Hamiltonian formalism for anisotropic cosmological perturbation theory

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Goal of the talk

- Develop a Hamiltonian approach to Cosmological Perturbation Theory (CPT) and its gauge group, gauge-fixing, partial gauge-fixing, spacetime reconstruction and gauge-invariant phase space.
- One of possible applications is to systematically study gauges, for instance, in universes with anisotropy. An interesting issue: the representations of gravitational waves (the tensor three-metric & three-momentum perturbations are not gauge-invariant in anisotropic universe).
- The Hamiltonian CPT provides a simple laboratory for quantizations of gravity. Such issues as quantization prescription, diffeomorphism invariance and time problem, semiclassical spacetime reconstruction, etc can be studied within this framework.

ADM formalism for cosmological perturbations

Split the geometric and matter variables in the ADM formalism:

$$\delta q_{ij} = q_{ij} - \bar{q}_{ij},\ \delta\pi^{ij} = \pi^{ij} - \bar{\pi}^{ij},\ N \mapsto N + \delta N,\ N^i \mapsto N^i + \delta N^i.$$

Expand the ADM Hamiltonian:

$$H_{ADM} = \int_\Sigma (N\mathbf{H}_0 + N^i\mathbf{H}_i)\ \mathrm{d}^3x,$$

where the constraints are first-class:

$$\{\vec{\mathbf{H}}(\vec{f}), \vec{\mathbf{H}}(\vec{g})\} = -\vec{\mathbf{H}}(\mathcal{L}_{\vec{f}}\vec{g}),\ \ \{\vec{\mathbf{H}}(\vec{f}), \mathbf{H}_0(g)\} = -\mathbf{H}_0(\mathcal{L}_{\vec{f}}g),$$
$$\{\mathbf{H}_0(f), \mathbf{H}_0(g)\} = -\vec{\mathbf{H}}(h(f, g, q_{ij})).$$

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Expand the ADM Hamiltonian:

$$H_{ADM} = N\mathbf{H}_0^{(0)} + \int_\Sigma \left(N\mathcal{H}_0^{(2)} + \delta N \delta \mathbf{H}_0 + \delta N^i \delta \mathbf{H}_i \right),$$

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where the constraints are first-class (up to first order):

$$\{\delta\mathbf{H}_i, \delta\mathbf{H}_j\} = 0, \ \ \{\delta\mathbf{H}_j, \delta\mathbf{H}_0\} = 0,$$

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where the constraints are first-class (up to first order):

$$\{\delta\mathbf{H}_i, \delta\mathbf{H}_j\} = 0,\quad \{\delta\mathbf{H}_j, \delta\mathbf{H}_0\} = 0,$$

$$\{\mathbf{H}_0^{(0)} + \int_\Sigma \mathcal{H}^{(2)}, \delta\mathbf{H}_0\} = -ik^j\delta\mathbf{H}_j,\quad \{\mathbf{H}_0^{(0)} + \int_\Sigma \mathcal{H}^{(2)}, \delta\mathbf{H}_i\} = 0.$$

Dirac method for cosmological perturbations

Reduction of a constrained system = establishing a reduced phase space with a true Hamiltonian.

At the background level:

$$\mathbf{H}_0^{(0)} = 0, \quad \mathbf{t}^{(0)} = t, \quad \{\mathbf{H}_0^{(0)}, \mathbf{t}^{(0)}\} \neq 0$$

At the perturbation level:

$$\delta\mathbf{H}_\mu = 0, \quad \delta\mathbf{C}_\nu = 0, \quad |\{\delta\mathbf{H}_\mu, \delta\mathbf{C}_\nu\}| \neq 0$$

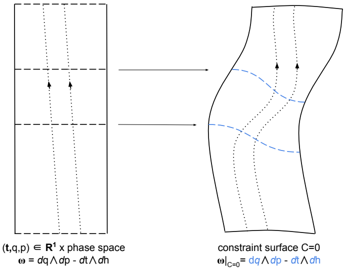
The Dirac bracket:

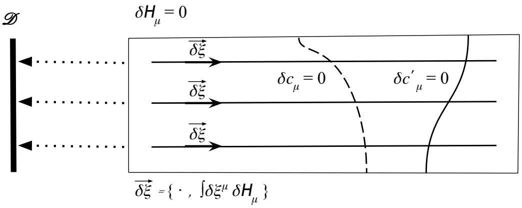
$$\{\cdot,\cdot\}_D = \{\cdot,\cdot\} - \{\cdot, \delta\mathbf{\Phi}_\mu\}\{\delta\mathbf{\Phi}_\mu, \delta\mathbf{\Phi}_\nu\}^{-1}\{\delta\mathbf{\Phi}_\nu, \cdot\}, \;\; \delta\mathbf{\Phi}_\nu \in \{\delta\mathbf{H}_0, \dots, \delta\mathbf{C}_0, \dots\}$$

The choice of physical variables:

$$H_{ADM} \quad \longrightarrow \quad H_{phys} = H_{phys}^{(0)}(v_{phys}) + \int_\Sigma \mathcal{H}_{phys}^{(2)}(\delta v_{phys}).$$

Phase space picture





Dirac method: gauge-invariant description

Dirac observables:

$$\{\delta D_I, \delta \mathbf{H}_\mu\} \approx 0 \quad \text{for all } \mu.$$

Express them in terms the physical variables δv_{phys}

$$\delta D_I + \xi_I^\mu \delta \mathbf{C}_\mu + \zeta_I^\mu \delta \mathbf{H}_\mu = \delta v_{phys,I},$$

$$\begin{aligned}\{\delta D_I, \delta D_J\} &= \{\delta D_I, \delta D_J\}_D \\ &= \{\delta D_I + \xi_I^\mu \delta \mathbf{C}_\mu + \zeta_I^\mu \delta \mathbf{H}_\mu, \delta D_J + \xi_J^\mu \delta \mathbf{C}_\mu + \zeta_J^\mu \delta \mathbf{H}_\mu\}_D \\ &= \{\delta v_{phys,I}, \delta v_{phys,J}\}_D.\end{aligned}$$

$$\{\cdot,\cdot\}_D = \{\cdot, \delta D_I\}\{\delta D_I, \delta D_J\}^{-1}\{\delta D_J, \cdot\}, \text{ where } \{\delta D_J, \delta \mathbf{C}_\nu\} = 0$$

Substitute:

$$H^{(2)}_{phys}(\delta v_{phys,I}) \quad \longrightarrow \quad H^{(2)}_{phys}(\delta D_I)$$

Spacetime reconstruction

The stability of gauge-fixing conditions:

$$\{\delta \mathbf{C}_\nu, H\} = 0 \;\Rightarrow\; \frac{\delta N^\mu}{N} = -\{\delta \mathbf{C}_\nu, \delta \mathbf{H}_\mu\}^{-1}\{\delta \mathbf{C}_\nu, \mathbf{H}_0^{(0)} + H^{(2)}\} \;\Rightarrow\; \frac{\delta N^\mu}{N}(\delta D_I)$$

Reconstruction of the three-surfaces:

$$(\delta \mathbf{C}_\nu, \delta \mathbf{H}_\mu, \delta D_I) \;\leftrightarrow\; (\delta q_{ij}, \delta \pi^{ij})$$

$$\begin{aligned} \Rightarrow\; & \delta q_{ij}(\delta \mathbf{C}_\nu = 0, \delta \mathbf{H}_\mu = 0, \delta D_I) = \delta q_{ij}(\delta D_I) \\ & \delta \pi^{ij}(\delta \mathbf{C}_\nu = 0, \delta \mathbf{H}_\mu = 0, \delta D_I) = \delta \pi^{ij}(\delta D_I) \end{aligned}$$

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Reconstruction of the three-surfaces:

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$$\delta\pi^{ij}(\delta\mathbf{C}_\nu = 0, \delta\mathbf{H}_\mu = 0, \delta D_I) = \delta\pi^{ij}(\delta D_I)$$

CAN IT BE IMPROVED?

