Hot topics in Modern Cosmology XIV, Cargèse

Bianchi IX gravitational collapse of matter inhomogeneities

Alexander Yu. Kamenshchik

University of Bologna and INFN, Bologna

May 8 - May 14, 2022

Based on:

Leonardo Giani, Oliver F. Piattella and
Alexander Yu. Kamenshchik,

Bianchi IX gravitational collapse of matter inhomogeneities,

Journal of Cosmology and Astroparticle Physics, 03, 028
(2022).

Content

1. Introduction and Motivations
2. The model. Bianchi IX field equations
3. Relation with other analytical models of gravitational collapse
4. Impact of small anisotropies on the spherical collapse
5. Impact of small anisotropies on the Zeldovich solution
6. Conclusions

Introduction and Motivations

- The cosmological principle lying in the basis of the Friedmann-Lemaître models and defined as the assumption that the Universe is homogeneous and isotropic on sufficiently large scales, is one of the pillars of the cosmological standard model ΛCDM.
- On the other hand, anisotropies and inhomogeneities are of the utmost importance when we focus our attention on smaller scales.
- The inhomogeneities, born in the very early universe as a result of quantum fluctuations, are responsible for the formation of the large-scale structure of the universe.
- The description of the gravitational collapse of these matter inhomogeneities with analytical methods becomes difficult once we enter in the nonlinear regime.
- One can obtain some analytical results using simplifying symmetry assumptions.

Introduction and Motivations

- One such an example is the Top Hat Spherical Collapse (THSC) model, which describes the collapse of an initially slightly overdense spherical shell of nonrelativistic matter.
- The hypothesis of a perfectly spherical symmetric collapse may be unrealistic, and it is sensible to ask what happens when this assumption is relaxed.
- An interesting description of anisotropic collapse is given by the Zeldovich solution, which describes the gravitational collapse triggered by a 1-dimensional overdense perturbation of a flat Friedmann universe.
- The Zeldovich solution predicts the formation of 2-dimensional structures called pancakes.

Introduction and Motivations

- It is known that most of the cosmic web is composed by filaments, i.e. 1-dimensional structures which must have generated from 2-dimensional anisotropic collapse.
- This motivates us to explore more general forms of gravitational collapse.
- Our goal is to describe an inhomogeneity which is initially expanding with the background, and then detaches from it and begin to collapse.
- It is reasonable to demand that the geometry of such an inhomogeneity is spatially closed.
- There are eleven different homogeneous but anisotropic three-dimensional spaces, classified by L. Bianchi.

Introduction and Motivations

- The only closed Bianchi space is the Bianchi IX.
- The intensive study of the Bianchi IX cosmology has led to the discovery of the oscillatory approach to the cosmological singularity by Belinsky, Khalatnikov and Lifshitz.
- We use the Bianchi IX geometry as a toy model to describe the anisotropic collapse of a matter inhomogeneity.
- The Bianchi IX model contains as limiting cases both the spherical collapse and the Zeldovich solution.
- It may describe within the same framework the evolution of filaments, pancakes and spherical objects composing the cosmic web.
- In this work we assess the impact of small anisotropies, constrained by the Bianchi IX potential, on the THSC model and the Zeldovich solutions.

The model. Bianchi IX field equations

We consider spacetime with the following metric

$$ds^2 = -dt^2 + a^2(t)\omega^1 \otimes \omega^1 + b^2(t)\omega^2 \otimes \omega^2 + c^2(t)\omega^3 \otimes \omega^3 \ ,$$

where a, b and c are scale factors and ω^1, ω^2 and ω^3 are the Maurer-Cartan basis 1-forms for the Bianchi IX space:

$$\begin{aligned}
\omega_1 &= -\sin x_3 dx_1 + \sin x_1 \cos x_3 dx_2 \ , \\
\omega_2 &= \cos x_3 dx_1 + \sin x_1 \sin x_3 dx_2 \ , \\
\omega_3 &= \cos x_1 dx_2 + dx_3 \ .
\end{aligned}$$

