Anomalies in the topology of the fluctuations in the Cosmic Microwave Background

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Outline

- Background Geometry and Topology

    - ➢Minkowski functionals (Euler characteristic/genus)
    - ➢Homology (co-homology)
    - ➢Hierarchical homology (persistent homology)

- Topological characteristics of CMB fluctuations

    - ➢Temperature
        - ➢Full sky
        - ➢Hemispheres

    - ➢Polarization (full sky)

- Conclusion

Minkowski Functionals

• Predominantly Geometric quantities : include the notion of Volume, surface area, contour length of a manifold M.

• $(D+1)$ quantifiers for D-dimensional sets

• Go by various names and orderings: quermassintegrals, Dehn and Steiner functionals, curvature integrals, intrinsic volumes, Minkowski functionals, and Lipschitz-Killing curvatures.

•We need only MFs and LKCs for our purpose, which when properly defined are related by

$$Q_j(M) = j!\,\omega_j \mathcal{L}_{D-j}(M), \qquad j = 0, \dots, D,$$

$$\omega_k = \pi^{k/2}\,\Gamma\left(\frac{1+k}{2}\right) : \text{volume of k} - \text{dim. unit ball}$$

Pranav et al, MNRAS, 485 (3), 4167-4208

Minkowski Functionals

- A useful way of defining these quantities is via the *Steiner formula* or *Weyl's tube formula :*

$$V_D(\{x \in R^D : \min \| x - y \|\} \leq \rho, y \in M) = \sum_{j=0}^{D} \frac{\rho^j}{j!} Q_j(M) = \sum_{j=0}^{D} \omega_{D-j} \rho^{D-j} \mathcal{L}_j(M)$$

- The set in the LHS is known as the "tube around M", and ρ is small.

- Trivial to check Q_0 and $\mathcal{L}_D$ measure D-dimensional volume (set $\rho = 0$ above).

- Q_1 and $2\mathcal{L}_2$ measure surface area.

- Other functionals are harder to define, but always a true and deep result that:

$$\chi(M) = \mathcal{L}_0(M) = \frac{1}{D!\,\omega_D} Q_D(M).$$

- In 3D, this only leaves Q_2 and $\mathcal{L}_1$. If the manifold M is convex, $\mathcal{L}_1(M) = Q_2(M)/2\pi$ is twice the *caliper diameter* of M.

Pranav et al, MNRAS, 485 (3), 4167-4208, 2019

Geometry and Topology

- Theorema Egrerium (latin for remarkable theorem) of Gauss states that the Gaussian curvature of a surface is an *intrinsic invariant*, meaning it is a constant irrespective of how the surface is bent (or twisted) in space

- Leads to the Gauss-Bonnet theorem

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi\chi(M)$$

- K : Gaussian curvature of M, k_g: geodesic curvature of the boundary of M

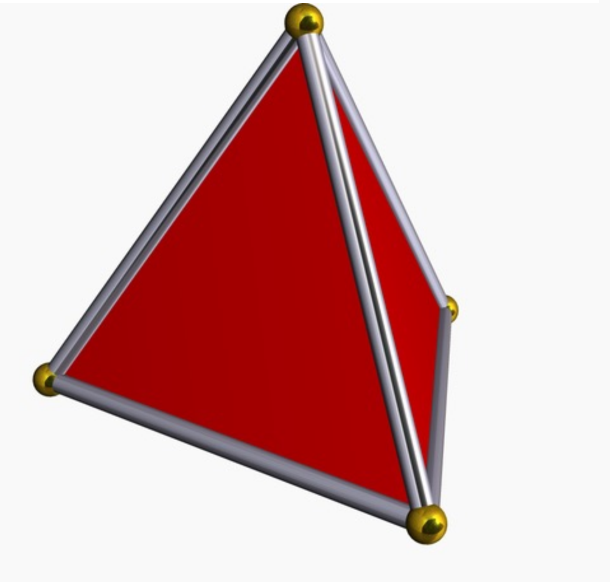
- The theorem is remarkable because it links and proves that a topological invariant (EC) can be computed purely from geometrical properties.

Pranav et al, MNRAS, 485 (3), 4167-4208, 2019

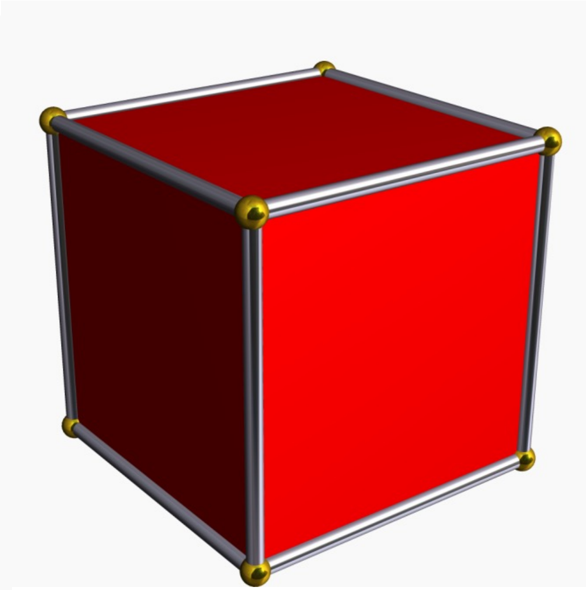
Euler characteristic

- Originally defined for polyhedra

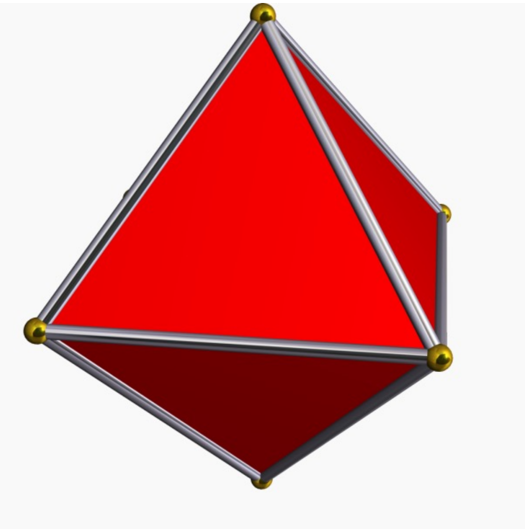
$$\chi = V - E + F$$



Tetrahedron
V = 4, E = 6, F = 4



Cube
V = 8, E = 12, F = 6



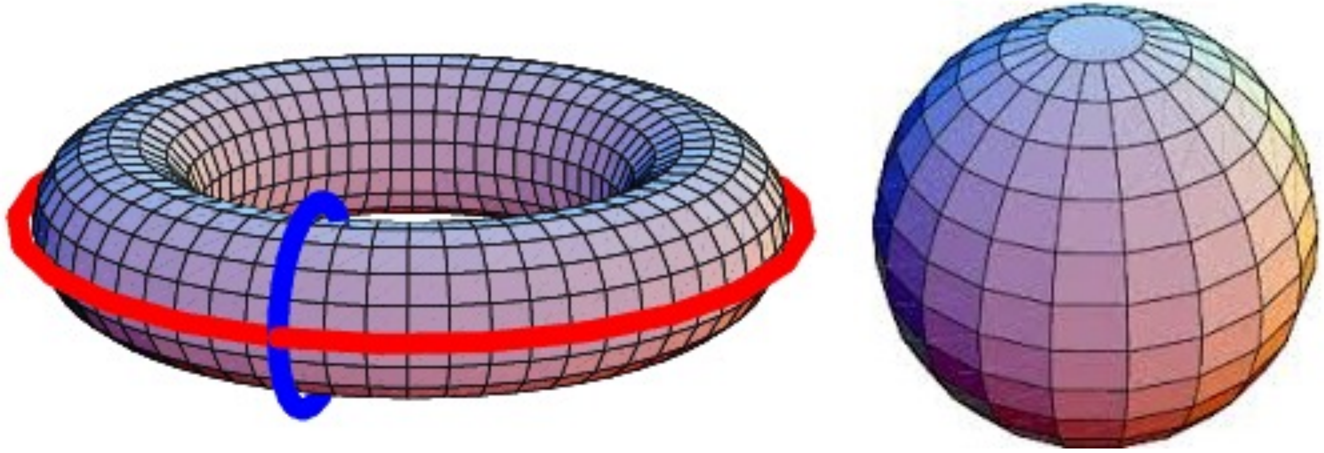
octahedron
V = 6, E = 12, F = 8

- Modern definition through algebraic topology, specifically *Homology*

Genus

- **For a connected, orientable surface, the Genus has a linear relationship with the maximal number of independent simple closed curves that can be drawn on the surface without rendering it disconnected**

- **Number of handles attached to a surface**



Why the Euler characteristic?

SIMPLICIAL TOPOLOGY

Simplices, complexes, cycles, numbers of simplices, Betti numbers

$$\sum_k (-1)^k \#\{k\text{-dimensional simplices}\}$$

INTEGRAL GEOMETRY

Convexity, convex ring
kinematic formulae
Minkowski functionals

$$\mathcal{M}_k(M) = c_{dk} \int_{\mathrm{Graff}(d,d-k)} \chi(M \cap V)\, d\mu^d_{d-k}(V)$$

$$\chi$$

$$\sum_k (-1)^k \beta_k$$

$$\sum_k (-1)^k \#\{\text{critical points of index } k\}$$

$$\int_M \mathrm{Tr}(R^{m/2}) \mathrm{Vol}_g$$

ALGEBRAIC TOPOLOGY

Homology, homotopy, dimensions of groups, Betti numbers, persistence

DIFFERENTIAL TOPOLOGY

Curvature, forms, Betti numbers, Morse theory, integration, Lipschitz-Killing curvatures

Genus, Euler & Betti

- Euler – Poincare formula

Relationship between Betti Numbers & Euler Characteristic χ:

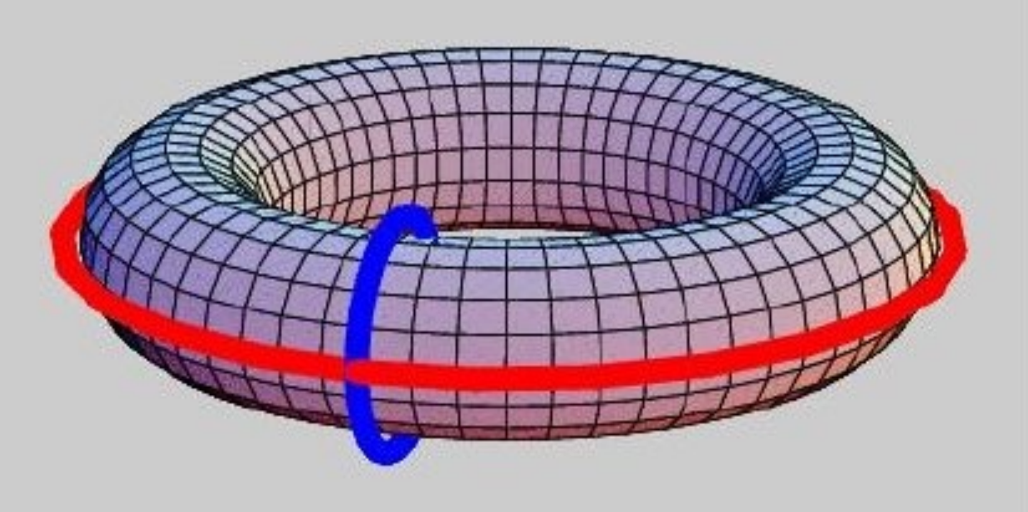
$$\chi = \sum_{k=0}^{d} (-1)^k \beta_k$$

Pranav et al, MNRAS, 465 (4), 4281-4310, 2017

Homology

Topology:
Study of connectivity and spatial relations that remain invariant under homeomorphisms (= continuous mapping between two topological objects)

Homology:
- Description of topology of a space in terms of cycles/boundaries.
- Fundamental lemma : Boundary of a boundary is necessarily empty.