Kuchař parametrization

In kinematical phase space introduce a **canonical** parametrization:

$$\underbrace{(\delta q_{ij}, \delta\pi^{ij})}_{ADM} \mapsto \underbrace{(\delta \mathbf{H}_\mu, \delta \mathbf{C}_\mu, \overbrace{\delta Q_I, \delta P_I}^{\delta D_J})}_{Kuchar}$$

The total Hamiltonian is given by $H_K = H_{ADM} + K$.

$$H_K = N \int_\Sigma \underbrace{\mathcal{H}^{(2)}_{phys}(\delta Q_I, \delta P_I)}_{\text{physical part}} + \underbrace{\left(\lambda_{1I\mu}\delta Q^I + \lambda_{2I\mu}\delta P^I + \lambda_{3\mu\nu}\delta \mathbf{H}^\nu + \lambda_{4\mu\nu}\delta \mathbf{C}^\nu\right)\delta \mathbf{H}^\mu}_{\text{weakly vanishing part}}$$

1. λ_1 and λ_2 depend on $\delta \mathbf{C}_\mu$,

$$\left.\frac{\delta N_\mu}{N}\right|_{\delta \mathbf{C}_\mu} = \left.\frac{\partial H_K}{\partial \delta \mathbf{H}^\mu}\right|_{\delta \mathbf{C}_\mu} \approx -\lambda_{1I\mu}\delta Q^I - \lambda_{2I\mu}\delta P^I.$$

2. λ_3 is completely irrelevant and can be disregarded.
3. λ_4 is implied by the constraint algebra (gauge-invariant).

Gauge transformations

Consider a **gauge transformation**:

$$(\delta \mathbf{H}_\mu, \delta \mathbf{C}_\mu, \delta Q_I, \delta P_I) \mapsto (\delta \mathbf{H}_\mu, \delta \tilde{\mathbf{C}}_\mu, \delta \tilde{Q}_I, \delta \tilde{P}_I).$$

Hence,

$$\{\delta \tilde{\mathbf{C}}_\mu - \delta \mathbf{C}_\mu, \delta \mathbf{H}_\nu\} = 0,$$

implying

$$\delta \tilde{\mathbf{C}}_\mu = \delta \mathbf{C}_\mu + \alpha_{\mu I} \delta P^I + \beta_{\mu I} \delta Q^I + \gamma_{\mu\nu} \delta \mathbf{H}^\nu.$$

$\Rightarrow$ The local gauge group is $\mathbb{G} = \mathbb{R}^{4n}$, where $n =$ the number of Dirac observables. The group is **abelian**. Fix a **gauge frame**, choose any gauge-fixing condition with $\alpha_{\mu I}$ and $\beta_{\mu I}$ at each moment of time.

$$\gamma_{\mu\nu} = \frac{1}{2} \left(\alpha_{\nu I} \beta_\mu{}^I - \alpha_{\mu I} \beta_\nu{}^I \right), \quad \delta \tilde{Q}_I = \delta Q_I + \alpha_{\mu I} \delta \mathbf{H}^\mu, \quad \delta \tilde{P}_I = \delta P_I - \beta_{\mu I} \delta \mathbf{H}^\mu.$$

Lapse and shift transformations

The stability of the new gauge:

$$\left.\frac{\delta N_\mu}{N}\right|_{\delta\tilde{\mathbf{C}}_\mu} - \left.\frac{\delta N_\mu}{N}\right|_{\delta\mathbf{C}_\mu} \approx$$

$$\left(\frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta Q^I\partial\delta Q^J}\alpha_{J\mu} - \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta Q^I\partial\delta P^J}\beta_{\mu J} - \dot{\beta}_{\mu I} + \lambda_{4\mu\nu}\beta_{\nu I}\right)\delta Q^I$$

$$+\left(\frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta P^I\partial\delta Q^J}\alpha_{\mu J} - \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta P^I\partial\delta P^J}\beta_{J\mu} - \dot{\alpha}_{\mu I} + \lambda_{4\mu\nu}\alpha_{\nu I}\right)\delta P^I.$$

$\Rightarrow$ The transformation is determined by the coefficients α and β. Makes use of the physical Hamiltonian $\mathcal{H}^{(2)}_{phys}$ and the constraint algebra coefficient λ_4.

Three-surface transformations

With the preferred gauge you define:

$$\begin{bmatrix} \delta q_{ij} \\ \delta \pi^{ij} \end{bmatrix} = M \begin{bmatrix} \delta \mathbf{C}_\mu \\ \delta \mathbf{H}^\mu \\ \delta Q^I \\ \delta P^I \end{bmatrix}$$

Then the three-surface in a new gauge reads:

$$\begin{bmatrix} \delta \tilde{q}_{ij} \\ \delta \tilde{\pi}^{ij} \end{bmatrix} = M \begin{bmatrix} -\alpha_{\mu I} \delta P^I - \beta_{\mu I} \delta Q^I \\ 0 \\ \delta Q^I \\ \delta P^I \end{bmatrix}$$

Partial gauge-fixing (e.g., Synchronous Gauge)

Replace all or some of the **gauge-fixing conditions** with **conditions on the lapse and shift functions**.

