Feynman Integrals and EFT Approach to Gravitational Interactions

4, 5, 6, 7, …
all-order Post-Newtonian static corrections

Pierpaolo Mastrolia

Hot Topics in Modern Cosmology
Spontaneous Workshop XVIII
IECS, Cargese, France
September 15, 2026

In collaboration with: **G. Brunello**, M. Mandal, **R. Patil, M. Pegorin**, J. Ronca, **S. Smith**, J. Steinhoff, W. Torres Bobadilla

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Istituto Nazionale di Fisica Nucleare

Outline

- **Feynman Calculus**
  - Integral relations: Integration-by-parts identities and Differential Equations
- **Amplitudes and Diagrams for Gravitational Wave Physics**
  - Post-Minkowskian Corrections
  - Post-Newtonian Corrections
    - Diagrammatic approach
    - Factorisation Theorem
- **6-Loop gravitational interactions at 6PN**
  - At the frontier of Multi-loop calculus
- **Vector Space Structure of Feynman (*Twisted Period) Integrals***
  - De Rahm co-homology groups and D-module
- **Conclusions**

Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x)\, \partial_x^{(n-1)} f(x) + \ldots + p_1(x)\, \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

Differential Equations

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$$\partial_x^{(n)} f(x) + p_{n-1}(x)\, \partial_x^{(n-1)} f(x) + \ldots + p_1(x)\, \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

- **Linear relations**

$$f_n(x) + a_{n-1}\, f_{n-1}(x) + \ldots + a_1\, f_1(x) + a_0\, f_0(x) = 0$$

Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x)\, \partial_x^{(n-1)} f(x) + \ldots + p_1(x)\, \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

$$\partial_x^{(n)} f(x) = - p_{n-1}(x)\, \partial_x^{(n-1)} f(x) - \ldots - p_1(x)\, \partial_x^{(1)} f(x) - p_0(x) f(x)$$

- **Linear relations**

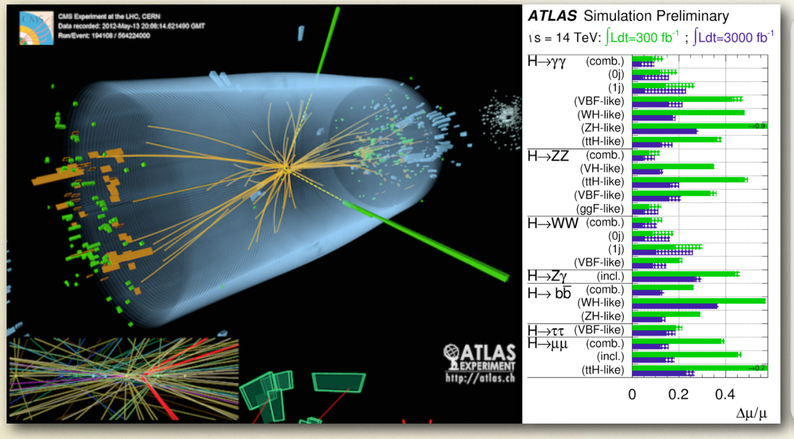
$$f_n(x) + a_{n-1} f_{n-1}(x) + \ldots + a_1 f_1(x) + a_0 f_0(x) = 0$$

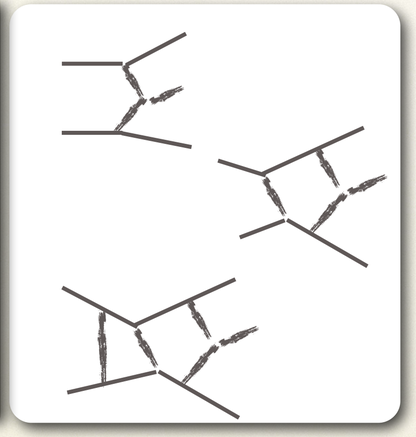
$$f_n(x) = - a_{n-1} f_{n-1}(x) - \ldots - a_1 f_1(x) - a_0 f_0(x)$$

Decomposition formulas
in terms
of **n *independent* elements**

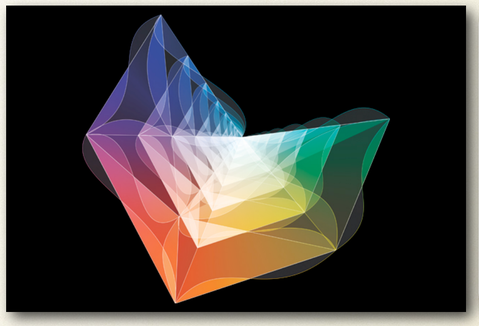
Impact of Scattering Amplitudes & Multiloop Calculus / Frontier of Theoretical Physics

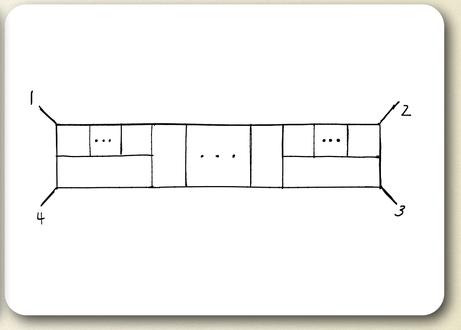
Collider Phenomenology



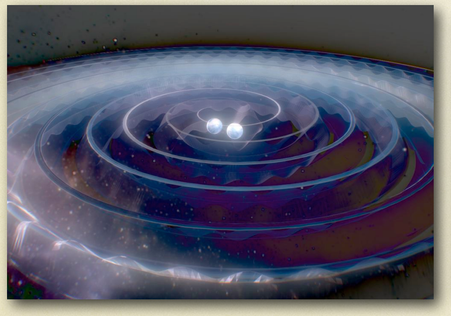


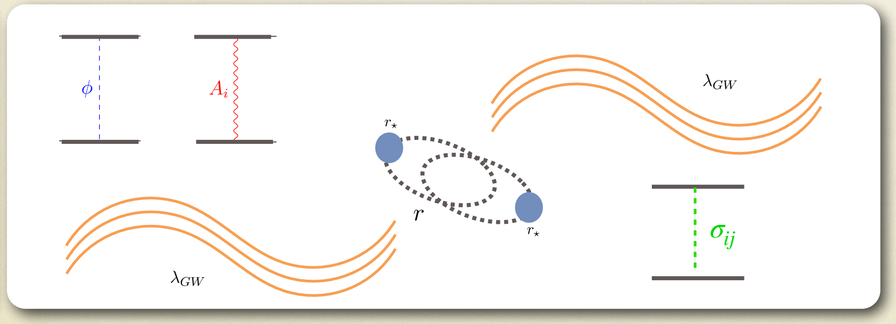
Geometry of Quantum Field Theory



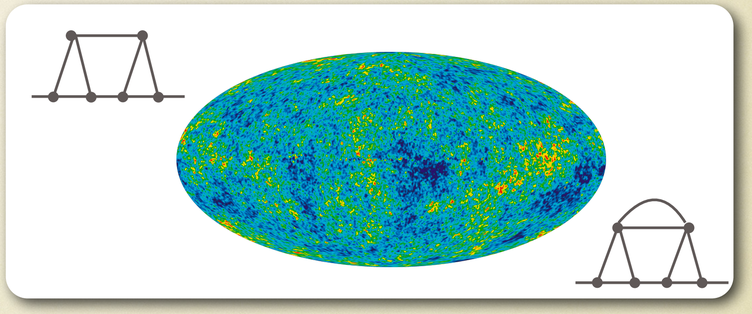


EFT Classical General Relativity



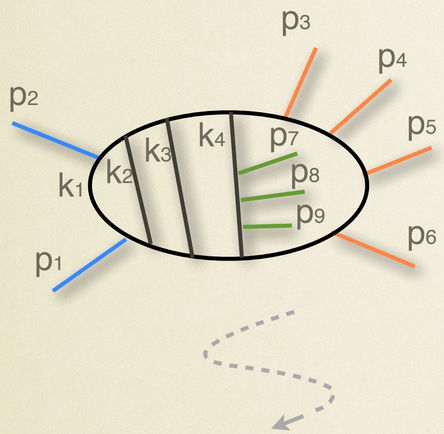


Cosmology



Feynman Integrals

Momentum-space Representation



$$= I^{[d]}_{a_1,\ldots,a_N} = \int \prod_{i=1}^{L} d^d k_i \left(\prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right)$$

N-denominator
generic Integral

L loops, $E{+}1$ external momenta,

$N = LE + \frac{1}{2}L(L{+}1)$ (generalised) denominators

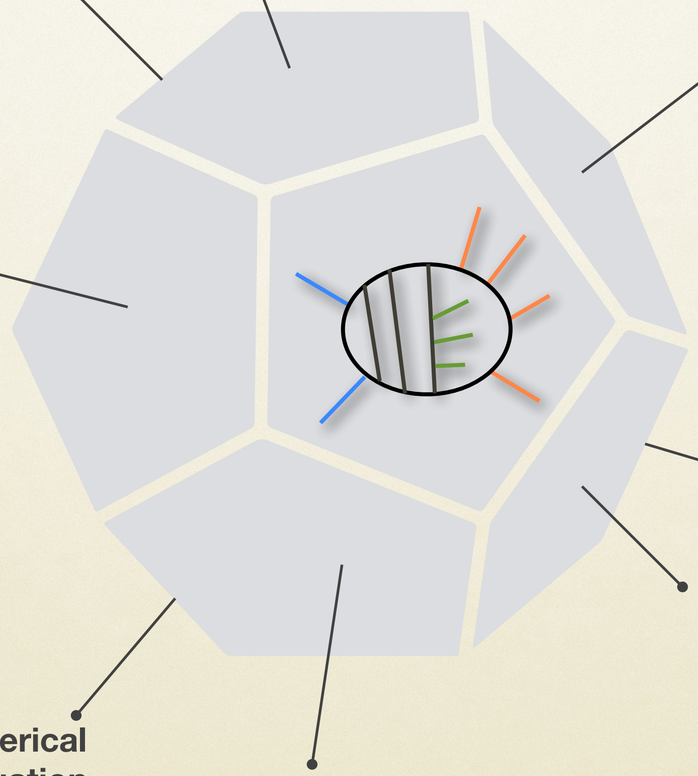
total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman

$$D_n = (p_1 \pm p_2 \pm \ldots \pm k_1 \pm k_2 \pm \ldots)^2 - m_n^2$$

[terminology: **sector** = $\{ D_1, D_2, \ldots, D_N \}$]

Feynman Integrals / (a few) Evaluation Methods



Integration-by-parts identities

Chetyrkin, Tkachov
Remiddi, Laporta; Laporta
Zhang, Larssen
……..

Direct Integral Evaluation

Passarino, Veltman, t'Hooft, Veltman
Beneke, Smirnov, Tausk
Binoth, Heinrich,
Anastasiou, Melnikov
Weinzierl
Czakon
……..

Unitarity-based and on-shell methods

Bern, Dixon, Dunbar, Kosower
Britto, Cachazo, Feng, Witten
Brandhuber, Spence, Travaglini
Britto, Buchbinder, Cachzao, Feng + P.M.
Glover, Badger & P.M.
Anastasiou, Britto, Kunszt & P.M.
Forde; Badger
Beger, Bern, Dixon, Forde, Kosower, Ita, Maitre
Ellis, Giele, Kunszt, Melnikov, Zanderighi
……..

Integrand Decomposition & Polynomial division

Ossola Papadopoulos Pittau
Ossola Papadopoulos Pittau & P.M.
Bevilacqua, Garzelli, Papadopoulos, Pittau, Worek
Ossola, Reiter, Tramontano, & P.M.
Mirabella, Ossola, Peraro & P.M.
Ossola & P.M.
Gluza, Kajda, Kosower
Badger, Frellsvig, Zhang
Zhang
Peraro, Primo & P.M.
Abreu, Febres Cordero, Ita, De Laurentis
……..

Difference Equations

Tarasov
Laporta
Lee
……..

Differential Equations

Barucchi, Ponzano, Regge
Bern, Dixon, Kosower
Kotikov, Remiddi,
Gehrmann Remiddi
Bonciani, Remiddi, & P.M.
Czakon
Henn,
Papadopoulos,
Liu, Ma, Peking
……..

Numerical evaluation

(py)-Secdec
Fiesta
AMFlow
Line
Tropical MC
Summertime
……..

What's next?

Feynman Integrals

- **Integration-by-parts Identities (IBPs)**

$$\int \prod_{i=1}^{L} d^d k_i \; \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0 \qquad v_\mu = v_\mu(p_i, k_j)$$

arbitrary

- IBP equations
- Contiguity relations

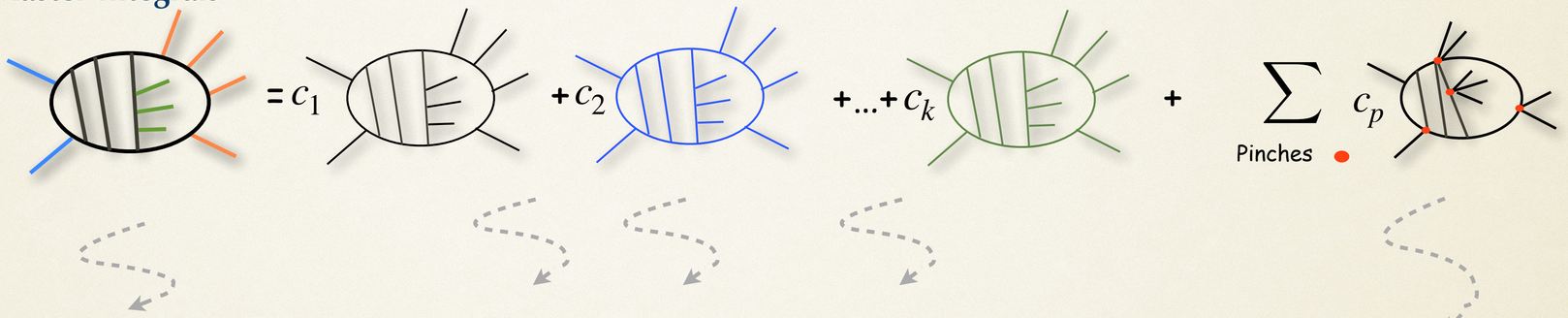
$$\sum_i b_i \, I^{[d]}_{a_1,\ldots,a_i \pm 1,\ldots,a_N} = 0$$

- **Generating an *overdimensioned (sparse) systems of linear equations***
- **Solutions:**

☑ Gauss' Elimination
☑ Groebner Bases
☑ Syzygy Equations
☑ **Finite Fields + Chinese Remainder Theorem + Rational Functions Reconstruction**

Linear relations for Feynman Integrals

- **Decomposition in terms of independent Master Integrals**

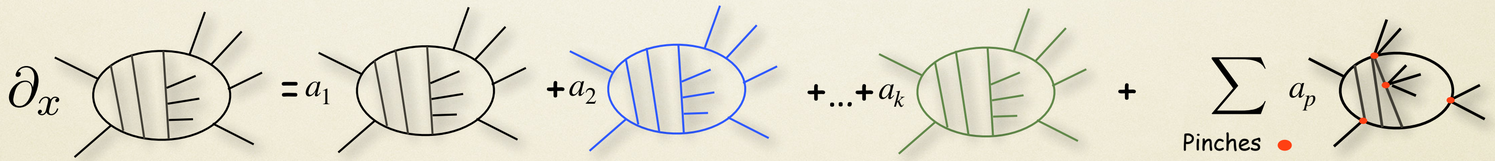


N-denominator
generic Integral

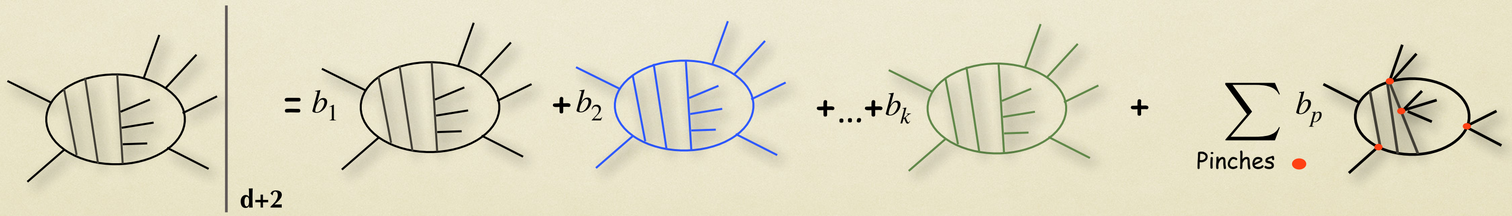
N-denominator
Master Integrals
[top sectors]

(n < N)-denominator
Master Integrals
[subsectors]

- **1st order Differential Equations for MIs**

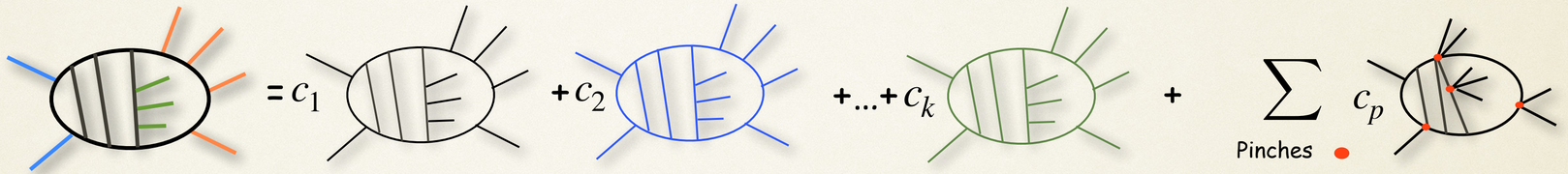


- **Dimension-Shift relations and Gram determinant relations**

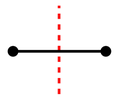


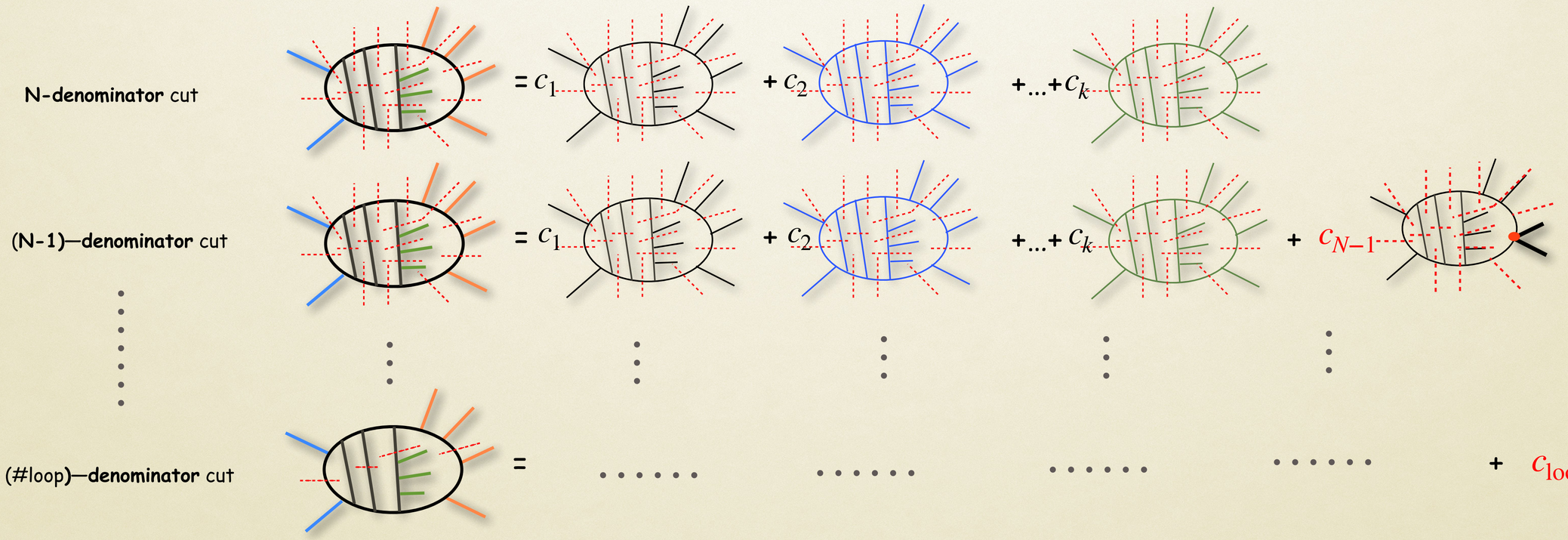
Linear relations for Feynman Integrals

- **Relations among Integrals in dim. reg.**



- **Unitarity-based and on-shell methods** $\dfrac{1}{D_i} \to \delta(D_i)$





- ☑ Novel **integrand generation**: product of tree-amplitudes/diagrams; **complex momenta** across the cut
- ☑ Novel **complex-integration** techniques: (see 1loop 4ple-cut, 3ple-cut, 2ple-cut, …)

Evaluating Master Integrals / Differential Equations and Theory Special Functions

Barucchi, Ponzano, Regge
Kotikov; Remiddi, Gerhmann;
Bern, Dixon, Kosower, Anastasiou, Melnikov, Weinzierl, Henn; Lee;
Bonciani, Remiddi **& P.M.**
Argeri, diVita, Mirabella, Schubert, Tancredi, Schlenck **& P.M.**; …

1st order Differential Equations for MIs

∂_x = + + … + + $\sum$ Pinches

System of 1st ODE

$$\partial_x \begin{pmatrix} \\ \\ \vdots \end{pmatrix} = A(d,x) \begin{pmatrix} \\ \\ \vdots \end{pmatrix}$$

kinematic variable (s,t,u, masses)

space-time dimensions

Solution

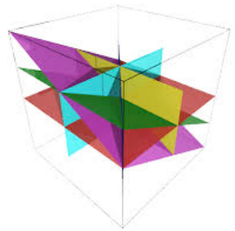
$$\begin{pmatrix} \\ \\ \vdots \end{pmatrix} = \mathrm{e}^{\Omega(d,x)} \begin{pmatrix} \\ \\ \vdots \end{pmatrix}$$

boundary term (simpler integral)

Dyson/Magnus Series: Iterated Integrals and Geometry

$$\mathrm{e}^{\Omega} = \; \uparrow \; - \frac{1}{2} \; + \frac{1}{4} \; + \frac{1}{12} \; + \cdots ,$$

