Gradient expansion and stochastic approach to inflation

Diego Cruces

ICC-Universitat de Barcelona
SW14-IESC

Based on 2203.13852, 2107.12735, 1807.09057 with C. Germani and T. Prokopec.

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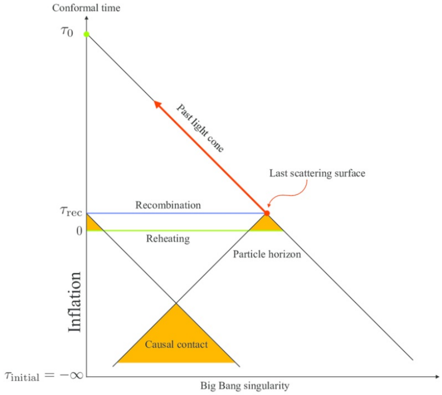
- Solves many previous problems of the standard Big Bang evolution of the universe.
- $(aH)^{-1}$ exponentially decreases $\rightarrow$ negative pressure fluid.



$$S = \int d^4x\sqrt{-g}[\tfrac{1}{2}R + \tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

$$H^2 = \rho \qquad \ddot{\phi} + 3H\dot{\phi} + V_\phi = 0$$

$$H = \frac{\dot{a}}{a}$$

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2} \ll 1$$

- There are inhomogeneities both in the scalar field and in the metric which are important due to the exponential expansion of the universe.
- How do we study them?
  Cosmological perturbation theory.

$$\phi \simeq \bar{\phi} + \delta\phi\,, \qquad g_{\mu\nu} \simeq \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

$$ds^2 = -(1+2A)dt^2 + 2a\partial_i B dx^i dt + a^2\left[(1+2D)\delta_{ij} - 2E^s_{ij}\right] dx^i_p dx^j\,,$$

Well known

BUT

dangerous when studying perturbations large enough.
(PHB's?)

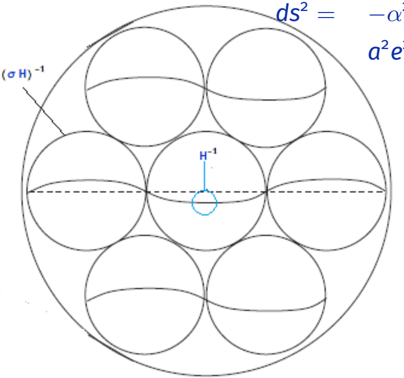
- Alternatives?

Gradient expansion

- The characteristic scale of inhomogeneities is larger that the Hubble horizon scale $L \gg H^{-1}$

$$L \sim \mathcal{O}\left(\frac{1}{\sigma}\right) \rightarrow L \sim \frac{1}{\sigma H}$$

  with $\sigma \ll 1$
- Each spatial derivative introduces a $\mathcal{O}(\sigma)$.



$$ds^2 = -\alpha^2 dt_{BG}^2 + a^2 e^{2\zeta} \tilde{\gamma}_{ij} \left(dx_{BG}^i + \beta^i dt_{BG}\right) \left(dx_{BG}^j + \beta^j dt_{BG}\right).$$

$$ds^2 = -dt_p^2 + a(t_p)^2 \delta_{ij} dx_p^i dx_p^j + \mathcal{O}(\sigma)$$

$$_{(0)}\alpha(t_{BG}) \sim \mathcal{O}(\sigma^0)$$
$$_{(0)}\zeta(t_{BG}) \sim \mathcal{O}(\sigma^0)$$
$$_{(0)}\beta^i(x_{BG}^i, t_{BG}) \sim \mathcal{O}(\sigma^{-1})$$
$$\tilde{\gamma}_{ij} - \delta_{ij} \sim \mathcal{O}(\sigma)$$

- Are those patches completely independent at leading order in gradient expansion?