It is "partial" because $\frac{\delta N_\mu}{N}\Big|_{\delta\tilde{\mathbf{C}}_\mu} - \frac{\delta N_\mu}{N}\Big|_{\delta\mathbf{C}_\mu} \approx 0$ implies

$$\dot{\alpha}_{\mu I} = \alpha_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta Q^J \partial\delta P^I} - \beta_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta P^J \partial\delta P^I} + \lambda_{4\mu\nu}\alpha^\nu{}_I,$$

$$\dot{\beta}_{\mu I} = \alpha_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta Q^J \partial\delta Q^I} - \beta_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta P^J \partial\delta Q^I} + \lambda_{4\mu\nu}\beta^\nu{}_I.$$

$\Rightarrow$ For any $(\alpha_{\mu I}(t_0), \beta_{\mu I}(t_0))$ a unique solution $t \mapsto (\alpha_{\mu I}(t), \beta_{\mu I}(t))$ exists. Hence, there is complete freedom in fixing $\delta\tilde{\mathbf{C}}_\mu(t_0)$.

$\Rightarrow$ Phase space picture: once $\delta\tilde{\mathbf{C}}_\mu(t_0)$ is fixed at one time, it is determined at all times.

$\Rightarrow$ Spacetime picture: $\delta\tilde{\mathbf{C}}_\mu(t_0)$ are needed in order to unambiguously move from the Kuchař to the ADM parametrization of the (intrinsic and extrinsic) geometry. Hence, once an arbitrary initial three-surface with space coordinates is chosen, it is propagated uniquely through the four-dimensional spacetime if the lapse and shifts are fixed everywhere.

Perturbed Bianchi I universe

The background metric of the Bianchi Type I model reads:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \sum a_i^2(\mathrm{d}x^i)^2, \quad a = (a_1 a_2 a_3)^{\frac{1}{3}},$$

where the coordinates $(x^1, x^2, x^3) \in [0,1)^3$ are assumed.

The canonical perturbation variables read

$$\delta q_{ij} = q_{ij} - a_i^2\delta_{ij}, \quad \delta\pi^{ij} = \pi^{ij} - p^i\delta^{ij}, \quad \delta\phi = \phi - \bar{\phi}, \quad \delta\pi_\phi = \pi_\phi - \bar{\pi}_\phi,$$

The Fourier transform of a perturbation variable δX,

$$\delta\check{X}(\underline{k}) = \int_\Sigma \delta X(\overline{x}) e^{-ik_i x^i}\mathrm{d}^3x,$$

yields

$$\{\delta\check{\phi}(\underline{k}), \delta\check{\pi}^\phi(\underline{k}')\} = \delta_{\underline{k},-\underline{k}'}, \quad \{\delta\check{q}_{ij}(\underline{k}), \delta\check{\pi}^{lm}(\underline{k}')\} = \delta_{(i}^{\ l}\delta_{j)}^{\ m}\delta_{\underline{k},-\underline{k}'},$$

($\Sigma \simeq \mathbb{T}^3$, $k_i = 2\pi n_i$, $n_i \in \mathbb{Z}$).

Perturbed Bianchi I universe

(Fix conformal metric: $\gamma_{ij} = a^{-2}\bar{q}_{ij}$):

$$A^1_{ij} = \gamma_{ij}, \quad A^2_{ij} = \hat{k}_i\hat{k}_j - \frac{1}{3}\gamma_{ij},$$
$$A^3_{ij} = \frac{1}{\sqrt{2}}\left(\hat{k}_i\hat{v}_j + \hat{v}_i\hat{k}_j\right), \quad A^4_{ij} = \frac{1}{\sqrt{2}}\left(\hat{k}_i\hat{w}_j + \hat{w}_i\hat{k}_j\right),$$
$$A^5_{ij} = \frac{1}{\sqrt{2}}\left(\hat{v}_i\hat{w}_j + \hat{w}_i\hat{v}_j\right), \quad A^6_{ij} = \frac{1}{\sqrt{2}}\left(\hat{v}_i\hat{v}_j - \hat{w}_i\hat{w}_j\right).$$

$$\delta q_n = \delta\check{q}_{ij}A^{ij}_n, \quad \delta\pi^n = \delta\check{\pi}^{ij}A^n_{ij}.$$