The non-vanishing component of the four-dimensional Ricci tensor are

$$R_0^0 = \left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} \right) ,$$

$$R_1^1 = \frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) + \frac{(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)}{2a^2 b^2 c^2} ,$$

$$R_2^2 = \frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c} \right) + \frac{(b^2 + a^2 - c^2)(b^2 + c^2 - a^2)}{2a^2 b^2 c^2} ,$$

$$R_3^3 = \frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\left(\frac{\dot{b}}{b} + \frac{\dot{a}}{a} \right) + \frac{(c^2 + b^2 - a^2)(c^2 + a^2 - b^2)}{2a^2 b^2 c^2} .$$

We consider a dust perfect fluid with an energy-momentum tensor:

$$T^{\mu}_{\nu} = \rho u^{\mu} u_{\nu} \ ,$$

where u^{μ} is the four-velocity satisfying $u^{\mu} u_{\mu} = -1$.
It is convenient to express the scale factors with the Misner parametrization:

$$a(t) = e^{\Omega + \frac{\beta_+}{2} + \frac{\sqrt{3}}{2}\beta_-} \ ,$$
$$b(t) = e^{\Omega + \frac{\beta_+}{2} - \frac{\sqrt{3}}{2}\beta_-} \ ,$$
$$c(t) = e^{\Omega - \beta_+} \ ,$$

where Ω is related to the volume ($abc = e^{3\Omega}$), and $\beta_{\pm}$ parametrize deviations from isotropy.

The field equations become:

$$\dot{\Omega}^2 = \frac{\rho}{3} + \frac{1}{4}\left(\dot{\beta}_+^2 + \dot{\beta}_-^2\right) + \frac{\mathcal{K}}{3}e^{-2\Omega} ,$$
$$\ddot{\Omega} = -\frac{\rho}{2} - \frac{3}{4}\left(\dot{\beta}_+^2 + \dot{\beta}_-^2\right) - \frac{\mathcal{K}}{3}e^{-2\Omega} ,$$
$$\ddot{\beta}_- + 3\dot{\Omega}\dot{\beta}_- + \mathcal{K}_{\beta_-}(\beta_+, \beta_-, \Omega) = 0 ,$$
$$\ddot{\beta}_+ + 3\dot{\Omega}\dot{\beta}_+ + \mathcal{K}_{\beta_+}(\beta_+, \beta_-, \Omega) = 0 ,$$

where

$$\mathcal{K} = \frac{1}{4}\left(e^{-4\beta_+} + e^{2\beta_+ - 2\sqrt{3}\beta_-}\right.$$
$$\left. -2e^{-\beta_+ + \sqrt{3}\beta_-} - 2e^{-\beta_+ - \sqrt{3}\beta_-} - 2e^{2\beta_+} + e^{2\beta_+ + 2\sqrt{3}\beta_-}\right) ,$$

$$\mathcal{K}_{\beta_+} = \frac{1}{3}e^{-2\Omega}\left(-2e^{-4\beta_+} - 2e^{2\beta_+} + e^{2\beta_+ - 2\sqrt{3}\beta_-} + e^{-\left(\beta_+ - \sqrt{3}\beta_-\right)}\right.$$
$$\left. + e^{-\left(\beta_+ + \sqrt{3}\beta_-\right)} + e^{2\beta_+ + 2\sqrt{3}\beta_-}\right) ,$$

$$\mathcal{K}_{\beta_-} = \frac{e^{-2\Omega}}{\sqrt{3}}\left(e^{-\left(\beta_+ + \sqrt{3}\beta_-\right)} + e^{2\left(\beta_+ + \sqrt{3}\beta_-\right)}\right.$$
$$\left. - e^{-\left(\beta_+ - \sqrt{3}\beta_-\right)} - e^{2\left(\beta_+ - \sqrt{3}\beta_-\right)}\right) .$$

Noticing that

$$\mathcal{K}_{\beta_\pm} = \frac{2}{3}e^{-2\Omega} d\mathcal{K}/d\beta_\pm ,$$

we can arrive to the continuity equation for the dust fluid

$$\dot{\rho} + 3\dot{\Omega}\rho = 0 ,$$

which shows that the density correctly dilute with the volume.

