The Hawking Energy in Linearly Perturbed FLRW

Spontaneous Cosmology Workshop XIV
Cargèse

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Outline

- Motivation & context
- Hawking energy E_H & properties
- Cosmological set-up
- E_H on lightcones
- Linearly perturbed FLRW & relation to observations
- Summary & outlook

→ DS Class.Quant.Grav.(2021) arXiv 2003.13583
→ DS Class.Quant.Grav.(2021), arXiv 2010.07896

→ Durrer & Stock in prep.

Motivation & Context

- Challenge of defining energy in GR:
  - Equivalence Principle: can locally set the gravitational field to zero $\rightarrow$ standard (**local**) definition of gravitational energy momentum via $T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}$ not working
  - Go quasi-local: assign energy to finitely-sized spacetime domains, connecting to asymptotic mass/energy definitions

  Question: definition in a restricted, yet relevant set-up possible?
- Set-up $\rightarrow$ **lightcone:** cosmological data collected on that hypersurface, unique geometrical structure
- E_H defined in terms of expansion scalars of null congruences $\Rightarrow$ intuitive choice when applied to lightcone generating null congruence

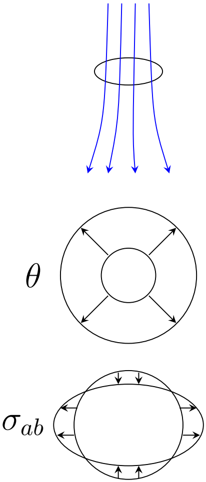
"Study E_H as tentative measure of the observable universe for a given observer to address cosmological questions"

Governing Equations

- $C^{-}(p)$ is a null hypersurface generated by the past null geodesic congruence issued at p with tangent vectors l^a
- The shape of cross sections of null congruence described by **expansion** θ and **shear** tensor σ_{ab}
- Their change along l^a is given by the optical equations:

$$\dot{\theta} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}l^a l^b \qquad (1)$$

$$\dot{\sigma}_{ab} = -\theta\sigma_{ab} - C_{acbd}l^c l^d \qquad (2)$$



Definition of the Hawking Energy

- Given a spacetime (M, g), take a spacelike (topol.) 2-sphere S with area $A(S) = \int_S dS$.
- $\exists$ past-directed outgoing and ingoing null geodesic congruences $\perp S$ with expansion scalars θ_+ & θ_-.
- Idea: energy in 3-volume surrounded by S affects the light bending on S.

Def. Hawking Energy E (Hawking 1968)

Given a spacelike 2-sphere S, the Hawking energy E is defined as

$$E_H(S) := \frac{\sqrt{A(S)}}{(4\pi)^{3/2}} \left[2\pi + \frac{1}{4} \int_S \theta_+ \theta_- \, dS \right] \tag{3}$$

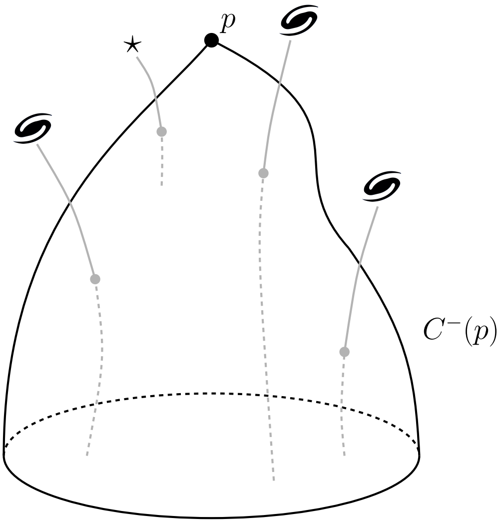
- Many useful properties & connections to energy definitions in special cases were established

Properties of E_H

- In the limit of S degenerating to a point, $E(S) \to 0$.
- Positivity for small sphere of radius r around $p \in M$ in the limit $r \to 0$ (Horowitz & Schmidt 1982):
  - non-vacuum: $E \sim r^3 T_{ab} t^a t^b \geq 0$ if DEC holds
  - vacuum: $E \sim B_{abcd} t^a t^b t^c t^d \geq 0$

  T_{ab}: energy-momentum tensor, B_{abcd}: Bel-Robinson tensor, t^a unit timelike vector
- For a section of the Kerr-Newman horizon: $E = M_{\text{irr}}$.
- For large spheres near $\mathcal{I}^+$: $E \to E_{\text{Bondi-Sachs}}$ (Hawking 1968).
- For large spheres near i^0: $E \to E_{\text{ADM}}$ (Eardley 1979).
- In spherical symmetry: $E = E_{\text{Misner-Sharp}}$ (Hayward 1996).