The Poisson bracket now reads

$$\{\delta\check{\phi}(\underline{k}), \delta\check{\pi}^{\phi}(\underline{k}')\} = \delta_{\underline{k},-\underline{k}'}, \quad \{\delta q_n(\underline{k}), \delta\pi^m(\underline{k}')\} = \delta_n{}^m\delta_{\underline{k},-\underline{k}'}.$$

<u>A^n_{ij}'s and A^{ij}_n's are in general time-dependent as γ_{ij} and $(\hat{k}_i, \hat{v}_i, \hat{w}_i)$ evolve.</u>

The Fermi-Walker basis

What is the polarization mode?

The Fourier transform fixes a slicing of the spatial coordinate space (x^1, x^2, x^3) with the wavefronts of plane waves. In the physical space, the wavefronts are not fixed but being continuously tilted and anisotropically contracted or expanded. The tangent basis $(\hat{v}, \hat{w})$ can be Fermi-Walker transported along the (future-oriented) null vector field $\vec{p}$ whose spatial component is dual to the wavefront $\underline{k}$ of a gravitational wave,

$$\vec{p} = \overline{k} + |\overline{k}|\partial_\eta \ ,$$

where $\nabla_{\vec{p}}\vec{p} = 0$. Field $\vec{p}$ may be identified with tangents to null geodesics associated with rays of gravitational waves in the eikonal approximation (i.e., for large wavenumbers):

$$\frac{\mathrm{d}\hat{v}^j}{\mathrm{d}\eta} = -\sigma_{vv}\hat{v}^j - \sigma_{vw}\hat{w}^j, \quad \frac{\mathrm{d}\hat{w}^j}{\mathrm{d}\eta} = -\sigma_{ww}\hat{w}^j - \sigma_{wv}\hat{v}^j,$$

(unfortunately, not suitable for quantization).

Reduction of the ADM formalism

Set gauge-fixing functions (flat slicing gauge):

$$\delta \mathbf{C}_1 := \delta q_1, \quad \delta \mathbf{C}_2 := \delta q_2, \quad \delta \mathbf{C}_3 := \delta q_3, \quad \delta \mathbf{C}_4 := \delta q_4.$$

A complete set of second-class constraints:

$$\delta \mathbf{\Phi}_\rho = \{\delta \mathbf{C}_1, \delta \mathbf{C}_2, \delta \mathbf{C}_3, \delta \mathbf{C}_4, \delta \mathbf{H}_0, \delta \mathbf{H}_k, \delta \mathbf{H}_v, \delta \mathbf{H}_w\}, \quad \det\{\delta \mathbf{\Phi}_\rho, \delta \mathbf{\Phi}_\sigma\} \neq 0.$$

Introduce Dirac's bracket:

$$\{\cdot, \cdot\}_D = \{\cdot, \cdot\} - \{\cdot, \delta \mathbf{\Phi}_\rho\}\{\delta \mathbf{\Phi}_\rho, \delta \mathbf{\Phi}_\sigma\}^{-1}\{\delta \mathbf{\Phi}_\sigma, \cdot\}.$$

Reduce the Hamiltonian:

$$H_{phys} = \int_\Sigma \left(N\mathcal{H}_0^{(2)} + \delta N^\mu \delta \mathbf{H}_\mu\right)\Big|_{\delta \mathbf{C}=0} = \int_\Sigma N\mathcal{H}_0^{(2)}\Big|_{\delta \mathbf{C}=0}.$$

By removing $(\delta q_i, \delta \pi_i)$, $i = 1, 2, 3, 4$, we obtain

$$\delta \dot{q} = N\frac{\partial \mathcal{H}_0^{(2)}\big|_{\delta \mathbf{C}}}{\partial \delta \pi}, \quad \delta \dot{\pi} = -N\frac{\partial \mathcal{H}_0^{(2)}\big|_{\delta \mathbf{C}}}{\partial \delta q},$$

where $(\delta q, \delta \pi) \in \{(\delta q_5, \delta \pi_5), (\delta q_6, \delta \pi_6), (\delta \phi, \delta \pi_\phi)\}$.