Torus: one 0-cycle: rank group H_0: 1
two 1-cycles: rank group H_1: 2
one 2-cycle rank group H_2: 1

p-chain: sum of p-simplices
p-cycle: boundary of (p+1) chain
0-cycle: closed component
1-cycle: closed loop of edges, or finite union
2-cycle: closed surface, or finite union

Mutually homologous p-cycles $\longrightarrow$ p-class

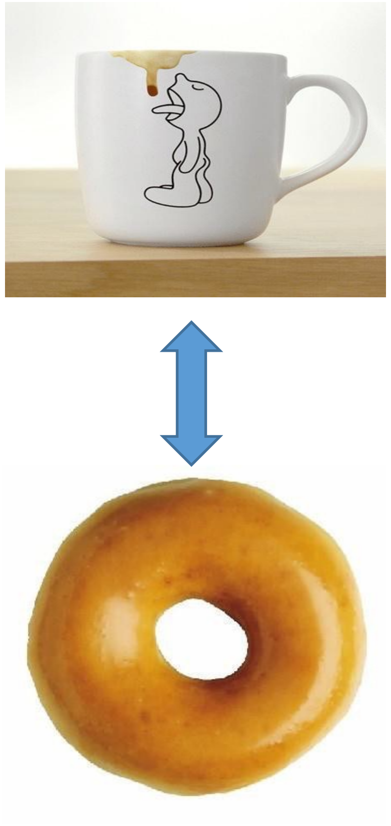
collection of independent p-classes :
Homology group H_p

Betti numbers denote the rank of the groups

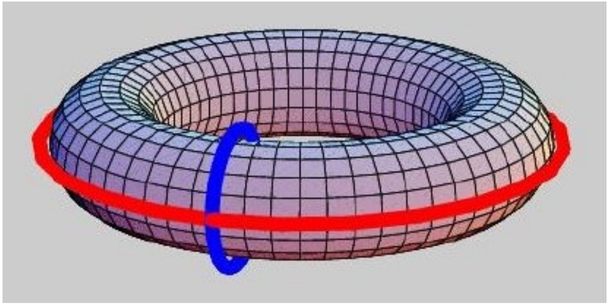
Topological cycles and holes

- intuitive interpretation

0 dimensional holes : gaps between connected objects



1 dimensional holes : loops/tunnels

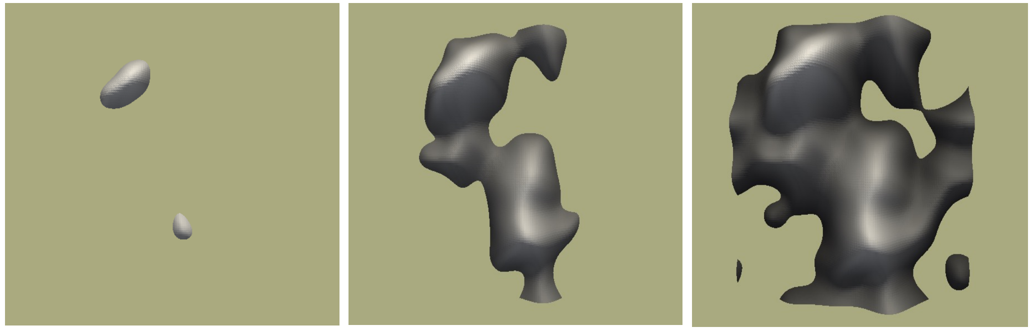


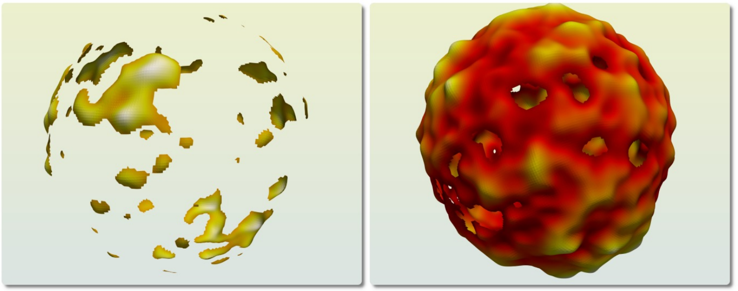
2 dimensional holes : voids

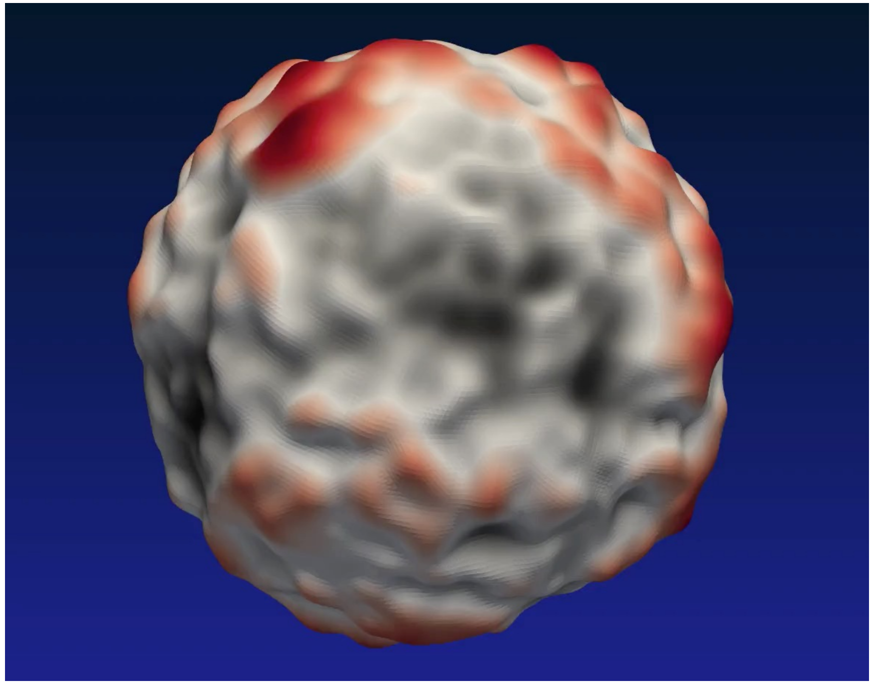
Critical points and filtration

birth and death of topological cycles

- Study the change in topology of a manifold w.r.t. the growing excursion sets of the function f
- *Topology only changes at critical points of the function*
- Addition of a critical point with index k, either creates a k-dimensional hole, or it destroys a (k-1)-dimensional hole