Observable Universe & Cosmological Set-up



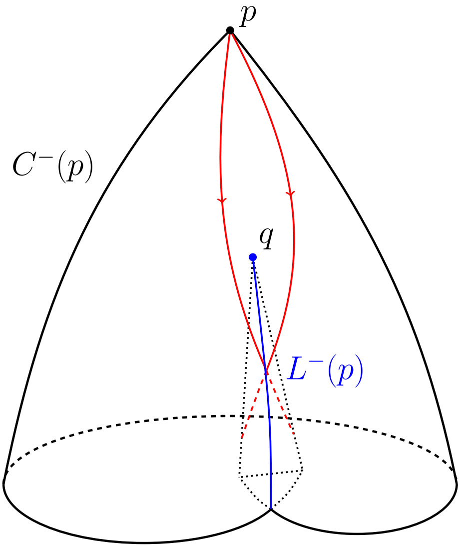
- Observable Universe at p (observer) given by the past lightcone $C^-(p)$
- $C^-(p)$ unique geometric object
- Approximate observations to happen instantaneously at p, because $t_{\mathrm{obs}} \ll t_{\mathrm{cosm}}$
- **Weak gravitational lensing regime**: $C^-(p)$ deformed, but no multiple imaging $\Rightarrow C^-(p) \simeq \mathbb{R} \times S^2$

Strong Gravitational Lensing

- Multiple imaging of the same source

- Self-intersections of $C^-(p)$
- Topology of $C^-(p)$ changes

Mathematically rigorous description in terms of **conjugate & cut points**



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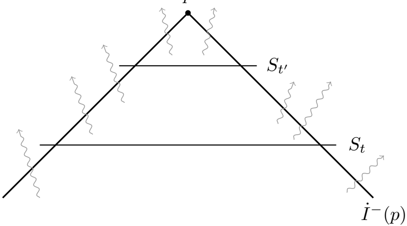
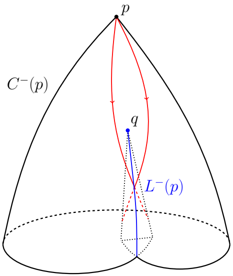
The Hawking Energy on the Lightcone (I)

Goal: to study behaviour of the Hawking energy down the past lightcone for a given affine parameter slicing S_λ.

Monotonicity intuitively clear:



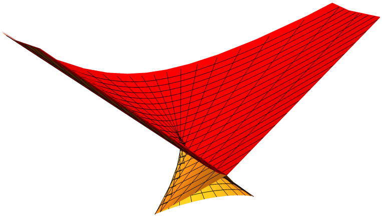
Take only the exterior of the lightcone $C^-(p) \cap \dot{I}^-(p)$!



$\Rightarrow$ Study $E(S_\lambda)$ with $S_\lambda \simeq S^2$ and $C^-(p) \cap \dot{I}^-(p) = \cup_\lambda S_\lambda$.

The Hawking Energy on the Lightcone (II)

- **Weak Lensing regime:** monotonicity intuitively clear, can be shown rigorously (Eardley 1978)
- **Strong Lensing regime:** E well-defined for (isolated) stable strong lenses; in general not monotonic anymore



$$\dot{E}(S) = \frac{E(S)}{2A(S)} \int_S \theta_+ \, \mathrm{d}S + \frac{\sqrt{A(S)}}{(4\pi)^{3/2}} \int_S \Big\{ -\left(\theta_- \sigma^+_{ab} \sigma_+^{ab} + \theta_+ \sigma^-_{ab} \sigma_+^{ab}\right) - 8\pi \left(\theta_- T_{ll} + \theta_+ \left[T_{ln} + \frac{1}{6} T \right] \right) + \theta_+ D_a \Omega^a \Big\} \, \mathrm{d}S$$

