Gyroscopic systems in Cosmology

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Outlines

- *Quadratic Lagrangians with 2 dof*
- *Gyroscopic Lagrangians: presence of* $\phi\,\dot{\phi}$
- *Classification of mixings*

  (Time=field with t-dep. vev, Space= field with $\vec{x}$*-dep. vev)*

  $$\begin{cases} \text{Time} - \text{Time} \\ \text{Space} - \text{Space} \\ \text{Space} - \text{Time} \end{cases}$$
- *Stability*
- *Applications*

Gyroscopic Equations of motion

linear conservative gyroscopic systems are described by the equation

$$\ddot{x} + \mathcal{D}\,\dot{x} + \mathcal{M}\,x = 0$$

$\mathcal{D}^t = -\mathcal{D}, \;\; \mathcal{M}^t = \mathcal{M}$

- The equilibrium is stable if $\mathcal{M} > 0$
- The equilibrium is unstable if $\mathcal{D} = 0$ and $\mathcal{M} < 0$
- The equilibrium is stable if $\mathcal{M} < 0$ but $\mathcal{D} > \mathcal{D}_{critical}$

$$\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 + x \cdot \mathcal{D} \cdot \dot{x} - \phi \cdot \mathcal{M} \cdot x\right)$$

Quadratic Lagrangians

Quadratic Lagrangian (only time derivatives ∂_t)

- *One dof:*

$$\mathcal{L} = k(t)\,\dot{\phi}^2 + \underbrace{d(t)\,\phi\,\dot{\phi}}_{0} - \underbrace{m(t)}_{m \Rightarrow m_{eff}}\,\phi^2$$

$$\int dt\; d\;\phi\;\dot{\phi} = -\int dt\;\frac{\dot{d}}{2}\;\phi^2 + b.c. \Rightarrow m_{eff} = m + \frac{\dot{d}}{2}$$

- *$n \geq 2$ dof:*

$$\mathcal{L} = \dot{\phi}\cdot\mathcal{K}\cdot\dot{\phi} + \phi\cdot\mathcal{D}\cdot\dot{\phi} - \phi\cdot\mathcal{M}\cdot\phi, \qquad \mathcal{K} = \mathcal{K}^t,\;\; \mathcal{M} = \mathcal{M}^t$$

$$\int dt\,\mathcal{D}_{ij}\;\phi_i\;\dot{\phi}_j - \mathcal{M}_{ij}\;\phi_i\;\phi_j = \int dt\left(\underbrace{\frac{(\mathcal{D}_{ij} - \mathcal{D}_{ij})}{2}}_{\mathcal{D}^t_{eff} = -\mathcal{D}_{eff}}\;\phi_i\;\dot{\phi}_j - \underbrace{(\mathcal{M}_{ij} + \frac{(\dot{\mathcal{D}}_{ij} + \dot{\mathcal{D}}_{ij})}{2})}_{\mathcal{M}_{eff}}\;\phi_i\;\phi_j\right) + b.c.$$

Quadratic Lagrangians

$$\mathcal{L}_{Bare} = \dot{\phi} \cdot \mathcal{K}_0 \cdot \dot{\phi} + \phi \cdot \mathcal{D}_0 \cdot \dot{\phi} - \phi \cdot \mathcal{M}_0 \cdot \phi$$

- Operations: $\pm$ *total derivative* terms
  $\mathcal{K}_0 = \mathcal{K}_0^t, \;\; \mathcal{M}_0 = \mathcal{M}_0^t, \;\; \mathcal{D}_0^t = -\mathcal{D}_0$
- Operations: *Point Transformations* $\phi \to F[\phi]$
  - Diagonalization Matrix $\mathcal{K} \to \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$
  - Antisymmetric Matrix $\mathcal{D} \to \; d(t) \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$
  - Diagonalization Matrix $\mathcal{M} \to \begin{vmatrix} m_1(t) & 0 \\ 0 & m_2(t) \end{vmatrix}$

Field Content

splitting of fields: background configuration ϕ plus fluctuation φ

$$\Phi = \phi + \varphi(t, \vec{x})\,.$$

Homogeneous and isotropic space time

- Fields with zero vev $\phi = 0$ we denote their fluctuations as $\varphi^A \equiv Z(t, \vec{x})$
- Time-fields: Fields with temporal vev $\phi = \phi(t)$ we denote their fluctuations as $\varphi \equiv T(t, \vec{x})$
- Space-Fields: Field with spatial vev $\phi^i = x^i$ we denote his fluctuations as $\varphi^i \equiv S^i(t, \vec{x}) = \frac{\partial_i}{\sqrt{\vec{\nabla}^2}} S + \hat{S}^i$ where $\partial_i \hat{S}^i = 0$.

Lagrangian formulation of hydrodynamical equation

- Building Blocks: four scalar fields $\varphi^A \quad A = 0,1,2,3$

  Stuckelberg fields for spont. broken space-time symm.
  $\varphi^{a=1,2,3}$ comoving coordinates of the continuous medium
  φ^0 internal time of the medium

$$\varphi^A(t,\vec{x}) = x^A + \underbrace{\pi^A(t,\vec{x})}_{phonons}$$

- Internal Symmetries in scalars space: Shift, $SO(3)_\varphi$, Non Linear extensions

$$Symmetries \leftrightarrow \begin{cases} Mechanical \begin{cases} (Super)Fluids \\ (Super)Solids \end{cases} \\ Thermodynamical \begin{cases} adiabatic \\ isentropic \\ barotropic, etc. \end{cases} \end{cases}$$

Operators on FRW

Symmetries: homogeneity and isotropy $SO(3)$

Space fields protected by shift symmetry: $\Phi^i \to \Phi^i + c_i \Rightarrow \mathcal{L}(\partial S,\ \partial T,\ T,\ t)$

- Operators with no derivatives:

$$\mathcal{O}_0 = T^2$$

- Operators with one derivative:

$$\begin{array}{rl} \text{spatial} & \mathcal{O}_x = \partial_i S^i \, T \ \propto k, \\ \text{temporal} & \mathcal{O}_t = T \, \dot{T} \end{array}$$

- Operators with two derivatives:

$$\begin{array}{rl} \text{both spatials} & \mathcal{O}_{xx} = \partial_i S^i \, \partial_j S^j, \ \ \partial_i S^j \, \partial_j S^i, \ \ \partial_i T \, \partial_j T \ \propto k^2 \\ \text{spatial and temporal} & \mathcal{O}_{tx} = \partial_i S^i \, \dot{T} \ \propto k, \\ \text{both temporals} & \mathcal{O}_{tt} = \dot{S}^i \dot{S}^i, \quad \dot{T} \dot{T} \end{array}$$