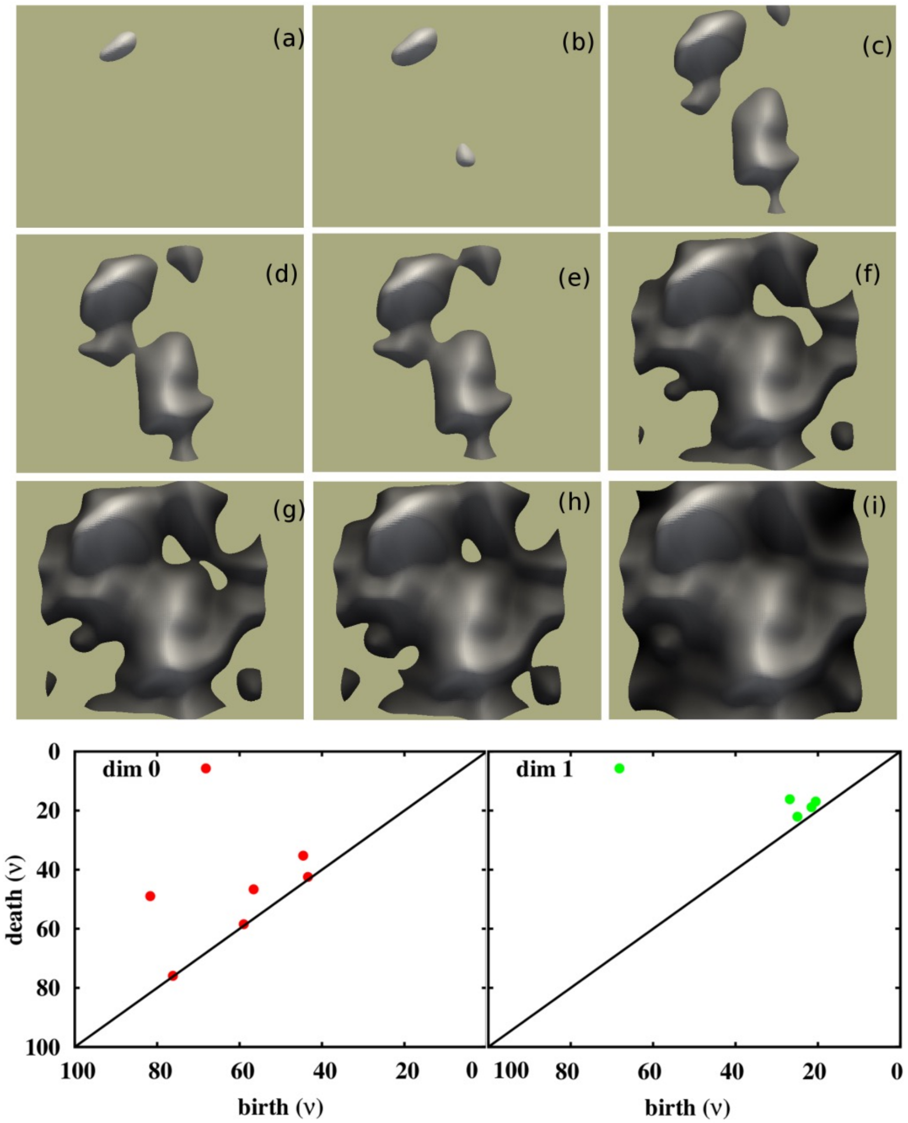






$$\emptyset = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

Birth, death and life-time(persistence): hierarchical topology



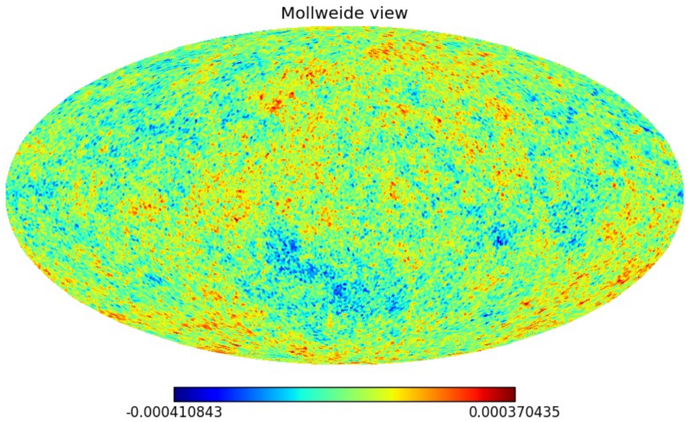
- **Representation of multi-scale topology**
- **Dots in the diagram record birth and death**
- **A diagram for each ambient dimension of the manifold**
  - **0-dimensional diagram: representation of merger of isolated objects (merger trees)**
  - **1-dimensional diagrams: formation and filling up of loops (percolation)**
  - **2-dimensional diagrams: formation and destruction of topological voids (voids)**

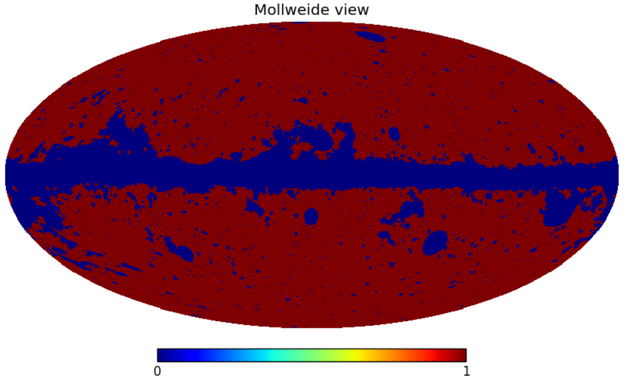
Pranav et al, MNRAS, 465 (4), 4281-4310, 2017

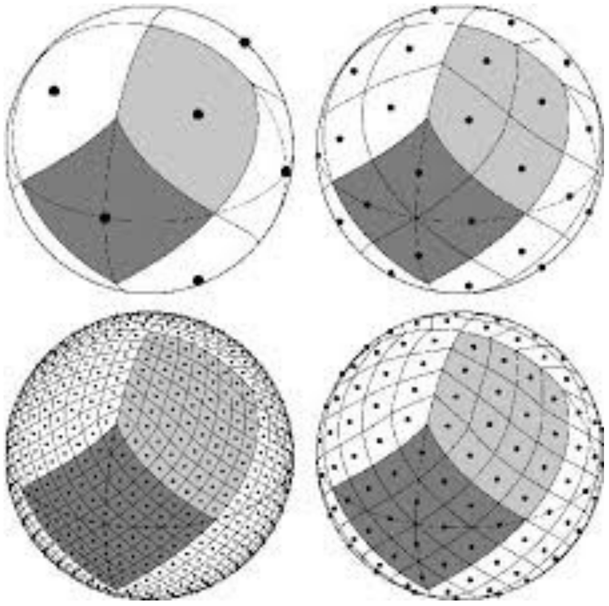
Topology of the CMB

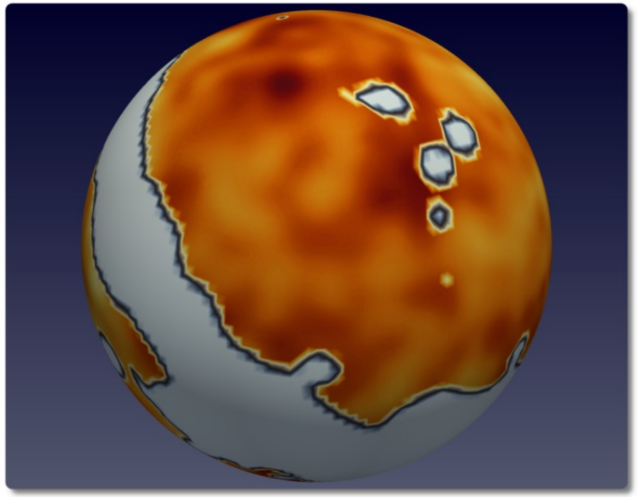
Planck Data

- Specified on S2, as the deviation from the background average (HEALPIX format)
- Measurement unreliable in some parts: foregrounds
- Unreliable parts masked
- Field converted to N (0,1) using unmasked pixels only