Hyperplanes



Polylogs

Higher-genus



Elliptic int's



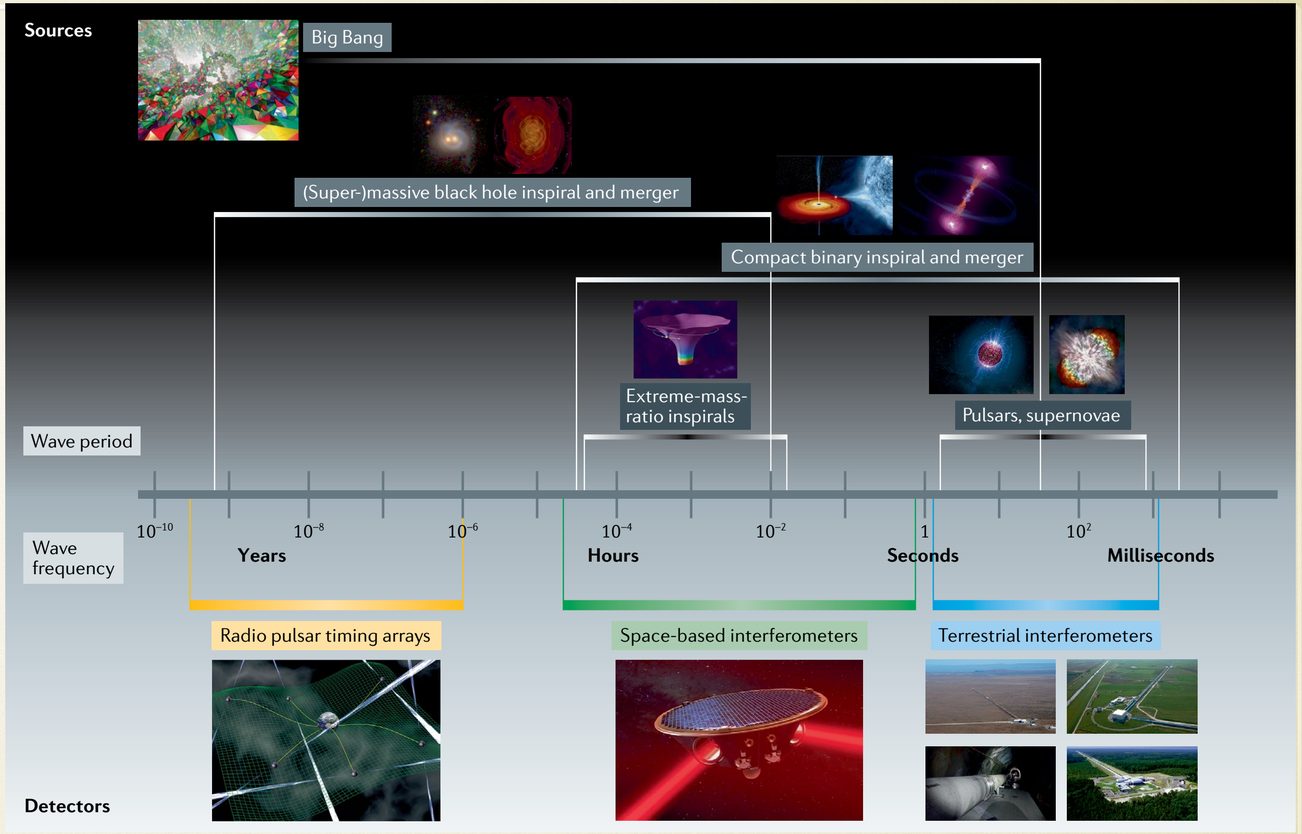


Calaby-Yau int's

GR EFT and Amplitudes/Diagrammatic approach

Motivation

- Gravitational Waves a new window on the Universe
- Two-body dynamics and radiative effects to exploring the most extreme conditions of spacetime and matter
- Next generation detectors, ground-based and in space, need of accurate waveform templates
- Precision Physics vs Precision Calculations: Multi-Loop Calculus, Scattering Amplitudes and General Relativity



[Bailes et al. 2021]

Based on collaborations with:

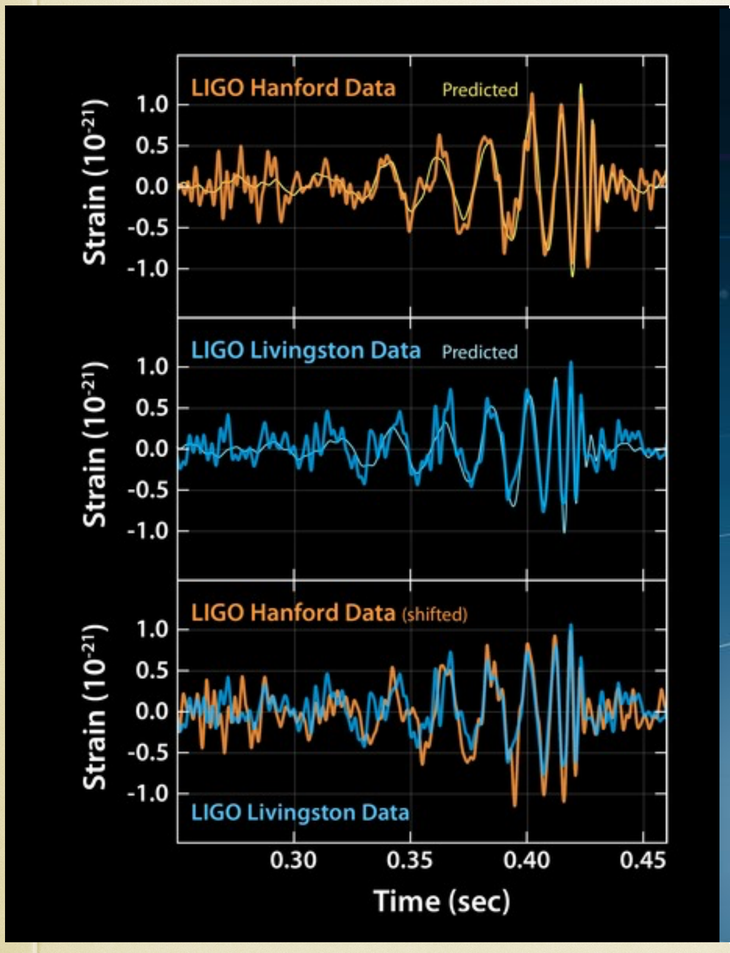
G. Brunello, **M. Mandal, R. Patil, M. Pegorin,**
J. Ronca, S. Smith, J. Steinhoff, **W. Torres Bobadila**

M. Mandal, R. Patil, H. Silva, J. Steinhoff,

D. Bini, T. Damour, A. **Geralico**, S. Laporta

S. Foffa, R. Sturani, C. Sturm, **W. Torres Bobadilla**

GW Detection



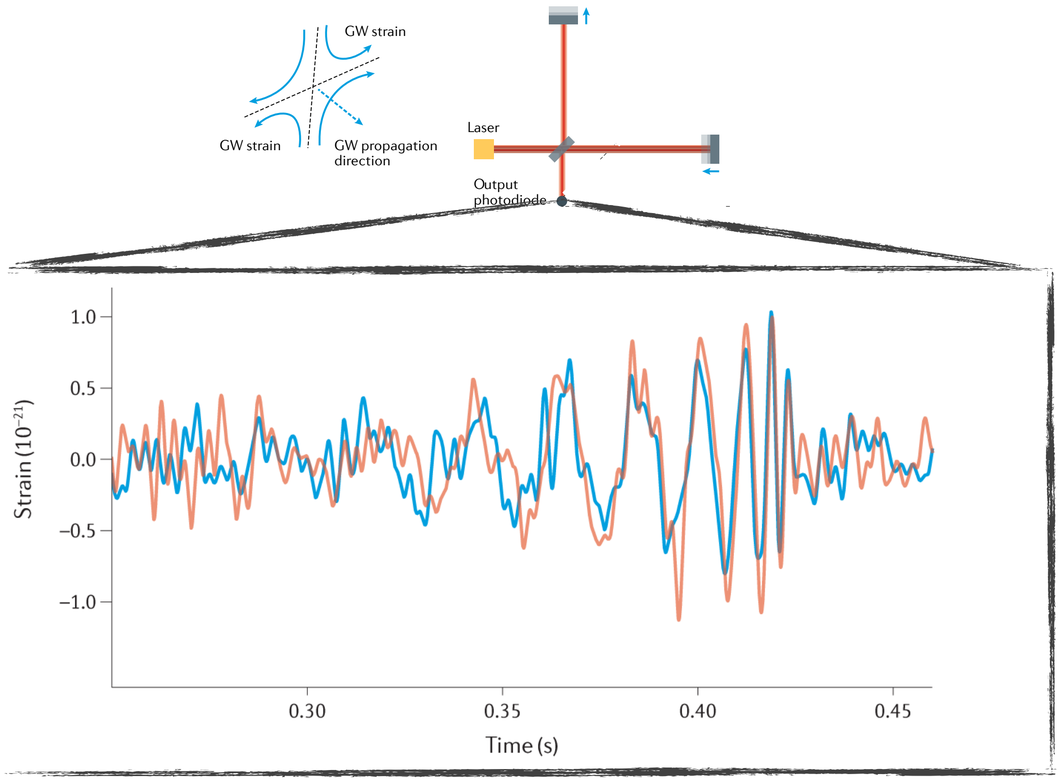


LIGO-Virgo Detection: **GW150914**

Cargese is here

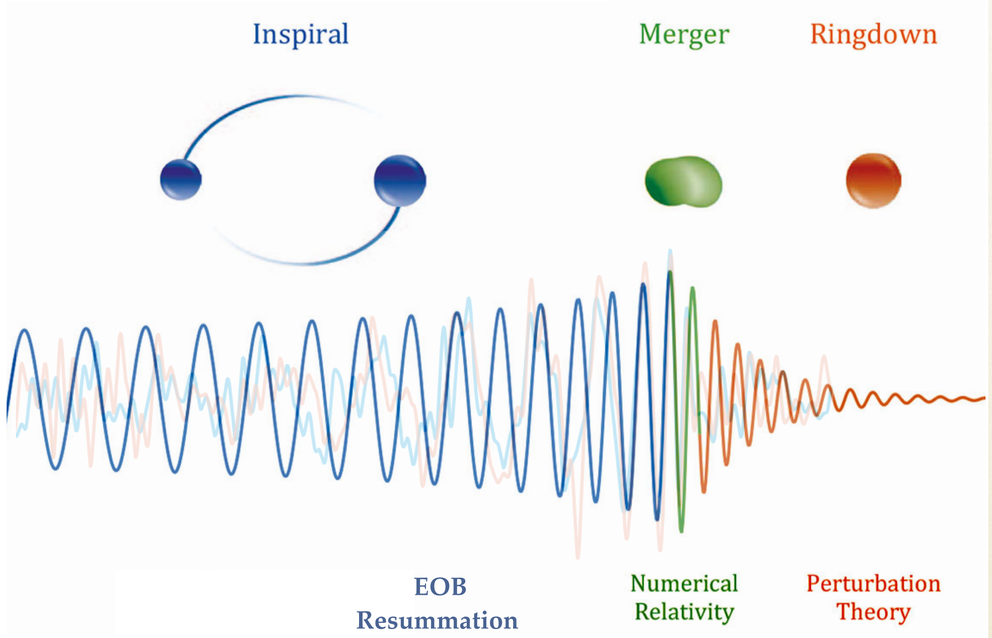
Two-body dynamics and GW signal

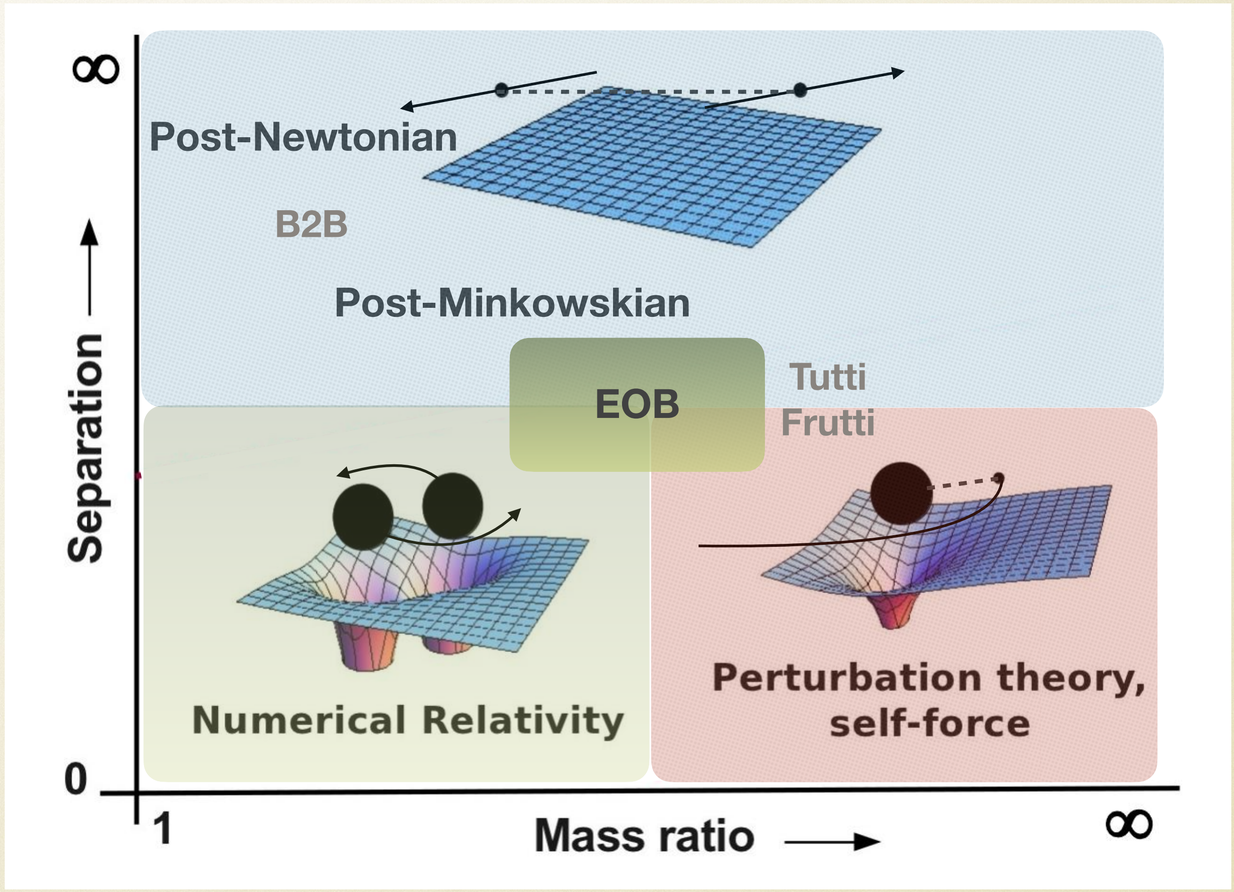
- Real Event



Two-body dynamics and GW signal

- Waveform Model and Computing Techniques

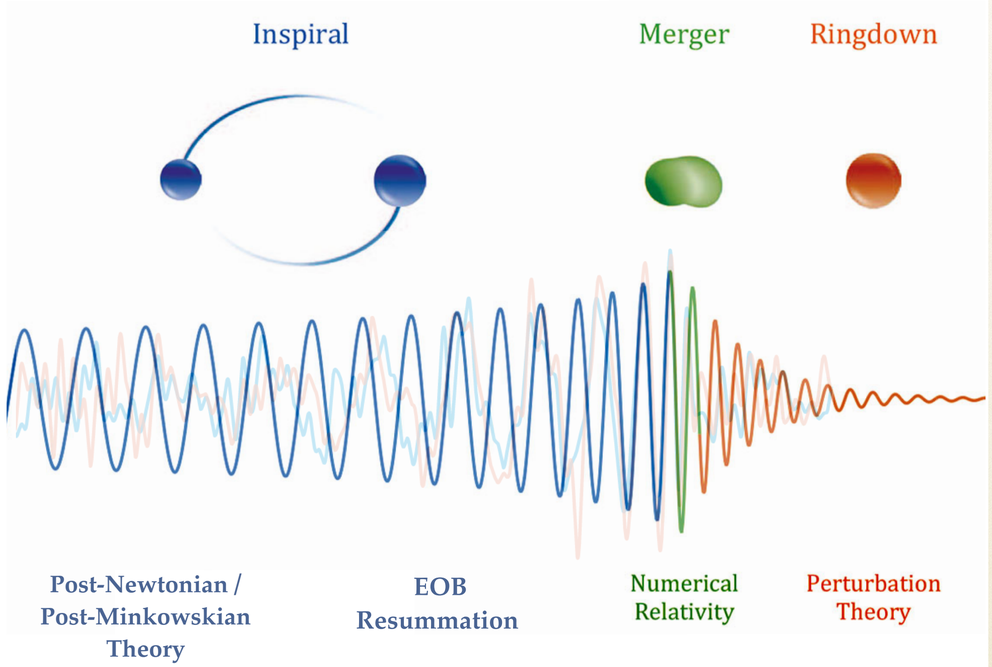




[adapted from: Barak]

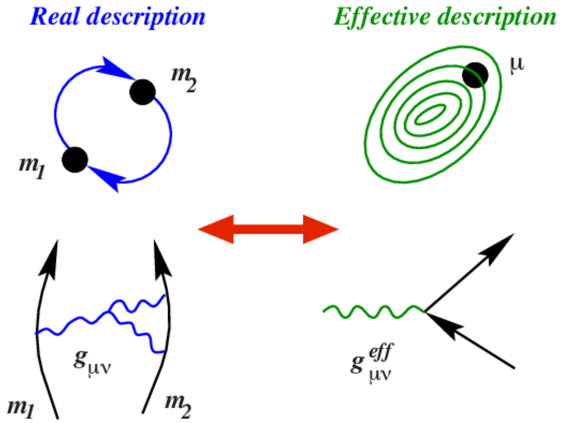
Two-body dynamics and GW signal

▸ **Waveform Model and Computing Techniques**



Effective One Body (EOB) Formalism [Buonanno Damour]

the contributions coming from different kinematic regions for combined and calibrated with Numerical Relativity



- Post-Minkowskian Expansion [relativistic scattering]

$$G_N \frac{m}{r} << v^2 \sim 1$$

Expansion in powers of G_N

- Post-Newtonian Expansion [non relativistic system]

$$G_N \frac{m}{r} \sim v^2 << 1$$

Expansion in powers of v/c

- BH perturbation theory / self force

Expansion for small metric deformation

$$\delta g_{\mu\nu} \sim \epsilon = m_2/m_1 << 1$$

GR Effective Field Theory (GREFT)

[Goldberger, Rothstein]

- **GREFT ACTION** $S_{tot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g]$

- **Einstein Hilbert + gauge fixing**

$$S_{GR}[g] = 2\Lambda^2 \int d^{d+1}x \sqrt{-g}\left(R - \frac{1}{2}\Gamma^\mu \Gamma_\mu\right)$$

$$\Lambda^{-1} = \sqrt{32\pi G_N}$$

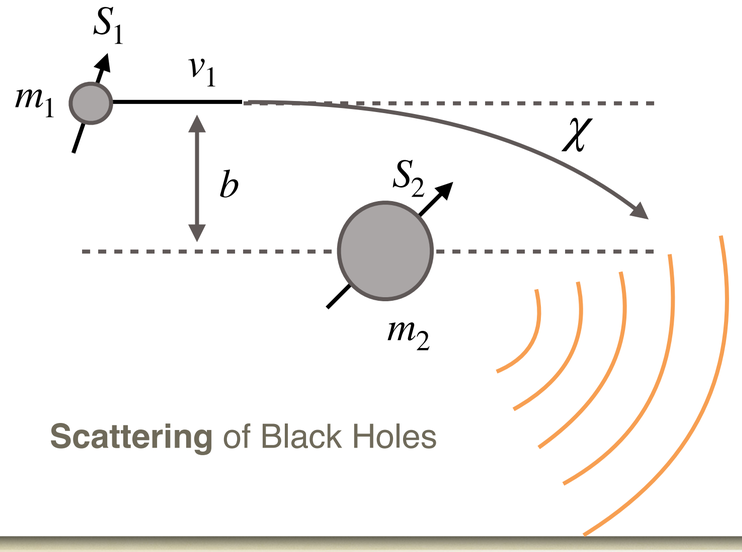
- **Source/Worldline** [internal structure: spin, tidal, …]

$$S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

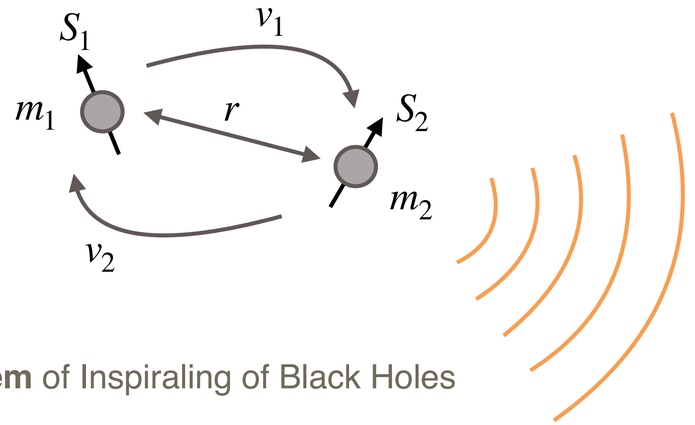
$$= -m_a \int d\tau_a = -m_a \int dt \sqrt{-g_{\mu\nu}(x_a)\dot{x}_a^\mu \dot{x}_a^\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}$$

GREFT@Two-body problem

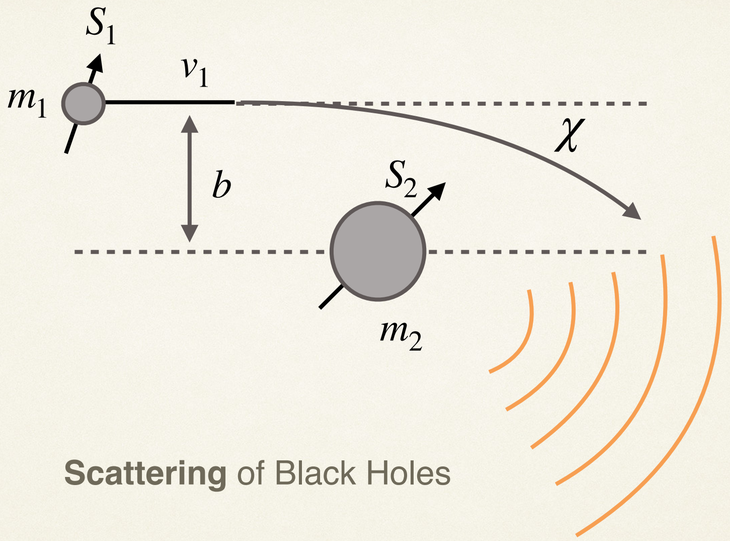


Scattering of Black Holes



Bound system of Inspiraling of Black Holes

GREFT / PM Corrections



Scattering of Black Holes

Corrections to the Newtonian potential:

- relativistic velocities:

$$G_N \frac{m}{r} << v^2 \sim 1$$

Expansion in powers of G_N

- Dynamics in **Post-Minkowskian** perturbative scheme

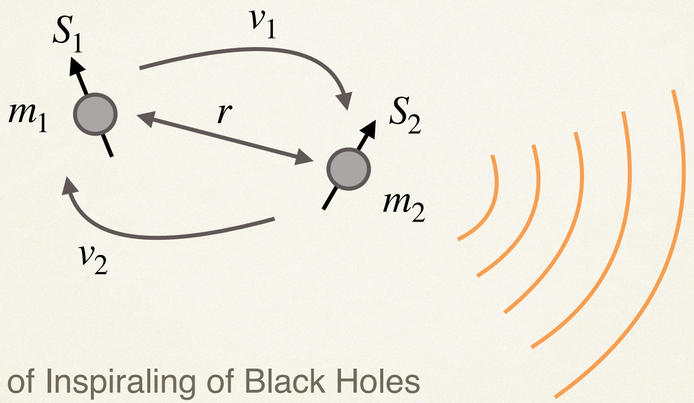
- At nPM order: G_N^n

Astrophysicists/Cosmologists' whishlist

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	
	G	$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots)$						1PM
1979-81 →	G^2	$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots)$						2PM
2019 →	G^3	$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots)$						3PM
2021 →	G^4	$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots)$						4PM
2025,26 →	G^5	$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots)$						5PM
	G^6	$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots)$						6PM
	G^7	$(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \dots)$						7PM
	G^8	$(1 + \dots)$						8PM

…Westphal, Damour, Cheung, Rothstein, Solon, Bern, Roiban, Shen, Zeng, Parra-Martinez, Ruf, Hermann, Buonanno, Porto, Dlapa, Kalin, Liu, Neef, Bjerrum-Bohr, Vanhove, Plante, Cristofoli, Damgaard, Guevara, Ochirov, Vines, Di Vecchia, Veneziano, Heisenberg, Russo, Plefka, Jakobsen, Mogull, Brandhuber, Travaglini, De Angelis, Accetulli-Huber, Luna, Kosmopoulos, Sid Smith, and collaborators…

GREFT / PN Corrections



Bound system of Inspiraling of Black Holes

Corrections to the Newtonian potential:

- Non-relativistic velocities:
- Virial theorem: $G_N \frac{m}{r} \sim v^2 << 1$

  Expansion in powers of v/c
- Dynamics in **Post-Newtonian** perturbative scheme
- At nPN order: $G_N^{n-\ell} v^{2\ell}$

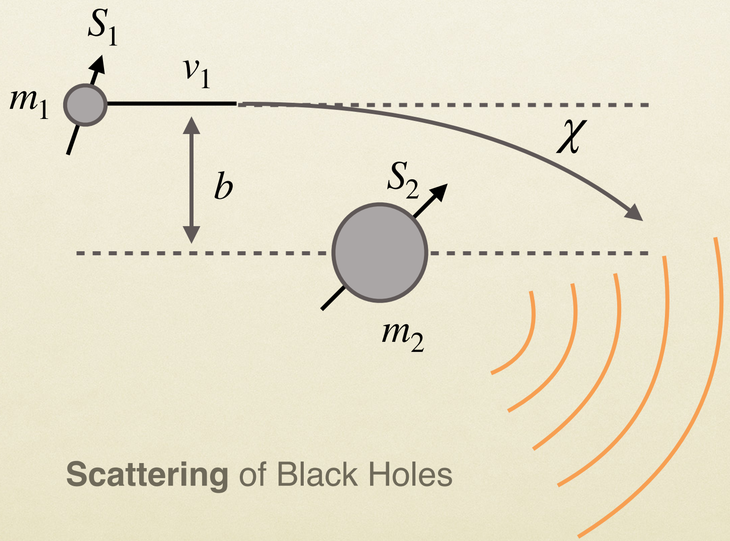
Astrophysicists/Cosmologists' whishlist

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN	
G	$(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10}$	$+ v^{12}$	$+ v^{14} + \cdots)$	1PM
G^2		$(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10}$	$+ v^{12} + \cdots)$	2PM
G^3			$(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8$	$+ v^{10} + \cdots)$	3PM
G^4				$(1$	$+ v^2$	$+ v^4$	$+ v^6$	$+ v^8 + \cdots)$	4PM
G^5					(1)	$+ v^2$	$+ v^4$	$+ v^6 + \cdots)$	5PM
G^6						(1)	$+ v^2$	$+ v^4 + \cdots)$	6PM
G^7							(1)	$+ v^2 + \cdots)$	7PM
G^8								$(1$ + …)	8PM

1687 1938 1980 2000 2014 2021 2025 2026?

…Jaranowski, Schaefer, Damour, Blanchet, Faye, Porto, Rothstein, Goldberger, Foffa, Sturani, Bini, Buonanno, Geralico, Sturm, Torres Bobadilla, Bluemlein, Maier, Marquard, Levi, Steinhoff, Vines, Antonelli, Kavanagh, Khalil, Galley, von Hippel, McLeod, Edison, Kim, Morales, Yin, Mandal, Patil, Teng, Brunello, Pegorin, Ronca, Smith, **P.M.** ….

GR EFT for PM corrections / Diagrammatic approach



Scattering of Black Holes

PM Corrections / Scattering Amplitudes based approach

- **GR EFT Feynman rules / on-shell / double-copy / spinor-formalism / recurrence relations**
- **Amplitudes based approach**

Cheung, Rothstein, Solon

Bern, Cheung, Hermannn, Parra Martinez, Roiban, Schen Solon, Smith, Zeng, …

(see Correia et al.'s review)

$$i\mathcal{M}_0 = \quad = 16\pi G \frac{(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)}{t} \equiv 16\pi G \frac{m_1^2 m_2^2 (2\sigma^2 - 1)}{t}\,, \qquad \sigma \equiv \frac{p_1\cdot p_2}{m_1 m_2}$$

- **Pushed to 5PM !!!**

$$i\mathcal{M} = \quad + \quad + \quad + \quad + \quad + \ldots + \quad + \quad + \ldots$$

$\mathcal{O}(G)$ $\mathcal{O}(G^3)$ $\mathcal{O}(G^3)$

- **Potential V from Lippman-Schwinger equation**

Correia, Isabella

$$T = V + VGV + VGVGV + \cdots = V\frac{1}{1 - GV}$$

$$= \quad + \quad$$

$$= \quad = 0$$

$$\mathrm{Im}\, G(\mathbf{p},\mathbf{k}) = \quad \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2)$$

$$G(\mathbf{p},\mathbf{k}) = \frac{1}{|\mathbf{k}|^2 - |\mathbf{p}|^2 - i\epsilon}$$

PM Corrections / Classical Observables based approach (KMOC)

Kosower, Maybee,o'Connel

Hermann, Parra-Martinez, Zeng

- **Asymptotic states**

$$|\psi\rangle_{in} = \int d\phi(p_1) d\phi(p_2)\ \phi_1(p_1)\phi_2(p_2) e^{i(b_1\cdot p_1 + b_2\cdot p_2)}\ |p_1,p_2\rangle_{in} \qquad |\psi\rangle_{out} = S\,|\psi\rangle_{in} \qquad S = 1 + i\,T$$

On-shell phase space integral — wavefunction — Two particle momentum eigenstates

- **Expectation value of a Physical Observable:**

$$\Delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_{out} - \langle\mathcal{O}\rangle_{in} = {}_{out}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{out} - {}_{in}\langle\psi|\,\mathcal{O}\,|\psi\rangle_{in} = {}_{in}\langle\psi|\,S^\dagger[\mathcal{O},S]\,|\psi\rangle_{in}$$

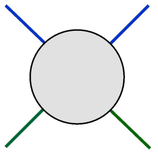
>> [in-in formalims]

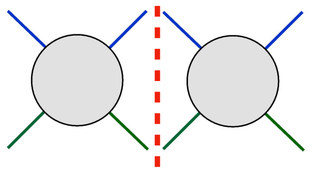
- **Impulse as *Fourier Transform* of Scattering Amplitudes**

$$d\mu = \prod_{i=1}^{2} \hat{d}^D q_i\, \delta(2p_i\cdot q_i + q_i^2)\, e^{ib_i\cdot q_i}$$

$$\Delta\langle p_1^\mu\rangle = \int d\mu\, \hat{\delta}^D(q_1+q_2)\Big(q^\mu \;-\; i\int d(LIPS)\; \mathscr{O}^\mu \Big)\,.$$

Fourier Transform — 4-point amplitude — Iteration terms



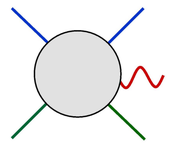


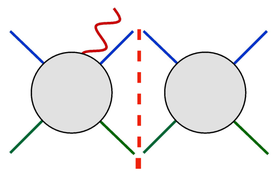
Caron Huot, Giroux, Hanesdottir, Mizera

- **Waveform as *Fourier Transform* of Scattering Amplitudes**

$$\Delta\langle\mathscr{W}_h\rangle(u,\vec{n}) = \frac{1}{4\pi r}\int_0^\infty \hat{d}\omega \int d\mu \Big\{ \hat{\delta}^D(q_1+q_2-k)e^{-i\,\omega u}\Big(- i\int d(LIPS)\Big) + c\,.\,c.\Big\}$$

Fourier Transform — 5-point amplitude — Iteration terms





Brunello, De Angelis

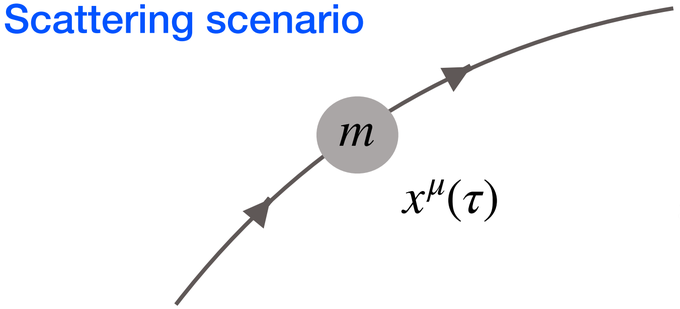
[credit: Brunello]

PM Corrections / QFT(EFT) Worldlines formalism

Driesse, Mogull, Plefka, Jakobsen, Sauer, Steinhoff, Usovitsch

Dlapa, Kallin, Liu, Porto

- **Scattering scenario**



$$x_i^\mu(\tau) = \underbrace{b_i^\mu + v_i^\mu\,\tau}_{\text{straight line: „in“ state}} + \underbrace{\sum_{n=1}^{\infty} G^n z_i^{(n)\,\mu}(\tau)}_{\text{deflections}}$$

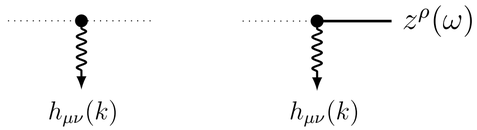
$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\underbrace{\sum_{n=0}^{\infty} G^n h_{\mu\nu}^{(n)}(x)}_{\text{emitted radiation}}$$

>> [in-in formalims]

- **Feynman rules**

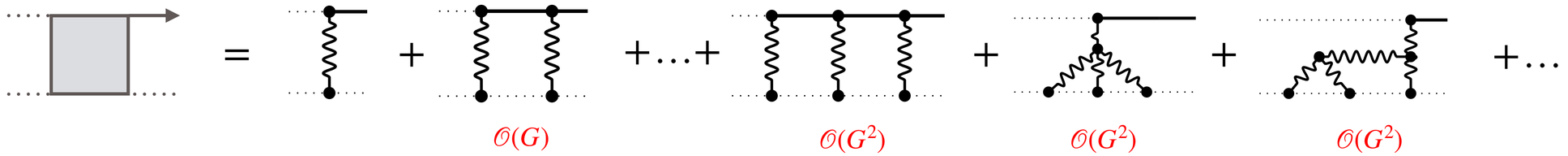


$$\langle z^\mu(\omega) z^\nu(-\omega)\rangle = -\frac{i}{m}\frac{\eta_{\mu\nu}}{(\omega+i0)^2}$$



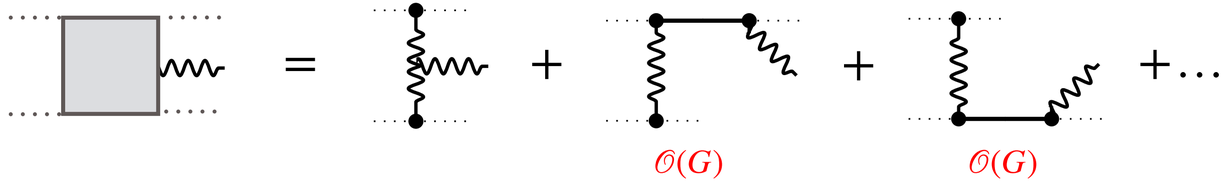
- **Impulse**

$$\langle \Delta p_1^\mu \rangle =$$



- **Waveform**

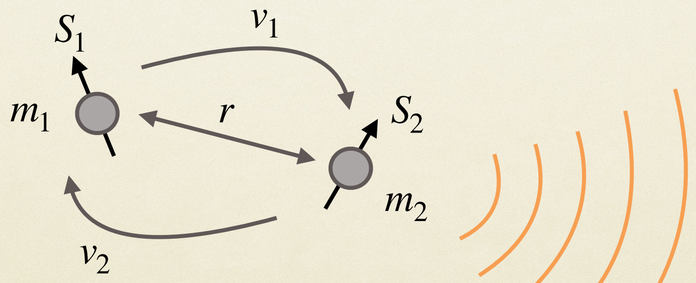
$$\langle h_{\mu\nu} \rangle =$$



- **Pushed to 5PM !!!**

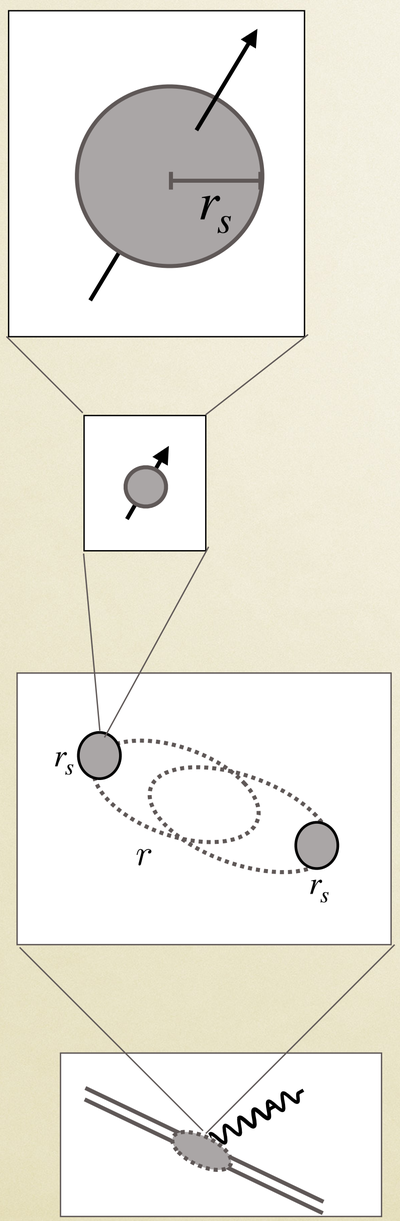
Driesse, Mogull, Plefka, Jakobsen, Sauer, Steinhoff, Usovitsch

GR EFT for PN corrections / Diagrammatic approach



Bound system of Inspiraling of Black Holes

Scales Hierarchies



$\sim r_s$ Black hole / Neutron star

$\gg r_s$ Point particle + internal structure

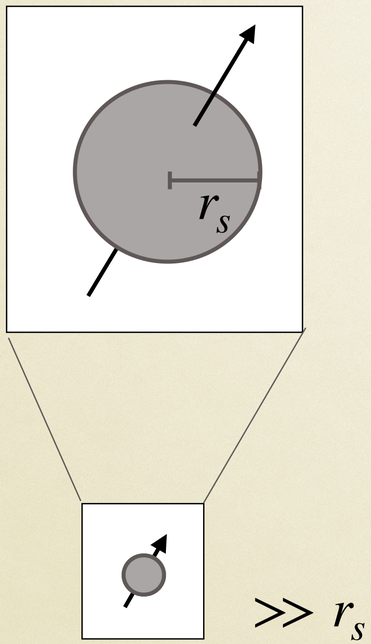
$\sim r$ Binary + Near zone + Far zone

$\gg r$ GW emission + Multipoles

[credit: Patil]

Point particle modelling

compact object as a **point particle** with internal structure parametrised **Wilson coefficients**



Mass: $\int dt\ m\sqrt{g^L_{\mu\nu}u^\mu u^\nu}$

Spin: $\int dt \left\{ -\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} - \frac{1}{2mc}\left(C^{(0)}_{\mathrm{ES}^2}\right)\frac{E_{\mu\nu}}{u}\left[S^\mu S^\nu\right]_{\mathrm{STF}} + \cdots \right\}$

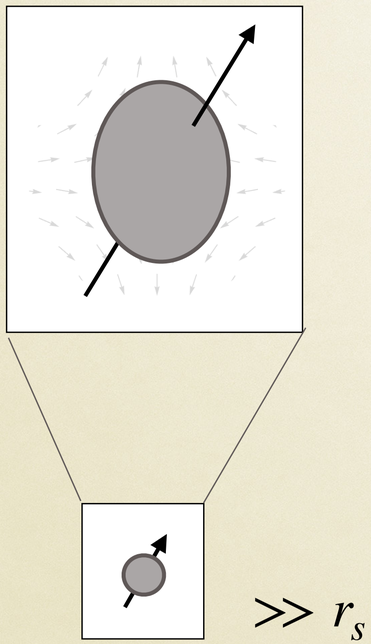
Spin induced quadrupole

[Porto, Levi, Steinhoff, …]

[credit: Patil]

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Spin induced quadrupole

[Porto, Levi, Steinhoff, …]

Adiabatic Tides: $\int dt \left\{ \lambda\frac{z}{4}E_{\mu\nu}E^{\mu\nu} + \lambda\kappa\frac{G_N^2 m^2}{c^6}\frac{z}{2}\frac{dE_{\mu\nu}}{dt}\frac{dE^{\mu\nu}}{dt} + \cdots \right\}$

Post-adiabatic Love number

Static Love number

[Flanagan, Hinderer, Damour, …]

Dynamic Tides: $\int dt \left\{ \frac{z}{4\lambda\omega_f^2}\left[\frac{c^2}{z^2}\dot{Q}_{\mu\nu}\dot{Q}^{\mu\nu} - \omega_f^2 Q_{\mu\nu}Q^{\mu\nu}\right] - \frac{z}{2}Q^{\mu\nu}E_{\mu\nu} + \cdots \right\}$

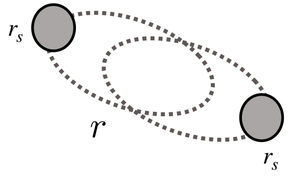
[Hinderer, Steinhoff, …]

Fundamental mode frequency

[credit: Patil]

Coalescing Binary System

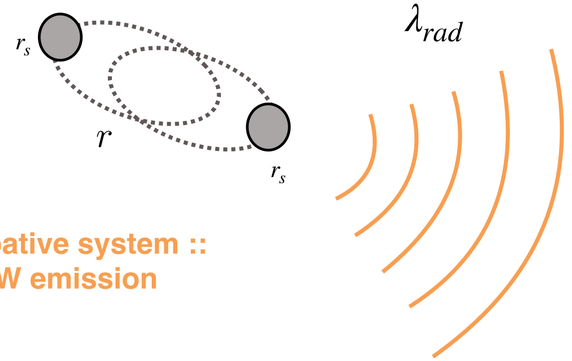
- Double Hierarchy



Conservative system ::
~~GW emission~~

$$r_s \ll r \ll \lambda_{rad}$$

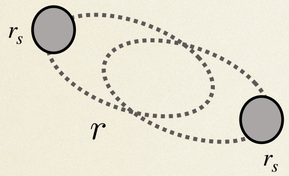
$$r_s \ll r \ll \lambda_{rad}$$



Dissipative system ::
GW emission

Coalescing Binary System

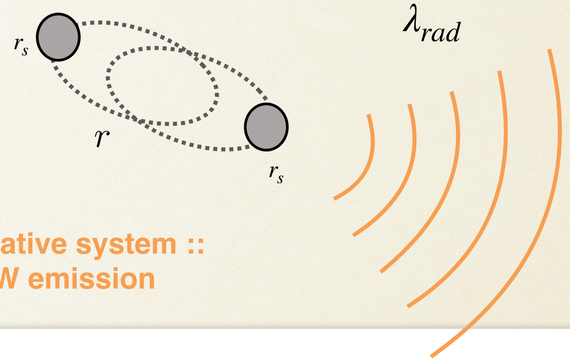
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Conservative system ::
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Dissipative system ::
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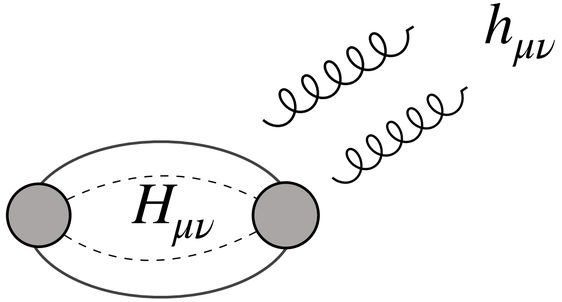
- Non-relativistic approximation [method of regions]: [Beneke Smirnov]
  - Weak field expansion:

$$v \ll 1 \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} = H_{\mu\nu} + \bar{h}_{\mu\nu}$$

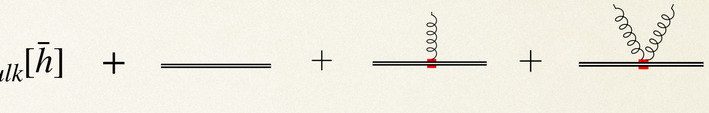
- Potential gravitons $H_{\mu\nu}$: $(k_0, \mathbf{k}) \sim \left(\frac{v}{r}, \frac{1}{r}\right)$ - - - - - -
- Radiation gravitons $\bar{h}_{\mu\nu}$: $(k_0, \mathbf{k}) \sim \left(\frac{v}{r}, \frac{v}{r}\right)$ ~~~~~
- Worldline/BH x_a : ———

- Effective action by integrating out gravitons:

$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH \; e^{iS_{tot}[x_a, H, \bar{h}]}$$