It is possible to map the Bianchi IX field equations in those for a closed FLRW Universe filled with two, non-minimally coupled and interacting scalar fields. When $\beta_{\pm} \ll 1$ the two scalar fields decouple and this results in two independent Klein-Gordon equations.

Relation with other analytical models of gravitational collapse

The equations for the Bianchi IX model contain as limiting cases both the Top Hat Spherical Collapse and the Zeldovich solution.
Choosing $\beta_\pm = 0$ we obtain the equations for the closed Friedmann universe, which therefore describe THSC.

The Zeldovich solution gives the evolution of a 1-dimensional perturbation of a flat FLRW spacetime with line element:

$$ds^2 = -dt^2 + a^2(t)\,(1 - \lambda(t))^2\,dx^2 + a^2(t)\left(dy^2 + dz^2\right) \ .$$

The field equations for the latter are

$$\dot{H} + H^2 = -\frac{\rho_{hom}}{6} \ ,$$

$$\ddot{\lambda} + 2H\dot{\lambda} - 4\pi G\rho_{\mathrm{hom}}\lambda = 0 \ ,$$

where $\rho = \rho_{hom}\,(1-\lambda)^{-1}$ and $\rho_{\mathrm{hom}} = \rho_0 a^{-3}$.

The equations for the Bianchi IX universe are reduced to the above equations when we set $\beta_- = 0$ and define the new variables $\alpha = e^{\Omega+\beta_+/2}$, $H = \dot{\alpha}/\alpha = \dot{\Omega} + \dot{\beta}_+/2$ and $\lambda = 1 - e^{-3\beta_+/2}$.

Zeldovich solution and Heckmann-Schucking solution

For the Bianchi I universe with the metric

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2,$$

filled with dust, exists the general solution of the Einstein equations discovered by
Heckmann and Schucking:

$$
\begin{aligned}
a^2(t) &= a_0 t^{p_1}(t_0+t)^{\frac{2}{3}-p_1},\\
b^2(t) &= b_0 t^{p_2}(t_0+t)^{\frac{2}{3}-p_2},\\
c^2(t) &= a_0 t^{p_1}(t_0+t)^{\frac{2}{3}-p_3},
\end{aligned}
$$

where the Kasner indicesp_1, p_2 and p_3 satisfy the relations

$$
\begin{aligned}
p_1 + p_2 + p_3 &= 1,\\
p_1^2 + p_2^2 + p_3^2 &= 1.
\end{aligned}
$$

One can easily check that the Heckmann-Schucking solution coincides with the Zeldovich solution if the following triplet of the Kasner indices is chosen:

$$
p_1 = -\frac{1}{3},\ p_2 = \frac{2}{3},\ p_3 = \frac{2}{3}.
$$

Impact of small anisotropies on the spherical collapse

Let us define:

$$\Omega \equiv \log R \,, \qquad \dot{\Omega} = \frac{\dot{R}}{R} \equiv H \,.$$

In this definition, R is some sort of average scale factor. If we assume the β's to be small, we can write the following Taylor expansions for $\mathcal{K}, \mathcal{K}_{\beta_\pm}$:

$$\mathcal{K} = \frac{1}{4}\left[-3 + 6\beta_+^2 + 6\beta_-^2 + O(\beta^3)\right] \,,$$

$$\mathcal{K}_{\beta_\pm} = \frac{2}{R^2}\beta_\pm + O(\beta^2) \,.$$

One can see that these terms are additive and can be treated as two non-interacting potentials for two scalar fields.
We then have:

$$
\begin{aligned}
H^2 &= \frac{\rho}{3} - \frac{1}{4R^2} + \frac{1}{4}\left(\dot{\beta}_+^2 + \dot{\beta}_-^2\right) + \frac{1}{2R^2}\left(\beta_+^2 + \beta_-^2\right) \ , \\
\dot{H} &= -\frac{\rho}{2} + \frac{1}{4R^2} - \frac{3}{4}\left(\dot{\beta}_+^2 + \dot{\beta}_-^2\right) - \frac{1}{2R^2}\left(\beta_+^2 + \beta_-^2\right) \ , \\
&\ddot{\beta}_\pm + 3H\dot{\beta}_\pm + \frac{2}{R^2}\beta_\pm = 0 \ .
\end{aligned}
$$

We can treat the problem as a spherical collapse for the system including dust and two perfect fluids.

Let us define:

$$\rho_{\beta_\pm} = \frac{3}{4}\dot{\beta}_\pm^2 + \frac{3}{2}\frac{\beta_\pm^2}{R^2} \ ,$$

then the first Friedmann equation becomes:

$$H^2 = \frac{\rho + \rho_{\beta_+} + \rho_{\beta_-}}{3} - \frac{1}{4R^2} \ .$$

The acceleration equation becomes:

$$\dot{H} = -\frac{\rho}{2} + \frac{1}{4R^2} - \sum_{i=\pm}\left(\frac{3}{4}\dot{\beta}_i^2 + \frac{1}{2}\frac{\beta_i^2}{R^2}\right) \ ,$$

hence, the pressures are:

$$p_{\beta\pm} = \frac{3}{4}\dot{\beta}_\pm^2 - \frac{\beta_\pm^2}{2R^2}.$$

The equation of state parameters for the $\beta_\pm$ effective fluids are:

$$w_{\beta_\pm} = \frac{\frac{3}{4}\dot{\beta}_\pm^2 - \frac{\beta_\pm^2}{2R^2}}{\frac{3}{4}\dot{\beta}_\pm^2 + \frac{3}{2}\frac{\beta_\pm^2}{R^2}} \ ,$$

they are never phantom $w_\beta < -1$ or super-stiff $w_\beta > 1$.

Linear growth and "String gas"

We suppose that that initially the volume of the almost spherical Bianchi IX spacetime follows the background matter dominated FLRW evolution, than $H \approx 2/3t$. Inserting this in the Klein-Gordon Equation for $\beta_\pm$ we obtain:

$$\ddot{\beta}_\pm + \frac{2}{t}\dot{\beta}_\pm + \frac{2}{a_0^2} t^{-\frac{4}{3}} \beta_\pm = 0 \ .$$

The general solution is a combination of Bessel functions of order $1/2$, which we can write as:

$$\bar{\beta}(t) = \frac{\bar{\beta}_0}{t} \cos\left(\omega t^{\frac{1}{3}} + \psi_0\right) + \frac{\bar{\beta}_1}{t^{\frac{2}{3}}} \cos\left(\omega t^{\frac{1}{3}} + \psi_1\right) \ .$$

It shows that small anisotropies generated during the linear evolution of the inhomogeneity oscillate and are smoothed out by the cosmological expansion.

Neglecting the $\bar{\beta}_0$ mode, which decays faster, we are left with

$$\bar{\beta}(t) \approx \bar{\beta}_1 \cos\left(\omega t^{1/3} + \psi\right) a^{-1}.$$

Since in this regime the kinetic energy of the scalar field decays as

$$\dot{\beta}^2 \propto t^{-3},$$

while the potential energy goes as

$$\beta^2/R^2 \propto a^{-4} \propto t^{-8/3},$$

we conclude that the scalar fields become potential dominated.
As a result, in this approximation, the Eos parameter of the anisotropic fluid is

$$w_{\beta_\pm} \approx -1/3.$$

This fluid is called String gas.
Then

$$\rho_{\beta_\pm} \propto R^{-2} \ .$$

It shows that, during the linear stage of the evolution, the effects of the anisotropic fluids on the averaged spherical collapse is to effectively shift the value of the spatial curvature term.