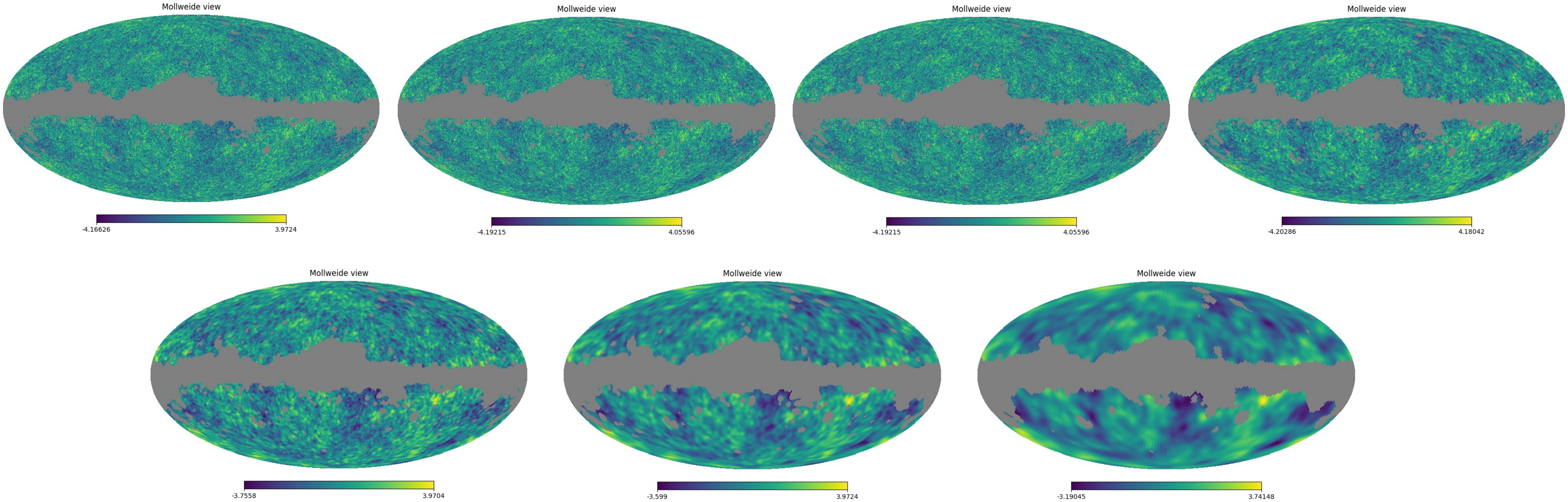




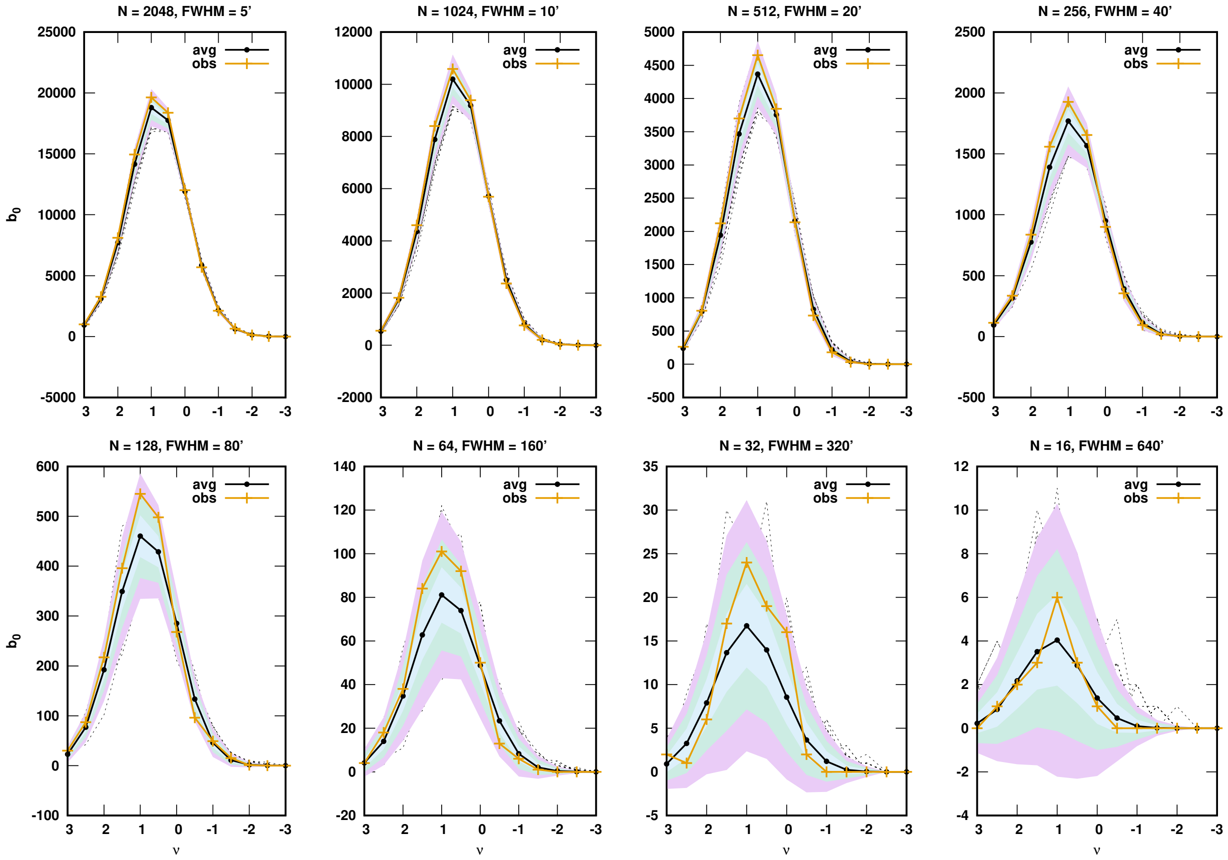


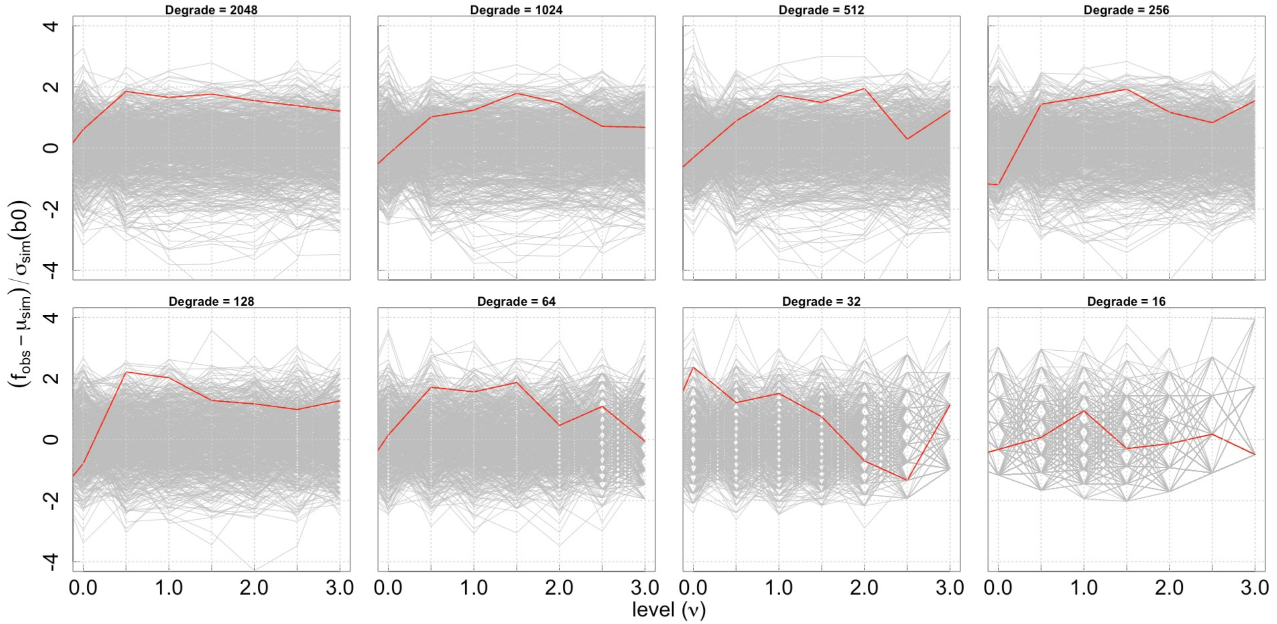
Temperature
Full Sky

Masked degraded maps (multi-scale analysis)



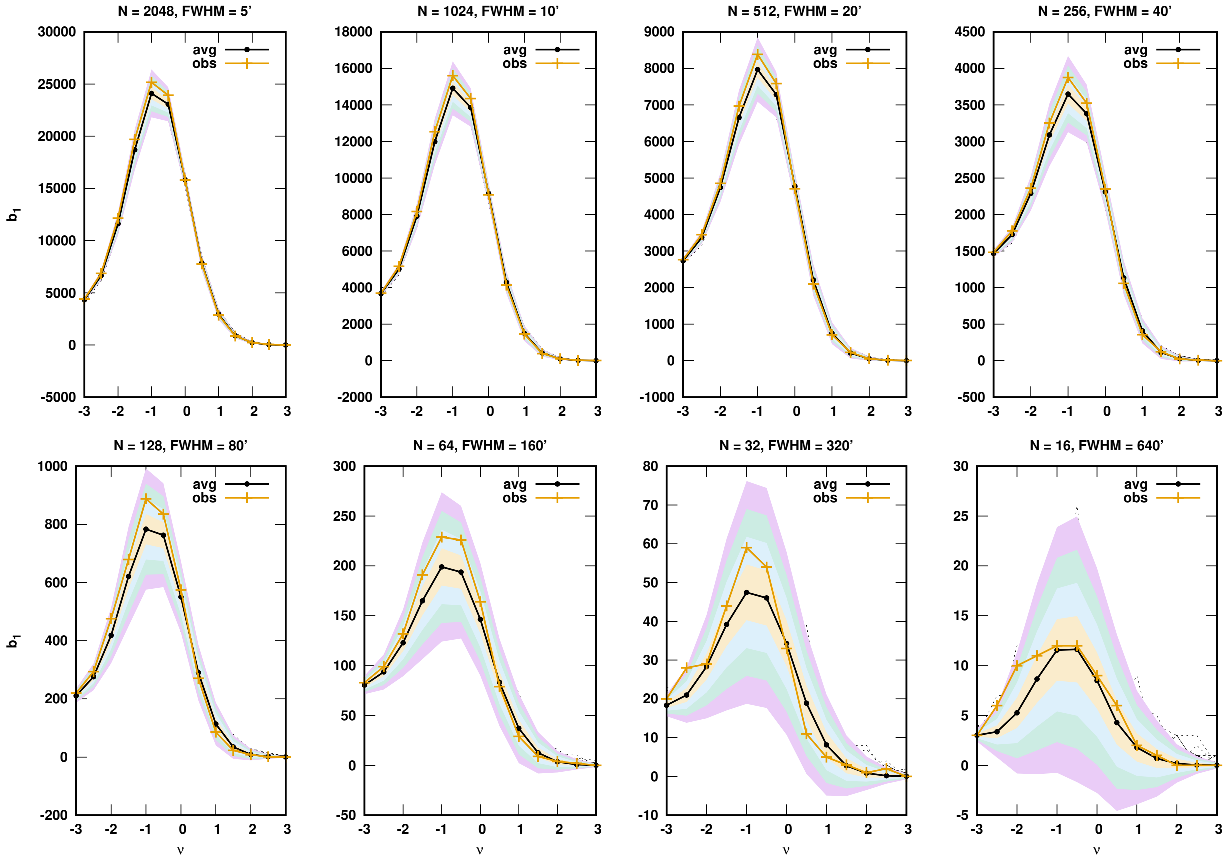
- Maps degraded to N_side = 2048, 1024, 512, 256, 128, 64, 32 and 16 (not shown), corresponding to FWHM = 5’, 10’, 20’, 40’, 80’, 160’, 320’, 640'
- Binary Mask degraded similarly (converts it to non-binary) – reconverted to binary by setting the threshold 0.9 (as done by Planck coll.)

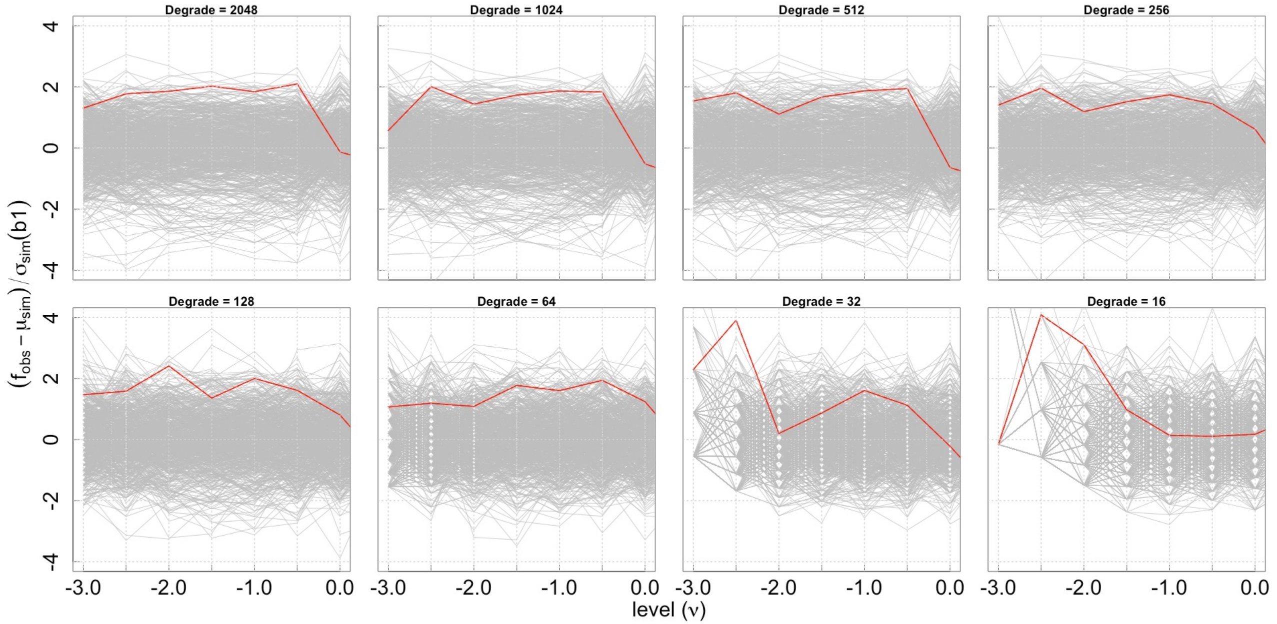




Isolated objects (betti 0)

P.Pranav, A&A 659, A115 (2022)





Loops (betti 1)