Linear momentum constraint

$$D^j\tilde{A}_{ij} - \frac{2}{3}D_iK - J_i = 0$$

$$\downarrow$$

$$0 + 2\partial_i\left(-HA + \dot{D} + \frac{1}{3}\nabla^2\dot{E} + \frac{\dot{\phi}}{2M_{PL}^2}\delta\phi\right) = 0$$

$$\downarrow$$

$$0 + 2\cancel{\partial_i}\left(-HA + \dot{D} + \frac{1}{3}\nabla^2\dot{E} + \frac{\dot{\phi}}{2M_{PL}^2}\delta\phi\right) = 0$$

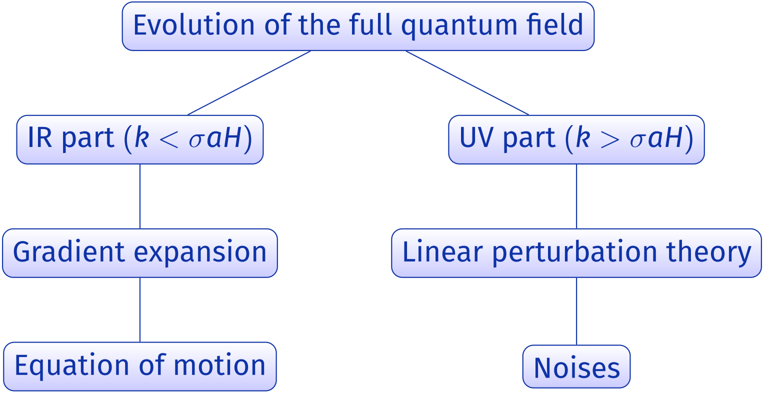
$$-HA + \dot{D} + \frac{1}{3}\nabla^2\dot{E} + \frac{\dot{\phi}}{2M_{PL}^2}\delta\phi = 0$$

Contribution at large scales!
Generically $\mathcal{O}(\epsilon_1)$

- It is no perturbative in terms of the amplitude of the inhomogeneities so:
  - It mixes scalar, vector and tensor terms.
  - The momentum constraint no longer has an overall derivative.
- Initial conditions for the inhomogeneities are no defined.

Stochastic approach

A. A. Starobinsky, Lect. Notes Phys. **246** (1986), 107-126



- In the stochastic formalism commonly used in the literature:
  - The momentum constraint is ignored.
  - Uniform-N gauge is used ($N = \int H_l dt_l = \int H^b dt^b$) $\rightarrow$ Scalar perturbations are given in terms only of the field.

$$\pi^{IR} = \frac{\partial \phi^{IR}}{\partial N} + \xi_1 \,,$$

$$\frac{\partial \pi^{IR}}{\partial N} + \left(3 - \frac{\left(\pi^{IR}\right)^2}{2M_{PL}^2}\right)\pi^{IR} + \left(3M_{PL}^2 - \frac{\left(\pi^{IR}\right)^2}{2}\right)\frac{V_{\phi(\phi^{IR})}}{V\left(\phi^{IR}\right)} = -\xi_2 \,,$$

where ξ_1 is the noise for the field ξ_2 is the noise for the momentum.

New stochastic formalism

- The absence of the momentum constraint has important consequences such as the incompatibility of ξ_2 with the rest of the system.

  Tomislav Prokopec, Gerasimos Rigopoulos Phys. Rev. D **104** (2021) no.8, 083505

  DC, C. Germani and T. Prokopec, JCAP **03** (2019), 048

- In order to solve these problems one can simply add a stochastic equation for the momentum constraint to the system:

  DC and C. Germani, Phys. Rev. D **105** (2022) no.2, 023533

- The new stochastic system is:

$$\pi^{IR} = \frac{\partial \phi^{IR}}{\partial N} + \xi_1 \,,$$

$$\frac{\partial \pi^{IR}}{\partial N} + \left(3 - \frac{\left(\pi^{IR}\right)^2}{2M_{PL}^2}\right)\pi^{IR} + \left(3M_{PL}^2 - \frac{\left(\pi^{IR}\right)^2}{2}\right)\frac{V_{\phi(\phi^{IR})}}{V\left(\phi^{IR}\right)} = -\xi_2 \,,$$

$$\partial_i\left(\frac{\partial}{\partial N}\left(\frac{1}{3}\nabla^2 E^{IR}\right)\right) - \frac{\partial_i \alpha^{IR}}{\alpha^{IR}} + \frac{\partial \phi^{IR}}{\partial N}\frac{\partial_i \phi}{2M_{PL}^2} = -\partial_i \xi_4 \,,$$