GREFT Action / Near & Far zones

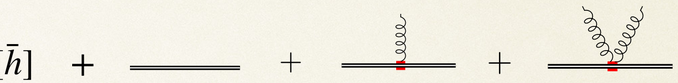
$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH\, e^{iS_{tot}[x_a,H,\bar{h}]} = \int D\bar{h}\, e^{\left\{ iS_{bulk}[\bar{h}] + \right.}$$  $$\left. + \ldots \right\}$$

- **Near zone (r)**

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$$



GREFT Action / Near & Far zones

$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH\, e^{iS_{tot}[x_a,H,\bar{h}]} = \int D\bar{h}\, e^{\left\{ iS_{bulk}[\bar{h}] \right.} +$$  $$\left. + \ldots \right\}$$

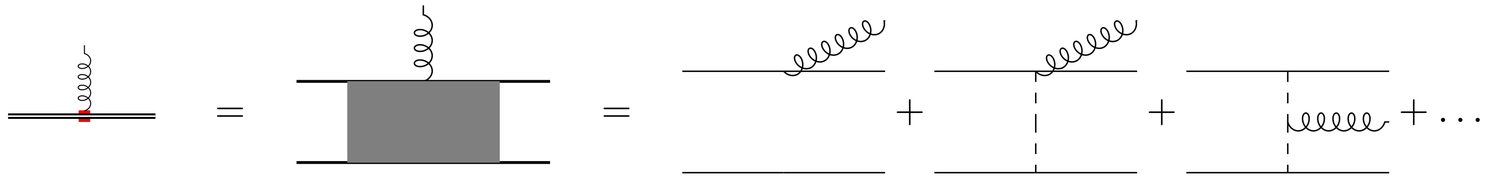
▸ **Near zone (r)**

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$$



▸ **Far zone (λ_{rad})**

$$S_{rad}[g, \{Q_i\}] = S_{GR}[g] + S_{mult}[g, \{Q_i\}] \qquad g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$$



PN-GREFT Diagrammar

[Goldberger, Rothstein]
[Gilmore, Ross]
[Foffa, Sturani]

▸ **Action**

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

▸ **Kaluza-Klein parametrization:**

[Kol Smolkin] [Blanchet Damour]

$$\gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} \qquad c_d = 2\frac{d-1}{d-2}$$

$$g_{\mu\nu} = e^{2\phi/\Lambda}\begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\frac{\phi}{\Lambda}}\gamma_{ij} - A_iA_j/\Lambda^2 \end{pmatrix}$$

Graviton = Scalar + Vector + Sym. Tensor

10 = 1 + 3 + 6

$$g_{\mu\nu} \Rightarrow \phi \quad A^i \quad \sigma^{ij}$$

▸ **Feynman rules for:** $x_a \quad \phi \quad A^i \quad \sigma^{ij}$

x_a ——————

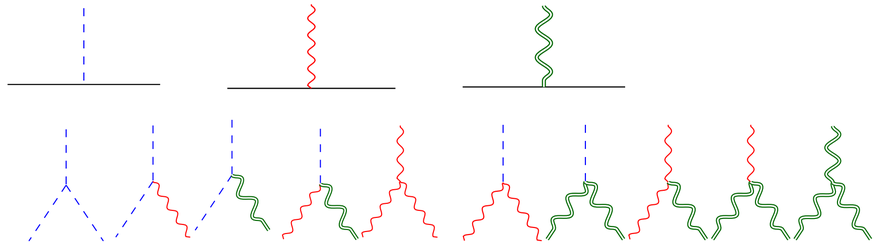
Static / non-propagating source

ϕ - - - - - - - - - - -

A^i ~~~~~~~~~~~~~~

σ^{ij} ~~~~~~~~~~~~~~

Propagating fields



PN-GREFT Diagrammar

[Goldberger, Rothstein]
[Gilmore, Ross]
[Foffa, Sturani]

▸ **Action**

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

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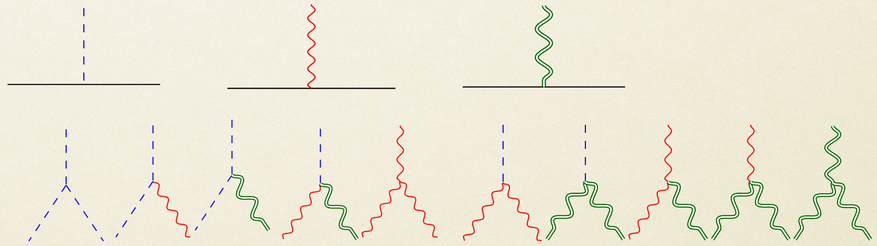
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x_a ——————

Static / non-propagating source

ϕ - - - - - - - - - -
A^i ~~~~~~~~~~
σ^{ij} ~~~~~~~~~~

Propagating fields



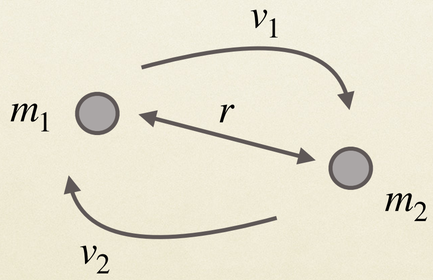
▸ **Newton Potential**

$$\mathscr{M}_{0PN} = \quad = \frac{im_1m_2}{2c_d\Lambda^2}\frac{1}{\mathbf{p}^2}$$

▸ **Fourier transform: from amplitude to the effective action:**

$$\mathscr{L}_{0PN} = -i\lim_{d\to 3}\int \frac{d^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}(x_1-x_2)} \left(\right) = \frac{G_N m_1 m_2}{r}$$

GR EFT for PN corrections / near zone spinless



Bound system of Inspiraling of Black Holes

PN-Corrections / GREFT Potential

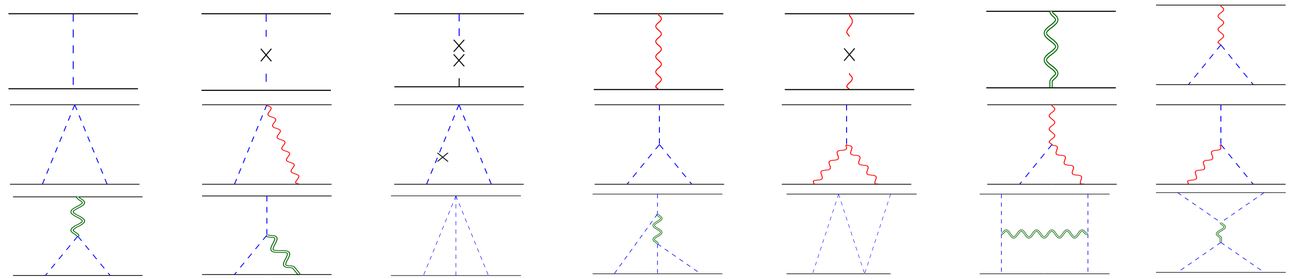
- 1PN corrections:

  Einstein, Infeld, Hoffman (1938)



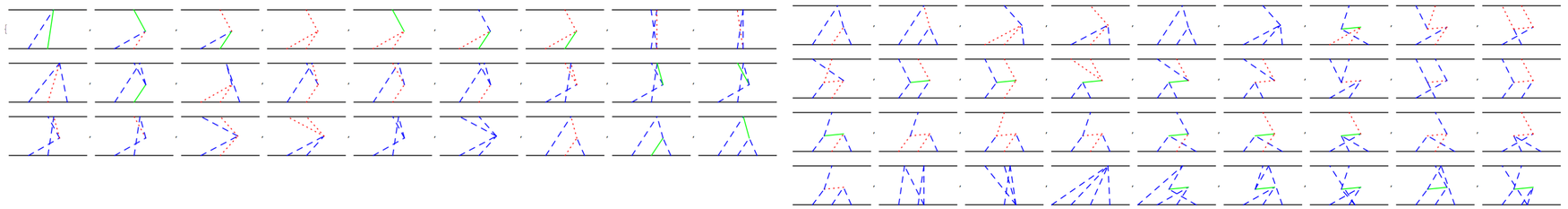
- 2PN corrections:

  Ohta-Okamura-Kimura-Hiida (1974)
  Gilmore, Ross (2008)



- 3PN corrections:

  Jaranowski,Schaefer (1997); Damour, Jaranowski, Schaefer (1997); Blachę, Faye (2000); Damour, Jaranowski Schaefer (2001); Foffa Sturani (2011)



- 4PN: corrections:

  Damour, Jaranowski, Schaefer (2014);
  Bernard, Blanchet, Bohe, Faye, Marsa (2016);
  Foffa, Sturani, Sturm & P.M. (2016);
  Foffa, Porto, Rothstein, Sturani (2019)
  Blumlein, Maier, Marquard,Schaefer (2020)

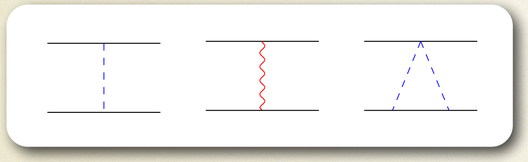
- 5PN: corrections:

  Bini, Damour, Geralico (2019);
  Foffa, Sturani, Sturm, Torres Bobadilla & P.M. (2019);
  Blumlein, Maier, Marquard, Schaefer (2020,2021)

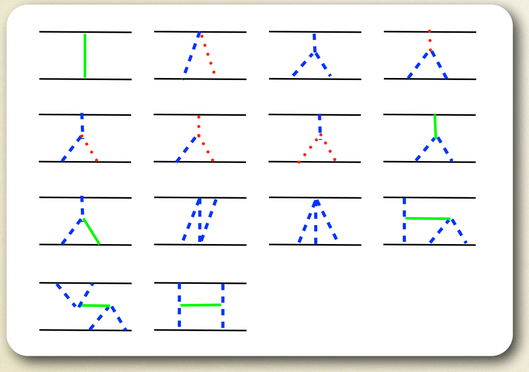
A closer look at the 4PN-Corrections

- Loop nr. $0 \leq \ell \leq n-1$

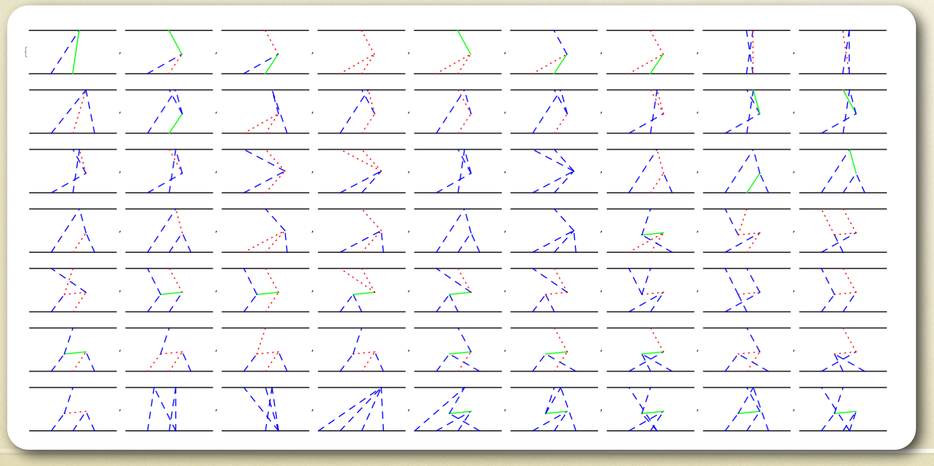
1PN



2PN

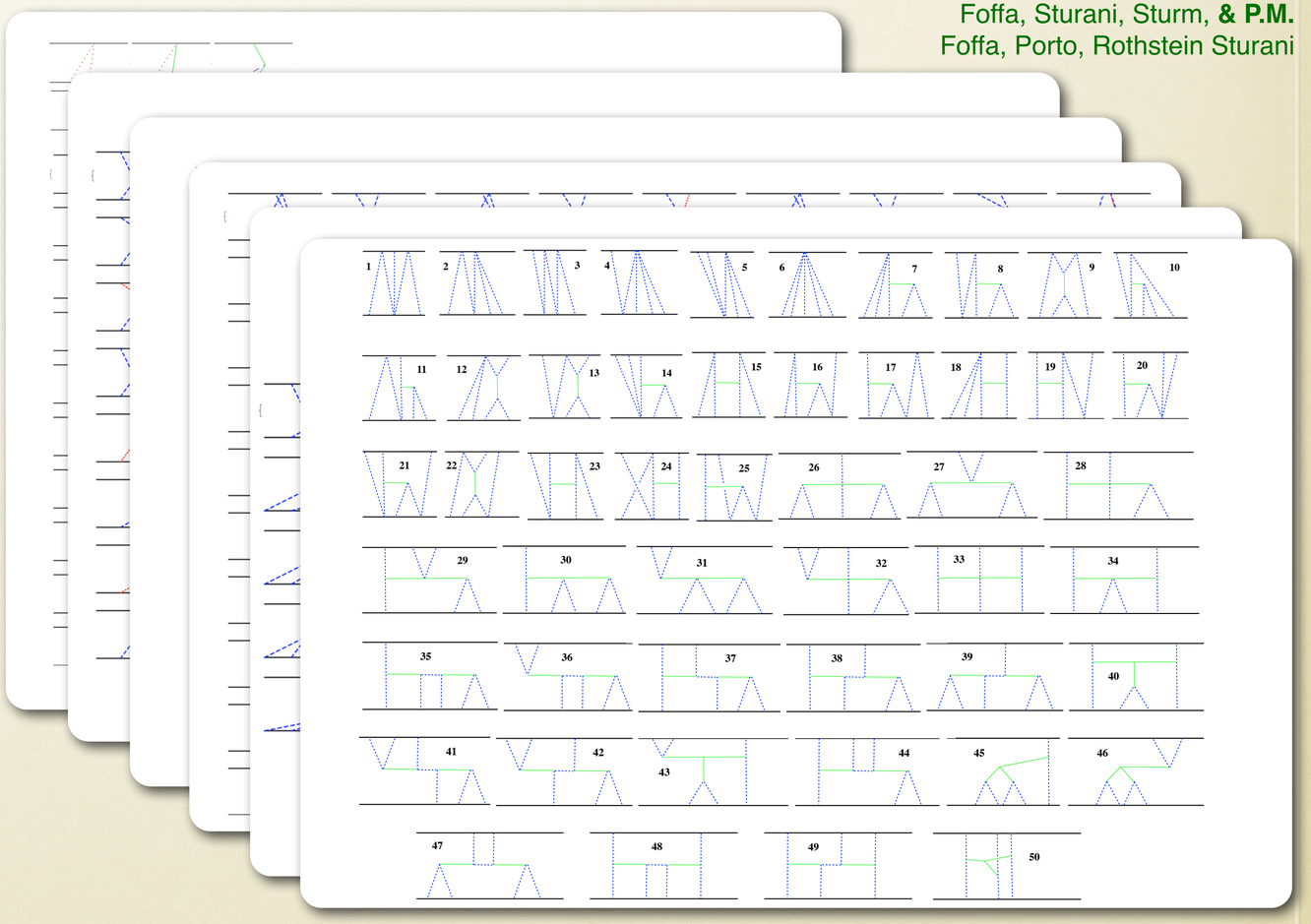


3PN



4PN : 605 GREFT diagrams (up-to 4-loops)

Foffa & Sturani
Foffa, Sturani, Sturm, **& P.M.**
Foffa, Porto, Rothstein Sturani



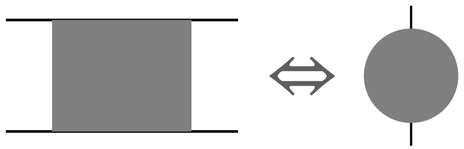
GREFT Diagrams vs 2-point QFT Integrals / a key observation

Kol, Shir (2013)

Foffa, Sturani, Sturm, & **P.M.** (2016)

Computational techniques:

- From Effective diagrams to QFT Amplitudes:
- World-lines are not propagating
- EFTGravity Amplitudes of order G_N^{ℓ} mapped into $(\ell - 1)$—loop 2-point functions with massless internal lines:
- Amplitudes evaluation with QFT multi-loop techniques



$$\mathscr{M} = \sum_i c_i \, I_i^{MI}$$

- **Dimensional Regularization** $d = 3 + \epsilon$
- **Integration-by-parts (IBP) decomposition**
- **Master Integrals evaluation**

- From QFT Amplitudes to Effective Lagrangians:

$$\mathscr{L}_{eff}[x_a] = -\, i \lim_{d \to 3} \int \frac{d^d \mathbf{p}}{(2\pi)^d} e^{i \mathbf{p}\cdot\mathbf{r}} \left(\right.$$



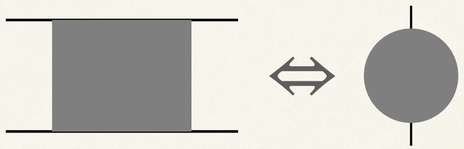
$$\left. \right)$$

GREFT Diagrams vs 2-point QFT Integrals / a key observation

Foffa, Sturani, Sturm, & **P.M.** (2016)

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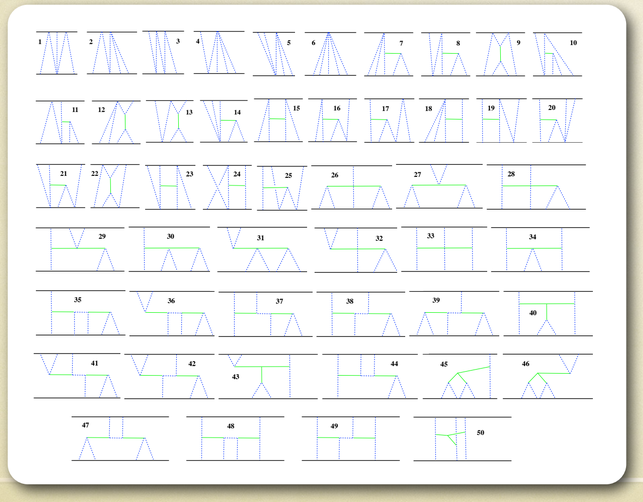
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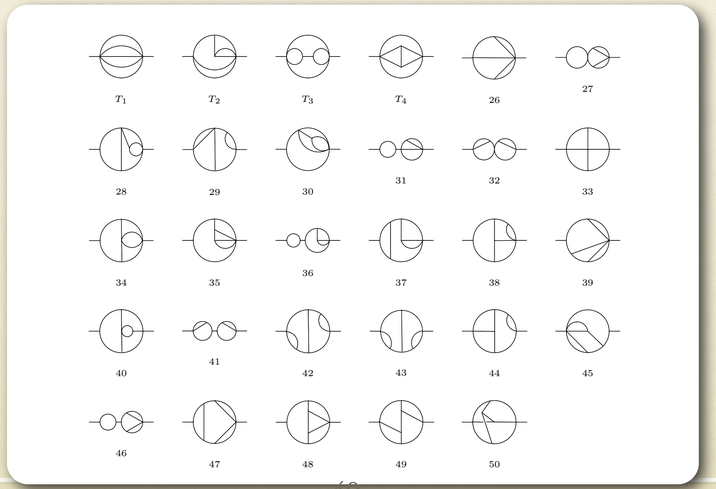
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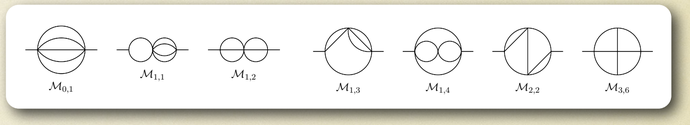
4PN static O(G^5): 50 4-loop GREFT diagrams



29 4-loop QFT diagrams



7 4-loop Master Integrals

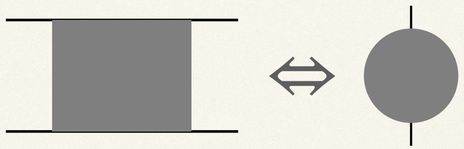


GREFT Diagrams vs 2-point QFT Integrals / a key observation

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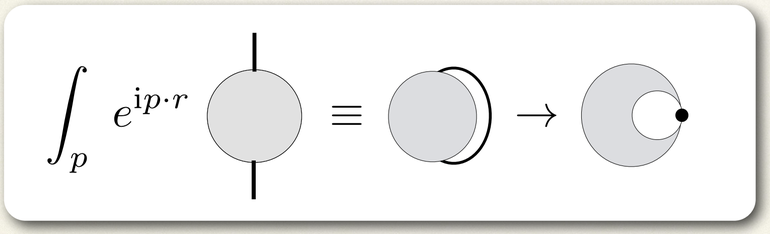
4PN static O(G^5): 50 4-loop GREFT diagrams

$$V_{\text{static}}^{(4PN)} = \frac{3}{8}\frac{G_N^5 m_1^5 m_2}{r^5} + \frac{31}{3}\frac{G_N^5 m_1^4 m_2^2}{r^5} + \frac{141}{8}\frac{G_N^5 m_1^3 m_2^3}{r^5} + (m_1 \leftrightarrow m_2)$$

41

GREFT Diagrams vs 2-point QFT Integrals / Factorization Theorem

Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (2019)

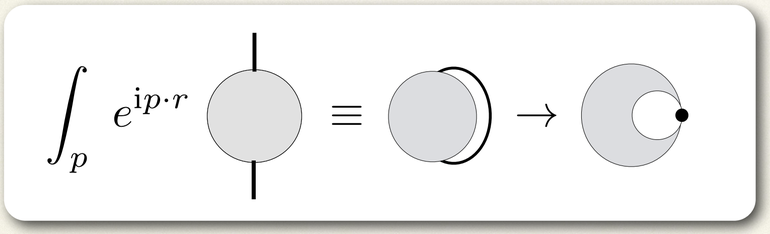


Newton Potential (reloaded):

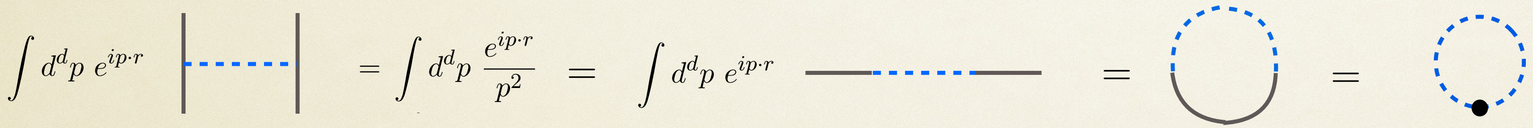
$$\int d^d p\; e^{ip\cdot r}$$  $$= \int d^d p\; \frac{e^{ip\cdot r}}{p^2} = \int d^d p\; e^{ip\cdot r}$$ $$=$$  $$=$$ 