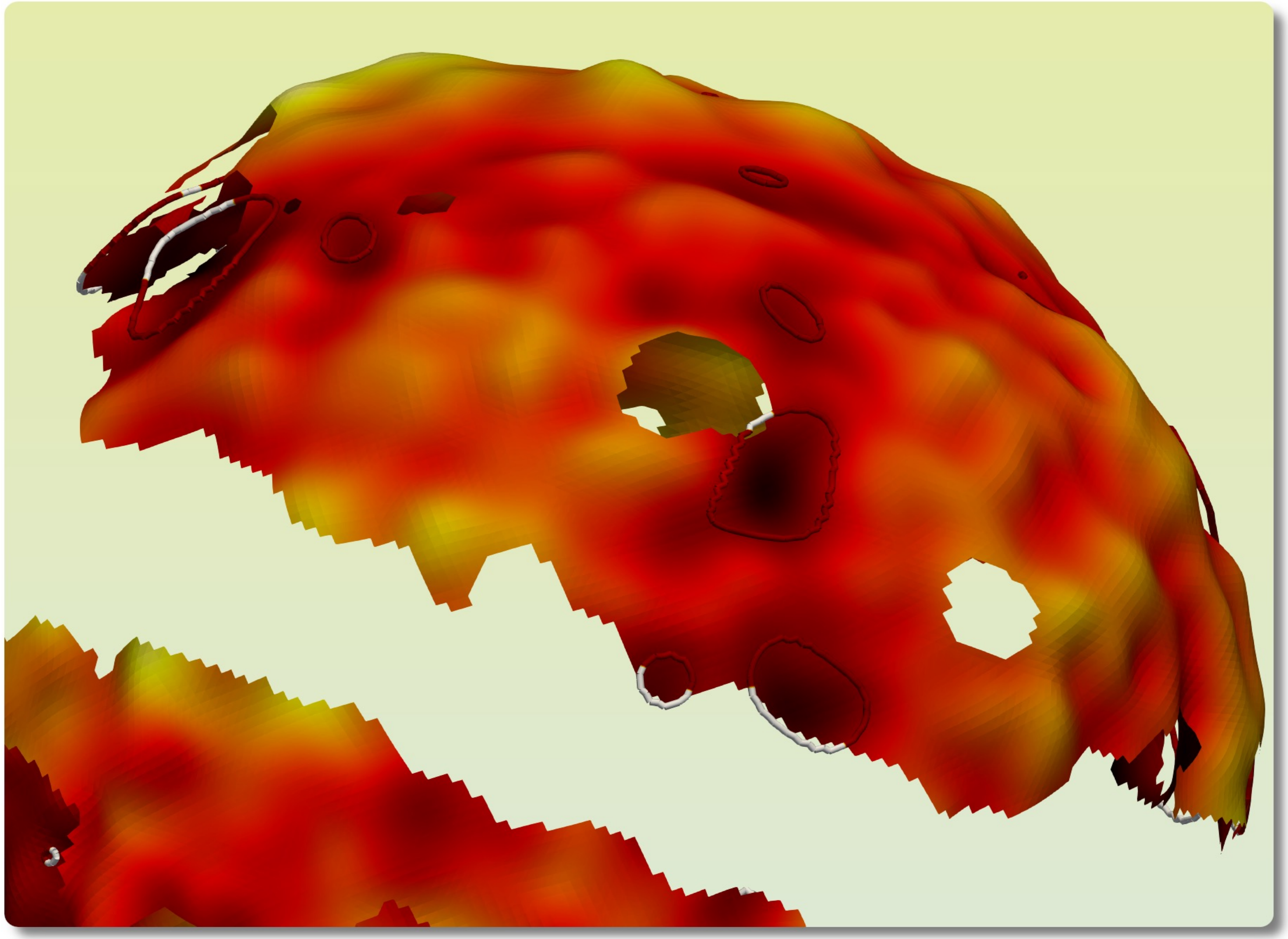
P.Pranav, A&A 659, A115 (2022)

Relative homology										
		χ^2 (theoretical)			χ^2 (empirical)			Tukey Depth		
Res	FWHM	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathsf{EC}_{\mathrm{rel}}$
threshold = 0.90										
2048	5	0.708	0.569	0.858	0.676	0.584	0.851	0.630	0.318	0.912
1024	10	0.649	0.475	0.383	0.628	0.476	0.363	0.672	0.352	**0.000**
512	20	0.156	0.417	0.203	0.161	0.408	0.183	0.332	0.308	**0.000**
256	40	0.398	0.596	0.772	0.403	0.600	0.756	0.362	0.560	0.642
128	80	0.356	0.204	0.563	0.383	0.186	0.555	**0.000**	0.312	0.730
64	160	0.398	0.494	0.768	0.401	0.468	0.746	0.465	0.718	0.755
32	320	**0.020**	**0.002**	**0.001**	**0.028**	**0.013**	**0.001**	**0.000**	**0.000**	**0.000**
16	640	0.974	**0.001**	0.203	0.981	**0.030**	0.222	0.983	**0.000**	0.440
summary	NA	0.478	**0.023**	**0.045**	0.290	0.076	**0.052**	0.803	**0.000**	**0.000**

Table 1: Table displaying the two-tailed p-values for relative homology obtained from parametric (Mahalanobis distance) and non-parametric (Tukey depth) tests, for different resolutions and smoothing scales for the `NPIPE` dataset. The last entry is the p-value for the summary statistic computed across all resolutions. Marked in boldface are p-values 0.05 or smaller.

P.Pranav, A&A 659, A115 (2022)

CMB Loops



P.Pranav, A&A 659, A115 (2022)

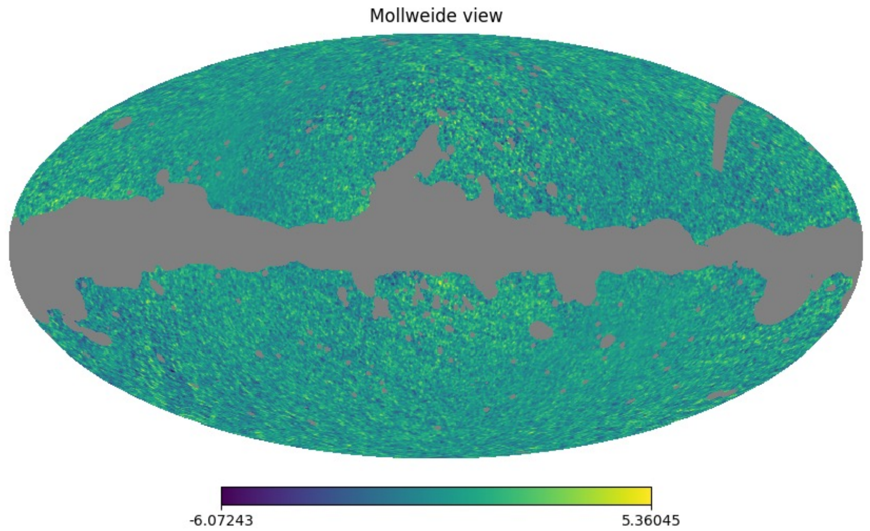
Polarization

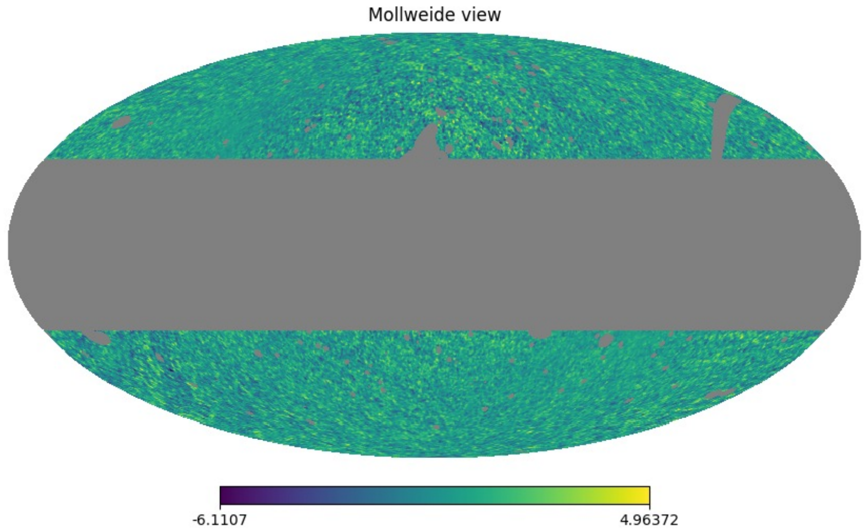
Experimental set up

- Use full sky to generate alms from TQU maps (prevents E/B leakage)

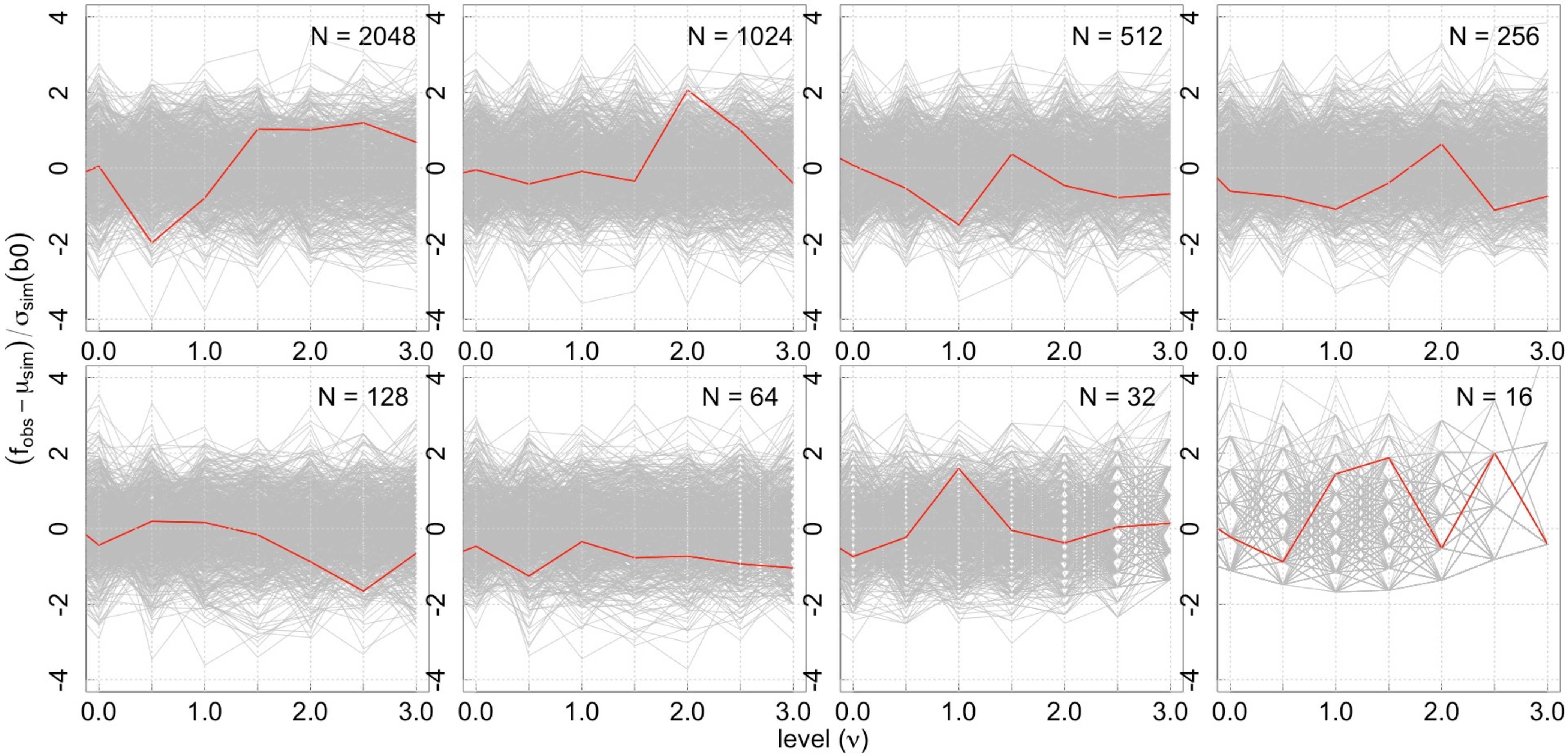
- Use the grad and curl-like like elements to synthesize the E and B maps