→ DS, Class.Quant.Grav. 2021, arXiv:**2003.13583**

Energy in Spatially Flat FLRW Spacetimes

- Matter: perfect fluid with density $\rho(z) = \rho_0 \exp\left[3\int_0^z \frac{1+w(\tilde{z})}{1+\tilde{z}}\mathrm{d}\tilde{z}\right]$, EOS $P(z) = w(z)\rho(z)$, satisfying DEC $|w| \leq 1$,
- E_H for const. redshift lightcone slices:

$$E(z) = \frac{4\pi}{3}\rho(z)D^3(z) \quad ,$$

with area/angular diameter distance $D(z) = \frac{1}{1+z}\int_0^z \frac{\mathrm{d}z'}{H(z')}$

Montonicity for $z \to \infty$:

$$E'(z) = \underbrace{\frac{4\pi\rho(z)}{(1+z)^3}\left(\int_0^z \frac{dz'}{H(z')}\right)^2}_{=:\alpha\geq 0} \cdot \left[\frac{w(z)}{1+z}\underbrace{\int_0^z \frac{dz'}{H(z')}}_{\geq \frac{z}{H(z)}} + \frac{1}{H(z)}\right]$$

$$\geq \frac{\alpha}{H(z)}\left(1 - \frac{z}{z+1}\right) \geq 0$$

Linearly Perturbed FLRW

- Standard framework to analyse & interpret cosmological observations & data

⇒ *Can E be linked to observables?*

- Here: scalar perturbations in longitudinal gauge for simplicity:

$$ds^2 = a^2(t)\left[-(1+2\Psi)\,dt^2 + (1-2\Phi)\left(dr^2 + r^2\,d\Omega\right)\right] \quad ,$$

with Bardeen potentials Ψ and Φ Energy-momentum tensor:

$$T^0_0 = -\bar{\rho}(1+\delta) \quad , \quad T^i_j = \bar{P}\left[(1+\pi_L)\delta^i_j + \Pi^i_j\right]$$
$$T^j_0 = -(\bar{\rho}+\bar{P})v^j \quad , \quad T^0_j = (\bar{\rho}+\bar{P})v_j$$

with density perturbation δ, isotropic stress perturbation π_L, anisotropic stress Π^i_j, velocity perturbation v^j

E in linearly perturbed FLRW

(R. Durrer et D. Stock in prep.)

$$E(z) = \bar{E}\left(1 + \frac{1}{2\pi}\int_{S_z}\left[3\frac{\delta D}{\bar{D}} + \frac{3}{4}(\Psi + \Phi) + \frac{1}{2}\delta - \frac{3}{4}\bar{w}\,\Pi^r_r\right]\,d\bar{\Omega}\right)$$

Terms appearing:

- $\bar{E} = \frac{4\pi}{3}\rho D^3$: energy of the FLRW background
- $\frac{\delta D}{\bar{D}}$: (rel.) area distance fluctuations
- $\Psi + \Phi$: Sachs-Wolfe effect
- δ : matter density fluct.
- Π_r : (radial) anisotropic stress

$\Rightarrow$ E is a gauge-invariant quantity based on observables!

How to measure the contributions?

- For perfect fluids: $\Pi^r_r = 0$.
- Area distance fluctuation: $\frac{\delta D}{\bar{D}} = \frac{\delta D_L}{\bar{D}_L}$, D_L: luminosity distance
- δ related to galaxy number count fluctuations Δ and volume perturbation δV (Bonvin, Durrer):

$$\Delta(\theta, \phi, z) = b\,\delta(\theta, \phi, z) + \frac{\delta V(\theta, \phi, z)}{\bar{V}(z)}$$

b: bias factor
Given two tracer populations "1" and "2", we find:

$$\Delta_1 - \Delta_2 = (b_1 - b_2)\,\delta$$

- $\Psi + \Phi$: standard lensing term in LSS analysis

In Practice: Variance of E_H

- Found that: $E(z) = \bar{E}(1 + \int \epsilon d\bar{\Omega})$; write $E(z) = \bar{E}(z) + \delta E(z)$
- Cannot measure background quantities, only expectation values $\langle\, . \,\rangle$ in the statistical ensemble
- Assuming that $\langle E \rangle = \bar{E}$:

$$\langle \delta E \rangle = 0 \quad \text{but} \quad \langle \delta E^2 \rangle \neq 0$$

- $\int \epsilon d\bar{\Omega}$ just contains a monopole
- Can study the angular decomposition of the "energy surface density" $\epsilon(z, \theta, \varphi)$

Summary & Outlook

- Hawking energy E_H interesting & well-behaved physical quantity on the lightcone
- Monotonicity & positivity can be established in the weak lensing regime (no caustics); also for flat FLRW spacetimes with matter obeying the DEC
- E_H is a gauge invariant quantity and within linear perturbation theory can be directly related to standard cosmological observables

$\Rightarrow$ Next step: study magnitude of the different contributions

Thank you for your attention!

