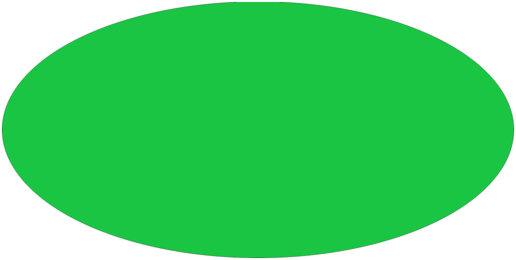
Based on collaboration with J. Grain & V. Vennin

Gauges in cosmological perturbation theory and in the separate-universe approach

Spontaneous workshop XIV, IESC Cargèse
09/05/2022

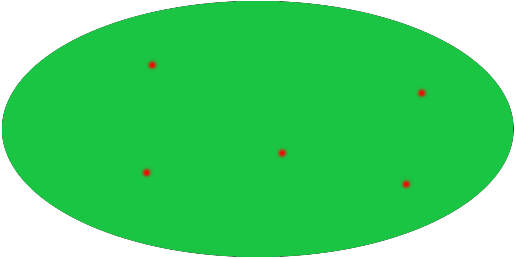
Danilo Artigas

Introduction



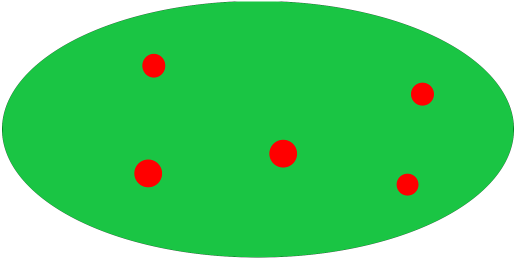
Friedmann-Lemaître-Robertson-Walker background (homogeneous and isotropic)

Introduction



Small (quantum) inhomogeneities and anisotropies

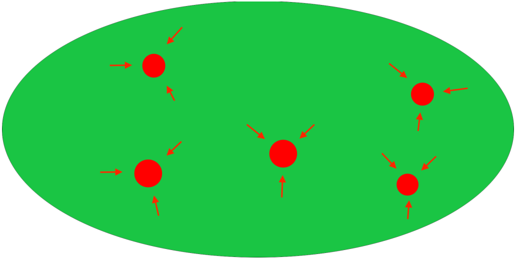
Introduction



Quantum fluctuations become classical at large scales

cosmological perturbation theory (CPT) $\rightarrow$ separate universe (SU) [Wands et al. (2000)]

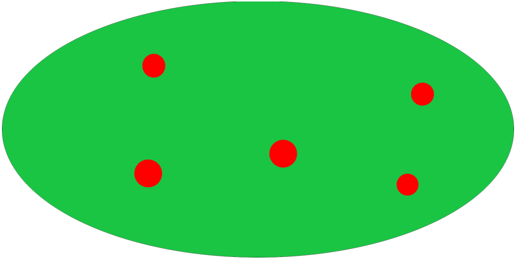
Introduction



Classical inhomogeneities and anisotropies backreact

SU $\rightarrow$ stochastic formalism [Cruces' talk on Tuesday]

Introduction



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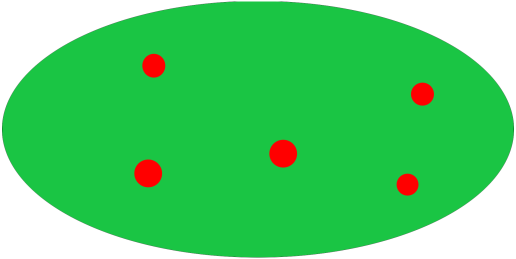

gauge fixing in CPT $=$ gauge fixing in SU ? [DA, Grain & Vennin (2022)]

Hamiltonian formalism for general relativity

Total (Hamiltonian) constraint of general relativity [Langlois (1994)]:

$$C(t,\vec{x}) := \int \mathrm{d}^3\vec{x} \left[N(t,\vec{x})\,\mathcal{S}(t,\vec{x}) + N^i(t,\vec{x})\,\mathcal{D}_i(t,\vec{x})\right] \tag{1}$$

where N is the lapse function and N^i is the shift vector.