Turnaround

If initially the inhomogeneity follows the background evolution, it will eventually slow down and cease its expansion.
The turnaround point is reached when the volume of the inhomogeneity $e^{3\Omega}$ reaches its maximum value, and therefore $H = 0$.
The Klein-Gordon equations in this regime become:

$$\ddot{\beta}_{\pm} + \frac{2}{R^2}\beta_{\pm} = 0 \ ,$$

i.e. the equation for a harmonic oscillator whose solution is:

$$\beta_{\pm} = \beta_{0\pm} \cos\left(\omega t + \psi\right) \ ,$$

where $\omega = \sqrt{2}/R$.
Then, from the Friedmann equation we obtain:

$$R^2 = \frac{3}{\rho}\left(\frac{1}{4} - \frac{1}{2}\beta_{0+}^2 - \frac{1}{2}\beta_{0-}^2\right) \ .$$

Taking into account the dilution of the energy density of dust one understand that the presence of the anisotropy fluids postopone the turnaround and makes it happen at higher value of the average scale factor of the universe.

Contraction and virialization

When the inhomogeneity evolves from the turnaround point to the contracting phase, i.e. when $H < 0$, the Klein-Gordon equations for the β's possess an anti-damping term, which will eventually lead to the instability of the anisotropies.
However, one can hope that there is some virialization mechanism which prevents an unlimited growth of the instabilities.
Furthermore, the geometry of the vacuum Bianchi IX model prevents the anisotropy to grow indefinitely because of the triangular potential wells, against which the system would eventually bounce off.
Some simple considerations permit to estimate the pressure of our effective fluids in the virialized halos:

$$p_{\beta_\pm}(R_{vir}) = e^{3\frac{M}{R_{vir}}}.$$

Here R_{vir} is the mean radius of the virialized halo and M is its total mass.

Impact on statistical large scale structures observables

To understand how our model modifies the statistical distribution of LSS we will make use of the Press-Schechter formalism for the description of the formation of the galaxies and clusters.
For a scale invariant power spectrum of primordial scalar fluctuations ($n_s = 1$), their model predicts that the number density n of haloes with mass between M and $M + \Delta M$ is

$$n\left(M,t\right) = \frac{\bar{\rho}}{M^2}\sqrt{\frac{2}{\pi}}\left|\frac{d\log\nu}{d\log M}\right|\nu e^{\frac{-\nu^2}{2}} \;,$$

where $\bar{\rho}$ is the background matter density and it was defined $\nu = \delta_c/\sigma_M$, in which σ_M is the mass variance and $\delta_c = 1.686D(t)$, with $D(t)$ the linear growth factor normalized to unity.

Within our approximations the effect of small anisotropies of the spherical collapse is to rescale the function ν as $\nu \rightarrow \tilde{\nu} = \nu \left(1 - 4\rho^0_{\beta_+} - 4\rho^0_{\beta_-}\right) = \kappa\nu$.

Accordingly, the number density becomes:

$$n\left(M, t\right) = \frac{\bar{\rho}}{M^2}\sqrt{\frac{2}{\pi}}\left|\frac{d\log\nu}{d\log M}\right|\kappa\nu e^{-\frac{(\kappa\nu)^2}{2}} \ .$$

Impact of small anisotropies on the Zeldovich solution

The Zeldovich equations under the assumption $\beta_- \ll 1$ become:

$$\dot{H} + H^2 = -\frac{\rho_{hom}}{6} \ ,$$

$$\ddot{\lambda} + 2H\dot{\lambda} - \frac{\rho_{hom}}{2}\lambda - \frac{3\dot{\beta}_-^2}{2}(1-\lambda) = 0 \ .$$