GREFT Diagrams vs 2-point QFT Integrals / Factorization Theorem

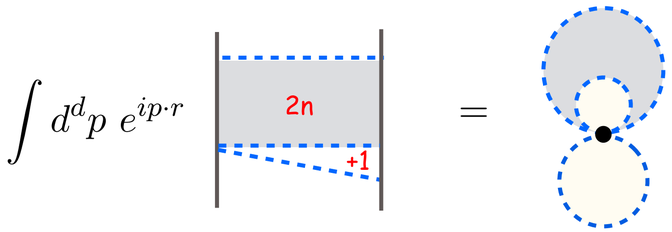
Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (2019)



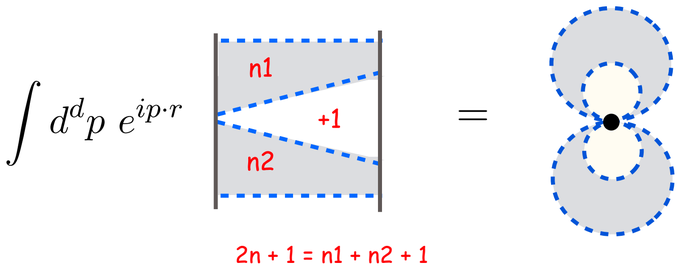
Newton Potential (reloaded):



(2n+1)-PN corrections: *Type-A*



(2n+1)-PN corrections: *Type-B*



- static (2n+1)-PN Potential as product of lower-PN Potential terms

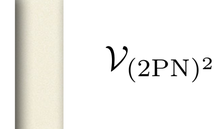
GREFT Diagrams vs 2-point QFT Integrals / Factorization Theorem

Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (2019)

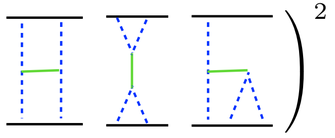
5PN static O(G^6): 154 5-loop GREFT diagrams

$$\mathcal{V}^{(5\mathrm{PN})}_{\mathrm{static}} = \mathcal{V}_{N^6} + \mathcal{V}_{N^3\times 2\mathrm{PN}} + \mathcal{V}_{N\times 4\mathrm{PN}} + \mathcal{V}_{(2\mathrm{PN})^2}$$

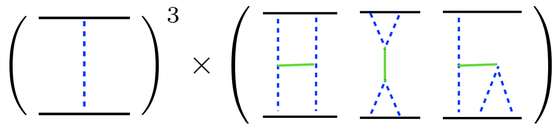
$\mathcal{V}_{N^6}$ =



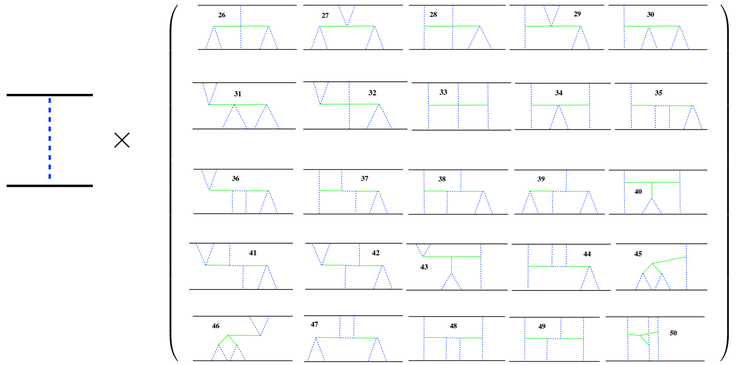
$\mathcal{V}_{(2\mathrm{PN})^2}$ =



$\mathcal{V}_{N^3\times 2\mathrm{PN}}$ =

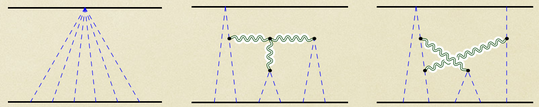


$\mathcal{V}_{N\times 4\mathrm{PN}}$ =



$$\mathcal{V}^{(5\mathrm{PN})}_{\mathrm{static}} = \frac{5}{16}\frac{G_N^6 m_1^6 m_2}{r^6} + \frac{91}{6}\frac{G_N^6 m_1^5 m_2^2}{r^6} + \frac{653}{6}\frac{G_N^6 m_1^4 m_2^3}{r^6} + (m_1 \leftrightarrow m_2).$$

- **Factorization Th'm: NO 5-loop Feynman diagram explicitly computed**
  - **Results confirmed and completed by explicit evaluation of 27582 5-loop Feynman Integrals**



Blümelein, Maier, Marquard, Schäfer (2019-21)

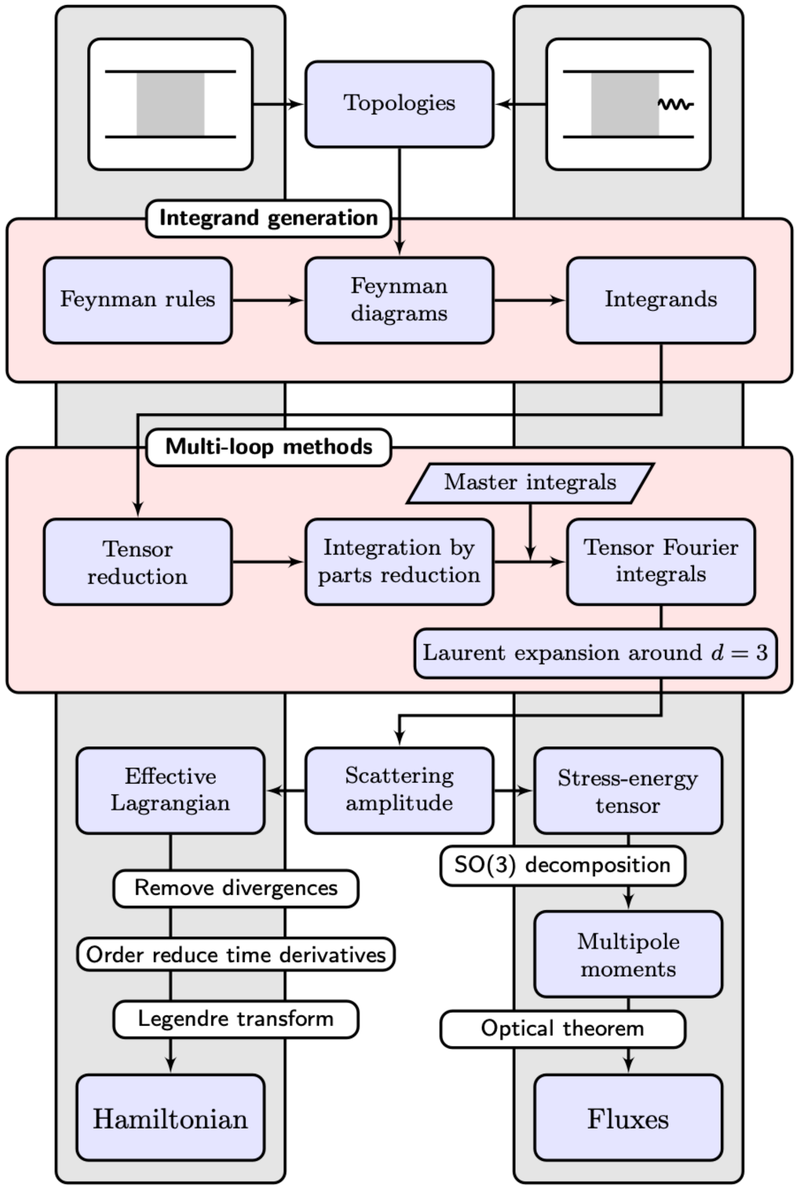
PN-GREFT Diagrammar & Multi-Loop Automation (PNTHR)

Mandal, **Patil**, Steinhoff, & PM

1. Diagrams Generation

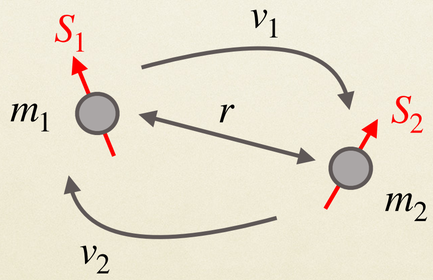
2. Integrals Evaluation

3. Applications



[credit: Patil]

GR EFT for PN corrections / near zone spinning



Bound system of Inspiraling of Black Holes

Near Zone with Spin / PN Corrections

Porto (2013)
Levi, Steinhoff (2015)
.....
Kim, Levi, Yin (2022)
Mandal, Patil, Steinhoff & P.M. (2022)
Levi, Morales, Yin (2022)
Levi, Yin (2022)

Action for Spinning compact object

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

$$S_{m_a}[x_a, g] = \sum_{a=1,2} \int d\tau \left(-m_{(a)} c \sqrt{u^2_{(a)}} - \frac{1}{2} S_{(a)\mu\nu} \Omega^{\mu\nu}_{(a)} - \frac{S_{(a)\mu\nu} u^\nu_{(a)}}{u^2_{(a)}} \frac{du^\mu_{(a)}}{d\tau} + \mathcal{L}^{(R)}_{(a)} + \mathcal{L}^{(R^2)}_{(a)} + \dots \right) \qquad u^\mu_{(a)} \equiv \dot{x}^\mu_a$$

Wilson coefficients that describe the internal structure

$$\mathcal{L}^{(R)}_{(a)} = -\frac{1}{2m_{(a)}c} \left(C^{(0)}_{\mathrm{ES}^2} \right)_{(a)} \frac{E_{\mu\nu}}{u_{(a)}} \left[S^\mu_{(a)} S^\nu_{(a)} \right]_{\mathrm{STF}} + \dots .$$

$$\mathcal{L}^{(R^2,S^0)}_{(a)} = \frac{1}{2} \left(C^{(2)}_{\mathrm{E}^2} \right)_{(a)} \frac{G_N^2 m_{(a)}}{c^5} \frac{E_{\mu\nu} E^{\mu\nu}}{u^3_{(a)}} S^2_{(a)} + \dots .$$

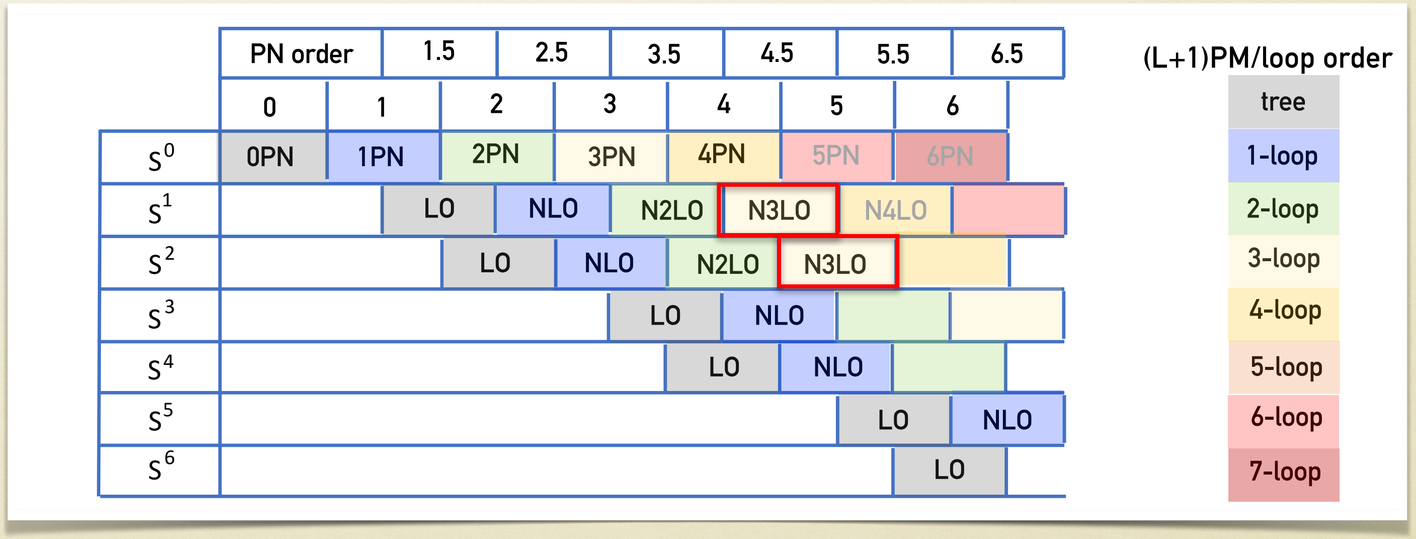
$$\mathcal{L}^{(R^2,S^2)}_{(a)} = \frac{1}{2} \left(C^{(0)}_{\mathrm{E}^2\mathrm{S}^2} \right)_{(a)} \frac{G_N^2 m_{(a)}}{c^5} \frac{E_{\mu\alpha} E_\nu{}^\alpha}{u^3_{(a)}} \left[S^\mu_{(a)} S^\nu_{(a)} \right]_{\mathrm{STF}} + \dots .$$

Electric and Magnetic components of Riemann tensor

$$E_{\mu\nu} \equiv R_{\mu\alpha\nu\beta} u^\alpha u^\beta$$

$$B_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu} u^\gamma u^\delta$$

STF = Symmetrized Trace-Free



[credit: Vines/Roiban]

GREFT Diagrams vs 2-point QFT Integrals

Kim, Levi, Yin (2022)

Mandal, Patil, Steinhoff & P.M. (2022)

S^0

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	1	1
		0	3
2PN	21	2	5
		1	10
		0	6
3PN	130	3	8
		2	75
		1	38
		0	9

(a) Non-spinning sector

S^1

Order	Diagrams	Loops	Diagrams
LO	2	0	2
NLO	13	1	8
		0	5
N^2LO	100	2	56
		1	36
		0	8
N^3LO	894	3	288
		2	495
		1	100
		0	11

(b) Spin-orbit sector

S^2

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	7	1	3
		0	4
N^2LO	58	2	27
		1	24
		0	7
N^3LO	553	3	125
		2	342
		1	76
		0	10

(a) Spin1-Spin2 and $Spin1^2$ ($Spin2^2$) sector

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	4	1	1
		0	3
N^2LO	25	2	7
		1	12
		0	6
N^3LO	168	3	15
		2	101
		1	43
		0	9

(b) ES^2 sector

Order	Loops	Diagrams
LO	1	1

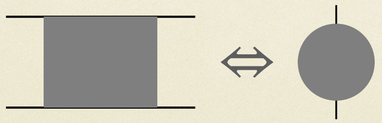
(c) E^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

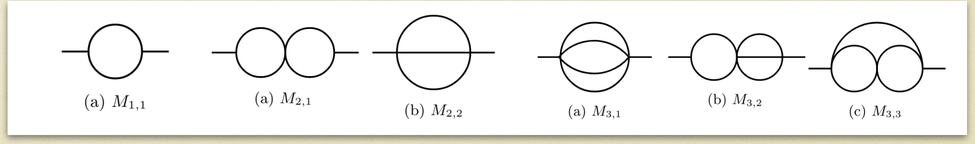
▸ **Mapping** to 2-point Functions

$$\mathscr{L}_{eff}[x_a, \dot{x}_a, \ddot{x}_a, \ldots, S_a, \dot{S}_a, \ldots] = -i \lim_{d\to 3} \int \frac{d^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{r}} \Bigg(\quad \Bigg)$$



$$\mathscr{M} = \sum_i c_i \, I_i^{MI}$$

- **Dimensional Regularization** $d = 3 + \epsilon$
- **Integration-by-parts (IBP) decomposition**
- **Master Integrals evaluation**



GREFT Diagrams vs 2-point QFT Integrals

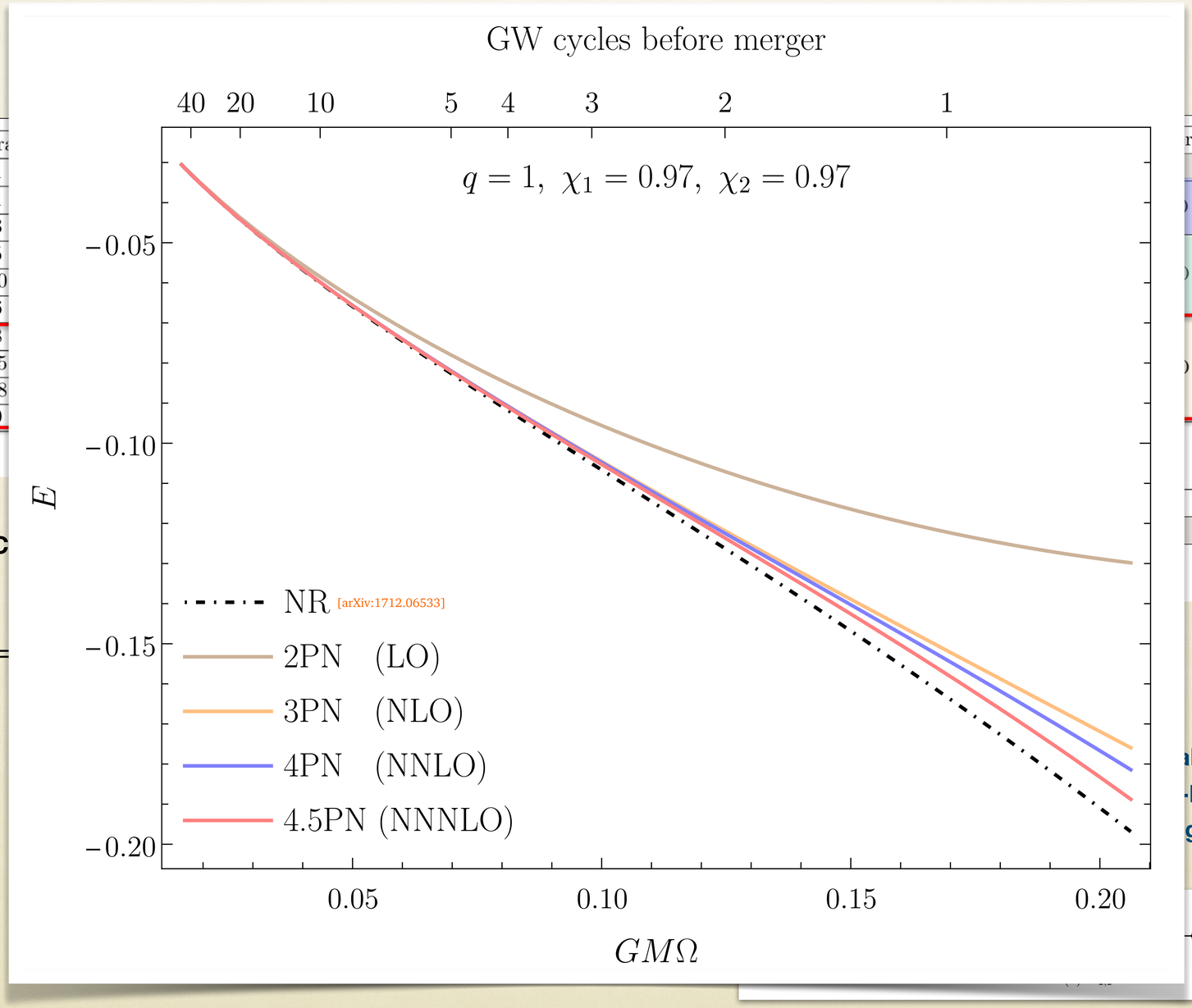
Kim, Levi, Yin (2022)

Mandal, Patil, Steinhoff & P.M. (2022)

S^0

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(a) Non-spinning sector



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		1	12
		0	6
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		2	101
		1	43
		0	9

(b) ES^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

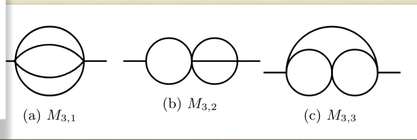
▸ Mapping to 2-point Func

$\mathscr{L}_{eff}[x_a, \dot{x}_a, \ddot{x}_a, \ldots, S_a, \dot{S}_a, \ldots] =$

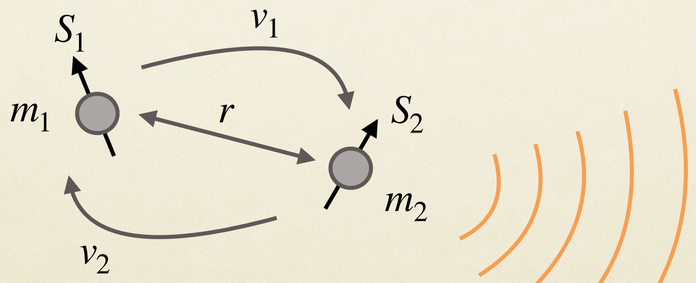
l Regularization $d = 3 + \epsilon$

-by-parts (IBP) decomposition

grals evaluation



GR EFT for PN corrections / far zone spinning



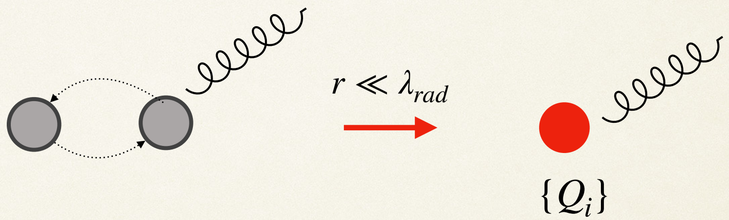
Bound system of Inspiraling of Black Holes

Far Zone / GREFT diagrammar

Thorne (1980)
Goldberger, Rothstein (2005)
Goldberger, Ross (2009)
Galley, Tiglio (2009,2012)
Foffa, Sturani (2012); Ross (2012)
Galley, Leibovich, Porto, Ross (2015)
Leibovich, Maia, Rothstein, Yang (2019)
Blanchet et al.(2021)
.....

$$S_{rad}[g,\{Q_i\}] = S_{GR}[g] + S_{mult}[g,\{Q_i\}]$$

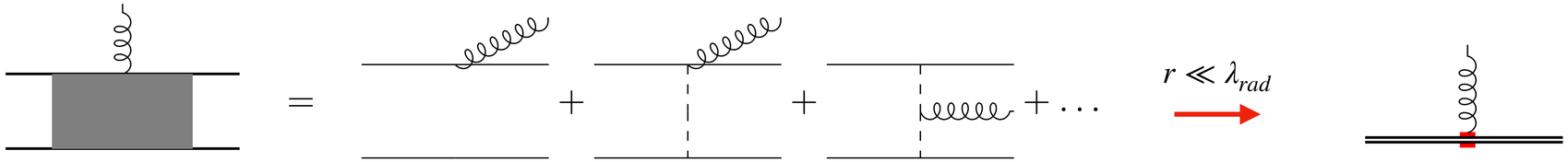
▸ **Far zone contributions to the conservative dynamics are needed, starting at $4PN$ order**



Multipole source emitting gravitons

▸ **Long wavelength (λ_{rad}) EFT**

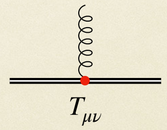
$$g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$$