We will compare:

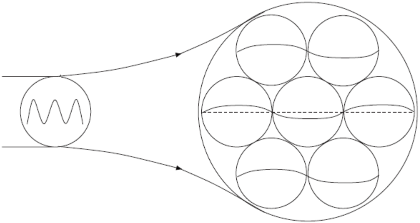
- Real space correlator computed in linear theory

$$\langle Q^{IR}_{lin}(N) Q^{IR}_{lin}(N)\rangle = \int_{\sigma H}^{\sigma a(N) H} \frac{dk}{k} \mathcal{P}_Q(k, N)$$

- Stochastic real space correlator with Markovian white noises (linear limit):

$$\langle \Delta Q^{IR}_{sto}(N) \Delta Q^{IR}_{sto}(N)\rangle = Var\left(Q^{IR}(N)\right)$$

- **Problem:** noises must be computed in a stochastic background.



- **Solution:**

$$ds^2 = -(1+2A)dt_p^2 + 2a\partial_i B dx_p^i dt_p + a^2\left[(1+2D)\delta_{ij} - 2E_{ij}^s\right] dx_p^i dx_p^j,$$

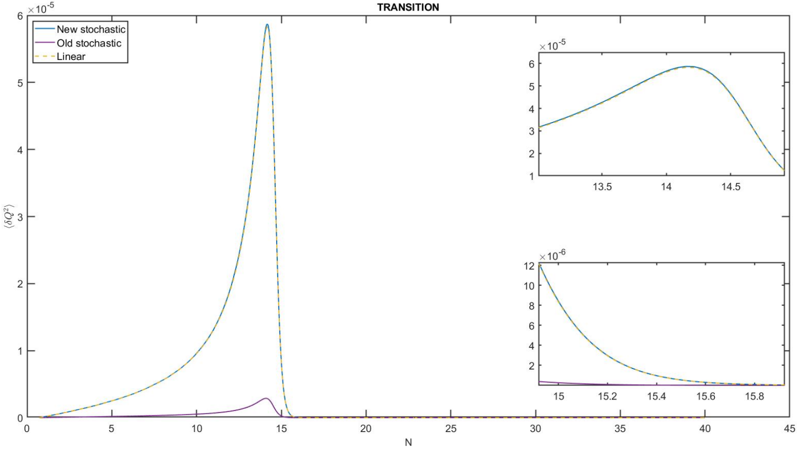
$$\downarrow$$

$$ds^2 = -(1+2A)dt_{BG}^2 + 2a\partial_i B dx_{BG}^i dt_{BG} + a^2\left[(1+2D)\delta_{ij} - 2E_{ij}^s\right] dx_{BG}^i dx_{BG}^j,$$

- **Price to pay:** $Y^{IR}X^{UV} = Y^b X^{UV} + \cancel{O((X^{UV})^2)} \Longrightarrow Y^{IR} - Y^b = \mathcal{O}(X^{UV})$
  Linear perturbation theory!

SR-USR-SR TRANSITION $V(\phi) = V_0 \left(1 + \beta (\phi - \phi_0)^3\right)$

Why? Inflection point $\rightarrow$ PBHs.

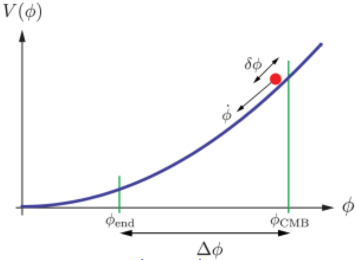


- The stochastic formalism, as it is usually used in the literature resides in a main approximation
  1. Absence of the momentum constraint $\rightarrow$ makes the model only trustworthy in Slow-Roll.
- By solving this limitation, we have checked that stochastic formalism with the Markovian construction is indeed linear perturbation theory, as expected.

Thank you!

