} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}} - G_{\mathrm{S}} \right] \right\} \\
&= \lim_{\Delta t \to 0} \left\{ \int_{\omega=0}^{\infty} \omega e^{\mathrm{i}\omega \Delta t} \, d\omega \left[ \sum_{\ell=0}^{\infty} (2\ell+1) \left| \psi_{\omega\ell}(r) \right|^2 + \frac{\omega}{4\pi^2 f} \right] \right\} \\
&\quad + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} \\
&= \lim_{\epsilon \to 0} \left\{ \int_{\omega=0}^{\infty} d\omega \, \mathrm{e}^{\mathrm{i}\omega\epsilon} \mathcal{G}_{\omega}(r) \right\} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r}
\end{aligned}$$

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Renormalized VP

$$
\begin{aligned}
\langle \hat{\Phi}^2 \rangle_{\rm ren} &= \lim_{\Delta t \to 0} \left\{ -{\rm i} \left[ G_{\rm F} - G_{\rm S} \right] \right\} \\
&= \lim_{\Delta t \to 0} \left\{ \int_{\omega=0}^{\infty} \omega e^{{\rm i}\omega \Delta t} \, d\omega \left[ \sum_{\ell=0}^{\infty} (2\ell+1) \left| \psi_{\omega\ell}(r) \right|^2 + \frac{\omega}{4\pi^2 f} \right] \right\} \\
&\quad + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} \\
&= \lim_{\epsilon \to 0} \left\{ \int_{\omega=0}^{\infty} d\omega \, e^{{\rm i}\omega\epsilon} \mathcal{G}_{\omega}(r) \right\} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r}
\end{aligned}
$$

- Integral over $\omega$ fails to converge in the usual sense

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Renormalized VP

$$\begin{aligned}\langle\hat{\Phi}^2\rangle_{\text{ren}} &= \lim_{\Delta t\to 0}\left\{-\mathrm{i}\left[G_{\mathrm{F}} - G_{\mathrm{S}}\right]\right\} \\ &= \lim_{\Delta t\to 0}\left\{\int_{\omega=0}^{\infty}\omega e^{\mathrm{i}\omega\Delta t}\,d\omega\left[\sum_{\ell=0}^{\infty}(2\ell+1)\left|\psi_{\omega\ell}(r)\right|^2+\frac{\omega}{4\pi^2 f}\right]\right\} \\ &\quad + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} \\ &= \lim_{\epsilon\to 0}\left\{\int_{\omega=0}^{\infty} d\omega\, e^{\mathrm{i}\omega\epsilon}\mathcal{G}_{\omega}(r)\right\} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r}\end{aligned}$$

- Integral over $\omega$ fails to converge in the usual sense
- Growing oscillations of wavelength $\nu$ as $\omega$ increases

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Renormalized VP

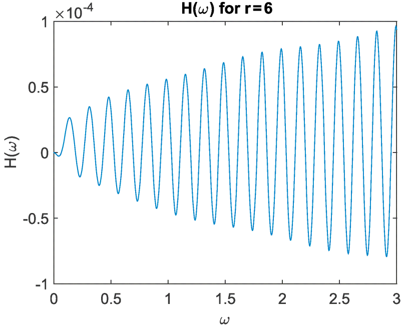
$$
\begin{aligned}
\langle \hat{\Phi}^2 \rangle_{\text{ren}} &= \lim_{\Delta t \to 0} \left\{ -\mathrm{i} \left[ G_{\mathrm{F}} - G_{\mathrm{S}} \right] \right\} \\
&= \lim_{\Delta t \to 0} \left\{ \int_{\omega=0}^{\infty} \omega e^{\mathrm{i}\omega \Delta t} \, d\omega \left[ \sum_{\ell=0}^{\infty} (2\ell + 1) \left| \psi_{\omega\ell}(r) \right|^2 + \frac{\omega}{4\pi^2 f} \right] \right\} \\
&\quad + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r} \\
&= \lim_{\epsilon \to 0} \left\{ \int_{\omega=0}^{\infty} d\omega \, e^{\mathrm{i}\omega\epsilon} \mathcal{G}_{\omega}(r) \right\} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r}
\end{aligned}
$$

- Integral over $\omega$ fails to converge in the usual sense
- Growing oscillations of wavelength $\nu$ as $\omega$ increases
- Replace with a generalized integral which cancels the oscillations

[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Generalized integrals

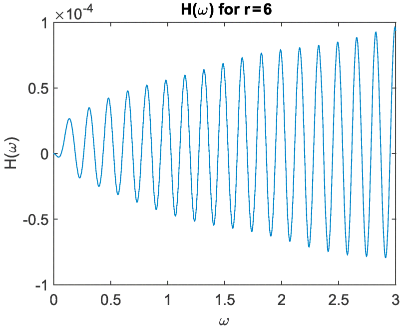
$$\mathcal{H}(\omega) = \int_{\varpi=0}^{\omega} d\varpi \, e^{i\varpi\epsilon} \mathcal{G}_{\varpi}(r)$$