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Invariance of the theory under time reparametrisation is ensured by the scalar constraint:

$$\mathcal{S} = 0\,. \tag{2}$$

Invariance under space reparametrisation is ensured by the diffeomorphism constraint:

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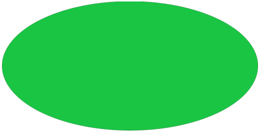
The dynamics is obtained by the use of the Poisson bracket $\dot{z} = \{z, C\}$, where we defined:

$$\{F, G\} = \sum_A \left[\frac{\partial F}{\partial q_A}\frac{\partial G}{\partial p_A} - \frac{\partial G}{\partial q_A}\frac{\partial F}{\partial p_A} \right], \tag{4}$$

for two arbitary functionals of the phase space F and G, with the configuration variables q_A and their momenta p_A.

Cosmological perturbation theory



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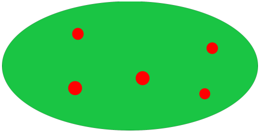
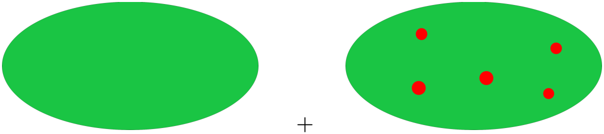


(v, θ), (φ, π_φ)
N
$\mathcal{S}^{(0)} = 0$

$(\delta\gamma_1, \delta\pi_1)$, $(\delta\gamma_2, \delta\pi_2)$, $(\delta\varphi, \delta\pi_\varphi)$
δN, δN_1
$\mathcal{S}^{(1)} = \mathcal{D}^{(1)} = 0$
$\mathcal{S}^{(2)}$

Cosmological perturbation theory



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Fixing a gauge: e.g. $\delta\gamma_1 = \delta\gamma_2 = 0$.

Separate Universe

CPT	separate-universe approach
$\delta\gamma_1$	$\overline{\delta\gamma_1}$
$\delta\pi_1$	$\overline{\delta\pi_1}$
$\delta\gamma_2$	0
$\delta\pi_2$	0
$\delta\varphi$	$\overline{\delta\varphi}$
$\delta\pi_\varphi$	$\overline{\delta\pi_\varphi}$
δN	$\overline{\delta N}$
δN_1	0
$\mathcal{S}^{(1)}, \mathcal{S}^{(2)}$	$\overline{\mathcal{S}}^{(1)}, \overline{\mathcal{S}}^{(2)}$
$\mathcal{D}^{(1)}$	$\overline{\mathcal{D}}^{(1)} \neq 0$

A geometrical framework: CPT

Let's arrange the perturbed quantities in a 6-dimensional vector:

$$\vec{\delta z} = (\delta\varphi, \delta\gamma_1, \delta\gamma_2, \delta\pi_\varphi, \delta\pi_1, \delta\pi_2)^T \, . \tag{5}$$

The phase space is described in terms of a 6-dimensional vector space:

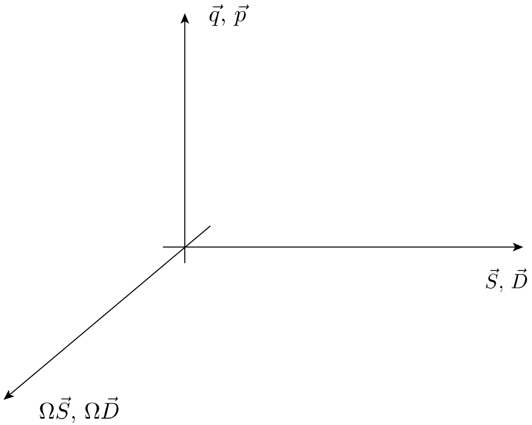
$$\text{Constraints:} \qquad \vec{S}^{(1)}\, , \, \vec{D}^{(1)} \tag{6}$$

$$\text{Gauges:} \qquad \Omega\vec{S}^{(1)}\, , \, \Omega\vec{D}^{(1)} \tag{7}$$

$$\text{Physical dof.:} \qquad \vec{q}\, , \, \vec{p} := \Omega\vec{q} \tag{8}$$

where $\Omega = \begin{pmatrix} 0 & \mathbb{I}_3 \\ -\mathbb{I}_3 & 0 \end{pmatrix}$.