Different regimes of inflation ($ad\tau = dt$)

- Slow roll inflation (SR)



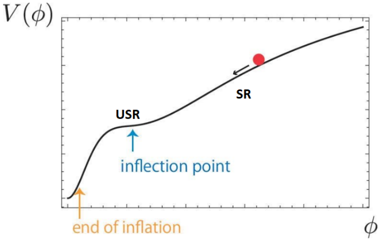
$$\cancel{\ddot\phi}+3H\dot\phi+V_\phi = 0$$

$$\dot\phi \sim \text{constant}$$

$$\epsilon_i \ll 1$$

$$\epsilon_{i+1} = \frac{1}{H\epsilon_i}\frac{d}{dt}\epsilon_i$$

- Ultra slow-roll inflation (USR)



$$\ddot\phi+3H\dot\phi+\cancel{V_\phi} = 0$$

$$\dot\phi \simeq e^{-3Ht}$$

$$\epsilon_1 \ll 1$$

$$\epsilon_2 \simeq -6$$

Study of inhomogeneities (ADM formalism)

ADM (or 3+1) formalism:

RL Arnowitt, S Deser, CW Misner (1959)

3+1 Metric:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt); \qquad \gamma_{ij} = a(t)^2 e^{2\zeta} \tilde{\gamma}_{ij}$$

3+1 Action with a single scalar field:

$$S = \frac{1}{2} \int \sqrt{\gamma} \left[\alpha R^{(3)} + \alpha (K_{ij} K^{ij} - K^2) - 2\alpha V + \alpha^{-1} (\dot{\phi} - \beta^i \partial_i \phi)^2 - \alpha \gamma^{ij} \partial_i \phi \partial_j \phi \right],$$

Extrinsic curvature:

$$K_{ij} \equiv -\nabla_i n_j = -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - D_i \beta_j - D_j \beta_i) = \frac{\gamma_{ij}}{3} K + a^2 e^{2\zeta} \tilde{A}_{ij}$$

$$n_\mu = (-\alpha, 0, 0, 0)$$

ADM EQUATIONS

- The lapse function α and the shift vector β_i act as Lagrange multipliers, generating the Hamiltonian and momentum constraints:

$$R^{(3)} - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = 2E; \qquad E \equiv T_{\mu\nu}n^\mu n^\nu$$

$$D^j\tilde{A}_{ij} - \frac{2}{3}D_iK = J_i; \qquad J_i \equiv T_{\mu\nu}n^\mu \gamma_i^\nu$$

- The spatial metric γ_{ij} and the extrinsic curvature K_{ij} are the dynamical variables so it exists an equation of motion for each one of the variables ($\zeta, \tilde{\gamma}_{ij}, K$ and $\tilde{A}_{ij}$)

ADM equations

- Evolution equations for the spatial metric:

$$(\partial_t - \beta^k\partial_k)\zeta + \frac{\dot{a}}{a} = -\frac{1}{3}(\alpha K - \partial_k\beta^k),$$

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k,$$

- Evolution equations for the extrinsic curvature:

$$(\partial_t - \beta^k\partial_k)\tilde{A}_{ij} = \frac{e^{-2\zeta}}{a^2}\left[\alpha\left(R^{(3)}_{ij} - \frac{\gamma_{ij}}{3}R^{(3)}\right) - \left(D_iD_j\alpha - \frac{\gamma_{ij}}{3}D_kD^k\alpha\right)\right]$$
$$+ \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k,$$

$$(\partial_t - \beta^k\partial_k)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - D_kD^k\alpha + 2\alpha\left(E + S^k_k\right),$$

where $\quad S^k_k \equiv \gamma^{lk}T_{lk}$

Linear perturbation theory

Expansion under the assumption that the inhomogeneities are very small in amplitude $\rightarrow$ perturbation theory. VF Mukhanov, HA Feldman, RH Brandenberger (1992)

$$\alpha \rightarrow 1 + A$$

$$\beta_j \rightarrow 0 + aB_j \rightarrow 0 + a\partial_j B$$

$$e^{2\zeta} \rightarrow 1 + 2D$$

$$\tilde{\gamma}_{ij} \rightarrow \delta_{ij} + E_{ij} \rightarrow \delta_{ij} + \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) E$$

$$\phi \rightarrow \phi^b + \delta\phi$$

Homogeneous equations

+

Perturbative equations for the scalar part.