Structure of *bare* matrices $\mathcal{K}_0$, $\mathcal{D}_0$, $\mathcal{M}_0$,

$$\mathcal{L} = \underbrace{\mathcal{K}\cdot\mathcal{O}_{tt}}_{\text{2 time derivatives}} + \left(\underbrace{\mathcal{D}^{(0)}\cdot\mathcal{O}_t + \mathcal{D}^{(x)}\cdot\mathcal{O}_{tx}}_{\text{1 time derivatives}}\right) - \left(\underbrace{\mathcal{M}^{(0)}\cdot\mathcal{O}_0 + \mathcal{M}^{(x)}\cdot\mathcal{O}_x + \mathcal{M}^{(xx)}\cdot\mathcal{O}_{xx}}_{\text{0 time derivatives}}\right)$$

- $\mathcal{K}$: *Kinetic matrix,*
- $\mathcal{D}$: *Gyroscopic matrix,*
- $\mathcal{M}$: *Mass matrix.*

Shift symmetry allows only operators with two derivatives

$$\mathcal{L}^{shift} = \mathcal{K}\cdot\mathcal{O}_{tt} + \mathcal{D}^{(tx)}\cdot\mathcal{O}_{tx} - \mathcal{M}^{(xx)}\cdot\mathcal{O}_{xx}$$

Fluids, Thermodynamics and Shift Symmetry

- $\boxed{Minkowski\ space - time \Leftrightarrow shift\ symmetry}$

- $\boxed{Bunch\ Davis\ vacuum}$ in deSitter $\Leftrightarrow$ Minkowski Lag. in the early time- short length limit ($k\,t \to -\infty$) (equivalence principle)

- Presence of a Thermodynamical Fluid $\Leftrightarrow$ Spontaneously broken space-time translations $\Leftrightarrow$ $\boxed{Goldstone\ fields}$

Examples: TT

Time-Time: $\Phi_1 = \phi_1(t) + T_1\,, \qquad \Phi_2 = \phi_2(t) + T_2\,; \quad \underbrace{\phi_i(t) = t}_{Mink.}$

$$X_i = \partial\Phi_i \cdot \partial\Phi_i, \quad i = 1, 2, \quad X_3 = \partial\Phi_1 \cdot \partial\Phi_2$$

$$\int d^2x\sqrt{g}\; U[X_{j=1,2,3}, \boxed{\begin{matrix}\Phi^{i=1,2}\\ FRW\end{matrix}}]$$

$$\mathcal{K}_0 = \begin{vmatrix} -2U_{X_1} + 4U_{X_1^2} + 4U_{X_1X_3} + U_{X_3^2} & \# \\ -U_{X_3} + 2U_{X_1X_3} + 4U_{X_1X_2} + U_{X_3^2} + 2U_{X_2X_3} & -2U_{X_2} + U_{X_3^2} + 2U_{X_2X_3} \end{vmatrix}$$

$$\mathcal{D}_0 = \mathcal{D}_0^{(t)} = \frac{1}{2}\left(\left(U_{\Phi_1X_3} - 2U_{\Phi_2X_1}\right) + \left(2U_{\Phi_1X_2} - U_{\Phi_2X_3}\right)\right) \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

$$\mathcal{M}_0 = -\begin{pmatrix} 2\,U_{X_1} & U_{X_3} \\ U_{X_3} & 2\,U_{X_2} \end{pmatrix} k^2 + \begin{vmatrix} \# & \# \\ \# & \# \end{vmatrix}$$

$\mathcal{D}_0$ is k indep. and Shift symmetry $\rightarrow U_{\Phi_i} = 0 \rightarrow \mathcal{D}_0^{(t)} = 0$

Examples: SS

Space-Space: $\Phi_1^i = x^i + \frac{\partial_i}{\sqrt{\vec{\nabla}^2}} S_1 \,, \qquad \Phi_2^i = x^i + \frac{\partial_i}{\sqrt{\vec{\nabla}^2}} S_2 \,.$

$$X_j = \sum_i \partial\Phi_j^i \cdot \partial\Phi_j^i, \;\; j = 1,2, \qquad X_3 = \sum_i \partial\Phi_1^i \cdot \partial\Phi_2^i \qquad \int d^2x \sqrt{g}\; U[X_{j=1,2,3}]$$

$$\mathcal{K}_0 = -\begin{pmatrix} 2\,U_{X_1} & U_{X_3} \\ -U_{X_3} & 2\,U_{X_2} \end{pmatrix}$$

$$\mathcal{D}_0 = 0$$

$$\mathcal{M}_0 = -k^2 \begin{vmatrix} 4\left(U_{X_1^2} + U_{X_1 X_{12}}\right) + U_{X_{12}^2} + 2U_{X_1} & \\ 4U_{X_1X_2} + 2\left(U_{X_1X_{12}} + U_{X_2X_{12}}\right) + U_{X_{12}^2} + U_{X_{12}} & 4\left(U_{X_2^2} + U_{X_2X_{12}}\right) + U_{X_{12}^2} + 2U_{X_2} \end{vmatrix}$$

We have always $\mathcal{D}_0 = 0$

Examples: TS

Time-Space: $\Phi_1 = \phi(t) + T\,, \qquad \Phi_2^i = x^i + \frac{\partial_i}{\sqrt{\vec{\nabla}^2}} S\,.$

$$X_1 = \partial\Phi_1 \cdot \partial\Phi_1, \qquad X_2 = \sum_i \partial\Phi_2^i \cdot \partial\Phi_2^i, \quad X_3 = \sum_i (\partial\Phi_1 \cdot \partial\Phi_2^i)^2,$$

$$\int d^2x \sqrt{g}\; U[X_{j=1,2,3},\, \Phi_1]$$

$$\mathcal{K}_0 = 2\begin{vmatrix} 2\,U_{X_1^2} - U_{X_1} & 0 \\ 0 & (U_{X_3} - U_{X_2}) \end{vmatrix}$$

$$\mathcal{D}_0^{(x)} = \boxed{k}\,(2\,U_{X_1X_2} - U_{X_3})\,\mathcal{J}$$

$$\mathcal{M}_0 = \begin{vmatrix} -2\,k^2\left(U_{X_1} + U_{X_3}\right) + \# & k\,\# \\ k\,\# & 2\,k^2\left(U_{X_2} + 2U_{X_2^2}\right) \end{vmatrix}$$