- Mask : PR3 common mask (+ b)





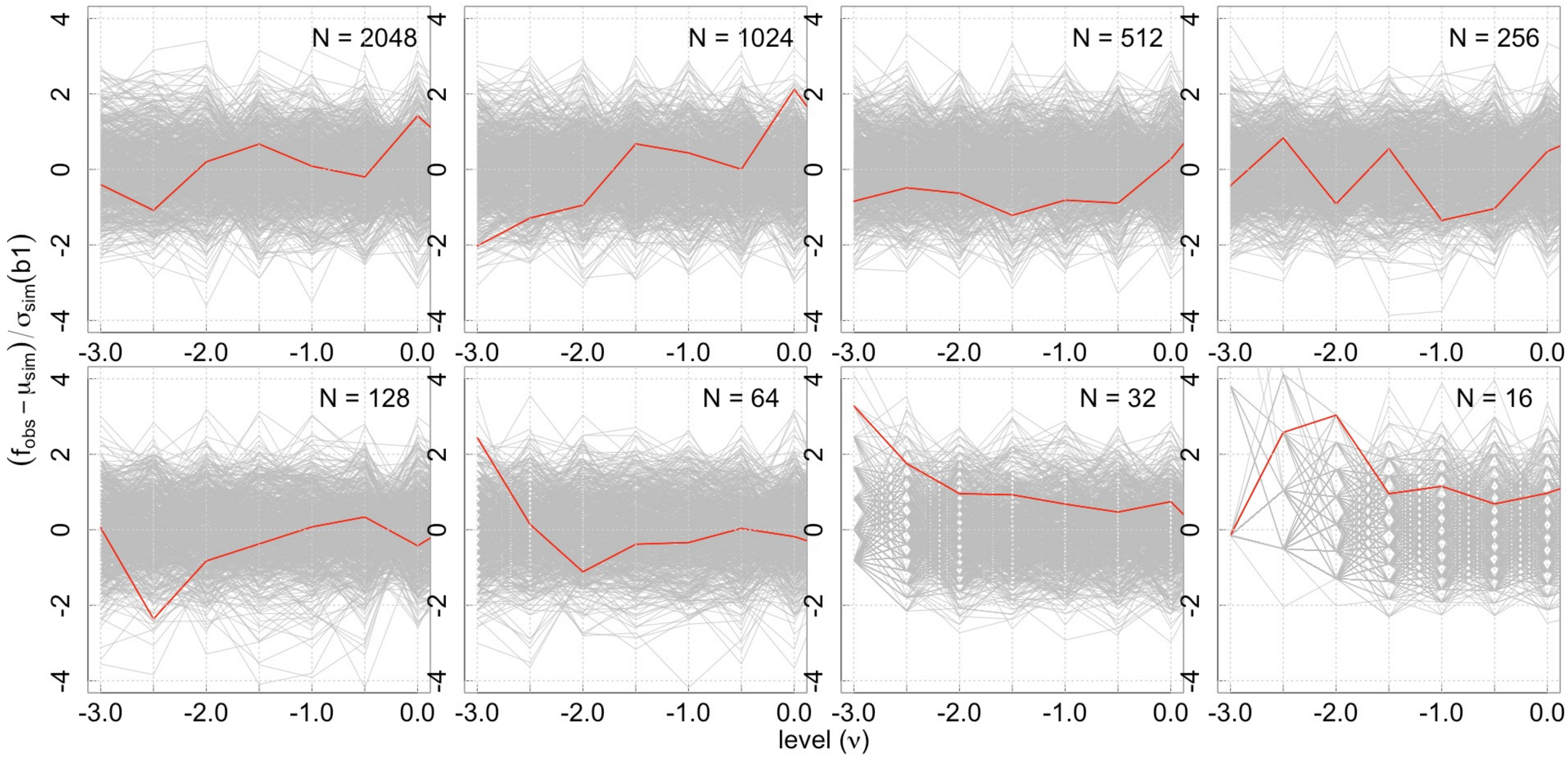
NPIPE : Planck 2018 common Mask



E-mode maps

Isolated objects (betti 0)

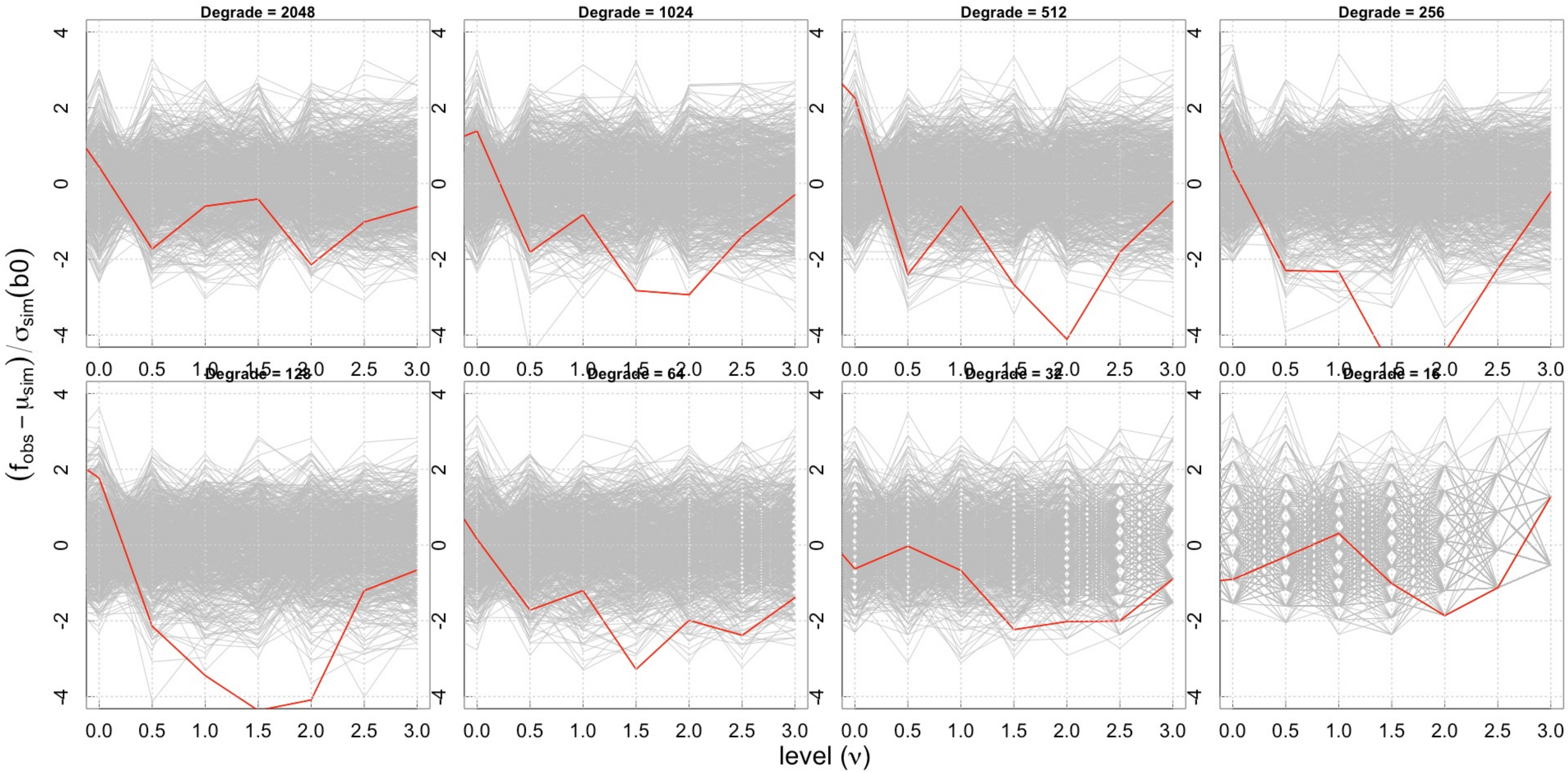
NPIPE : Planck 2018 common Mask



E-mode maps

Loops(betti 1)