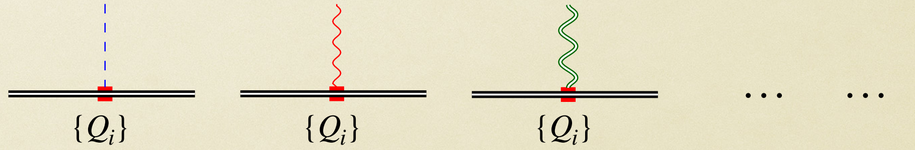
▸ **Multipole Action EFT *matching* Far zone**

Binary system as a linear source $T_{\mu\nu}$ of size r emitting $\bar{h}_{\mu\nu}$:

$$S_{mult} = -\frac{1}{2}\int d^4x T^{\mu\nu}\bar{h}_{\mu\nu}$$



$$S_{mult}[\bar{h},\{Q_i\}] = \int dt\left[\frac{1}{2}E\bar{h}_{00} - \frac{1}{2}\epsilon_{ijk}L^i\bar{h}_{0j,k} - \frac{1}{2}Q^{ij}\mathscr{E}_{ij} - \frac{1}{6}O^{ijk}\mathscr{E}_{ij,k} - \frac{2}{3}J^{ij}B_{ij} + \dots\right]$$



- $\mathscr{E}_{ij}$, B_{ij} **are the electric and magnetic components of the Riemann tensor**
- $\{Q_i\}$: **multipole moments** $E, L^i, Q^{ij}, O^{ijk}, J^{ij}$

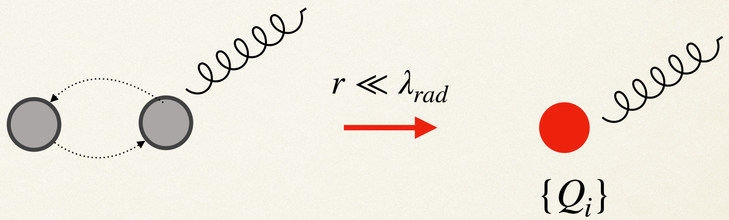
$$\mathscr{E}_{ij} = R_{0i0j} \approx -\frac{1}{2}\left(\bar{h}_{00,ij} + \ddot{\bar{h}}_{ij} - \dot{\bar{h}}_{0i,j} - \dot{\bar{h}}_{0j,i} + \mathcal{O}(\bar{h}^2)\right) \qquad B_{ij} = \frac{1}{2}\epsilon_{ikl}R_{0jkl} \approx \frac{1}{4}\epsilon_{ikl}\left(\dot{\bar{h}}_{jk,l} - \dot{\bar{h}}_{jl,k} + \dot{\bar{h}}_{0l,jk} - \dot{\bar{h}}_{0k,jl} + \mathcal{O}(\bar{h}^2)\right)$$

Far Zone / GREFT diagrammar

Thorne (1980)
Goldberger, Rothstein (2005)
Goldberger, Ross (2009)
Galley, Tiglio (2009,2012)
Foffa, Sturani (2012); Ross (2012)
Galley, Leibovich, Porto, Ross (2015)
Leibovich, Maia, Rothstein, Yang (2019)
Blanchet et al.(2021)
.....

$$S_{rad}[g,\{Q_i\}] = S_{GR}[g] + S_{mult}[g,\{Q_i\}]$$

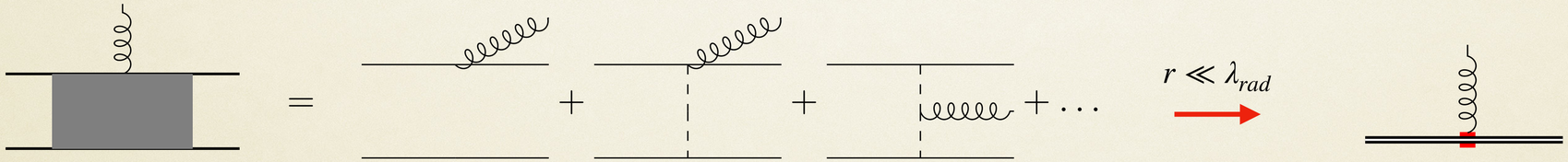
- Far zone contributions to the conservative dynamics are needed, starting at $4PN$ order



Multipole source emitting gravitons

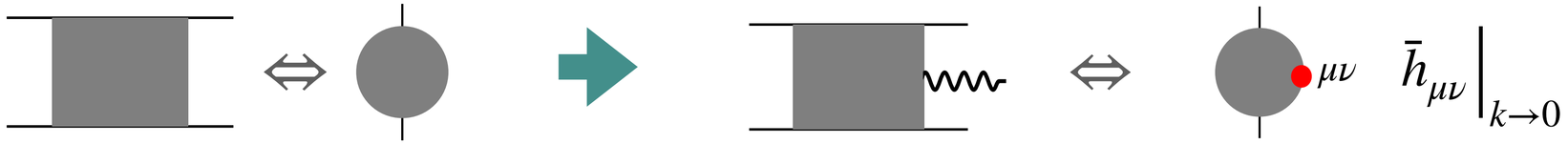
- Long wavelength (λ_{rad}) EFT

$$g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$$



- Multipole Action EFT matched to the Far zone

- Mapping to 2-point function Mandal, Patil, Steinhoff & P.M. (2024)



No radiation

With radiation

$$\mathscr{M} = \sum_i c_i \, I_i^{MI}$$

- Dimensional Regularization $d = 3 + \epsilon$
- Integration-by-parts (IBP) decomposition
- Master Integrals evaluation

Far Zone / Flux

Mandal, **Patil**, Steinhoff, & PM

PN orders	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	...
Point Particle	**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	**N^3LO +NNLO (Her)**	**N^3LO (Her)**	**N^4LO +N^4LO (Her)**	**N^5LO (Her)**	
Spin-orbiit				**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	N^3LO +NNLO (Her)	
Spin^2					**LO**		**NLO**	**LO (Her)**	**NNLO**	NLO (Her)	
Spin^3								**LO**		NLO	
Spin^4									LO		

Disconnected
Tree
1 Loop
2 Loop
3 Loop

$$
\begin{aligned}
\mathscr{F}^{\rm pp}_{cir} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_{\rm E} - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_{\rm E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \\
& \quad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_{\rm E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \quad + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\Bigg]x^4 \\
& + \left[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_{\rm E} - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\right]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}
\end{aligned}
$$

[Blanchet, Faye, Henry, Larrouturou, Trestini (2023)]

$$x = \left(GM\,\omega_{\rm orb}\right)^{2/3}$$

[credit: Patil]

Far Zone / Flux

Mandal, **Patil**, Steinhoff, & PM

PN orders	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	...
Point Particl	**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	**N^3LO +NNLO (Her)**	**N^3LO (Her)**	**N^4LO +N^4LO (Her)**	**N^5LO (Her)**	
Spin-orbiit				**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	N^3LO +NNLO (Her)	
Spin^2											
Spin^3											
Spin^4											

PN orders	0	0.5	1	1.5	2	2.5	3
$E_{\mu\nu}E^{\mu\nu}$	**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	N^3LO +NNLO (Her)
$B_{\mu\nu}B^{\mu\nu}$			**LO**		**NLO**	**LO (Her)**	NNLO
$\nabla_\alpha E_{\mu\nu}\nabla^\alpha E^{\mu\nu}$					**LO**		NLO
$\dot{E}_{\mu\nu}\dot{E}^{\mu\nu}$							LO

$$\mathcal{F}^{\rm pp}_{cir} = \frac{32c^5}{5G}\nu^2 x$$

$$\begin{aligned}
\mathcal{F}_E^{\rm AT} =& \left\{\left(\frac{768\nu^2}{5}+\frac{192\nu}{5}\right)\tilde{\lambda}_{(+)}+\frac{192}{5}\delta\tilde{\lambda}_{(-)}\right\}x^{10}\\
&+\left\{\left(-992\nu^3-\frac{9736\nu^2}{35}-\frac{1408\nu}{35}\right)\tilde{\lambda}_{(+)}+\left(-\frac{184\nu^2}{5}-\frac{1408\nu}{35}\right)\delta\tilde{\lambda}_{(-)}\right\}x^{11}\\
&+4\pi\left\{\left(\frac{768\nu^2}{5}+\frac{192\nu}{5}\right)\tilde{\lambda}_{(+)}+\frac{192}{5}\delta\tilde{\lambda}_{(-)}\right\}x^{23/2}\\
&+\left\{\left(3088\nu^4+\frac{63692\nu^3}{35}-\frac{1299706\nu^2}{945}+\frac{5344\nu}{45}\right)\tilde{\lambda}_{(+)}\right.\\
&\left.+\left(-\frac{11116\nu^3}{15}+\frac{149566\nu^2}{105}+\frac{5344\nu}{45}\right)\delta\tilde{\lambda}_{(-)}\right\}x^{12}+\mathcal{O}\left(x^{25/2}\right).
\end{aligned}$$

[Henry, Faye, Blanchet(2020), Mandal, Patil, **PM**, Steinhoff (2024)]

$$x = \left(GM\,\omega_{\rm orb}\right)^{2/3}$$

urou, Trestini (2023)]

orb)$^{2/3}$

$$+\left[\frac{26978667519}{745113600}-\frac{6848}{105}\gamma_{\rm E}-\frac{5424}{105}\ln(16\,x)+\left(\frac{2062241}{22176}+\frac{41}{12}\pi^2\right)\nu-\frac{153112905}{290304}\nu^2-\frac{5719141}{38016}\nu^3\right]\pi x^{9/2}+\mathcal{O}\left(x^5\right)\Bigg\}$$

[credit: Patil]

Advanced Feynman Calculus

6-loop gravitational interactions at the 6th-PN order

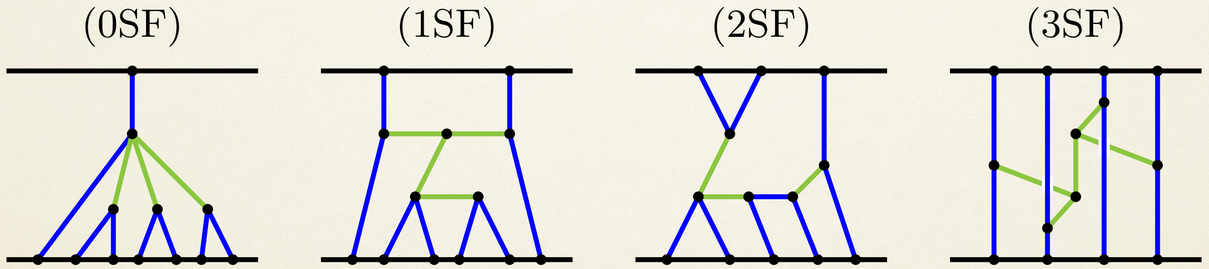
Static contribution to the 6PN potential

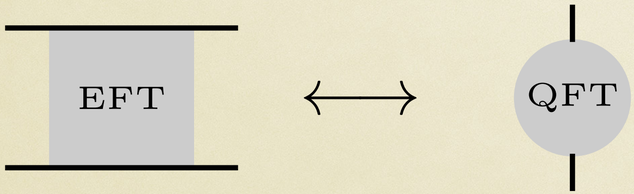
Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)

$$\mathcal{V}_{\rm eff} = \mathbf{i} \lim_{d\to 3} \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)}-\mathbf{x}_{(2)})} \times$$



1117 GREFT Graphs





$$I_{n_1,\ldots,n_{27}} = \int \prod_{i=1}^{6} \frac{d^d k_i}{(2\pi)^d} \prod_{i=1}^{27} D_i^{-n_i}$$

6-loop integrals

13 **denominators** + 14 **isp's**

rank: (up to) 6

[mass-dimension of numerator (up to) 12]

PN-GREFT Diagrammar & Multi-Loop Automation (PNTHR)

Mandal, **Patil**, Steinhoff, & PM

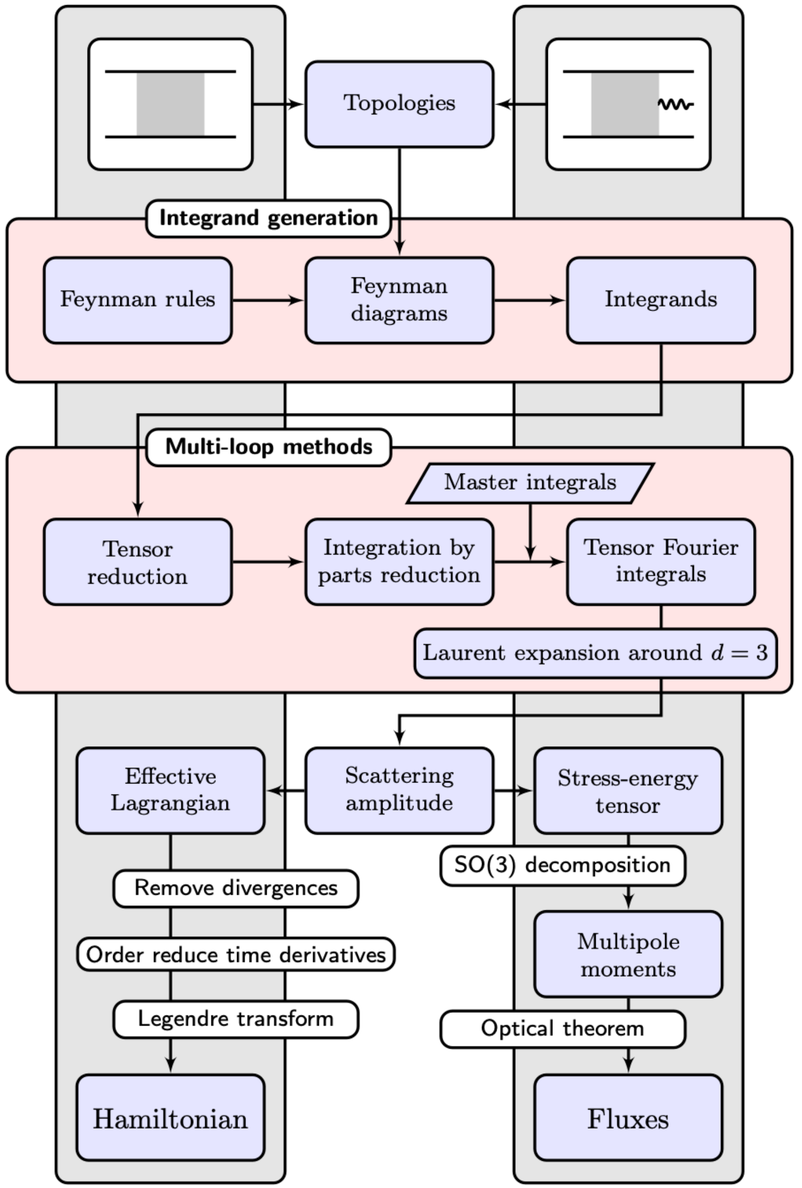
1. Diagram Generation

improved!

2. Integrals' Evaluation

improved!

3. Applications



[credit: Patil]

Integration-by-parts identities

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)

Topology Mapping ⇒ **minimising** the number of integral families subject to IBP decomposition

Seeding ⇒ **minimising** the number of IBP equations vs number of invalid integrals

Syzygies ⇒ **limiting** the number of IBP equations (avoiding those involving higher-exponent integrals)

Spanning-cuts ⇒ **decomposing** the complete IBP system into **smaller-b**lock systems

Finite Fields ⇒ **simplifying** the algebraic/arithmetic **operations** + **reconstructing *only*** the **final expressions** (not intermediate results)

[credit: Smith]

Integration-by-parts identities

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)

Topology Mapping

Seeding

Syzygies

Spanning-cuts

Finite Fields

⇒ **minimising** the number of IBP equations vs number of involved integrals

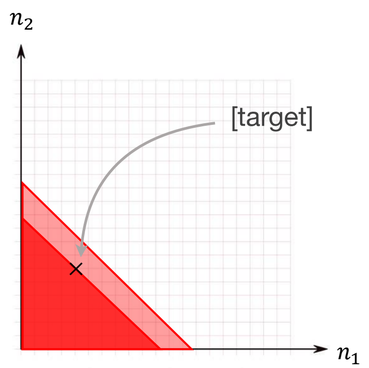
$$0 = \int_{\ell_1,\dots\ell_L} \sum_a \frac{\partial}{\partial \ell_a^\mu} \frac{v_a^\mu (D_1, \dots, D_N)}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$

IBP's are **symbolic equations** depending on the **vectors** v_a^μ and the **exponents** $\vec{n} = \{n_1, n_2, \dots, n_N\}$

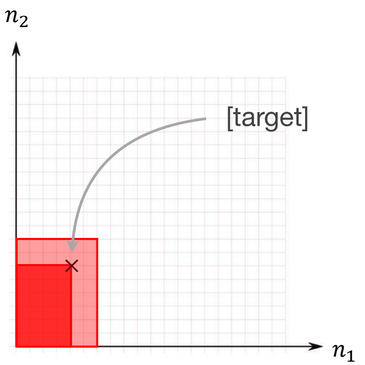
After choosing v_a^μ, **IBP's systems** of equations are **generated by seeding** $\vec{n}$, namely by assigning explicit integer variables to each n_i

Seeding determines the "**shape**" of the **IBP systems**

Ex.: 2 denominator sectors $\vec{n} = \{n_1, n_2\}$



Equilateral seeding



Rectangular seeding

[credit: Smith]

Integration-by-parts identities

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)

Topology Mapping

Seeding

Syzygies

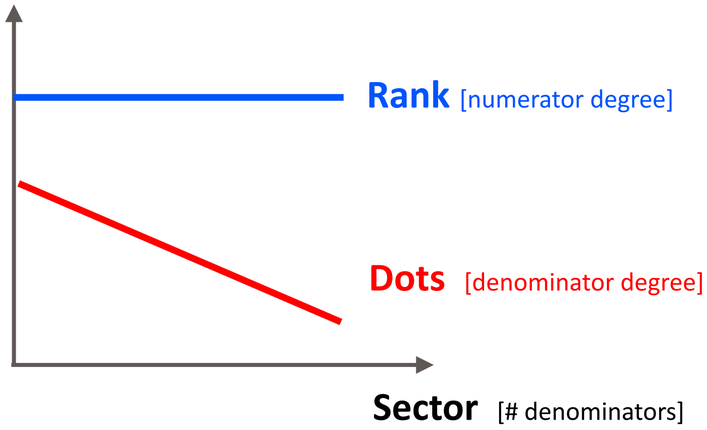
Spanning-cuts

Finite Fields

⇒ **minimising** the number of IBP equations vs number of involved integrals

Standard seeding@ Fixed Rank

⇒ **Increasing Dots** of **subsectors**

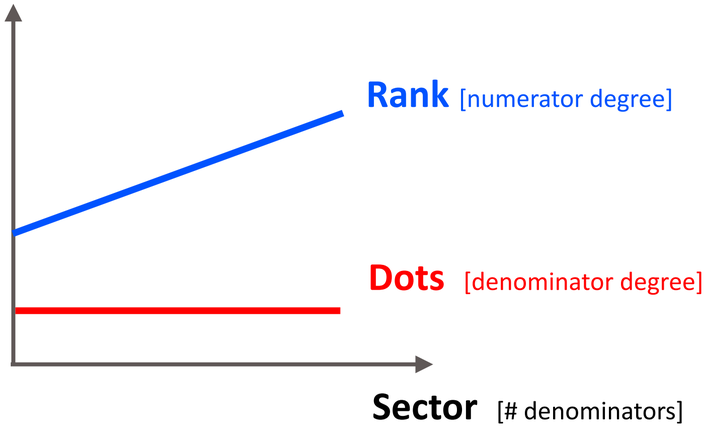


Laporta, Remiddi (1996)

Laporta (2000)

Improved Seeding @ Fixed Dots

⇒ **Decreasing Rank** of **subsectors**



Kosower, Gluza (2010

Larssen, Zhang (2015)

Driess, Jakobsen, Mogull, Plefka, Sauer

Guan, Liu, Ma, Wu (2024)

Bern, Hermann, Roiban, Ruf, Smirnov, Smirnov, Zeng (2024)

Smith, Zeng (2025)

[credit: Smith]

Integration-by-parts identities

@ the frontier of calculus

Brunello, Mandal, Patil, Pegorin, Ronca, Smith, Steinhoff, **Torres Bobadilla**, & PM (2025)

Topology Mapping

Seeding

Syzygies

Spanning-cuts

Finite Fields

⇒ **limiting** the number of **IBP equations** / avoiding those involving higher-exponent (Dots)

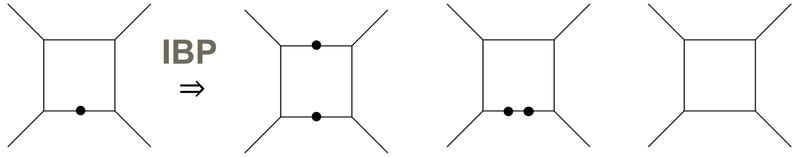
$$0 = \int_{\ell_1,\dots\ell_L} \sum_a \frac{\partial}{\partial \ell_a^\mu} \frac{v_a^\mu (D_1, \dots, D_N)}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$

Standard choice: $v_a^\mu \in \{\text{loop \& external momenta}\}$

$$0 = \int_{\ell_1,\dots\ell_L} \sum_a \left(\frac{\partial v_a^\mu}{\partial \ell_a^\mu} - \sum_i \frac{n_i}{D_i} v_a^\mu \frac{\partial D_i}{\partial \ell_a^\mu} \right) \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$

[increasing Dots]

⇒ higher-exponent integrals



Syzygy solution: $v_a^\mu \frac{\partial D_i}{\partial \ell_a^\mu} = f_i \, D_i$

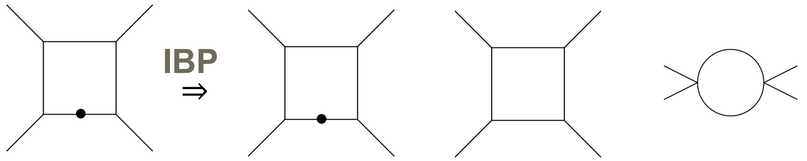
$$0 = \int_{\ell_1,\dots\ell_L} \sum_a \left(\frac{\partial v_a^\mu}{\partial \ell_a^\mu} - \sum_i \frac{n_i}{D_i} f_i \, D_i \right) \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$

[not increasing Dots]

@ SINGULAR

@ in-house Syzygy solver

⇒ **no** higher-exponent integrals



[credit: Smith]

Integration-by-parts identities

@ the frontier of calculus

Brunello, Mandal, Patil, Pegorin, Ronca, Smith, Steinhoff, **Torres Bobadilla**, & PM (2025)