- $\mathcal{K}_0$ is always diagonal
- The system is gyroscopic $\mathcal{D}_0 \neq 0$ for $U_{X_1X_2},\ U_{X_3} \neq 0$ with linear k dependence.
- The Mass matrix is diagonal for shift symmetric Time field

Lagrangian formulation of hydrodynamical equation

	$\mathcal{K}$	$\mathcal{D}^{(0)}$	$\mathcal{D}^{(x)} \propto k$	$\mathcal{M}^{(0)}$	$\mathcal{M}^{(x)} \propto k$	$\mathcal{M}^{(xx)} \propto k^2$
$\mathcal{L}_{Tot}$	#	#	#	#	#	#
$\mathcal{L}^{shift}_{Tot}$	#		#			#
LI	#					#
$\mathcal{L}_{TT}$	#	#		#		#
$\mathcal{L}^{shift}_{TT}$	#					#
$\mathcal{L}_{SS}$	#					#
$\mathcal{L}_{TS}$	#		#		#	#
$\mathcal{L}^{shift}_{TS}$	#		#			#

Table: We defined the total lagrangian as $\mathcal{L}_{Tot} \equiv \mathcal{L}_{TT} + \mathcal{L}_{TS} + \mathcal{L}_{SS}$, when a shift symmetry is present for all fields: $\mathcal{L}^{shift}_{Tot} \equiv \mathcal{L}^{shift}_{TT} + \mathcal{L}^{shift}_{TS} + \mathcal{L}_{SS}$, while the space fields are always subjected to shift symmetry $\mathcal{L}^{shift}_{SS} \equiv \mathcal{L}_{SS}$. Lagrangians $\mathcal{L}_{SS}$, $\mathcal{L}^{shift}_{TS}$, $\mathcal{L}^{shift}_{TT}$ are suitable for a Minkowski background descriptions and the corresponding matrices $\mathcal{K}$, $\mathcal{D}$, $\mathcal{M}$ will be constant in time, while $\mathcal{L}_{SS}$, $\mathcal{L}_{TT}$, $\mathcal{L}_{TS}$ can be used to describe fluctuations in a cosmogical setting and the corresponding matrices can be time dependent.

Transformation (1): Canonical Lagrangian

$$\mathcal{L}_{Bare} = \dot{\varphi}_0^t\,\mathcal{K}_0\,\dot{\varphi}_0 + \varphi_0^t\,\mathcal{D}_0\,\dot{\varphi}_0 - \varphi_0^t\,\mathcal{M}_0\,\varphi_0\,, \qquad \varphi_0 = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix};$$

Canonical Fields & Lagrangian

$$\varphi_0 = A_k\,\varphi_c, \qquad A_k = \mathcal{R}_k\,\mathcal{K}_d^{-1/2}\,, \quad \mathcal{R}_k^t\,\mathcal{K}_0\,\mathcal{R}_k = \mathcal{K}_d = \begin{vmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{vmatrix}$$

$$\mathcal{L}_c = \dot{\varphi}_c^t\,\mathcal{K}_c\,\dot{\varphi}_c + \varphi_c^t\,\mathcal{D}_c\,\dot{\varphi}_c - \varphi_c^t\,\mathcal{M}_c\,\varphi_c\,,$$

$$\mathcal{K}_c = \mathbf{I}\,, \qquad \mathcal{D}_c = d_c \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \qquad \boxed{d_c = \frac{d_0 - \dot{\theta}_k\,(\kappa_1+\kappa_2)}{\sqrt{\kappa_1\,\kappa_2}}}$$

$$\mathcal{M}_c = A^t\,\mathcal{M}_0\,A + \begin{vmatrix} \frac{(\kappa_1')^2}{4\kappa_1^2} - \frac{\kappa_1''}{2\kappa_1} - \frac{\kappa_2(\theta_k')^2}{\kappa_1} & \# \\ \left(\frac{(3\kappa_1+\kappa_2)\kappa_1'}{4\kappa_1^{3/2}\sqrt{\kappa_2}} + \frac{(-\kappa_1-3\kappa_2)\kappa_2'}{4\sqrt{\kappa_1}\kappa_2^{3/2}}\right)\theta_k' + \frac{(\kappa_1-\kappa_2)\theta_k''}{2\sqrt{\kappa_1}\sqrt{\kappa_2}} & \frac{(\kappa_2')^2}{4\kappa_2^2} - \frac{\kappa_2''}{2\kappa_2} - \frac{\kappa_1(\theta_k')^2}{\kappa_2} \end{vmatrix}$$

Transformation (1): Final Lagrangian

Normal Fields $\quad \varphi_c = \mathcal{R}_m \varphi_n = A_k^{-1} \varphi_0$

$$\mathcal{R}_m^t \, M_c \, \mathcal{R}_m = M_d = \begin{vmatrix} \tilde{m}_1 & 0 \\ 0 & \tilde{m}_2 \end{vmatrix}, \quad \mathcal{R}_m = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} , \quad \mathcal{R}_m^t \mathcal{R}_m = 1$$

Normal Lagrangian

$$\mathcal{L} = \dot{\varphi}^t \, \mathcal{K} \, \dot{\varphi} + \varphi^t \, \mathcal{D} \, \dot{\varphi} - \varphi^t \, \mathcal{M} \, \varphi \, ,$$

$$\begin{aligned} \mathcal{K} &= \mathbf{I}, \\ \mathcal{D} &= d \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \qquad d = \frac{d_0}{det[\mathcal{K}_0]^{1/2}} - \dot{\theta}_k \, \frac{tr[\mathcal{K}_0]}{det[\mathcal{K}_0]^{1/2}} - 2\,\dot{\theta}_m^2, \\ \mathcal{M} &= \mathcal{M}_d - 2\,\dot{\theta}_m^2 \, \mathbf{I} = \begin{vmatrix} \tilde{m}_1 - 2\,\dot{\theta}_m^2 & 0 \\ 0 & \tilde{m}_2 - 2\,\dot{\theta}_m^2 \end{vmatrix} \equiv \begin{vmatrix} m_1 & 0 \\ 0 & m_2 \end{vmatrix} \end{aligned}$$

with $\varphi = \mathcal{R}_m^{-1} \, K_d^{1/2} \, \mathcal{R}_k^{-1} \, \varphi_0$