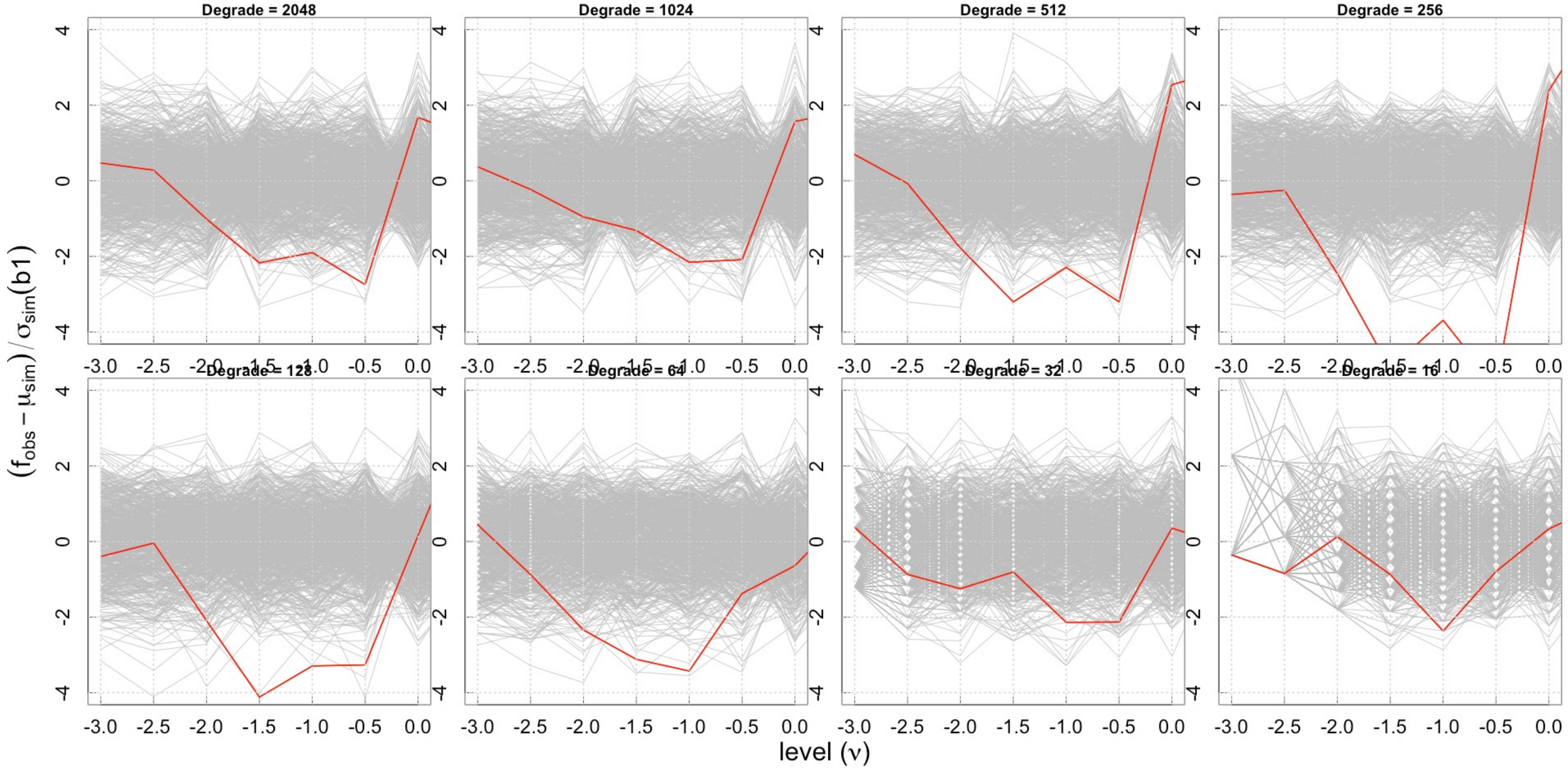
NPIPE : Planck 2018 common Mask



B-mode maps

Isolated objects (betti 0)

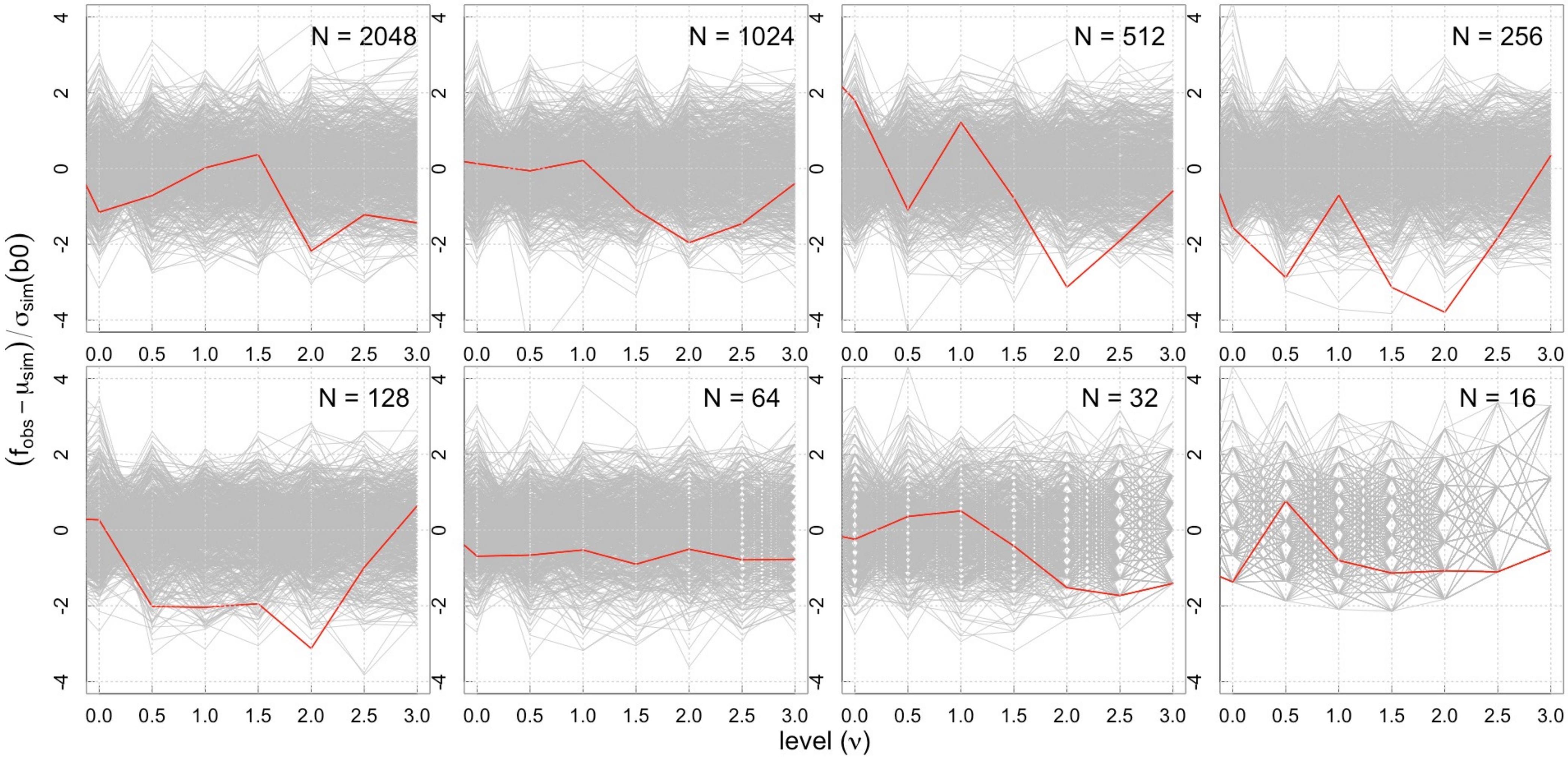
NPIPE : Planck 2018 common Mask



B-mode maps

Loops(betti 1)

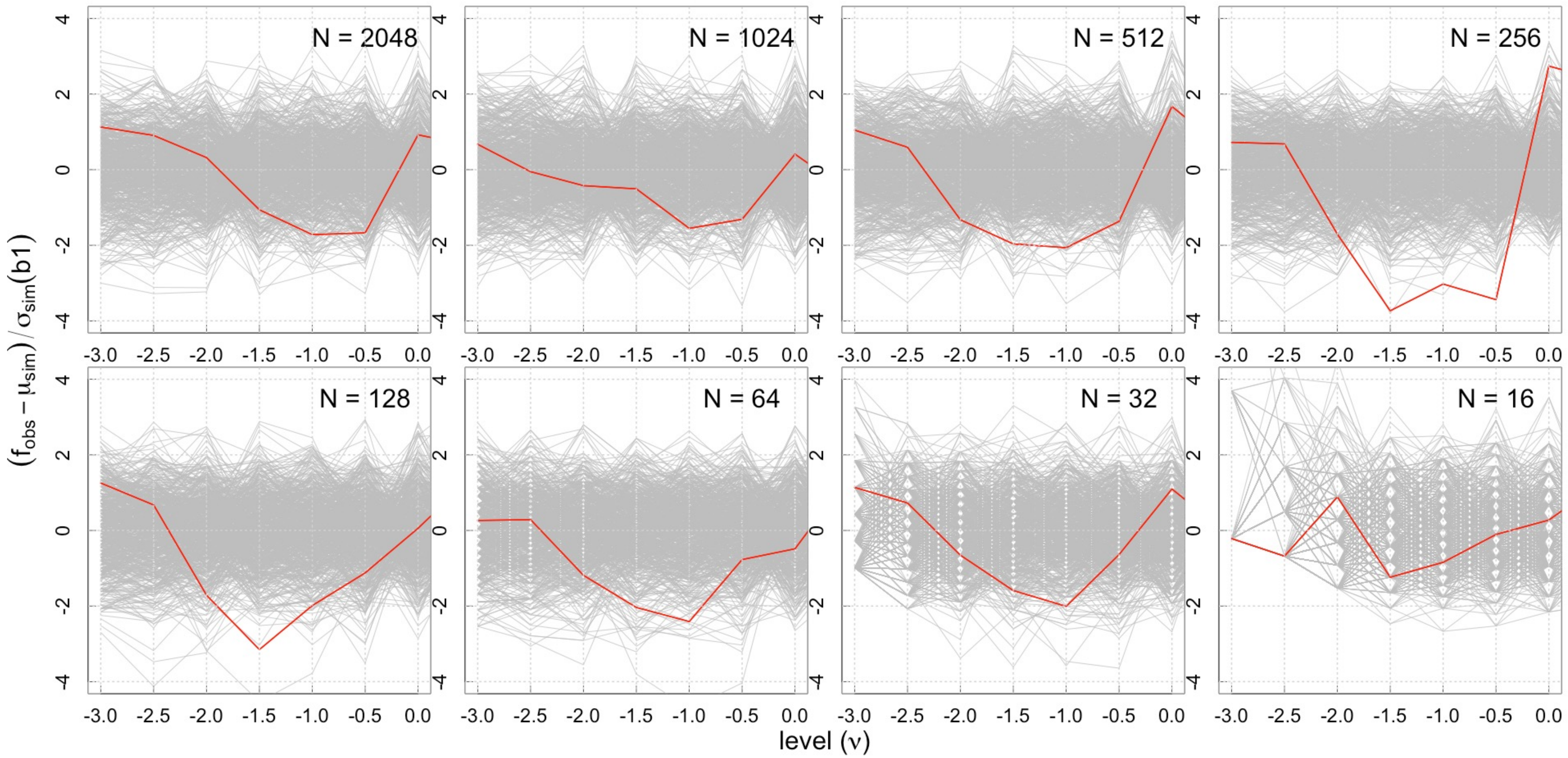
NPIPE : Planck 2018 common Mask + galcut 60