Inhomogeneities

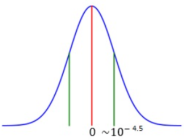
The correspondence between the homogeneous and perturbed space-times fixes a coordinate system $\to$ gauge choice.
One can construct a gauge invariant quantity that encompasses all the scalar perturbations: the curvature perturbation $\mathcal{R}$ (or the Mukhanov-Sasaki variable Q)

$$\mathcal{R} = \frac{\dot{\phi}^b}{H^b} Q = \frac{1}{a}\left(\frac{H^b}{\dot{\phi}^b}\delta\phi + D + \frac{1}{3}\nabla^2 E\right)$$

The quantity of interest is the power spectrum of $\mathcal{R}$

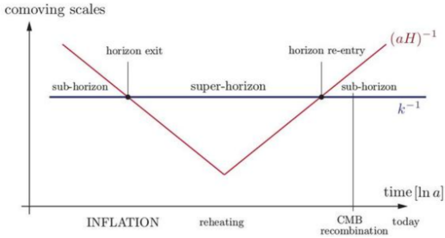
$$\langle 0 \left| \hat{\mathcal{R}}(\tau, \mathbf{x}) \hat{\mathcal{R}}(\tau, \mathbf{x} + r) \right| 0 \rangle = \int_0^\infty \frac{dk}{k} \frac{\sin kr}{kr} \mathcal{P}_{\mathcal{R}}(\tau, k) \qquad \mathcal{P}_{\mathcal{R}}(\tau, k) = \frac{k^3}{2\pi^2} \left|\zeta_{\mathbf{k}}\right|^2$$

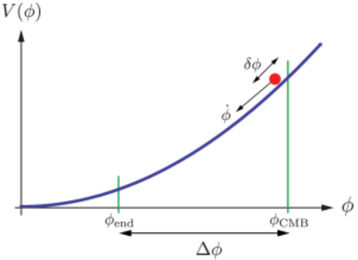
What does the power spectrum tell us?



$$\mathcal{P}_{\mathcal{R}}(k) \sim 10^{-9}$$

Succes of SR inflation





$$\cancel{\ddot{\phi}} + 3H\dot{\phi} + V_\phi = 0$$

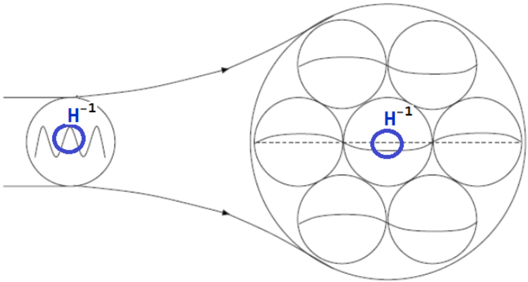
$$\dot{\phi} \sim \text{constant}$$

Long wavelength limit

We can solve the perturbation equations and apply the limit $\frac{k}{aH} \to 0$.

or

We can see the effect of the long-wavelength perturbation as a *independent* FLRW patch that differs perturbatively from the FLRW background $\left(\frac{k}{aH} = 0\right)$.



$k \to 0$ vs $k = 0$

- $k \to 0$
  The equation of motion for the MS variable Q is:

$$\ddot{Q} + 3H\dot{Q} + \left[-\frac{\nabla^2}{a^2} + H^2\left(-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\right]Q = 0$$

  Its long wavelength limit is obviously

$$\ddot{Q} + 3H\dot{Q} + \left[H^2\left(-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\right]Q = 0$$