Extensions & Generalisations of E_H

- For a metric 2-sphere in Minkowski: $E = 0$; not the case for general 2-sphere.
- Hayward (1993) generalised the Hawking energy to achieve a vanishing energy for any sphere in Minkowski space by including shear & twist terms; however, $E < 0$ for small spheres in vacuum
- Hawking energy can also be defined for higher genus surfaces (S, g):

$$E(S) = \frac{\pi^{3/2}}{32}\sqrt{A(S)}\left(8\pi(1-g) + \int_S \theta_+\theta_-\right)$$

- Domain additivity:
  Assume two disconnected pieces $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$, then

$$\frac{E(S)}{\sqrt{A(S)}} = \frac{E(S_1)}{\sqrt{A(S_1)}} + \frac{E(S_2)}{\sqrt{A(S_2)}} \quad \Rightarrow \quad E(S) > E(S_1) + E(S_2)$$

$\Rightarrow$ Superadditivity!

Appendix A: Comparing (In)homogeneous Domains

- Consider a spacetime filled with an ideal fluid, and S a non-trapped sphere with area radius D.
- The Hawking energy of S with $A(S) =: 4\pi D^2$ can also be written as

$$E(S) = D^3 \left(\frac{4\pi}{3} \langle \rho \rangle_S + \frac{1}{2} \langle \sigma^+_{ab} \sigma^{ab}_- \rangle_S \right) \quad .$$

- Comparing the energies of a general and a shear-free domain of **equal size** D **and average density** $\langle \rho \rangle_S \Rightarrow \delta E = \frac{D^3}{2} \langle \sigma^+_{ab} \sigma^{ab}_- \rangle_S$
- Integrating the Gauss equation yields a bound on the shear term:

$$\langle \sigma^+_{ab} \sigma^{ab}_- \rangle_S \leq \frac{4\pi}{A} - \frac{8\pi}{3} \langle \rho \rangle_S$$

If $\langle \rho \rangle_S > \frac{3}{2A}$, a shear-free domain maximises the energy for given size A and $\langle \rho \rangle_S$.

Appendix B: Towards Addressing the Fitting Problem

RW reference slices can be constructed based on E and A for e.g. given affine parameter λ:

- From $E(S_\lambda) = E(\bar{S}_\lambda) \quad \Rightarrow \quad \bar{\rho}(\lambda) = \langle \rho \rangle_{S_\lambda} + \frac{3}{8\pi} \langle \sigma^{+}_{ab} \sigma^{ab}_{-} \rangle_{S_\lambda}$
- From $A(S_\lambda) = A(\bar{S}_\lambda)$ and equality of first and second derivative $\Rightarrow$

$$\bar{\rho}\,(1+\bar{w})\,(1+\bar{z})^2 = \underbrace{\langle \rho(1+w)(1+z)^2 \rangle_{S_\lambda}}_{\geq 0} + \frac{1}{4\pi} \underbrace{\langle \sigma^{+}_{ab} \sigma^{ab}_{+} \rangle_{S_\lambda}}_{\geq 0} + \frac{1}{8\pi} \underbrace{\left(\langle \theta_+ \rangle^2_{S_\lambda} - \langle \theta^2_+ \rangle_{S_\lambda} \right)}_{\leq 0}$$

- Slicewise procedure is a step towards fitting problem based on geometric quantities
- Construction of FLRW reference lightcone possible, if **either** A **or** E are fixed $\Rightarrow$ full fitting problem

Appendix C: Fitting Problem Based on A or E_H

A whole FLRW-reference lightcone can in principle be constructed by fixing either energy **or** area by solving

$$H\left[(1+z)^2 HD'\right]' = -4\pi\rho(1+w)D \tag{4}$$

recalling that $H = \sqrt{\frac{8\pi}{3}\rho - K(1+z)^2}$ and $1+w = \frac{1}{3}(1+z)\frac{\rho'}{\rho}$

- For given area distance D: $\Rightarrow$ solve for $\rho(z)$
- For given energy E, use $\rho = \frac{3}{4\pi}\frac{E}{D^3}$
  $\Rightarrow$ solve for $D(z)$

Fitting problem based on lightcones can be addressed geometrically