Sources for the Effective Gyroscopic term

$$d = \frac{d_0}{det[\mathcal{K}_0]^{1/2}} - \dot\theta_k\, \frac{tr[\mathcal{K}_0]}{det[\mathcal{K}_0]^{1/2}} - 2\,\dot\theta_m^2$$

Sources:

- the bare matrix coefficient, $d_0 \neq 0$ (Non shift symmetric TT models, ST models)
- A non trivial time dependence of the kinetic and canonical mass matrix: $\dot\theta_k \neq 0,\ \dot\theta_m \neq 0$

Cases where $\dot\theta_\lambda = 0$ ($\Lambda = |\lambda_{ik}(t)|,\ \tan(2\,\theta_\lambda) = \frac{2\,\lambda_{1,2}}{\lambda_{2,2}-\lambda_{1,1}}$)

- Static matrix $\dot\lambda_{ij} = 0$
- Diagonal matrix $\lambda_{1,2} = 0$
- Overall time dependence for the full matrix: $\Lambda = f(t)\,\bar\Lambda$ where $\bar\Lambda$ is constant in time.
- $\lambda_{1,1} - \lambda_{2,2} = x\,\lambda_{1,2}$ where x=const
- For $x = 0$ we have $\lambda_{1,1} = \lambda_{2,2}$

Quadratic Lagrangians

By a Lagrangian field redefinition, we can set $\mathcal{D} = 0$.
$\varphi = L\,\tilde{\varphi}$, with $\dot{L} = \mathcal{D} \cdot L$

$$L = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}, \qquad \dot{\theta} = d\,.$$

$$\mathcal{M}_F = \begin{vmatrix} d^2 + m_1^2 \cos(d\ t)^2 + m_2^2 \sin(d\ t)^2 & \frac{1}{2}\left(m_1^2 - m_2^2\right)\sin(2\ d\ t) \\ \frac{1}{2}\left(m_1^2 - m_2^2\right)\sin(2\ d\ t) & d^2 + m_2^2 \cos(d\ t)^2 + m_1^2 \sin(d\ t)^2 \end{vmatrix}.$$

When m_i, d are *const*, the time dependence is of the Floquet type where the Hamiltonian is periodic $H(t) = H(t+T)$ with $T = \frac{\pi}{d}$

$$\mathcal{L} = \dot{\phi} \cdot \mathbf{I} \cdot \dot{\phi} - \phi \cdot \mathcal{M}_F \cdot \phi$$

Time reversal

time reversal requires a transformation of phase space that reverses momenta and preserves position

$$\mathbf{T}\ (\phi(t),\ \pi(t)) = (\phi(-t),\ -\pi(-t))$$

⋆ Time reversal conditions: $H(\phi,\ \pi) = H(\phi,\ -\pi) + const$
⋆ If the momentum of a system is proportional to its velocity, $\pi = m\ \dot{\phi}$, then the system is time reversal invariant.
In our case

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi} - \mathcal{D}\ \phi$$

and

$$\mathcal{H} = \pi\ \dot{\phi} - \mathcal{L} = (\pi -\ \phi\ \mathcal{D})(\pi + \mathcal{D}\ \phi) + \phi\ \mathcal{M}\ \phi$$

Lagrangian eqs of motion

Lagrangian Equation of motion (*Chetaev system*)

$$\mathbf{I} \cdot \ddot{\phi} + 2\,\mathcal{D} \cdot \dot{\phi} + \mathcal{M} \cdot \phi = 0$$

spectral analysis

$$\text{for } \phi \sim e^{i\,\omega\,t} \qquad \Rightarrow \qquad det\| -\mathbf{I}\,\omega^2 + 2\,i\,\mathcal{D}\,\omega + \mathcal{M}\| = 0$$

$$\boxed{\omega^2_{1,2} = \frac{1}{\sqrt{2}}\left(m_1 + m_2 + 4\,d^2 \pm \sqrt{(m_1 + m_2 + 4\,d^2)^2 - 4\,m_1\,m_2}\right)}$$

stability: $\omega_{1,2} \in \mathbf{R} \to \omega^2_{1,2} > 0$

Classical stability

$$\boxed{\omega_{1,2}^2 = \frac{1}{\sqrt{2}}\left(X \pm \sqrt{X^2 - 4\,m_1\,m_2}\right)}$$

$$X \equiv m_1 + m_2 + 4\,d^2$$

$$\omega_{1,2}^2 > 0 \quad \to \quad m_1\ m_2 > 0 \ \ \& \ \ X^2 > 4\ m_1\ m_2$$

$$\text{Stability} \begin{cases} \boxed{m_1 > m_2 > 0},\ d^2 > 0 & \Rightarrow \quad 0 < d^2 < \frac{(\omega_1 - \omega_2)^2}{4} \\ \boxed{m_1 < m_2 < 0},\ d^2 > \frac{(\sqrt{-m_1} + \sqrt{-m_2})^2}{4} & \Rightarrow \quad d^2 > \frac{(\omega_1 + \omega_2)^2}{4} \end{cases}$$

Equation of motion (*Chetaev system*)

$$\mathbf{I}\cdot\ddot{\phi} + 2\,(\underbrace{\underbrace{\mathcal{D}}_{\mathcal{D}^t = -\mathcal{D}}}_{\textit{Gyroscopic}} + \underbrace{\underbrace{\mathcal{G}}_{\mathcal{G}^t = \mathcal{G}}}_{\textit{Dissipative}})\cdot\dot{\phi} + \underbrace{\underbrace{\mathcal{M}}_{\mathcal{M} = \mathcal{M}^t}}_{\textit{Potential}}\cdot\phi = 0$$

- $\mathcal{M}$ positive defined.

  If the system is stable when $\mathcal{D} = \mathcal{G} = 0$, then it remains stable after the introduction of arbitrary gyroscopic and dissipative forces.
- $\mathcal{M}$ negative defined.
  - For $\boxed{\#_{dof} = \textit{even}}$ the system can be stabilised by $\mathcal{D} > \mathcal{D}_{critical}$
  - For $\boxed{\#_{dof} = \textit{odd}}$ the system cannot be stabilised
  - $\mathcal{G}$ is destabilising the system

  If a system with an unstable potential energy is stabilized with gyroscopic forces, then this stability is lost after the addition of arbitrarily small dissipation.