The Klein-Gordon equation for β_- becomes, at linear order:

$$\ddot{\beta}_- + \dot{\beta}_-\left(3H - \frac{\dot{\lambda}}{1-\lambda}\right) + \frac{2\beta_-}{\alpha^2}\left(\frac{2}{(1-\lambda)^2} - 1\right) = 0 \ .$$

Neglecting the term $\dot{\beta}_-^2$, the solution for λ in the matter dominated epoch is the same as the density contrast for nonrelativistic matter, i.e. $\lambda = \lambda_0 t^{2/3} \propto \alpha$.
Thus, we obtain:

$$\left(3H - \frac{\dot{\lambda}}{1-\lambda}\right) = H\left(\frac{3-4\lambda}{1-\lambda}\right) .$$

If the background is strongly anisotropic, i.e. $|\lambda| \gg 1$, then

$$\ddot{\beta}_- + \frac{8}{3t}\dot{\beta}_- - \frac{2}{\alpha_0^2 t^{\frac{4}{3}}}\beta_- = 0 .$$

It admits the following real solution:

$$\beta_- = \frac{10}{18}\bar{\beta}_- \frac{\left(\left(\frac{1}{3kt^{\frac{2}{3}}}+1\right)\sinh\left(3\sqrt{k}t^{\frac{1}{3}}\right) - \frac{\cosh\left(3\sqrt{k}t^{\frac{1}{3}}\right)}{\sqrt{k}t^{\frac{1}{3}}}\right)}{k^{\frac{3}{2}}t},$$

where $k = 2/\alpha_0^2$ and $\bar{\beta}_-$ is an integration constant.
This solution is unstable and implies that the growing β_- will at some point spoil the validity of the perturbative approach.

To see qualitatively what happens, note that for $|\lambda| \gg 1$:

$$\ddot{\lambda} + 2H\dot{\lambda} \approx \lambda \left(\frac{\rho_{hom}}{2} - \frac{3}{2}\dot{\beta}_-^2 \right) \, .$$

As soon as $3\dot{\beta}_-^2$ becomes bigger than ρ_{hom}, an effective force appears working against the growth of λ.
The growth of β_- triggers the appearance of a dynamical force working against the original perturbation λ.
This qualitative picture is in agreement with what would we expect for a Bianchi IX model.
Anisotropy along one direction cannot grow arbitrarily because of the triangular shape of the potential $V(\beta_+, \beta_-)$, so that the system will eventually bounce from one of the potential wells and change the direction of anisotropic contraction.

This result suggests that, within the Bianchi IX model of gravitational collapse for structure formation, the so called pancakes of the Zeldovich solution are deformed by the switching of the direction of contraction and expansion, and undergo oscillatory behavior.

Conclusions

- The BIX geometry contains as limiting cases both the THSC model (in the trivial case of vanishing anisotropy) and the Zeldovich solution for a 1-dimensional perturbation. This provides a common framework to describe spherical DM haloes and Zeldovich pancakes.
- For almost spherical inhomogeneities, before the collapse, our qualitative analysis shows that the anisotropies effectively change the value of the FLRW spatial curvature. The reason is that the anisotropic fields in this regime, curiously, mimic a fluid with an EoS parameter $w \approx -1/3$, whose energy density coincides with the one of the spatial curvature.
- We studied how the anisotropies affect the number density of collapsed objects. They result in a rescaling of the Gaussian peak of the distribution and of the total number of objects, while the mass dependence is unchanged.

- We studied anisotropic deviations from the Zeldovich solution. These turn out to be unstable, and work against the growth of the original 1-dimensional perturbation.
- Our qualitative understanding is that the final stage of the evolution are not the pancakes, but more complex objects evolved from a series of subsequent Zeldovich-like epochs before virialization. These, to us, seem a promising tool to mimic the rich variety of filaments weaving the cosmic web.
- The Bianchi IX geometry is a promising tool towards a better understanding of the physics of structure formation, and deserves further investigations.