B-mode maps

Isolated objects (betti 0)

NPIPE : Planck 2018 common Mask + galcut 60



B-mode maps

Loops(betti 1)

Relative homology – `NPIPE` E-mode							
		χ^2 (empirical)			χ^2 (theoretical)		
Res	FWHM	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$
threshold = 0.90							
2048	5	0.600	0.665	0.457	0.601	0.671	0.470
1024	10	0.498	0.815	0.495	0.504	0.806	0.506
512	20	0.173	0.638	0.215	0.180	0.631	0.214
256	40	0.582	0.187	0.388	0.596	0.190	0.407
128	80	0.447	0.992	0.705	0.476	0.989	0.699
64	160	0.953	0.962	0.993	0.936	0.960	0.991
32	320	0.358	0.027	0.018	0.371	0.016	0.015
16	640	0.670	0.130	0.088	0.764	0.198	0.058

Relative homology – `NPIPE` B-mode							
		χ^2 (empirical)			χ^2 (theoretical)		
Res	FWHM	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$
threshold = 0.90							
2048	5	0.140	0.708	0.473	0.154	0.708	0.470
1024	10	0.210	0.377	0.315	0.226	0.381	0.324
512	20	**0.000**	0.100	**0.008**	**0.000**	0.097	**0.006**
256	40	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
128	80	**0.005**	0.062	**0.002**	**0.003**	0.053	**0.000**
64	160	0.817	0.172	0.777	0.839	0.183	0.775
32	320	0.587	0.475	0.753	0.594	0.487	0.760
16	640	0.423	0.565	0.540	0.453	0.683	0.569

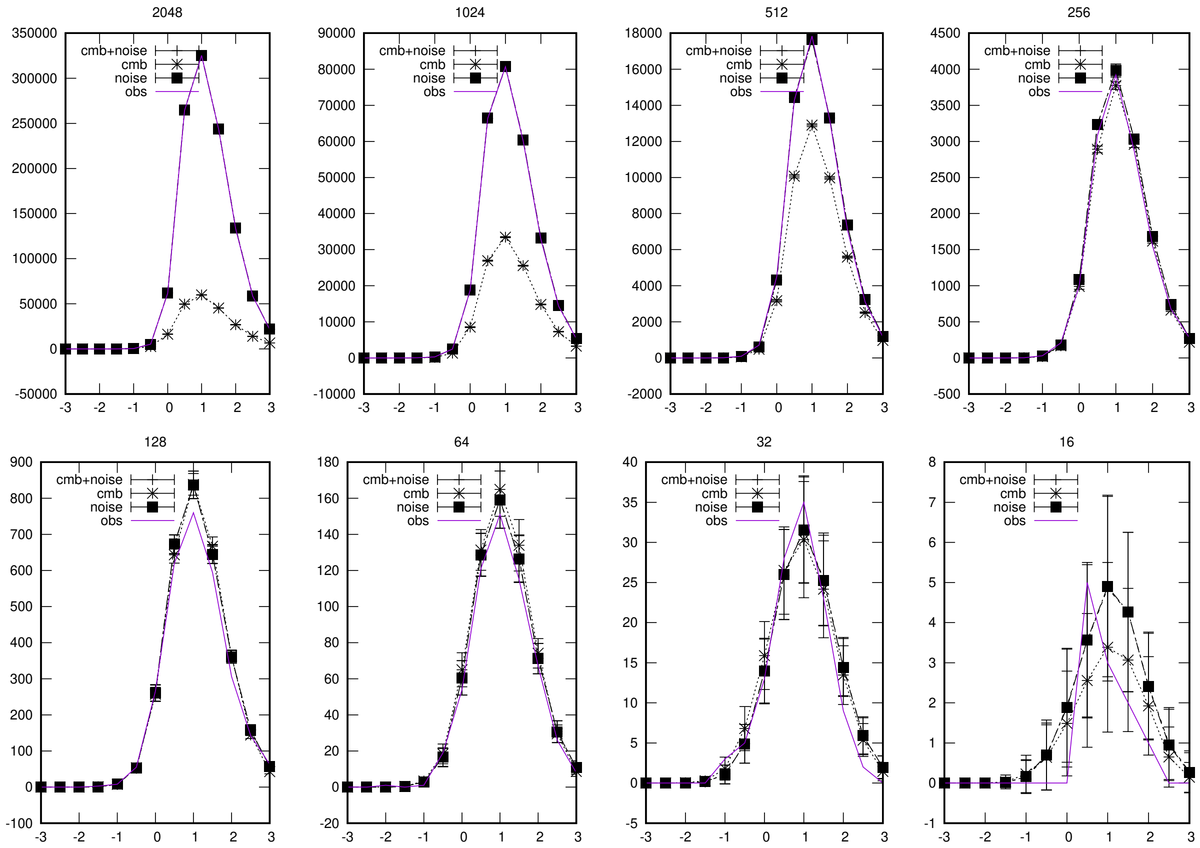
Relative homology										
		χ^2 (empirical)			χ^2 (theoretical)			Tukey Depth		
Res	FWHM	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$
threshold = 0.90										
2048	5	0.578	0.332	0.583	0.584	0.334	0.574	0.540	0.360	0.485
1024	10	0.517	0.957	0.838	0.503	0.950	0.830	0.513	0.948	0.870
512	20	0.480	0.363	0.812	0.468	0.376	0.799	0.487	0.338	0.795
256	40	**0.038**	0.102	0.090	**0.048**	0.097	0.088	**0.000**	**0.000**	**0.000**
128	80	0.178	0.168	0.430	0.201	0.169	0.427	**0.000**	**0.000**	0.462
64	160	0.103	0.503	0.707	0.102	0.494	0.698	**0.000**	0.505	0.750
32	320	0.802	0.587	0.825	0.781	0.617	0.828	0.780	0.558	0.798
16	640	0.178	0.103	0.228	0.177	0.061	0.217	**0.000**	**0.000**	**0.000**
summary	NA	0.582	**0.017**	**0.057**	0.569	**0.011**	**0.060**	**0.000**	**0.000**	**0.000**

Table 1: Table displaying the two-tailed p-values for relative homology obtained from parametric (Mahalanobis distance) and non-parametric (Tukey depth) tests, for different resolutions and smoothing scales for the `NPIPE` dataset Q component. The last entry is the p-value for the summary statistic computed across all resolutions. Marked in boldface are p-values 0.05 or smaller.

Relative homology										
		χ^2 (empirical)			χ^2 (theoretical)			Tukey Depth		
Res	FWHM	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$	b_0	b_1	$\mathrm{EC}_{\mathrm{rel}}$
threshold = 0.90										
2048	5	**0.045**	0.768	0.295	**0.049**	0.767	0.309	0.000	0.785	**0.000**
1024	10	0.593	0.275	0.192	0.620	0.264	0.171	0.625	0.252	**0.000**
512	20	0.593	0.150	0.283	0.602	0.145	0.285	0.577	**0.000**	**0.000**
256	40	0.298	0.747	0.932	0.314	0.747	0.925	0.335	0.730	0.887
128	80	0.933	0.700	0.760	0.936	0.690	0.755	0.953	0.567	0.832
64	160	0.823	**0.040**	0.298	0.831	**0.043**	0.302	0.787	**0.000**	**0.000**
32	320	0.682	0.435	0.447	0.679	0.439	0.444	0.502	0.362	**0.000**
16	640	0.312	0.177	0.667	0.320	0.139	0.700	0.255	0.355	0.827
summary	NA	**0.023**	**0.008**	**0.010**	**0.022**	**0.003**	**0.011**	**0.000**	**0.000**	**0.000**

Table 2: Table displaying the two-tailed p-values for relative homology obtained from parametric (Mahalanobis distance) and non-parametric (Tukey depth) tests, for different resolutions and smoothing scales for the `NPIPE` dataset U component. The last entry is the p-value for the summary statistic computed across all resolutions. Marked in boldface are p-values 0.05 or smaller.

Signal-to-Noise



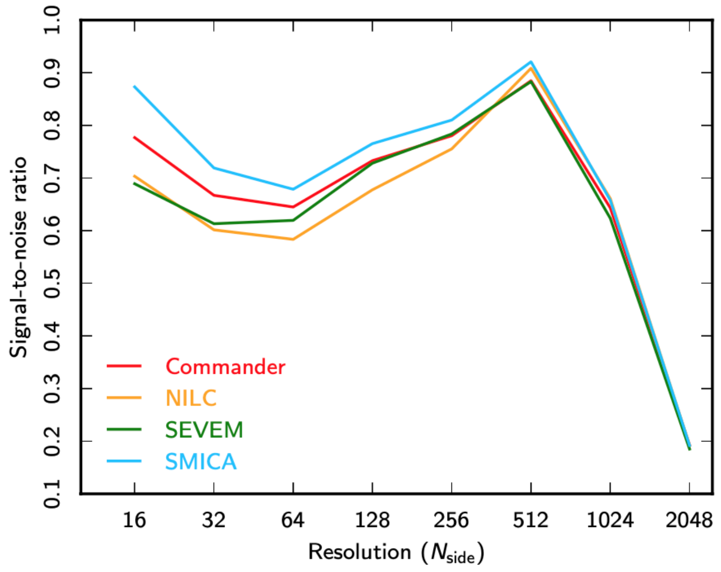


Fig. 5. Signal-to-noise ratio for the variance estimator in polarization for `Commander` (red), `NILC` (orange), `SEVEM` (green), and `SMICA` (blue), obtained by comparing the theoretical variance from the *Planck* FFP10 fiducial model with an MC noise estimate (right-hand term of Eq. (7)). Note that the same colour scheme for distinguishing the four component-separation maps is used throughout this paper.

Conclusions

- Topology and geometry powerful means of characterizing data.

- CMB data exhibits mild to significant anomaly in both temperature and polarization data.
    - Temperature: full sky (large-scale); hemisphere (Degree-scale)
    - Polarization: full-sky (degree-scale)

- Results are based on legitimate mathematical foundations, and not a case of over-exploitation of data through tailor-made statistics.

- Hints of violation of cosmological principal, or other late time mechanisms at play (including doppler boosting/dipolar modulation)?