A geometrical framework: CPT



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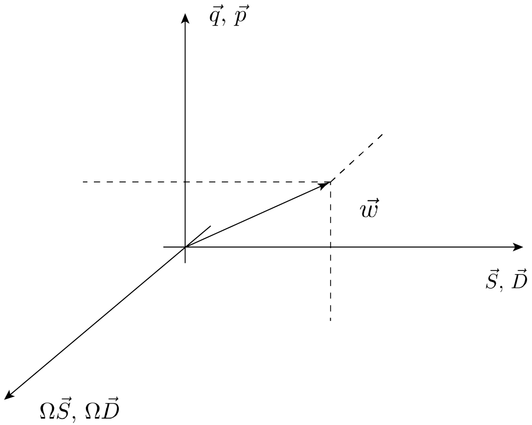
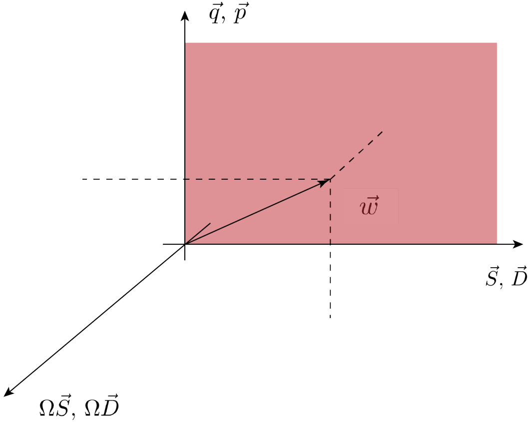


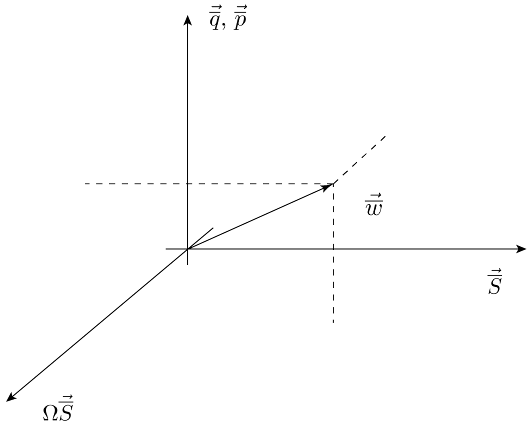
$$W := \vec{w} \cdot \vec{\delta z}$$

A geometrical framework: CPT



Gauge invariant variables

A geometrical framework: SU



$$\overline{W} := \vec{\vec{w}} \cdot \overline{\vec{\delta z}}$$

Gauge-invariant CPT

Why do we use only the Mukhanov-Sasaki variable and the curvature perturbation [Sasaki (1986)] [Mukhanov(1988)] [Lyth & Wands (2003)]?
Mathematically, it is the easiest one to find!

$$\Omega\vec{\mathcal{S}}^{(1)} = \left(-\frac{\pi_\varphi}{v}\,,\ \frac{\sqrt{3}}{M_{\rm Pl}^2}v^{2/3}\theta\,,\ 0\,,\ vV_{,\varphi}\,,\ -\frac{v^{1/3}}{\sqrt{3}}\left[\frac{\pi_\varphi^2}{v^2}-V+M_{\rm Pl}^2\frac{k^2}{v^{2/3}}\right]\,,\ \frac{M_{\rm Pl}^2}{\sqrt{6}}\frac{k^2}{v^{1/3}}\right) \tag{5}$$

$$\Omega\vec{\mathcal{D}}^{(1)} = \left(0\,,\ \frac{2}{\sqrt{3}}v^{2/3}\,,\ 2\sqrt{\frac{2}{3}}v^{2/3}\,,\ \pi_\varphi\,,\ \frac{1}{2\sqrt{3}}v^{1/3}\theta\,,\ -\sqrt{\frac{2}{3}}v^{1/3}\theta\right) \tag{6}$$

$$\vec{q}_{MS} = \left(1\,,\ \frac{M_{\rm Pl}^2\pi_\varphi}{\sqrt{3}\theta v^{5/3}}\,,\ -\frac{M_{\rm Pl}^2\pi_\varphi}{\sqrt{6}\theta v^{5/3}}\,,\ 0\,,\ 0\,,\ 0\right) \tag{7}$$

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But in SU:

$$\vec{\vec{q}}_{MS} = \left(1\,,\ \frac{M_{\rm Pl}^2\pi_\varphi}{\sqrt{3}\theta v^{5/3}}\,,\ 0\,,\ 0\right) \tag{8}$$

which is not a gauge-invariant variable of CPT.