Lancaster: Review (2012)

Hamiltonian diagonalization in symplectic formalism

$$\mathcal{L} = \dot{\varphi}^t \, \mathbf{I} \, \dot{\varphi} + \varphi^t \, \mathcal{D} \, \dot{\varphi} - \varphi^t \, \mathcal{M} \, \varphi \,,$$

Conjugate momentum: $\pi = \dot{\phi} - \mathcal{D} \cdot \phi$

$$H = \frac{1}{2} z^t \cdot \mathcal{H}_{4\times 4} \cdot z \qquad z = \begin{vmatrix} \varphi \\ \pi \end{vmatrix} \quad \text{with} \quad \varphi = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix}, \quad \pi = \begin{vmatrix} \pi_1 \\ \pi_2 \end{vmatrix}$$

$$\mathcal{H} = \begin{vmatrix} \mathbf{I}_{2\times 2} & \mathcal{D} \\ \mathcal{D}^t & (\mathcal{M} - \mathcal{D}^2) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & d \\ 0 & 1 & -d & 0 \\ 0 & -d & m_1 + d^2 & 0 \\ d & 0 & 0 & m_2 + d^2 \end{vmatrix}$$

Hamiltonian diagonalization

Eqs of motion in Hamiltonian form

$$\dot{z}(t) = \{z(t),\ H\} = \Omega \cdot \mathcal{H} \cdot z(t), \qquad \Omega = \begin{vmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{vmatrix}$$

$$z(t) = \mathbf{G}(t,\ t_0) \cdot z_0$$

$$\partial_t \mathbf{G}(t,\ t_0) = \Omega \cdot \mathcal{H} \cdot \mathbf{G}(t,\ t_0), \qquad \mathbf{G}(t_0,\ t_0) = \mathbf{I}_{4\times 4}$$

$$\mathbf{G}(t,\ t_0) = T_\tau\ e^{\int_0^t d\tau\ \Omega\cdot\mathcal{H}(\tau)}$$

For a constant, in time, Hamiltonian:

$$\mathbf{G}(t,\ 0) = e^{\Omega\cdot\mathcal{H}\ t}$$

All the eigenvalues of the matrix $\Omega \cdot \mathcal{H}$ are purely imaginary:
$\pm i\ \omega_{1,2} \qquad (\omega_{1,2} \in \mathbf{R})$

Diagonalization by a Simplectic S matrix (*congruence eigenvalues*)

$$S^t \cdot \mathcal{H} \cdot S = \mathcal{H}_{diag}$$

$$\mathcal{H}_{diag} = \begin{vmatrix} \frac{T_{44}\left(\omega_1^2-\omega_2^2\right)}{8\,d^2} & 0 & 0 & 0 \\ 0 & \frac{T_{33}\left(\omega_1^2-\omega_2^2\right)}{8\,d^2} & 0 & 0 \\ 0 & 0 & \frac{T_{22}}{2\,(\omega_1^2-\omega_2^2)} & 0 \\ 0 & 0 & 0 & \frac{T_{11}}{2\,(\omega_1^2-\omega_2^2)} \end{vmatrix}$$

with

$$T_{11} = 4\,d^2 + \chi + \omega_1^2 - \omega_2^2, \quad T_{22} = -4\,d^2 + \chi + \omega_1^2 - \omega_2^2,$$
$$T_{33} = 4\,d^2 - \chi + \omega_1^2 - \omega_2^2, \quad T_{44} = -4\,d^2 - \chi + \omega_1^2 - \omega_2^2$$

$\chi = (16\,d^4 - 8\,d^2\,\left(\omega_1^2+\omega_2^2\right) + \left(\omega_1^2-\omega_2^2\right)^2)^{1/2}$

$$T_{11} \geq \text{ and } T_{33} \geq 0 \text{ for } 0 \leq d^2 \leq \frac{(\omega_2-\omega_2)^2}{4} \;\&\; d^2 \geq \frac{(\omega_1+\omega_2)^2}{4}$$

$$T_{22} \geq 0 \text{ and } T_{44} \geq 0 \text{ for } 0 \leq d^2 \leq \frac{(\omega_1-\omega_2)^2}{4}$$

So, for $0 \leq d^2 \leq \frac{(\omega_1-\omega_2)^2}{4}$ the diagonal Hamiltonian is positive defined while in the region $d^2 \geq \frac{(\omega_1+\omega_2)^2}{4}$ it results indefinited, with two positive entries ($\propto\ T_{11},\ T_{33}$) and two negative one ($\propto\ T_{22},\ T_{44}$).

Diagonalization

Finally we can still perform a final canonical transformation that brings our Hamiltonian in the classical oscillator form, very useful for the quantization

$$\tilde{z} = \boldsymbol{N}^{(\pm)} \cdot z_c \,, \qquad \boldsymbol{N}^{(\pm)}_{4\times 4} = \frac{1}{\sqrt{2}} \begin{vmatrix} n^{(\pm)}_{2\times 2} & 0 \\ 0 & n^{-1\,(\pm)}_{2\times 2} \end{vmatrix}$$

where, we have to distinguish two different cases depending if T_{22} is positive or negative

$$n^{(\pm)} = \begin{vmatrix} \frac{\sqrt{2}\sqrt{\omega_1}\sqrt{\omega_1^2-\omega_2^2}}{\sqrt{T_{11}}} & 0 \\ 0 & \frac{\sqrt{2}\sqrt{\omega_2}\sqrt{\omega_1^2-\omega_2^2}}{\sqrt{\pm T_{22}}} \end{vmatrix}$$

$$H^{(\pm)}_c = z^+_c \; \Lambda^{(\pm)}_c \; z_c \,, \qquad \Lambda^{(\pm)}_c = \begin{vmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \pm\omega_2 & 0 & 0 \\ 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & \pm\omega_2 \end{vmatrix}$$

where, $\Lambda^{(+)}_c$ is defined in the region $0 \leq d^2 \leq \frac{(\omega_1-\omega_2)^2}{4}$ and it results definite positive while $\Lambda^{(-)}$ is defined in the region $d^2 \geq \frac{(\omega_1+\omega_2)^2}{4}$ and it results indefinite (note that always we have $\omega_{1,2} \geq 0$). The explicit expressions for the Hamiltonian

$$\underbrace{H^{(+)}_c = \sum_{i=1,2} \omega_i \; \left(\pi^2_{i,c} + \varphi^2_{i,c}\right),}_{0 \leq d^2 \leq \frac{(\omega_1-\omega_2)^2}{4}} \qquad H^{(-)}_c = \underbrace{\omega_1 \; \left(\pi^2_{1,c} + \varphi^2_{1,c}\right) - \omega_2 \; \left(\pi^2_{2,c} + \varphi^2_{2,c}\right)}_{d^2 \geq \frac{(\omega_1+\omega_2)^2}{4}}$$

shows that once we quantize $H^{(-)}_c$ result affected by ghost instabilities.