Physical Hamiltonian

After rescaling $(\delta q_5, \delta\pi_5)$, $(\delta q_6, \delta\pi_6)$, $(\delta\phi, \delta\pi_\phi)$:

$$H_{BI} = \frac{N}{2a}\Big[\delta\tilde{\pi}_5^2 + \delta\tilde{\pi}_6^2 + \delta\tilde{\pi}_\phi^2 + (k^2 + U_5)\delta\tilde{q}_5^2 + (k^2 + U_6)\delta\tilde{q}_6^2 + (k^2 + U_\phi)\delta\tilde{\phi}^2 + C_1\delta\tilde{q}_5\delta\tilde{q}_6 + C_2\delta\tilde{q}_5\delta\tilde{\phi} + C_3\delta\tilde{q}_6\delta\tilde{\phi}\Big].$$

Rename the dynamical variables:

$$H_{BI} = \frac{N}{2a}\Big[\delta P_1^2 + \delta P_2^2 + \delta P_3^2 + (k^2 + U_5)\delta Q_1^2 + (k^2 + U_6)\delta Q_2^2 + (k^2 + U_\phi)\delta Q_3^2 + C_1\delta Q_1\delta Q_2 + C_2\delta Q_1\delta Q_3 + C_3\delta Q_2\delta Q_3\Big],$$

where δQ_I and δP_I are Dirac observables s.t.:

$$\delta Q_1\big|_{\delta\mathbf{C}} = \delta\tilde{q}_5,\ \delta Q_2\big|_{\delta\mathbf{C}} = \delta\tilde{q}_6,\ \delta Q_3\big|_{\delta\mathbf{C}} = \delta\tilde{\phi},$$
$$\delta P_1\big|_{\delta\mathbf{C}} = \delta\tilde{\pi}_5,\ \delta P_2\big|_{\delta\mathbf{C}} = \delta\tilde{\pi}_6,\ \delta P_3\big|_{\delta\mathbf{C}} = \delta\tilde{\pi}_\phi.$$

Dirac observables

$$\delta Q_1 = \underbrace{\frac{1}{\sqrt{2}a}\delta q_5} + \frac{2P_{vw}}{aP_{kk}}\left(\delta q_1 - \frac{1}{3}\delta q_2\right),$$

$$\delta Q_2 = \underbrace{\frac{1}{\sqrt{2}a}\delta q_6} + \frac{P_{vv} - P_{ww}}{aP_{kk}}\left(\delta q_1 - \frac{1}{3}\delta q_2\right),$$

$$\delta Q_3 = \underbrace{a\delta\phi + \frac{p_\phi}{aP_{kk}}\left(\delta q_1 - \frac{1}{3}\delta q_2\right)},$$

$$\delta P_1 = \underbrace{\sqrt{2}a\delta\pi_5 + \frac{\frac{5}{6}(TrP) - P_{kk}}{\sqrt{2}a^3}\delta q_5} - \frac{2P_{vw}}{\sqrt{2}a^3 P_{kk}}\left(\frac{P_{vv} - P_{ww}}{2}\delta q_6 + P_{vw}\delta q_5\right),$$

$$+ \mathcal{F}(P_{vw}, P_{kv}P_{kw})\left(\delta q_1 - \frac{1}{3}\delta q_2\right) - \frac{P_{vw}}{a^3 P_{kk}}\left(3P_{kk}\delta q_1 + a^2 p_\phi \delta\phi\right) + \frac{\sqrt{2}}{a^3}\left(P_{kw}\delta q_3 + P_{kv}\delta q_4\right)$$

$$\delta P_2 = \underbrace{\sqrt{2}a\delta\pi_6 + \frac{\frac{5}{6}(TrP) - P_{kk}}{\sqrt{2}a^3}\delta q_6} - \frac{P_{vv} - P_{ww}}{\sqrt{2}a^3 P_{kk}}\left(\frac{P_{vv} - P_{ww}}{2}\delta q_6 + P_{vw}\delta q_5\right)$$