Gauge-invariant SU

- Gauge invariant quantities:

$$Q_{\overline{\varphi\gamma_1}} = \overline{\delta\varphi} + \frac{M_{\rm Pl}^2}{\sqrt{3}} \frac{\pi_\varphi}{\theta v^{5/3}} \overline{\delta\gamma_1} \tag{9}$$

$$Q_{\overline{\varphi\pi_\varphi}} = \overline{\delta\varphi} + \frac{\pi_\varphi}{v^2 V_{,\varphi}} \overline{\delta\pi_\varphi} \tag{10}$$

$$Q_{\overline{\varphi\pi_1}} = \overline{\delta\varphi} - \frac{\sqrt{3}\pi_\varphi}{v^{4/3}} \left[\frac{\pi_\varphi^2}{v^2} - V\right] \overline{\delta\pi_1} \tag{11}$$

$$Q_{\overline{\gamma_1\pi_\varphi}} = \overline{\delta\gamma_1} - \frac{\sqrt{3}}{M_{\rm Pl}^2} \frac{\theta}{v^{1/3} V_{,\varphi}} \overline{\delta\pi_\varphi} \tag{12}$$

$$Q_{\overline{\gamma_1\pi_1}} = \overline{\delta\gamma_1} + \frac{3}{M_{\rm Pl}^2} \frac{v^{1/3}\theta}{\frac{\pi_\varphi^2}{v^2} - V} \overline{\delta\pi_1} \tag{13}$$

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- Some of those are also gauge-invariant quantities for CPT ($Q_{\overline{\varphi\gamma_1}}$, $Q_{\overline{\varphi\pi_\varphi}}$, $Q_{\overline{\gamma_1\pi_\varphi}}$). Thus *e.g.* :

$$Q = \frac{1}{2} Q_{\overline{\varphi\gamma_1}} + \frac{1}{2} Q_{\overline{\varphi\pi_\varphi}} - \frac{M_{\mathrm{Pl}}^2}{2\sqrt{3} v^{5/3}\theta} \frac{2M_{\mathrm{Pl}}^2 \pi_\varphi v V_{,\varphi} + 3\pi_\varphi^2\theta}{2M_{\mathrm{Pl}}^2 v V_{,\varphi} - 3\pi_\varphi\theta} Q_{\overline{\gamma_1\pi_\varphi}} \tag{14}$$

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- Some of those are also gauge-invariant quantities for CPT ($Q_{\overline{\varphi\gamma_1}}$, $Q_{\overline{\varphi\pi_\varphi}}$, $Q_{\overline{\gamma_1\pi_\varphi}}$). Thus *e.g.* :

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Canonical transformations

$$\vec{w} \to M\vec{w}$$

- One can decompose the Hamiltonian $\mathcal{C}^{(2)}$ into $\mathcal{C}^{(2)}_{G.I.} + \mathcal{C}^{(2)}_{G}$.
  This might be relevant in the context of quantum gravity and for solving the problem of time [Isham (1993)].
- One may pass from a gauge choice to another by a canonical transformation.

Example of gauges

	CPT	SU
Spatially flat	$\delta\gamma_1 = \delta\gamma_2 = 0$	$k\delta N_1$ term is not neglected anymore...
Newtonian	$\delta\gamma_2 = \delta\pi_2 = 0$	$\overline{\delta N} = -\frac{N}{2\sqrt{3}} v^{2/3} \overline{\delta\gamma_1}$ and $\overline{\mathcal{D}}^{(1)} = 0$
Generalized synchronous	$\delta N = \delta N_1 = 0$ and $\theta\delta\gamma_2 = -2v^{1/3}\delta\pi_2$	$\overline{\delta N} = 0$ and $\overline{\mathcal{D}}^{(1)} = 0$
Uniform expansion	$\delta\gamma_1 = \delta N_1 = 0$ is pathological	$\overline{\delta\gamma_1} = 0$ is healthy

Conclusion

- We formulated the cosmological-perturbation theory (CPT) and the separate-universe (SU) approach in a Hamiltonian framework.
- At large-scales, the isotropic and anisotropic degrees of freedom decouple. The SU can be understood as a perturbed FLRW universe.
- One needs to find a systematic way to link gauges in CPT with gauges in SU.
  In order to do so, we are developing a geometrical framework that may be more convenient for solving such a problem.
- This might result in the discovery of some gauges or gauge-invariant quantities that are more suitable for formulating the stochastic formalism.