[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Generalized integrals

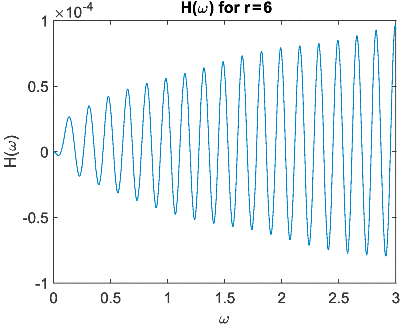
$$\mathcal{H}(\omega) = \int_{\varpi=0}^{\omega} d\varpi \, e^{i\varpi\epsilon} \mathcal{G}_{\varpi}(r) \qquad \mathcal{H}_{*}(\omega) = \frac{1}{2}\left[\mathcal{H}(\omega) + \mathcal{H}(\omega + \nu/2)\right]$$

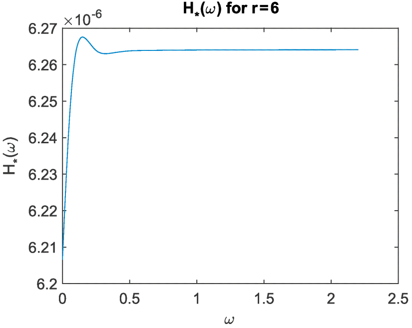



[ Levi & Ori *PRD* **91** 104028 (2015) ]

# Generalized integrals

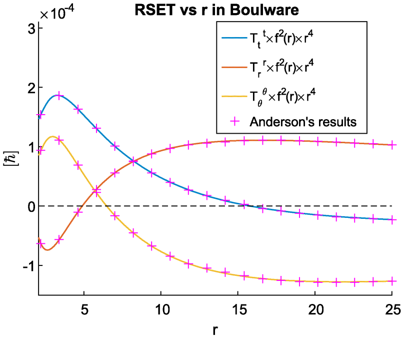
$$\mathcal{H}(\omega) = \int_{\varpi=0}^{\omega} d\varpi\, \mathrm{e}^{\mathrm{i}\varpi\epsilon} \mathcal{G}_{\varpi}(r) \qquad \mathcal{H}_{*}(\omega) = \frac{1}{2}\left[\mathcal{H}(\omega) + \mathcal{H}(\omega + \nu/2)\right]$$




[ Levi & Ori *PRD* **91** 104028 (2015) ]

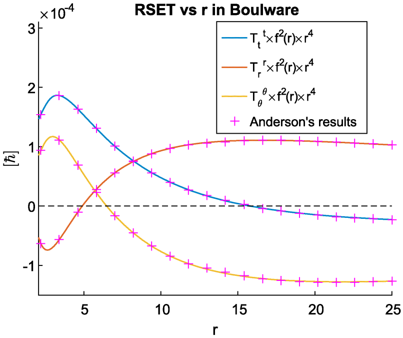
# RSET on Schwarzschild

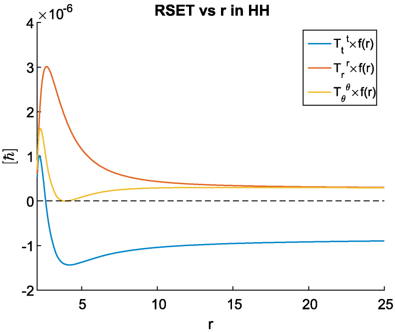



- Diverges on horizon
- Vanishes as $r \to \infty$

[ Levi *PRD* **95** 025007 (2017) ]

# RSET on Schwarzschild




- Diverges on horizon
- Vanishes as $r \to \infty$

- Regular on horizon
- Nonzero as $r \to \infty$

[ Levi *PRD* **95** 025007 (2017) ]

# Summary

# Summary

## WKB-based method

- Euclidean space-times
- Static, spherically symmetric black holes

# Summary

## WKB-based method

- Euclidean space-times
- Static, spherically symmetric black holes

## Extended coordinates method

- $\langle \hat{\Phi}^2 \rangle$ on four and higher-dimensional black holes
- Both conformal and more general couplings
- To do: $\langle \hat{T}_{\mu\nu} \rangle_{\rm ren}$

# Summary

## WKB-based method

- Euclidean space-times
- Static, spherically symmetric black holes

## Extended coordinates method

- $\langle \hat{\Phi}^2 \rangle$ on four and higher-dimensional black holes
- Both conformal and more general couplings
- To do: $\langle \hat{T}_{\mu\nu} \rangle_{\rm ren}$

## Pragmatic mode-sum method

- Lorentzian space-times
- Both static and stationary black holes
- $\langle \hat{\Phi}^2 \rangle$ and $\langle \hat{T}_{\mu\nu} \rangle_{\rm ren}$