Conjecture about Ghost

"We therefore conjecture that, if the classical dynamics of the system is benign, its quantum dynamics will also be benign, irrespectively of whether the spectrum has, or does not have, a bottom." (Smilga Damour 2022)

ask to Damiano or to Cedric.... :)

The 6-th Mode in Minkowski

$S^i = \frac{\partial_i}{\sqrt{\vec{\nabla}^2}} S + \hat{S}^i$

Transverse vectors (2 dof, $\partial_i \hat{S}^i = 0$) one gets $\quad (p = w\ \rho))$

$$L_{(\hat{S})} = \frac{1}{2}\,(c_1 + 1 + w)\ \hat{S}'^2 - k^2\ c_2\ \hat{S}^2\,.$$

Scalars $T,\ S$

$$\mathcal{L}_{T,S} = \underbrace{\frac{1}{2}\,(c_1 + w + 1)\ S'^2 + c_0\ T'^2}_{\mathcal{K}} + \underbrace{\frac{(c_1 - 2\,c_4)}{2}\ k\ \left(S\ T' - T\ S'\right)}_{\mathcal{D}} + \underbrace{(c_3 - c_2)\,k^2 S^2 + \frac{1}{2} c_1 k^2 T^2}_{\mathcal{M}}$$

canonization

$$\mathcal{K} = \mathbf{I}, \quad d = \frac{(c_1 - 2\,c_4)\,k}{2\sqrt{2}\sqrt{c_0}\sqrt{c_1 + w + 1}}, \quad \mathcal{M} = \begin{vmatrix} \frac{2(c_2 - c_3)k^2}{c_1 + w + 1} & 0 \\ 0 & -\frac{c_1 k^2}{2c_0} \end{vmatrix}$$

$$c_0 > 0, \ \ 1 + w + c_1 > 0, \ \ c_2 > 0$$

Branch one:

$$m_{1,2} > 0: \qquad -(1 + w) < c_1 < 0\,, \quad c_3 < c_2$$

for $w \to -1$ we have $c_1 \to 0$

Branch two:

$$m_{1,2} < 0: \qquad c_1 > Max[0,\ -(1 + w)]\,, \quad c_3 > c_2 > 0\,, \quad 0 < c_0 < \frac{(c_1 - 2c_4)^2}{4\,(c_3 - c_2))}, \quad + \ldots$$

for $w \to -1$ we have $c_4 < -\sqrt{c_0\ (c_3 - c_2)}$, $\ c_4 > c_1 + \sqrt{c_0\ (c_3 - c_2)}$ but $\mathcal{H}$ undefined...

Bunch Davis vacuum for Inflation

$$\mathcal{K}_0 = \begin{vmatrix} \bar{\kappa}_1 \; a^{\xi_1} & 0 \\ 0 & \bar{\kappa}_2 \; a^{\xi_2} \end{vmatrix}, \quad d_0 = \bar{d} \; a^{\varsigma + \frac{\xi_1+\xi_2}{2}} \; , \quad \mathcal{M}_0 = \left| \bar{m}_{ij} + \hat{m}_{ij} \; a^{\eta_{ij}} \right| \; , \quad \dot{\theta}_k = 0$$

Note: for shift symmetry $\bar{m}_{ij} = 0$. Then we fine tune $\eta_{11} = \xi_1$, $\eta_{22} = \xi_2$

$$\mathcal{M}_c = \begin{vmatrix} \frac{\hat{m}_{11}}{\bar{\kappa}_1} + a^{-3w-1} \left(\frac{\xi_1}{3w+1} - \frac{\xi_1^2}{(3w+1)^2} \right) & \# \\ \frac{1}{\sqrt{\kappa_2 \; \kappa_2}} \left(a^{\eta_{12} - \frac{\xi_1+\xi_2}{2}} \; \hat{m}_{12} + a^{-\frac{1+3w}{2} + \varsigma} \; d \; \left(\frac{\xi_1 - \xi_2}{3w+1} \right) \right) & \frac{\hat{m}_{22}}{\bar{\kappa}_2} + a^{-3w-1} \left(\frac{\xi_2}{3w+1} - \frac{\xi_2^2}{(3w+1)^2} \right) \end{vmatrix} + a^{-\frac{\xi_1+\xi_2}{2}} \; \bar{m}_{ij}$$

$$d = \frac{\bar{d}}{det[\bar{K}]^{1/2}} \; a^{\varsigma} - 2 \, \dot{\theta}_m^2$$

For Bunch Davis vacuum $a \to 0$ we need $\mathcal{M}_c \to const$ and $d \to const$ (i.e. $\dot{\theta}_m \to 0$):

$$w < -\frac{1}{3} \; , \quad \varsigma \geq 0 \, , \, \eta_{12} \geq \frac{\xi_1 + \xi_2}{2}$$

Lagrangian formulation of hydrodynamical equation

we reduce the mass matrix $\mathcal{M}$ to the following constant mass matrix

$$\mathcal{M}^{BD} = \begin{vmatrix} \frac{\hat{m}_{11}}{\bar{\kappa}_1} & \frac{\hat{m}_{12}}{\sqrt{\bar{\kappa}_1}\sqrt{\bar{\kappa}_2}}\,\delta_{\eta_{12}}^{(\xi_1+\xi_2)/2} \\ \frac{\hat{m}_{12}}{\sqrt{\bar{\kappa}_1}\sqrt{\bar{\kappa}_2}}\,\delta_{\eta_{12}}^{(\xi_1+\xi_2)/2} & \frac{\hat{m}_{22}}{\bar{\kappa}_2} \end{vmatrix} + \delta^0_{\xi_i} \begin{vmatrix} \frac{\bar{m}_{11}}{\bar{\kappa}_1} & \frac{\bar{m}_{12}}{\sqrt{\bar{\kappa}_1}\sqrt{\bar{\kappa}_2}} \\ \frac{\bar{m}_{12}}{\sqrt{\bar{\kappa}_1}\sqrt{\bar{\kappa}_2}} & \frac{\bar{m}_{22}}{\bar{\kappa}_2} \end{vmatrix}$$