Topology Mapping

Seeding

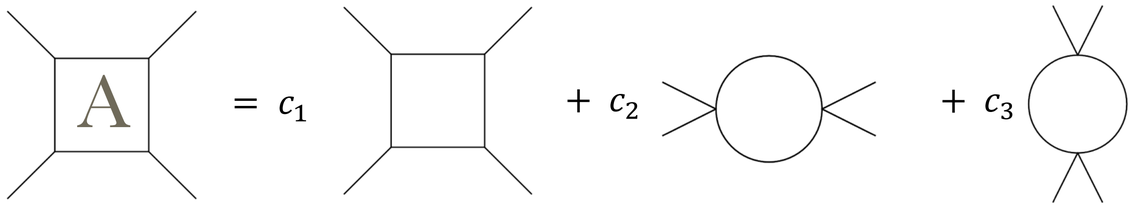
Syzygies

Spanning-cuts

Finite Fields

⇒ **decomposing** the complete IBP system into **smaller-subsystems**

Complete IBP

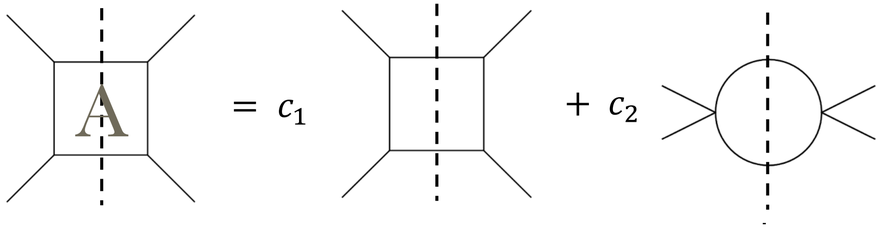


Complete set of coefficients c_i

On-shell cut condition

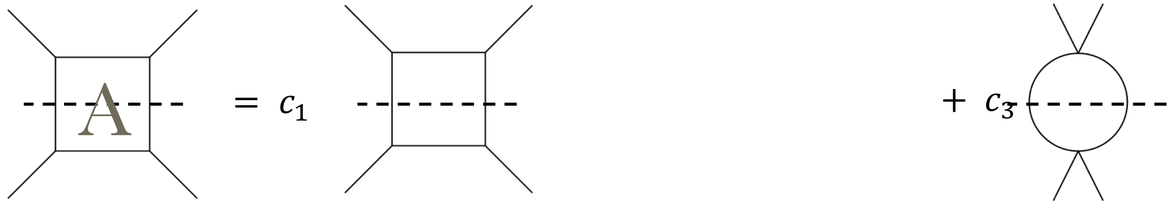
$$\frac{1}{D_i} \to \delta(D_i) \qquad I(n_1, \dots, n_N) \to I_C(n_1, \dots, n_N) = \begin{cases} I(n_1, \dots, n_N), & n_i < 0 \; \forall \; i \in C \\ 0, & \text{otherwise} \end{cases}$$

IBP on cut #1



Subset #1 of coefficients c_i

IBP on cut #2



Subset #2 of coefficients c_i

{Subset #1} ∪ **{Subset #2}** = **Complete set**

[credit: Smith]

Integration-by-parts identities

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)

Topology Mapping	⇒	**minimising** the number of integral families subject to IBP decomposition
Seeding	⇒	**minimising** the number of IBP equations vs number of invalid integrals
Syzygies	⇒	**limiting** the number of IBP equations (avoiding those involving higher-exponent integrals)
Spanning-cuts	⇒	**decomposing** the complete IBP system into **smaller-b**lock systems
Finite Fields	⇒	**simplifying** the algebraic/arithmetic **operations** + **reconstructing *only*** the **final expressions** (not intermediate results)

PRISM: Prime-field **R**eduction of **I**ntegrals using **S**yzygy **M**ethods — Brunello, Ronca, **Smith**, Torres Bobadilla, & PM (in progress)

[credit: Smith]

Integration-by-parts identities

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)

Topology Mapping ⇒ **minimising** the number of integral families subject to IBP decomposition

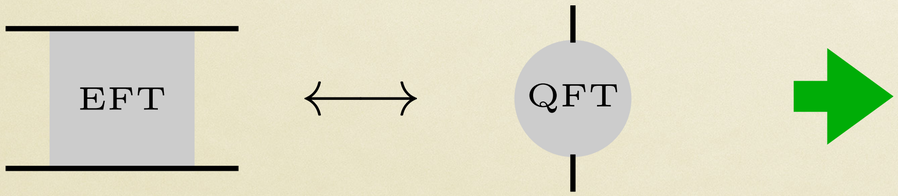
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Finite Fields ⇒ **simplifying** the algebraic/arithmetic **operations** + **reconstructing *only*** the **final expressions** (not intermediate results)

FiniteFlow [Peraro (2016,2019)]

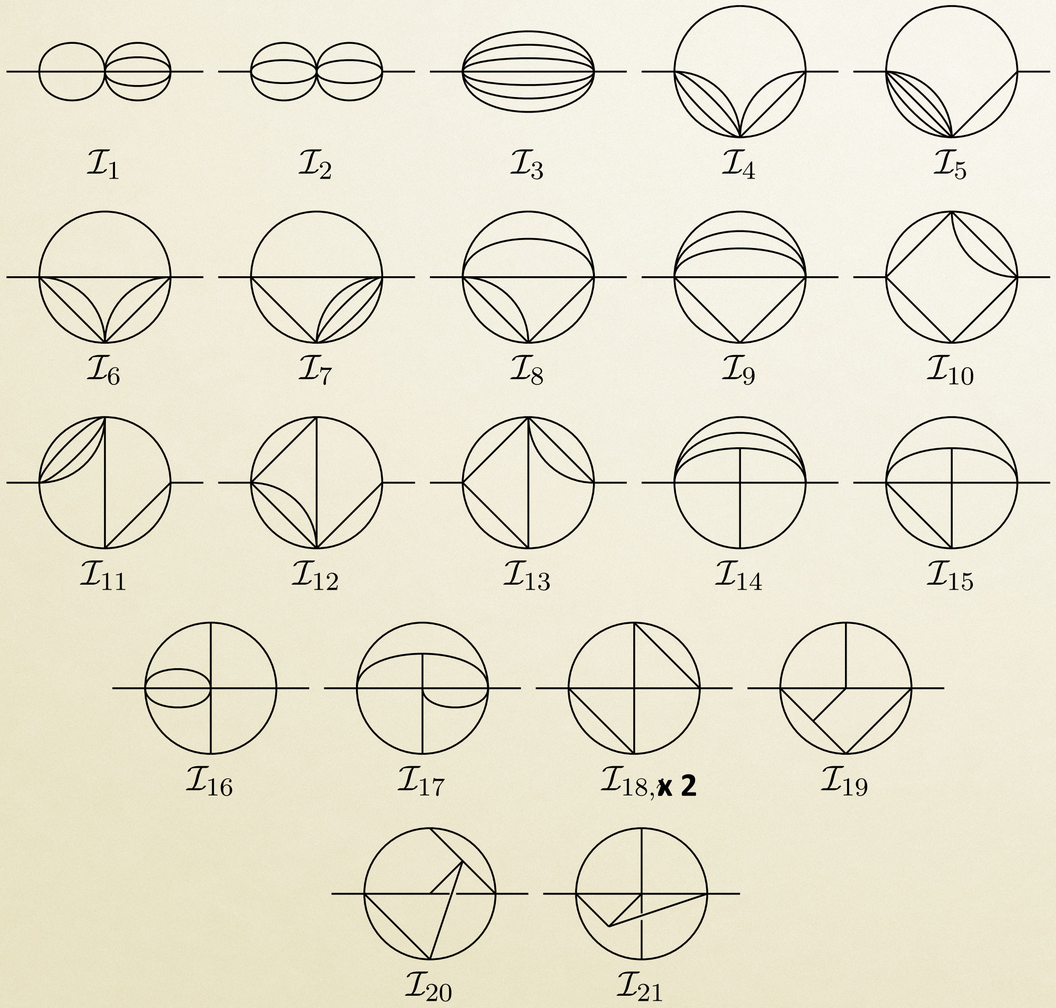


SF order	Diagrams	MIs
0	36	1
1	161	3
2	387	13
3	533	16
Total	1117	21

Master Integrals

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)



IBP decomposition of 6-loop 2-point diagrams
of planar and non-planar GREFT integrals
⇒ 39 Master Integrals

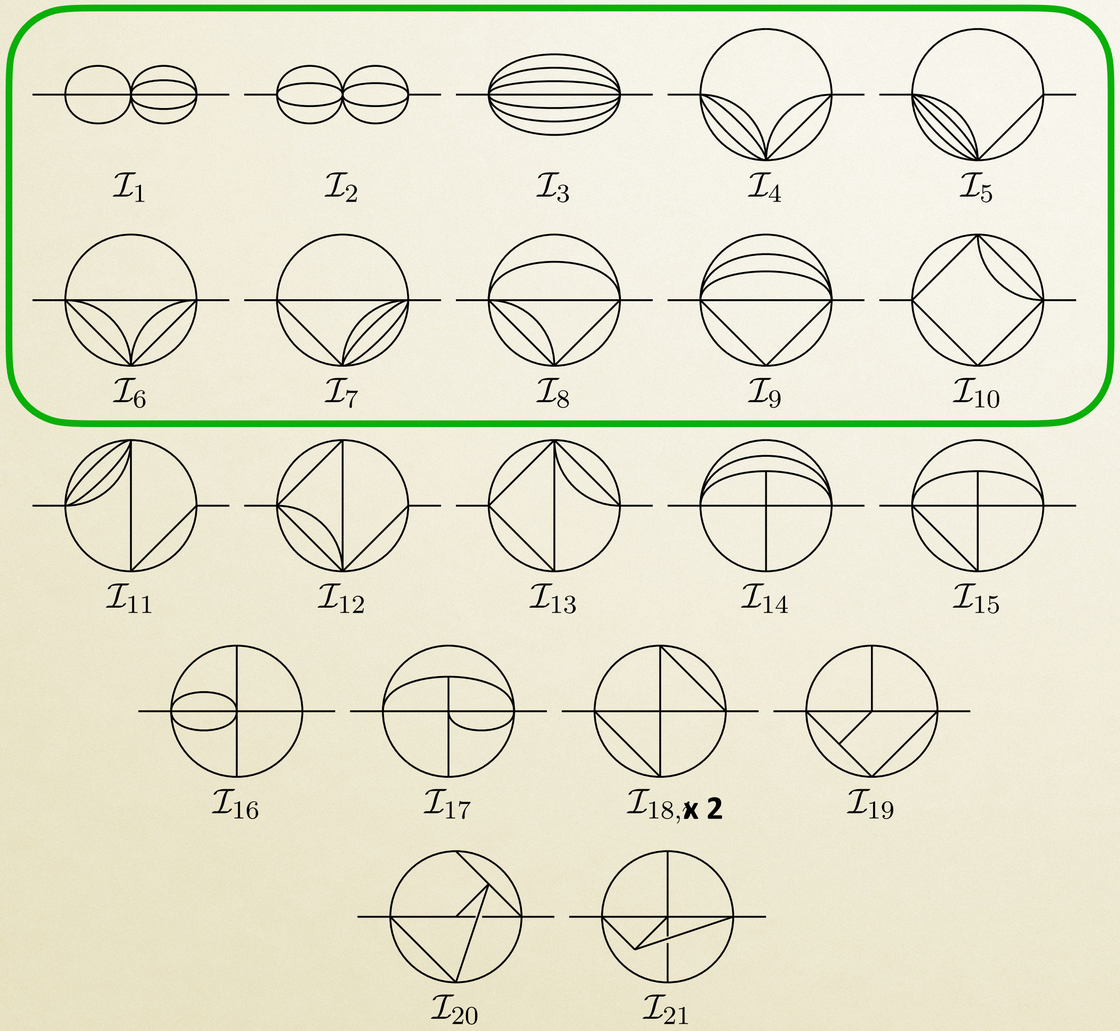
***only* 21 Master Integrals**
contribute to
the **conservative potential**

(*) Gauge theory/QFT version unknown

Master Integrals

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)



Analytic Evaluation: direct integration

Iterated 1-loop formula:

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^{\nu_1}[(k-p)^2]^{\nu_2}}$$

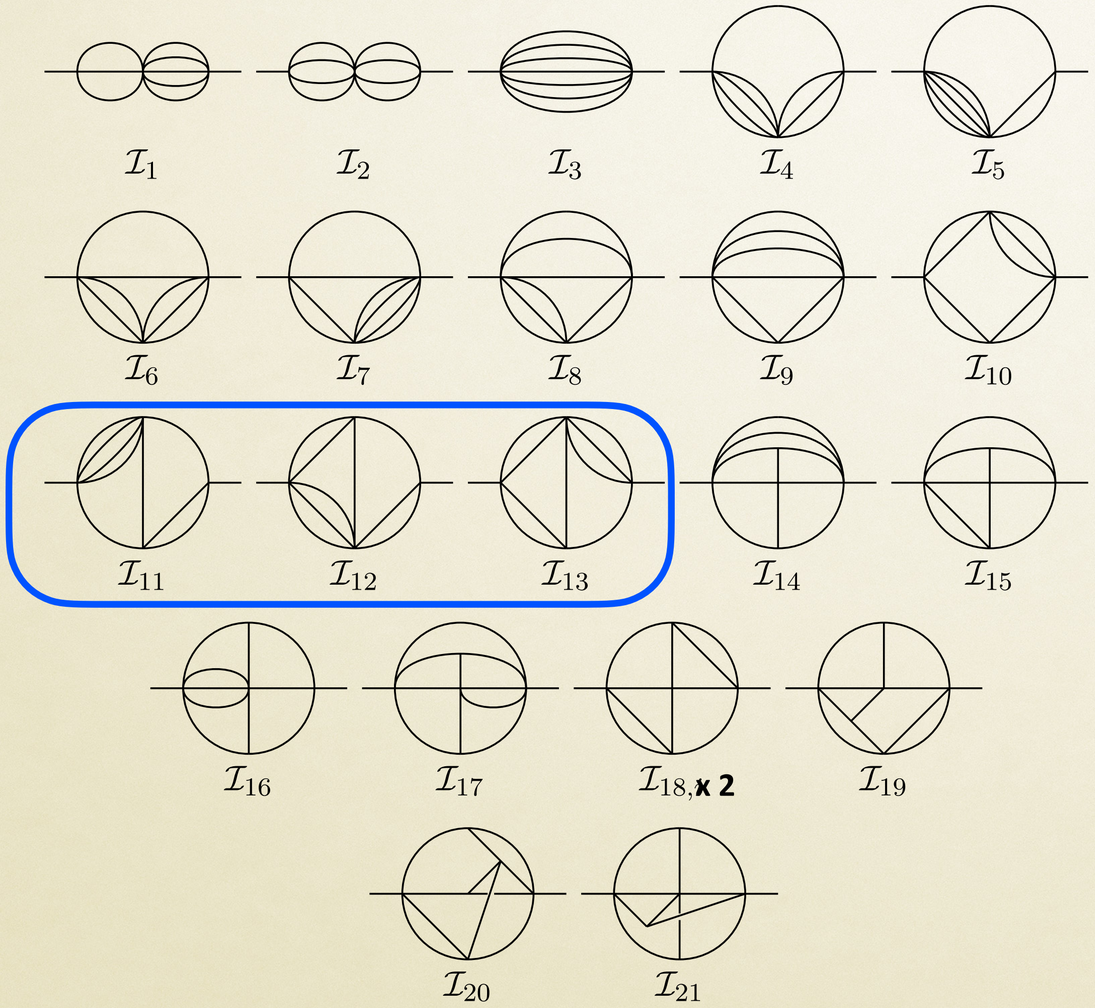
$$= \frac{(p^2)^{\frac{d}{2}-(\nu_1+\nu_2)}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d}{2}-\nu_1\right)\Gamma\left(\frac{d}{2}-\nu_2\right)\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(d-\nu_1-\nu_2)}$$

[credit: Brunello, Pegorin]

Master Integrals

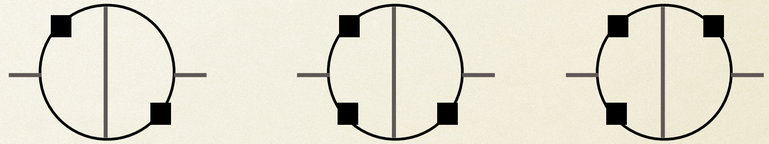
@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)



Kite Type Intergrals

Partial application of Iterated 1-loop formula:



2-loop kite diagrams
with non-integer exponents (■) :

- *Gegenbauer polynomials*

  Chetyrkin, Kataev, Tkachov (1980), Kotikov (1996)

- *Inverse Mellin Transform*

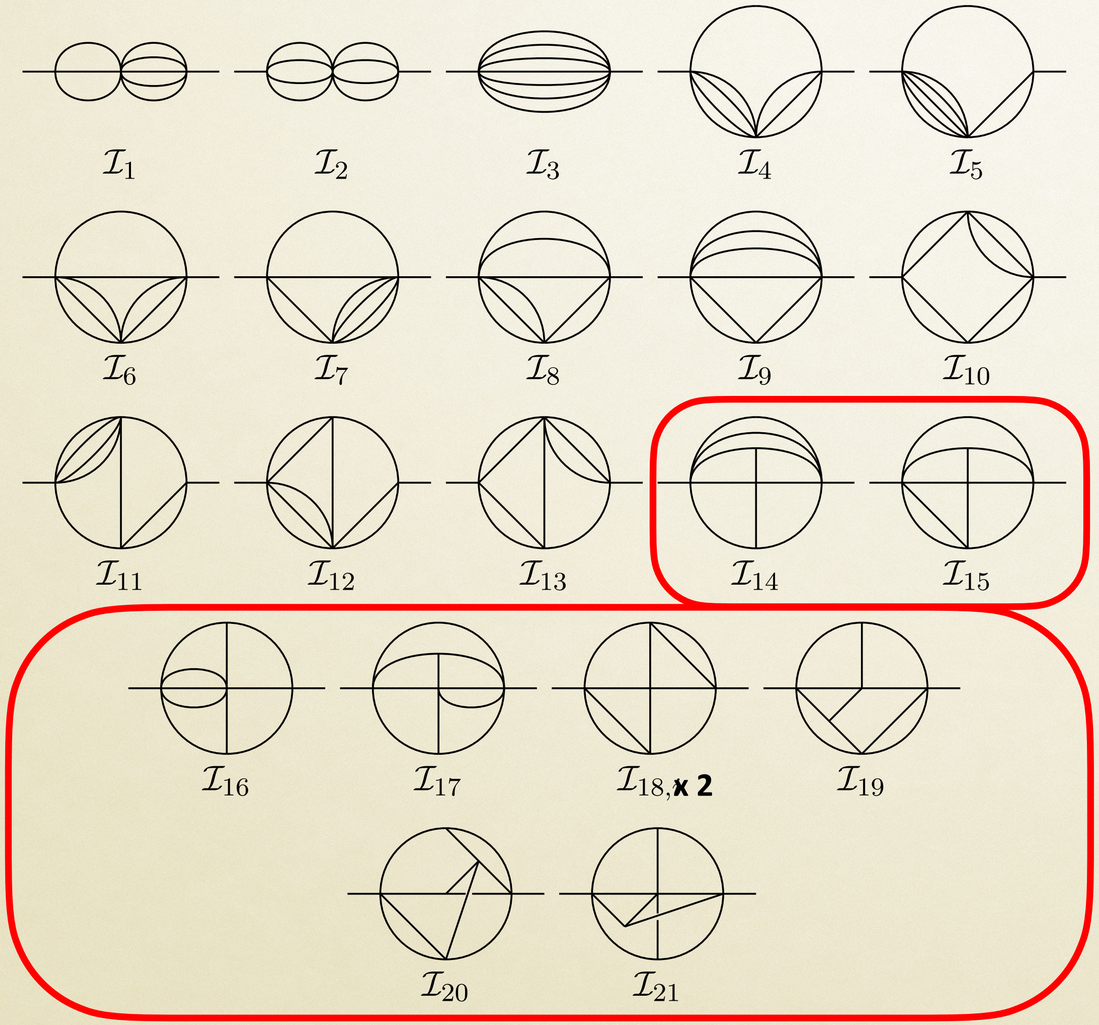
  Biernebaum, Weinzierl (2003)

[credit: Brunello, Pegorin]

Master Integrals

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)



Analytic Reconstruction

Numerical solution of ***Differential Equations***
by **AMFlow** Liu, Ma, Wang (2017,2022)

Integer Relation Finding Algorithm
fitting transcendental constants
by **PSLQ algorithm** and Mathematica
Ferguson, Bailey, Arno (1999)

checks:

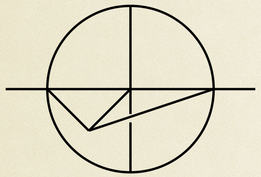
1. ***Numerical evaluation***
Tropical Integration of a ***finite basis of Master Integrals***
@ d=5 by **FeynTrop** Borinsky, Munch, Tellander (2023)

2. ***IBPs*** @ d=5: from the finite basis to the true basis

3. ***Dimensional Recurrence*** from d=5 to d=3
by ***Dream*** Lee, Mingulov (2017)

[credit: Brunello, Pegorin]

Experimental Mathematics



$\mathcal{I}_{21}$

$\epsilon = d - 3$

```
In[6]:= I21 = N[Series[I21, {eps, 0, -1}], 50] // Chop
```

Out[6]=

$$\frac{0.25000000000000000000000000000000000000000000000000}{\text{eps}^3} -$$

$$\frac{0.25000000000000000000000000000000000000000000000000}{\text{eps}^2} -$$

$$\frac{4.42527541260212737051573353123838916893565931942 88}{\text{eps}} + O[\text{eps}]^0$$

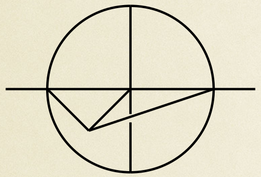
AMFlow numerical value

$\frac{1}{4}$

$-\frac{1}{4}$

?

Experimental Mathematics



$\mathcal{I}_{21}$

$\epsilon = d - 3$

In[6]:=
```
I21 = N[Series[I21, {eps, 0, -1}], 50] // Chop
```

Out[6]=

$$\frac{0.25000000000000000000000000000000000000000000000000}{\text{eps}^3} - \frac{0.25000000000000000000000000000000000000000000000000}{\text{eps}^2} - \frac{4.42527541260212737051573353123838916893565931942 88}{\text{eps}} + O[\text{eps}]^0$$

AMFlow numerical value

$\frac{1}{4}$

$-\frac{1}{4}$

?