$$+ \mathcal{F}\left(\frac{P_{vv} - P_{ww}}{2}, \frac{P_{kv}^2 - P_{kw}^2}{2}\right)\left(\delta q_1 - \frac{1}{3}\delta q_2\right) - \frac{P_{vv} - P_{ww}}{2a^3 P_{kk}}\left(3P_{kk}\delta q_1 + a^2 p_\phi \delta\phi\right) + \frac{\sqrt{2}}{a^3}\left(P_{kv}\delta q_3 - P_{kw}\delta q_4\right)$$

$$\delta P_3 = \underbrace{\frac{1}{a}\delta\pi_\phi - \frac{(TrP)P_{kk} + 3p_\phi^2}{6aP_{kk}}\delta\phi - \frac{3p_\phi}{2a^3}\delta q_1 + \frac{2(TrP)P_{kk}p_\phi - 6a^6 P_{kk}V_{,\phi} - 3p_\phi^3}{6a^3 P_{kk}^2}\left(\delta q_1 - \frac{1}{3}\delta q_2\right)}$$

$$- \frac{p_\phi}{\sqrt{2}a^3 P_{kk}}\left(\frac{P_{vv} - P_{ww}}{2}\delta q_6 + P_{vw}\delta q_5\right) - \frac{p_\phi\left((P_{vv} - P_{ww})^2 + 4P_{vw}^2\right)}{2a^3 P_{kk}^2}\left(\delta q_1 - \frac{1}{3}\delta q_2\right).$$

Dirac observables

The rotation of the Dirac observables in the $(\hat{v}, \hat{w})$-plane:

$$R_{\hat{k}}(\theta)\delta Q_1 = \cos(2\theta)\delta Q_1 - \sin(2\theta)\delta Q_2$$

$$R_{\hat{k}}(\theta)\delta P_1 = \cos(2\theta)\delta P_1 - \sin(2\theta)\delta P_2$$

$$R_{\hat{k}}(\theta)\delta Q_2 = \cos(2\theta)\delta Q_2 + \sin(2\theta)\delta Q_1$$

$$R_{\hat{k}}(\theta)\delta P_2 = \cos(2\theta)\delta P_2 + \sin(2\theta)\delta P_1$$

$$R_{\hat{k}}(\theta)\delta Q_3 = \delta Q_3$$

$$R_{\hat{k}}(\theta)\delta P_3 = \delta P_3$$

where $R_{\hat{k}}(\theta)$ is the rotation around $\hat{k} = \hat{v} \times \hat{w}$ by the angle θ.

Gravitational waves in anisotropic universe

Scalar gravity gauge:

$$\delta\tilde{\mathbf{C}}_1 = \delta q_2, \quad \delta\tilde{\mathbf{C}}_2 := \delta q_3, \quad \delta\tilde{\mathbf{C}}_3 := \delta q_4, \quad \delta\tilde{\mathbf{C}}_4 := \delta q_5.$$

$$\mathrm{Det}\,\{\delta\mathbf{H}, \delta\tilde{\mathbf{C}}\} = -\frac{8i\sqrt{2}k^2 P_{vw}}{a},$$

$$\delta Q_1\big|_{\delta\tilde{\mathbf{C}}} = \frac{2P_{vw}}{aP_{kk}}\delta q_1,$$
$$\delta Q_2\big|_{\delta\tilde{\mathbf{C}}} = \frac{1}{\sqrt{2}a}\delta q_6 + \frac{P_{vv} - P_{ww}}{aP_{kk}}\delta q_1,$$

$$\frac{\delta\tilde{N}_\mu}{N}\bigg|_{SG} - \frac{\delta N_\mu}{N}\bigg|_{FS} = A_\mu \delta Q_1 + B_\mu \delta P^1.$$

Conclusions

- The reduced phase space for anisotropic CPT can be easily derived with the Dirac method. Anisotropic CPT brings in some interesting issues: the dynamical triad $(\hat{k}, \hat{v}, \hat{w})$, ambiguous polarization modes, new gauges including the representation of a gravitational wave by a scalar metric perturbation, and richer dynamics of perturbations.
- The structure of the theory is conveniently displayed in the Kuchař parametrization. The finite gauge transformations can be easily deduced and the space of gauge-fixing conditions is given explicitly, spacetime reconstruction is easier once a gauge frame is established.
- Hamiltonian formalism for CPT can be a useful playground for quantizations of gravity. Such issues as quantization prescription, diffeomorphism invariance and time problem, semiclassical spacetime reconstruction, ... can be studied within this framework.