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Perturbed constraints

The resulting equations of motions for perturbations are

$$
\begin{cases}
\delta\dot{\gamma}_1 = -\frac{2}{\sqrt{3}} v^{2/3} k\delta N_1 - \frac{\sqrt{3}}{M_{\rm Pl}^2} v^{2/3}\theta\,\delta N - \frac{N}{M_{\rm Pl}^2}\left(2v^{1/3}\,\delta\pi_1 + \frac{\theta}{2}\,\delta\gamma_1\right), \\
\delta\dot{\pi}_1 = -\frac{v^{1/3}\theta}{2\sqrt{3}} k\delta N_1 + \frac{v^{1/3}}{\sqrt{3}}\left(\frac{\pi_\varphi^2}{v^2} - V + M_{\rm Pl}^2\frac{k^2}{v^{2/3}}\right)\delta N \\
\qquad +N\Bigg[-\frac{2}{3v^{1/3}}\left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\rm Pl}^2 k^2}{4v^{2/3}}\right)\delta\gamma_1 + \frac{\theta}{2M_{\rm Pl}^2}\,\delta\pi_1 \\
\qquad +\frac{\sqrt{3}}{2} v^{1/3}\left(\frac{\pi_\varphi}{v^2}\,\delta\pi_\varphi - V_{,\varphi}\,\delta\varphi\right) - \frac{\sqrt{2}}{12v} M_{\rm Pl}^2 k^2 \delta\gamma_2\Bigg],
\end{cases}
$$

$$
\begin{cases}
\delta\dot{\gamma}_2 = -2\sqrt{\frac{2}{3}} v^{2/3} k\delta N_1 + N\left(\frac{4v^{1/3}}{M_{\rm Pl}^2}\delta\pi_2 + \frac{\theta}{M_{\rm Pl}^2}\delta\gamma_2\right), \\
\delta\dot{\pi}_2 = \sqrt{\frac{2}{3}} v^{1/3}\theta k\delta N_1 - \frac{M_{\rm Pl}^2 k^2}{\sqrt{6}v^{1/3}}\delta N \\
\qquad +N\Bigg[-\frac{\theta}{M_{\rm Pl}^2}\delta\pi_2 - \frac{\sqrt{2}M_{\rm Pl}^2}{12v} k^2\delta\gamma_1 - \frac{2}{3v^{1/3}}\left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\rm Pl}^2 k^2}{8v^{2/3}}\right)\delta\gamma_2\Bigg],
\end{cases} \tag{15}
$$

$$
\begin{cases}
\delta\dot{\varphi} = \frac{\pi_\varphi}{v}\,\delta N + N\left(\frac{1}{v}\,\delta\pi_\varphi - \frac{\sqrt{3}}{2}\frac{\pi_\varphi}{v^{5/3}}\,\delta\gamma_1\right), \\
\delta\dot{\pi}_\varphi = -\pi_\varphi k\delta N_1 - vV_{,\varphi}\,\delta N - N\left[v\left(\frac{k^2}{v^{2/3}} + V_{,\varphi,\varphi}\right)\delta\varphi + \frac{\sqrt{3}}{2} v^{1/3} V_{,\varphi}\,\delta\gamma_1\right].
\end{cases}
$$

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The resulting equations of motions for perturbations are

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\left\{
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\delta\dot{\pi}_1 &= -\frac{v^{1/3}\theta}{2\sqrt{3}} k \delta N_1 + \frac{v^{1/3}}{\sqrt{3}} \left(\frac{\pi_\varphi^2}{v^2} - V + M_{\rm Pl}^2 \frac{k^2}{v^{2/3}} \right) \delta N \\
&\quad + N \Bigg[-\frac{2}{3 v^{1/3}} \left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\rm Pl}^2 k^2}{4 v^{2/3}} \right) \delta\gamma_1 + \frac{\theta}{2 M_{\rm Pl}^2}\, \delta\pi_1 \\
&\quad + \frac{\sqrt{3}}{2} v^{1/3} \left(\frac{\pi_\varphi}{v^2}\, \delta\pi_\varphi - V_{,\varphi}\, \delta\varphi \right) - \frac{\sqrt{2}}{12 v} M_{\rm Pl}^2 k^2 \delta\gamma_2 \Bigg],
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\delta\dot{\gamma}_2 &= -2\sqrt{\frac{2}{3}} v^{2/3} k \delta N_1 + N \left(\frac{4 v^{1/3}}{M_{\rm Pl}^2} \delta\pi_2 + \frac{\theta}{M_{\rm Pl}^2} \delta\gamma_2 \right), \\
\delta\dot{\pi}_2 &= \sqrt{\frac{2}{3}} v^{1/3} \theta k \delta N_1 - \frac{M_{\rm Pl}^2 k^2}{\sqrt{6} v^{1/3}} \delta N \\
&\quad + N \Bigg[-\frac{\theta}{M_{\rm Pl}^2} \delta\pi_2 - \frac{\sqrt{2} M_{\rm Pl}^2}{12 v} k^2 \delta\gamma_1 - \frac{2}{3 v^{1/3}} \left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\rm Pl}^2 k^2}{8 v^{2/3}} \right) \delta\gamma_2 \Bigg],
\end{aligned}
\right. \tag{15}
$$