In this limit we get $\dot{\theta}_m \to 0$ so that

$$\mathcal{L}^{BD} = \dot{\varphi}^t\, \mathbf{I}\, \dot{\varphi} + \frac{\bar{d}}{\sqrt{\bar{\kappa}_1\,\bar{\kappa}_2}}\, \delta^0_\varsigma\; \varphi^t\, \mathcal{J}\, \dot{\varphi} - \varphi^t\, \mathcal{M}^{BD}\, \varphi$$

So that the $\bar{d}$ term is present only for $\varsigma = 0$ and for Time-Time (with specific shift symm. breaking) or Space-Time fields.

$$\omega^2_{1,2} = \frac{1}{\sqrt{2}}\,(Tr[\mathcal{M}] + 4\,d^2 \pm \sqrt{(Tr[\mathcal{M}] + 4\,d^2)^2 - 4\,Det[\mathcal{M}]})\underbrace{=}_{k\to\infty} c_s^2\;k^2 + \underbrace{\dots}_{shift\;\;breaking}$$

to see the sound velocities in the $k \to \infty$ limit, we write $m_{i,j} \equiv c^2_{ij}\;k^2 + \tilde{m}_{i,j}$:

If $d \propto k$ as in ST models, it is essential for the BD vacuum

If $d \propto \mathcal{O}(1)$ as in TT models, it is NOT essential for the BD vacuum

For Supersolid inflation $w = -1,\ \xi_1 = -\xi_2 = 4,\ \eta_{12} = 1,\ \varsigma = 0$ and $a \to 0$

$$d^{BD} = \mathbf{d}\;k\,, \qquad \mathcal{M}^{BD} = k^2 \begin{vmatrix} c_1^2 & 0 \\ 0 & c_2^2 \end{vmatrix}$$

Conclusions

We studied quadratic Gyroscopic Lagrangians with two dof (i.e. with terms $\propto d\ \phi_1\ \dot{\phi}_2$)

- Lagrangian diagonalization (partial)
- Hamiltonian diagonalization (complete)
- Two main stability regimes on Minkowski
$$\begin{cases} m_i > 0 & d^2 > 0 & \mathcal{H} \text{ def. positive} \\ m_i < 0 & d^2 > \frac{\sqrt{-m_1}+\sqrt{-m_2}}{4} & \mathcal{H} \text{ NOT def. positive} \end{cases}$$
- Gyroscopic fluids on Minkowski: Supersolids
- BD vacuum

The 6-th Mode in Minkowski

The linear expansion of the medium action gives

$$S_\pi^{(1)} = \int d^4x \left[\sigma\ \dot{T} + (\sigma - p - \rho)\sqrt{\partial_i^2}S\right] = 0\,; \quad {\scriptstyle p,\ \rho,\ \sigma = const}$$

$$S_h^{(1)} = \int d^4x\ \left[T^{\mu\nu}_{Mink}\ h_{\mu\nu}\right] = 0, \quad {\scriptstyle T^{\mu\nu}_{Mink} = p\ \eta_{\mu\nu} + (\rho+p)\ u_\mu\ u_\nu = 0}$$

$p,\ \rho = 0 \to$ Ghost!!

1. $R \sim \frac{\rho,\, p}{M_{pl}^2} \ll k^2$ the flat space picture is adequate and the fluctuations of the spacetime metric can be neglected.
2. $R \sim \frac{\rho,\, p}{M_{pl}^2} \simeq k^2$ the background solution has to amended and the metric fluctuations are important.

Lagrangian formulation of hydrodynamical equation

The ground state breaks spatial translations (and Rotations)

$$< \varphi^i >= x^i \qquad i = 1, 2, 3$$

To have an homogeneous EMT we needs Internal symmetries

$$\mathcal{T}_{internal} : \varphi \to \varphi' = \varphi + c$$

The background is invariant under diagonal translation

$$\mathcal{T}_{diagonal} = \mathcal{T}_{spatial} + \mathcal{T}_{internal} : \begin{cases} x \to x - c \\ \varphi \to \varphi + c \end{cases}$$

To have an isotropic background we impose

$$\mathcal{R}_{internal} : \varphi^a \to \varphi'^a = R^a_b\, \varphi^b, \qquad R \in SO(3)$$

etc...

Fields $\boxed{g_{\mu\nu} \oplus \varphi^A}, \quad A = 0, 1, 2, 3$

Diffeormorphism invariance:

$$x'^{\mu} = x^\mu + \zeta^\mu \quad \begin{cases} \delta g_{\mu\nu} = \nabla_\nu \zeta_\mu + \nabla_\mu \zeta_\nu \\ \varphi^A = \bar{\phi}^A_{bg} + \pi^A, \quad \delta\pi^A = -\zeta^\alpha \partial_\alpha \bar{\phi}^A_{bg} \end{cases}$$

- Lagrangian in Newtonian gauge $\zeta^\alpha = 0$
  (2 tensors + 4 phonons)

$$\int \sqrt{g} \; (R(g) + U(\partial \varphi^A, g))$$

  # DoF: 2 (tensor gravitational modes) + 4 (scalar fields) =6

- Lagrangian in Unitary (Comoving) Gauge $\pi^A = 0 \rightarrow \varphi^A = x^A$:
  Massive Gravity ($C^{AB} = g^{AB}$)

$$\int \sqrt{g} \; (R + U(g^{00}, \; g^{0i}, \; g^{ij}))$$

  # DoF:(2 tensors + 2 vectors + 2 scalars) gravitational modes =6

Mass terms (Perturbative)

Unitary gauge: $\varphi^A \equiv x^A \rightarrow \partial_\mu \varphi^A = \delta^A_\mu$

$$U(C^{AB} = \partial_\alpha \varphi^A \, g^{\alpha\beta} \, \partial_\beta \varphi^B) \rightarrow U(g^{AB})$$

In FRW: $g_{\mu\nu} = a^2 \, (\eta_{\mu\nu} + h_{\mu\nu})$ and expansion at second order $\mathcal{O}(h)^2$ with $SO(3)$ invariance

$$\int \sqrt{g}\; U(g^{AB}) = \int M_0 \; h_{00}^2 + \overbrace{M_1}^{Vectors} \; h_{0i}^2 + \underbrace{M_2 \, h_{ij}^2 + M_3 \; h_{ii}^2}_{Graviton-mass} + M_4 \; h_{00}$$

$$M_i \propto M_i(U, \; \partial_k U, \; \partial^2_{k,r} U)\Big|_{\bar{\phi}}, \qquad k,r = b, Y, X, \tau_i, y_i$$