$k \to 0$ vs $k = 0$

- $k = 0$

$$3H_l^2 = \frac{\left(\frac{d}{dt_l}\phi_l\right)^2}{2} + V(\phi_l)$$

$$\frac{d}{dt_l}(H_l) + H_l^2 = -\frac{1}{3}\left(\left(\frac{d}{dt_l}\phi_l\right)^2 - V(\phi_l)\right)$$

$$H_l = H - HA + \dot{D} - \cancel{\frac{1}{3}\frac{\nabla^2}{a}B}$$

$$dt_l = (1+A)dt$$

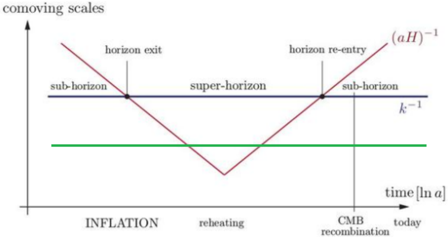
$$\phi_l = \phi^b + \delta\phi$$

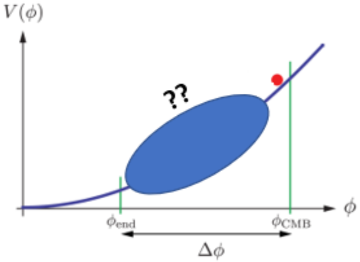
$$\bar{Q} = \delta\phi + \frac{\dot{\phi}}{H}\left(D + \cancel{\frac{1}{3}\nabla^2 E}\right)$$

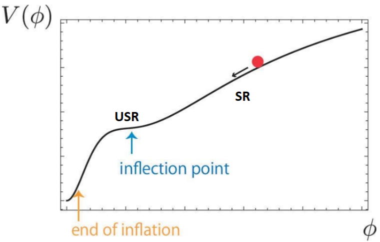
$$\ddot{\bar{Q}} + 3H\left(1 + \frac{\epsilon_1\epsilon_2}{3(3-\epsilon_1)}\right)\dot{\bar{Q}} + H^2\left(-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_2 - \frac{\epsilon_1\epsilon_2^2}{2(3-\epsilon_1)}\right)\bar{Q} = 0$$

Other regimes of inflation

We are actually not accessible to physics before CMB



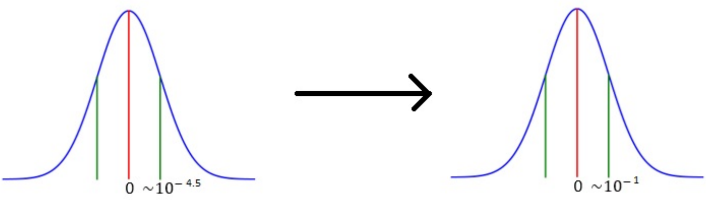




USR INFLATION

During a Ultra-Slow-Roll phase of inflation, the power spectrum for curvature perturbations is enhanced:

$$\ddot{\phi} + 3H\dot{\phi} + \cancel{V_\phi} = 0 \qquad \rightarrow \qquad \dot{\phi}\downarrow \qquad \rightarrow \qquad \mathcal{P}_{\mathcal{R}}(k)\uparrow$$



$\mathcal{R} \sim \mathcal{O}(0.1)$ are reached $\rightarrow$ PBHs ! B. J. Carr and S. W. Hawking, (1974),

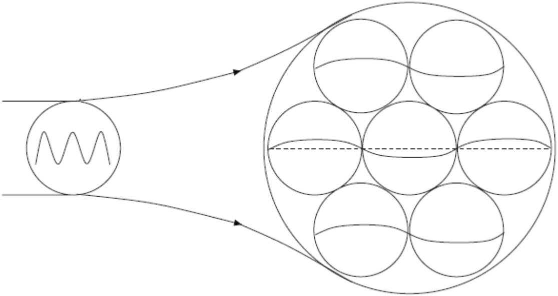
From UV to IR

$$X^{IR}(t,\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \Theta(\sigma a H - k) \mathcal{X}_{\mathbf{k}}(\mathbf{x}, t),$$

$$X^{UV}(t,\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \Theta(k - \sigma a H) \mathcal{X}_{\mathbf{k}}(\mathbf{x}, t),$$

NOISES
$k > \sigma a H$

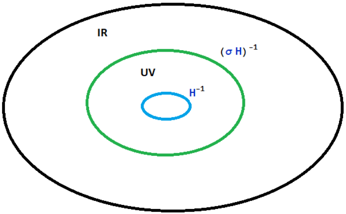
E.O.M
$k > \sigma a H$



Stochastic inflation

Each patch will follow an stochastic equation with:

1. A deterministic part which is the non-linear equation coming from gradient expansion.
2. A noise that takes into account the modes entering into the IR part.