$$
\left\{
\begin{aligned}
\delta\dot{\varphi} &= \frac{\pi_\varphi}{v}\, \delta N + N \left(\frac{1}{v}\, \delta\pi_\varphi - \frac{\sqrt{3}}{2} \frac{\pi_\varphi}{v^{5/3}}\, \delta\gamma_1 \right), \\
\delta\dot{\pi}_\varphi &= -\pi_\varphi k \delta N_1 - v V_{,\varphi}\, \delta N - N \left[v \left(\frac{k^2}{v^{2/3}} + V_{,\varphi,\varphi} \right) \delta\varphi + \frac{\sqrt{3}}{2} v^{1/3} V_{,\varphi}\, \delta\gamma_1 \right].
\end{aligned}
\right.
$$

The scalar part of the diffeomorphism constraint always acts through gradient terms. It can be neglected at large scales.

Perturbed constraints

The resulting equations of motions for perturbations are

$$
\begin{cases}
\delta\dot{\gamma}_1 = -\frac{2}{\sqrt{3}} v^{2/3} k\delta N_1 - \frac{\sqrt{3}}{M_{\rm Pl}^2} v^{2/3}\theta\,\delta N - \frac{N}{M_{\rm Pl}^2}\left(2v^{1/3}\,\delta\pi_1 + \frac{\theta}{2}\,\delta\gamma_1\right), \\
\delta\dot{\pi}_1 = -\frac{v^{1/3}\theta}{2\sqrt{3}} k\delta N_1 + \frac{v^{1/3}}{\sqrt{3}}\left(\frac{\pi_\varphi^2}{v^2} - V + M_{\rm Pl}^2\frac{k^2}{v^{2/3}}\right)\delta N \\
\qquad +N\Big[-\frac{2}{3v^{1/3}}\left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\rm Pl}^2 k^2}{4v^{2/3}}\right)\delta\gamma_1 + \frac{\theta}{2M_{\rm Pl}^2}\,\delta\pi_1 \\
\qquad +\frac{\sqrt{3}}{2} v^{1/3}\left(\frac{\pi_\varphi}{v^2}\,\delta\pi_\varphi - V_{,\varphi}\,\delta\varphi\right) - \frac{\sqrt{2}}{12v} M_{\rm Pl}^2 k^2 \delta\gamma_2\Big],
\end{cases}
$$

$$
\begin{cases}
\delta\dot{\gamma}_2 = -2\sqrt{\frac{2}{3}} v^{2/3} k\delta N_1 + N\left(\frac{4v^{1/3}}{M_{\rm Pl}^2}\delta\pi_2 + \frac{\theta}{M_{\rm Pl}^2}\delta\gamma_2\right), \\
\delta\dot{\pi}_2 = \sqrt{\frac{2}{3}} v^{1/3}\theta k\delta N_1 - \frac{M_{\rm Pl}^2 k^2}{\sqrt{6}v^{1/3}}\delta N \\
\qquad +N\Big[-\frac{\theta}{M_{\rm Pl}^2}\delta\pi_2 - \frac{\sqrt{2}M_{\rm Pl}^2}{12v} k^2\delta\gamma_1 - \frac{2}{3v^{1/3}}\left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\rm Pl}^2 k^2}{8v^{2/3}}\right)\delta\gamma_2\Big],
\end{cases} \tag{15}
$$