Lorentz Invariant $U(g^{\mu\alpha}\eta_{\alpha\nu})$: $M_0 = A + B, \; M_{1,2} = -A, \; M_{3,4} = B$

Pauli Fierz Lagrangian: $A = -B$

Ghost in Massive Gravity

Mass term for a $g_{\mu\nu} \rightarrow$ reference metric $\bar{f}_{\mu\nu}$: $U(g_{\mu\alpha}\bar{f}^{\alpha\nu})$
A Massive spin 2 on a Minkowski background has 5 DoF

- Fierz-Pauli action (1939) for single massive spin 2 particle in flat space

$$\int m^2 \left((h_{\mu\nu} h^{\mu\nu}) - \underbrace{\lambda}_{1} (h^{\mu}_{\mu})^2 \right), \qquad h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

- Boulware-Deser Ghost (1972): they analyzed a large class of various mass terms, showing the presence of sixth degree of freedom (extra scalar), but they did not consider the most general possible potential.
- C. de Rham, G. Gabadadze, and A. J. Tolley dRGT(2010) $U(\sqrt{g^{\mu\alpha}\eta_{\alpha\nu}})$ ghost free (only 5 DoF).

The fate of the ghost like 6-th Mode

General structure of the quadratic Lagrangian ($\int \sqrt{g}\,(R+U)$)
Phonon pert: $\Phi^0 \equiv x^0 + \pi^0$, $\Phi^i \equiv x^i + \pi_i$,
$\pi_i \equiv \partial_i \pi_L + v_i \;\; (\partial^i v_i = 0)$

Metric pert: $g_{\mu\nu} = \eta_{\mu\nu} + \Phi\, \delta^0_\mu\, \delta^0_\nu + \Psi\, \delta^i_\mu\, \delta^i_\nu$

$$\mathcal{L}_2 = \mathcal{L}_U(\pi^A) + \mathcal{L}_U(\pi^A,\ \Phi,\ \Psi) + \mathcal{L}_R(\Phi,\ \Psi)$$

General Behaviours

(1) Phonons on Minkowski: stable for $\rho + p \neq 0$

(2) Phonons + GR perturbaz on Minkowski: Ghost

(3) Phonons + GR perturbaz on FRW and
$k^2 (\sim \frac{1}{\lambda^2}) \gg \mathcal{H}^2$: stable (same stability conditions of (1))

Mechanical properties (Non Perturbative)

$$T_{\mu\nu} = \frac{\delta(\sqrt{g}\ U)}{\sqrt{g}\ \delta g^{\mu\nu}} = \underbrace{p\ g_{\mu\nu} + (p+\rho)\ u_\mu\ u_\nu}_{Perfect\ Fluid} + \underbrace{q_\mu\ u_\nu + q_\nu\ u_\mu + \overbrace{\Pi_{\mu\nu}}^{Solid}}_{Super\ Fluid/Solid}$$

EMT	Lagrangian	Medium	Masses
$q_\mu = 0,\ \Pi_{\mu\nu} = 0$	$U(b,\ Y)$	Perfect Fluid	$M_{1,\,2} = 0$
$q_\mu = 0,\ \Pi_{\mu\nu} \neq 0$	$U(b,\ \tau_n,\ Y)$	Solid	$M_1 = 0$
$q_\mu \neq 0,\ \Pi_{\mu\nu} \propto\ q_\mu\ q_\nu$	$U(b,\ Y,\ X)$	SuperFluid	$M_2 = 0$
$q_\mu \neq 0,\ \Pi_{\mu\nu} \neq 0$	$U(b,\ Y,\ X,\ \tau_n,\ y_n)$	SuperSolid	$M_{1,2} \neq 0$

Building the Lagrangian

$$\varphi^{A=0,1,2,3} \underbrace{\Rightarrow}_{\text{shift sym.}:\underline{\varphi^A \to \varphi^A + \partial c^A}} \partial\varphi^A$$

$$\underbrace{\Rightarrow}_{\text{Lorentz Scalar}} C^{AB} = \partial_\mu\varphi^A g^{\mu\nu} \partial_\nu\varphi^B \underbrace{\to}_{SO(3)_\Phi} \underbrace{C^{00} \equiv \mathbf{S}}_{Scalar}, \underbrace{C^{0a} \equiv \mathbf{V}}_{Vector}, \underbrace{C^{ab} \equiv \mathbf{T}}_{Tensor}:$$

$$\mathbf{S}, \; Tr[\mathbf{T}^{n=1,2,3}], \; \mathbf{V}\cdot\mathbf{T}^{n=0,1,2,3}\cdot\mathbf{V}$$

$$\underbrace{\Rightarrow}_{\text{Lorentz Vector},\ SO(3)_\Phi \text{ Scalar}} u^\mu \sim \epsilon^{\mu\alpha\beta\gamma} e_{abc}\partial_\alpha\varphi^a\partial_\beta\varphi^b\partial_\gamma\varphi^c \underbrace{\to}_{\text{Lorentz Scalar}} u^\mu\partial_\mu\varphi^0$$

$$\boxed{9 \; operators}$$

$A, B = 0, 1, 2, 3\,, \quad a, b, c = 1, 2, 3$

Lagrangian of 4 scalar + shift sym.+ Lorentz

- At lowest order in derivatives: $C^{AB} = \partial_\mu \varphi^A \, g^{\mu\nu} \, \partial_\nu \varphi^A$

$$S = \int d^4x \sqrt{-g} \, \left(M_{pl}^2 \, R \, + \, U(C^{AB})\right)$$

global internal spatial $SO(3)_\Phi$ symmetry

$$\varphi^0 \to \varphi^0, \quad \varphi^a \to R^a_b \, \varphi^b \qquad R \, R^T = I, \quad a = 1, 2, 3$$

$\underline{SO(3)_\Phi \text{ Tensors}}$: C^{ab}, C^{a0}, C^{00}

Operator	Definition
C^{AB}	$g^{\mu\nu} \, \partial_\mu \varphi^A \, \partial_\nu \varphi^B$, $A, B = 0, 1, 2, 3$
B^{ab}	$C^{ab} = g^{\mu\nu} \partial_\mu \varphi^a \, \partial_\mu \varphi^b$, $a, b = 1, 2, 3$
Z^{ab}	$C^{a0} \, C^{b0}$
X	C^{00}
W^{ab}	$B^{ab} - Z^{ab}/X$