- The evolution of many patches is equivalent to solve many times the stochastic equation with different random values in the noise.
- IR correlators $\rightarrow$ Statistical moments

Hamiltonian constraint

As an example we are going to derive the stochastic equation for the Hamiltonian constraint

$$R^{(3)} - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = \frac{2}{M_{PL}^2}T_{\mu\nu}n^\mu n^\nu ,$$

in uniform N gauge ($N = \int H_l dt_l = \int H^b dt^b$)

$$\zeta = 0, \quad \beta_i = 0; \qquad D = 0, \quad B = 0$$

$$\alpha \to \alpha^{IR} + \alpha^{UV}$$

$$\phi \to \phi^{IR} + \delta\phi$$

$$\ldots$$

Hamiltonian constraint

$$\left(\frac{H}{\alpha^{IR}}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\left(\frac{\dot{\phi}^{IR}}{\alpha^{IR}}\right)^2 + V\left(\phi^{IR}\right)\right) =$$

$$-\frac{1}{3\left(\alpha^{IR}\right)^3}\left(-6H^2\alpha^{UV} + \left(\dot{\phi}^{IR}\right)^2\alpha^{UV} - \alpha^{IR}\dot{\phi}^{IR}\delta\dot{\phi} - \left(\alpha^{IR}\right)^3 V_\phi\delta\phi\right) + \frac{\nabla^2}{3a^2}\nabla^2 E^{UV}$$

$$\left(\frac{H}{\alpha^{IR}}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\left(\frac{\dot{\phi}^{IR}}{\alpha^{IR}}\right)^2 + V\left(\phi^{IR}\right)\right)$$

$$= \boxed{\frac{\dot{\phi}^{IR}}{3\left(\alpha^{IR}\right)^2}\left(-\sigma aH\left(1-\epsilon_1\right)\int\frac{d\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma aH\right)\varphi_{\mathbf{k}}\right)} \quad \Rightarrow \frac{\dot{\phi}^{IR}}{3(\alpha^{IR})^2}\xi_1$$

$$\boxed{\begin{aligned}&+\int\frac{d\mathbf{k}}{(2\pi)^{3/2}}\Theta(k-\sigma aH)\left\{\frac{1}{3(\alpha^{IR})^3}\left(6H^2\mathcal{A}_{\mathbf{k}} + \alpha^{IR}\dot{\phi}^{IR}\dot{\varphi}_{\mathbf{k}}\ .\right.\right.\\ &\left.\left.-\left(\dot{\phi}^{IR}\right)^2\mathcal{A}_{\mathbf{k}} + V_\phi\left(\phi^{IR}\right)\varphi_{\mathbf{k}}\right) + \frac{k^4}{2a^2}\mathcal{E}_{\mathbf{k}}\right\}\end{aligned}} \quad \Longrightarrow \quad 0$$

Hamiltonian constraint

$$\left(\frac{H}{\alpha^{IR}}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\left(\frac{\dot{\phi}^{IR}}{\alpha^{IR}}\right)^2 + V\left(\phi^{IR}\right)\right) = \frac{\dot{\phi}^{IR}}{3\left(\alpha^{IR}\right)^2}\xi_1 \tag{1}$$

$$\xi_1 = -\sigma a H\left(1-\epsilon_1\right)\int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a H\right)\varphi_{\mathbf{k}}$$

$\varphi_{\mathbf{k}}$ is computed using stochastic equations!

$$\frac{1}{3\left(\alpha^{IR}\right)^3}\left\{6H^2\mathcal{A}_{\mathbf{k}} + \alpha^{IR}\dot{\phi}^{IR}\dot{\varphi}_{\mathbf{k}} - \left(\dot{\phi}^{IR}\right)^2\mathcal{A}_{\mathbf{k}} - V_\phi\left(\phi^{IR}\right)\varphi_{\mathbf{k}} + \frac{\nabla^2}{3a^2}\nabla^2\mathcal{E}_{\mathbf{k}}\right\} = 0$$

$$\ldots$$

Eq (1) represents a non-markovian process $\rightarrow$ very difficult to solve.