$$
\begin{cases}
\delta\dot{\varphi} = \frac{\pi_\varphi}{v}\,\delta N + N\left(\frac{1}{v}\,\delta\pi_\varphi - \frac{\sqrt{3}}{2}\frac{\pi_\varphi}{v^{5/3}}\,\delta\gamma_1\right), \\
\delta\dot{\pi}_\varphi = -\pi_\varphi k\delta N_1 - vV_{,\varphi}\,\delta N - N\left[v\left(\frac{k^2}{v^{2/3}} + V_{,\varphi,\varphi}\right)\delta\varphi + \frac{\sqrt{3}}{2} v^{1/3} V_{,\varphi}\,\delta\gamma_1\right].
\end{cases}
$$

At large scales, the isotropic degrees of freedom decouple from the anisotropic ones.

Perturbed constraints

The resulting equations of motions for perturbations are

$$\begin{cases} \delta\dot{\gamma}_1 = -\frac{2}{\sqrt{3}} v^{2/3} k \delta N_1 - \frac{\sqrt{3}}{M_{\mathrm{Pl}}^2} v^{2/3} \theta\, \delta N - \frac{N}{M_{\mathrm{Pl}}^2} \left(2 v^{1/3}\, \delta\pi_1 + \frac{\theta}{2}\, \delta\gamma_1 \right), \\ \delta\dot{\pi}_1 = -\frac{v^{1/3}\theta}{2\sqrt{3}} k \delta N_1 + \frac{v^{1/3}}{\sqrt{3}} \left(\frac{\pi_\varphi^2}{v^2} - V + M_{\mathrm{Pl}}^2 \frac{k^2}{v^{2/3}} \right) \delta N \\ \qquad + N \Big[-\frac{2}{3 v^{1/3}} \left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\mathrm{Pl}}^2 k^2}{4 v^{2/3}} \right) \delta\gamma_1 + \frac{\theta}{2 M_{\mathrm{Pl}}^2}\, \delta\pi_1 \\ \qquad + \frac{\sqrt{3}}{2} v^{1/3} \left(\frac{\pi_\varphi}{v^2}\, \delta\pi_\varphi - V_{,\varphi}\, \delta\varphi \right) - \frac{\sqrt{2}}{12 v} M_{\mathrm{Pl}}^2 k^2 \delta\gamma_2 \Big], \end{cases}$$

$$\begin{cases} \delta\dot{\gamma}_2 = -2\sqrt{\frac{2}{3}} v^{2/3} k \delta N_1 + N \left(\frac{4 v^{1/3}}{M_{\mathrm{Pl}}^2} \delta\pi_2 + \frac{\theta}{M_{\mathrm{Pl}}^2} \delta\gamma_2 \right), \\ \delta\dot{\pi}_2 = \sqrt{\frac{2}{3}} v^{1/3} \theta k \delta N_1 - \frac{M_{\mathrm{Pl}}^2 k^2}{\sqrt{6} v^{1/3}} \delta N \\ \qquad + N \Big[-\frac{\theta}{M_{\mathrm{Pl}}^2} \delta\pi_2 - \frac{\sqrt{2} M_{\mathrm{Pl}}^2}{12 v} k^2 \delta\gamma_1 - \frac{2}{3 v^{1/3}} \left(\frac{\pi_\varphi^2}{v^2} + \frac{V}{2} - \frac{M_{\mathrm{Pl}}^2 k^2}{8 v^{2/3}} \right) \delta\gamma_2 \Big], \end{cases} \tag{15}$$

$$\begin{cases} \delta\dot{\varphi} = \frac{\pi_\varphi}{v}\, \delta N + N \left(\frac{1}{v}\, \delta\pi_\varphi - \frac{\sqrt{3}}{2} \frac{\pi_\varphi}{v^{5/3}}\, \delta\gamma_1 \right), \\ \delta\dot{\pi}_\varphi = -\pi_\varphi k \delta N_1 - v V_{,\varphi}\, \delta N - N \left[v \left(\frac{k^2}{v^{2/3}} + V_{,\varphi,\varphi} \right) \delta\varphi + \frac{\sqrt{3}}{2} v^{1/3} V_{,\varphi}\, \delta\gamma_1 \right]. \end{cases}$$

This motivates the formulation of the separate-universe (or quasi-isotropic) approach at large scales, where we focus only on the isotropic degrees of freedom. See *e.g.* [Wands et al. (2000)]