In[13]:=
```
test = N[Coefficient[I21, eps, -1], 50]
```

Out[13]=
```
-4.42527541260212737051573353123838916893565931942 88
```

In[14]:=
```
vars = {1, π, π^2, Log[2]}
```

Basis Ansatz

Out[14]=
```
{1, π, π^2, Log[2]}
```

In[15]:=
```
guess = {test, vars} // Flatten
```

Out[15]=
```
{-4.42527541260212737051573353123838916893565931942 88, 1, π, π^2, Log[2]}
```

In[16]:=
```
fit = FindIntegerNullVector[guess]
```

Integer Relation Finding Algorithm

Out[16]=
```
{-32, -112, 0, -3, 0}
```

In[17]:=
```
res = -1 / fit[[1]] * Sum[fit[[i]] * guess[[i]], {i, 2, Length[guess]}] // Expand
```

Out[17]=

$$-\frac{7}{2} - \frac{3\,\pi^2}{32}$$

Reconstructed result

In[18]:=
```
N[%, 50]
```

Out[18]=
```
-4.42527541260212737051573353123838916893565931942 88
```

$$\mathcal{I}_{21} = \frac{c_\epsilon}{(p^2)^4}\left[\frac{1}{4\epsilon^3} - \frac{1}{4\epsilon^2} - \frac{1}{\epsilon}\left(\frac{7}{2} + \frac{3\pi^2}{32}\right) + \mathcal{O}(\epsilon^0)\right].$$

Master Integrals

@ the frontier of calculus

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)

$$\mathcal{I}_1 = \frac{c_\epsilon}{(p^2)}\left[-\frac{1}{48\epsilon^2}+\frac{35}{144\epsilon}-\frac{199}{108}+\frac{29\pi^2}{1152}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_2 = \frac{c_\epsilon}{(p^2)}\left[\frac{\pi^2}{16}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_3 = c_\epsilon\left[-\frac{1}{720\epsilon}+\frac{457}{21600}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_4 = \frac{c_\epsilon}{(p^2)}\left[-\frac{1}{72\epsilon^2}+\frac{37}{216\epsilon}-\frac{229}{162}+\frac{23\pi^2}{576}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_5 = \frac{c_\epsilon}{(p^2)}\left[-\frac{1}{432\epsilon^2}+\frac{13}{432\epsilon}-\frac{259}{972}+\frac{31\pi^2}{3456}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_6 = \frac{c_\epsilon}{(p^2)}\left[-\frac{1}{36\epsilon^2}+\frac{10}{27\epsilon}-\frac{527}{162}+\frac{37\pi^2}{864}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_7 = \frac{c_\epsilon}{(p^2)}\left[\frac{\pi^2}{72}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_8 = \frac{c_\epsilon}{(p^2)}\left[-\frac{1}{72\epsilon^2}+\frac{23}{108\epsilon}-\frac{173}{81}+\frac{61\pi^2}{1728}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_9 = \frac{c_\epsilon}{(p^2)}\left[\frac{\pi^2}{144\epsilon}-\frac{37\pi^2}{432}-\frac{1}{72}\pi^2\log(2)+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_{10} = \frac{c_\epsilon}{(p^2)^2}\left[\frac{\pi^2}{48\epsilon^2}-\frac{\pi^2}{24\epsilon}(5+\log(2))+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{11} = \frac{c_\epsilon}{(p^2)^2}\left[-\frac{1}{48\epsilon^2}+\frac{17}{48\epsilon}-\frac{47}{12}-\frac{\pi^2}{384}+\mathcal{O}(\epsilon)\right],$$

$$\mathcal{I}_{12} = \frac{c_\epsilon}{(p^2)^2}\left[-\frac{1}{48\epsilon^3}+\frac{5}{16\epsilon^2}-\frac{1}{24\varepsilon}\left(77-\frac{61\pi^2}{48}\right)+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{13} = \frac{c_\epsilon}{(p^2)^2}\left[\frac{\pi^2}{48\epsilon^2}-\frac{\pi^2}{24\epsilon}(5+\log(2))+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{14} = \frac{c_\epsilon}{(p^2)^3}\left[-\frac{1}{4\epsilon^2}-\frac{3}{2\epsilon}+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{15} = \frac{c_\epsilon}{(p^2)^3}\left[\frac{1}{48\epsilon^3}+\frac{3}{16\epsilon^2}+\frac{1}{24\epsilon}\left(29-\frac{47\pi^2}{16}\right)+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{16} = \frac{c_\epsilon}{(p^2)^3}\left[-\frac{1}{8\epsilon^3}+\frac{1}{8\epsilon^2}-\frac{1}{4\epsilon}\left(3-\frac{13\pi^2}{48}\right)+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{17} = \frac{c_\epsilon}{(p^2)^3}\left[\frac{1}{24\epsilon^3}+\frac{7}{24\epsilon^2}+\frac{1}{12\epsilon}\left(19-\frac{47\pi^2}{16}\right)+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{18,2} = \frac{c_\epsilon}{(p^2)^2}\left[\frac{\pi^2}{24\epsilon^2}-\frac{\pi^2}{3\epsilon}\left(1+\frac{\log(2)}{4}\right)+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{19} = \frac{c_\epsilon}{(p^2)^4}\left[\frac{5}{48\epsilon^3}-\frac{7}{24\epsilon^2}-\frac{1}{48\epsilon}\left(139+\frac{95\pi^2}{8}\right)+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{20} = \frac{c_\epsilon}{(p^2)^4}\left[\frac{1}{24\epsilon^3}-\frac{1}{24\epsilon}\left(41+\frac{59}{8}\pi^2\right)+\mathcal{O}(\epsilon^0)\right],$$

$$\mathcal{I}_{21} = \frac{c_\epsilon}{(p^2)^4}\left[\frac{1}{4\epsilon^3}-\frac{1}{4\epsilon^2}-\frac{1}{\epsilon}\left(\frac{7}{2}+\frac{3\pi^2}{32}\right)+\mathcal{O}(\epsilon^0)\right].$$

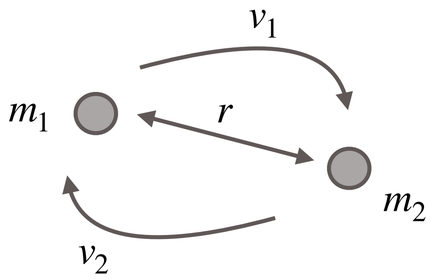
NEW

$$c_\epsilon = \frac{(p^2)^{2+3\epsilon}e^{3\gamma_E\epsilon}}{(4\pi)^{3(\epsilon+2)}}$$

$\epsilon = d - 3$

$\mathcal{O}(G_N^7\,, v^0)$ - 6PN static Corrections

Brunello, Mandal, **Patil, Pegorin, Ronca, Smith**, Steinhoff, **Torres Bobadilla**, & PM (2025)



Bound system of Inspiraling of Black Holes

$$\mathcal{V}_{\rm 6PN}^{\rm G_N^7} = -\frac{G_N^7}{r^7}\left(\frac{5}{16}m_1^7 m_2 + \frac{190}{9}m_1^6 m_2^2 + \frac{37651}{144}m_1^5 m_2^3 + \frac{5852}{9}\frac{m_1^4 m_2^4}{2} + (1 \leftrightarrow 2)\right)$$

NEW

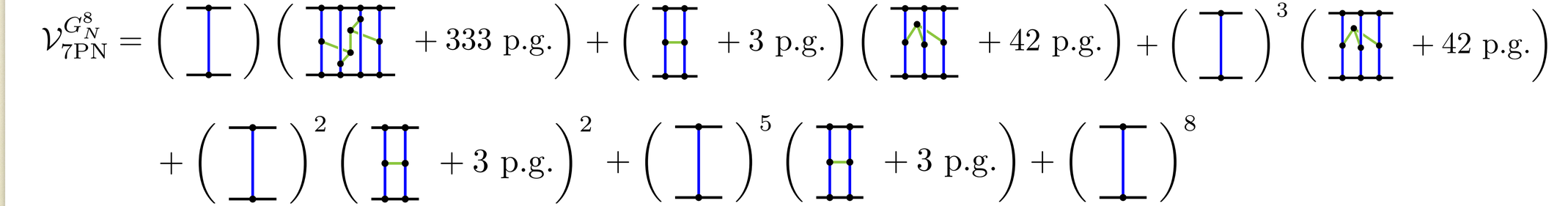
Strikingly simple!

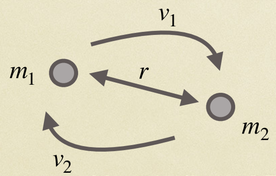
GREFT Diagrams vs 2-point QFT Integrals / Factorization Theorem

Brunello, Mandal, Patil, Pegorin,, Smith, Steinhoff, & PM (2026)

7PN static O(G^8): 3842 **7**-loop GREFT diagrams

$$\mathcal{V}^{G_N^8}_{7\mathrm{PN}} = \mathcal{V}_{0\mathrm{PN}\times 6\mathrm{PN}} + \mathcal{V}_{2\mathrm{PN}\times 4\mathrm{PN}} + \mathcal{V}_{(0\mathrm{PN})^3\times 4\mathrm{PN}} + \mathcal{V}_{(0\mathrm{PN})^2\times(2\mathrm{PN})^2} + \mathcal{V}_{(0\mathrm{PN})^5\times 2\mathrm{PN}} + \mathcal{V}_{(0\mathrm{PN})^8}$$

$\mathcal{V}^{G_N^8}_{7\mathrm{PN}} =$ 



Bound system of Inspiraling of Black Holes

$$\mathcal{V}^{G_N^8}_{7\mathrm{PN}} = \frac{G_N^8}{r^8}\left(\frac{35}{128}m_1m_2^8 + \frac{248}{9}m_1^2m_2^7 + \frac{1059}{2}m_1^3m_2^6 + \frac{89383}{36}m_1^4m_2^5 + (1\leftrightarrow 2)\right)$$

NEW

Strikingly simple!

- **Factorization Th'm:** NO 7-loop Feynman diagram explicitly computed

4PN, 5PN, 6PN, 7PN, … static corrections to the GR-Effective Potential

$$V^{(4PN)}_{\text{static}} = \frac{3}{8}\frac{G_N^5 m_1^5 m_2}{r^5} + \frac{31}{3}\frac{G_N^5 m_1^4 m_2^2}{r^5} + \frac{141}{8}\frac{G_N^5 m_1^3 m_2^3}{r^5} + (m_1 \leftrightarrow m_2)$$

Foffa, Sturani, Sturm & **PM** (2016)

$$\mathcal{V}^{(5\text{PN})}_{\text{static}} = \frac{5}{16}\frac{G_N^6 m_1^6 m_2}{r^6} + \frac{91}{6}\frac{G_N^6 m_1^5 m_2^2}{r^6} + \frac{653}{6}\frac{G_N^6 m_1^4 m_2^3}{r^6} + (m_1 \leftrightarrow m_2).$$

Foffa, Sturani, Sturm, Torres-Bobadilla & **PM** (2019)

$$\mathcal{V}^{\text{G}_\text{N}^7}_{6\text{PN}} = -\frac{G_N^7}{r^7}\left(\frac{5}{16}m_1^7 m_2 + \frac{190}{9}m_1^6 m_2^2 + \frac{37651}{144}m_1^5 m_2^3 + \frac{5852}{9}\frac{m_1^4 m_2^4}{2} + (1 \leftrightarrow 2)\right)$$

Brunello, Mandal, Patil, Pegorin, Ronca, Smith, Steinh**off**, Torres Bobadilla, & **PM** (2025)

$$\mathcal{V}^{G_N^8}_{7\text{PN}} = \frac{G_N^8}{r^8}\left(\frac{35}{128}m_1 m_2^8 + \frac{248}{9}m_1^2 m_2^7 + \frac{1059}{2}m_1^3 m_2^6 + \frac{89383}{36}m_1^4 m_2^5 + (1 \leftrightarrow 2)\right)$$

Brunello, Mandal, Patil, Pegorin,, Smith, Steinh**off**, & **PM** (2026)

Strikingly simple polynomial structure!

… All-order (nPN) structure of the static corrections to the GR-Effective Potential

Brunello, Mandal, Patil, **Pegorin**, Smith, Steinhoff, & PM (2026)

Correlation functions based approach to the Effective Potential

▸ **Einstein-Hilbert + De Donder g.f.** $$S_{\rm bulk} = S_{\rm EH} + S_{\rm GF} = -\frac{c^4}{16\pi G_d}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left(R - \frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu\right)$$

▸ **Point-Particle** $$S_{\rm pp} = \sum_{a=1,2}\int \mathrm{d}\tau\left(-m_a c\sqrt{g_{\mu\nu}u^\mu_{(a)}u^\nu_{(a)}}\right)$$

▸ **Static limit** $$S^{(v^0)}_{\rm bulk} = -\frac{c^4}{16\pi G_d}\int \mathrm{d}^{d+1}x\sqrt{\gamma}\left(\frac{c_d}{2\,c^4}\,\boldsymbol{\gamma^{ij}}\partial_i\boldsymbol{\phi}\partial_j\boldsymbol{\phi} - R[\boldsymbol{\gamma}] + \frac{1}{2}\left|\Gamma_i[\boldsymbol{\gamma}]\right|^2\right) \qquad S^{(v^0)}_{\rm pp} = \sum_{a=1,2}\int \mathrm{d}t\left(-m_a c^2 e^{\boldsymbol{\phi}(x_{(a)})/c^2}\right)$$

▸ **Path Integral Approach**

$$Z[\{\boldsymbol{x}^\mu_{(a)}\}] = e^{\frac{\mathbf{i}}{\hbar}\int \mathrm{d}t\,(\mathcal{T}-\mathcal{V}_{\rm eff})(\{\boldsymbol{x}^\mu_{(a)}\})} = \frac{1}{Z_0}\int \mathrm{D}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]e^{\frac{\mathbf{i}}{\hbar}(S_{\rm pp}+S_{\rm bulk})}\,, \qquad Z_0 = \int \mathrm{D}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]\,e^{\frac{\mathbf{i}}{\hbar}S_{\rm bulk}}$$

$$\left(\mathcal{T} - \mathcal{V}_{\rm eff}[\{\boldsymbol{x}^\mu_{(a)}\}]\right) = -\mathbf{i}\hbar\log\left(\langle e^{\frac{\mathbf{i}}{\hbar}\mathcal{S}_{\rm pp}}\rangle_{\rm bulk}\right)\,,$$

$$\mathcal{V}_{\rm eff} = \mathbf{i}\lim_{d\to 3}\int \frac{d^d\mathbf{p}}{(2\pi)^d}\,e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)}-\mathbf{x}_{(2)})}\times$$ 

$$\langle\mathcal{O}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]\rangle_{\rm bulk} = \frac{1}{Z_0}\int \mathrm{D}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]\,\mathcal{O}[\boldsymbol{\phi},\boldsymbol{\sigma}_{ij}]\,e^{\frac{\mathbf{i}}{\hbar}\mathcal{S}_{\rm bulk}}\,.$$

… All-order (nPN) structure of the static corrections to the GR-Effective Potential

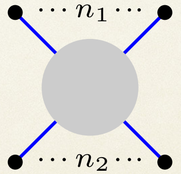
Brunello, Mandal, Patil, **Pegorin**, Smith, Steinhoff, & PM (2026)

$$\mathcal{S}_{\rm pp} = -\left(m_1\Phi_1 + m_2\Phi_2\right) \qquad \Phi_1 := e^{\phi_1}\,, \qquad \Phi_2 := e^{\phi_2}\,, \qquad \left\langle e^{\frac{i}{\hbar}\mathcal{S}_{\rm pp}}\right\rangle_{\rm bulk} = \sum_{a,b\geq 0}\left(-\frac{\mathbf{i}}{\hbar}\right)^{a+b}\frac{m_1^a}{a!}\frac{m_2^b}{b!}\,\langle\Phi_1^a\Phi_2^b\rangle_{\rm bulk}$$

$$W_{\rm bulk}(J_1,J_2) = -\mathbf{i}\hbar\log\left\langle\exp\left[\frac{\mathbf{i}}{\hbar}\Big(J_1\phi_1 + J_2\phi_2\Big)\right]\right\rangle_{\rm bulk} = \sum_{n_1,n_2\geq 1}\frac{J_1^{n_1}J_2^{n_2}}{n_1!n_2!}\Gamma_{n_1,n_2} \qquad \langle\Phi_1^a\Phi_2^b\rangle_{\rm bulk} = \exp\left(\frac{\mathbf{i}}{\hbar}W_{\rm bulk}\left(-\mathbf{i}\hbar\,a,-\mathbf{i}\hbar\,b\right)\right)$$

▸ **Connected Correlation functions**

$$\Gamma_{n_1,n_2} = \left(\frac{\mathbf{i}}{\hbar}\right)^{n_1+n_2-1}\left\langle\phi_1^{n_1}\phi_2^{n_2}\right\rangle_c =$$



$$\propto \hbar^0\left(\frac{G_N}{r}\right)^{n_1+n_2-1}$$

▸ **All-order static PN structure / Wick Theorem & Ursell series expansion**

$$\mathcal{S}_{\rm eff} = \mathcal{T} - \mathcal{V}_{\rm eff} = -\mathbf{i}\hbar\log\left[\sum_{a,b\geq 0}\left(-\frac{\mathbf{i}}{\hbar}\right)^{a+b}\frac{m_1^a}{a!}\frac{m_2^b}{b!}\exp\left(\sum_{\substack{n_1,n_2\geq 1\\ n_1+n_2\ \text{even}}}(-\mathbf{i}\hbar)^{n_1+n_2-1}\frac{a^{n_1}}{n_1!}\frac{b^{n_2}}{n_2!}\Gamma_{n_1,n_2}\right)\right]$$

$$\mathcal{V}_{k\rm PN}^{G_N^{k+1}} = \sum_{l=1}^{k+1} m_1^l m_2^{k-l+2}\frac{\widehat{\mathcal{V}}_{l,\,k-l+2}}{l!(k-l+2)!}\,, \qquad \widehat{\mathcal{V}}_{r,s} = -\lim_{\hbar\to 0}\frac{\partial^{r+s}}{\partial m_1^r\,\partial m_2^s}\left(\mathcal{S}_{\rm eff}\right)\Big|_{\substack{m_1=0\\ m_2=0}}$$

0,1,2,3,4,5,...

Tree-level coefficient

$$\mathcal{V}^{G_N}_{0\mathrm{PN}} = -m_1 m_2 \Gamma_{1,1}\,, \tag{A1}$$

$$\mathcal{V}^{G_N^2}_{1\mathrm{PN}} = \frac{1}{2} m_1 m_2^2 \Gamma_{1,1}^2 + (m_1 \leftrightarrow m_2)\,, \tag{A2}$$

2-loop coefficients

$$\mathcal{V}^{G_N^3}_{2\mathrm{PN}} = -m_1 m_2^3 \left(\frac{1}{6}\Gamma_{1,1}^3 + \frac{1}{6}\Gamma_{1,3}\right) - \frac{m_2^2 m_1^2}{2}\left(\Gamma_{1,1}^3 + \frac{1}{4}\Gamma_{2,2}\right) + (m_1 \leftrightarrow m_2)\,, \tag{A3}$$

$$\mathcal{V}^{G_N^4}_{3\mathrm{PN}} = m_1 m_2^4 \left(\frac{1}{24}\Gamma_{1,1}^4 + \frac{1}{6}\Gamma_{1,3}\Gamma_{1,1}\right) + m_1^2 m_2^3 \left(\Gamma_{1,1}^4 + \frac{1}{2}\Gamma_{1,3}\Gamma_{1,1} + \frac{1}{2}\Gamma_{2,2}\Gamma_{1,1}\right) + (m_1 \leftrightarrow m_2)\,, \tag{A4}$$

4-loop coefficients

$$\begin{aligned}
\mathcal{V}^{G_N^5}_{4\mathrm{PN}} = {} & m_1 m_2^5 \left(-\frac{1}{120}\Gamma_{1,1}^5 - \frac{1}{12}\Gamma_{1,3}{\Gamma_{1,1}}^2 - \frac{1}{120}\Gamma_{1,5}\right) \\
& + m_1^2 m_2^4 \left(-\frac{2}{3}{\Gamma_{1,1}}^5 - \frac{7}{6}\Gamma_{1,3}{\Gamma_{1,1}}^2 - \frac{1}{2}\Gamma_{2,2}{\Gamma_{1,1}}^2 - \frac{1}{48}\Gamma_{2,4}\right) \\
& + \frac{m_1^3 m_2^3}{2}\left(-\frac{9}{4}{\Gamma_{1,1}}^5 - \frac{3}{2}\Gamma_{1,3}{\Gamma_{1,1}}^2 - 2\Gamma_{2,2}{\Gamma_{1,1}}^2 - \frac{1}{36}\Gamma_{3,3}\right) \\
& + (m_1 \leftrightarrow m_2)\,,
\end{aligned} \tag{A5}$$

$$\begin{aligned}
\mathcal{V}^{G_N^6}_{5\mathrm{PN}} = {} & m_1 m_2^6 \left(\frac{1}{720}\Gamma_{1,1}^6 + \frac{1}{36}\Gamma_{1,3}\Gamma_{1,1}^3 + \frac{1}{120}\Gamma_{1,5}\Gamma_{1,1} + \frac{1}{72}\Gamma_{1,3}^2\right) \\
& + m_1^2 m_2^5 \left(\frac{1}{3}\Gamma_{1,1}^6 + \frac{4}{3}\Gamma_{1,3}\Gamma_{1,1}^3 + \frac{1}{3}\Gamma_{2,2}\Gamma_{1,1}^3 + \frac{1}{24}\Gamma_{1,5}\Gamma_{1,1} + \frac{1}{24}\Gamma_{2,4}\Gamma_{1,1} + \frac{1}{8}\Gamma_{1,3}^2 + \frac{1}{12}\Gamma_{1,3}\Gamma_{2,2}\right) \\
& + m_1^3 m_2^4 \left(3\Gamma_{1,1}^6 + \frac{49}{12}\Gamma_{1,3}\Gamma_{1,1}^3 + \frac{15}{4}\Gamma_{2,2}\Gamma_{1,1}^3 + \frac{1}{12}\Gamma_{2,4}\Gamma_{1,1} + \frac{1}{12}\Gamma_{3,3}\Gamma_{1,1} + \frac{1}{8}\Gamma_{2,2}^2 + \frac{1}{4}\Gamma_{1,3}\Gamma_{2,2} + \frac{1}{12}\Gamma_{1,3}^2\right) \\
& + (m_1 \leftrightarrow m_2)\,,
\end{aligned} \tag{A6}$$

…,6 & 7,…

$$
\begin{aligned}
\mathcal{V}^{G_N^7}_{6\mathrm{PN}} = m_1 m_2^7 \Bigg(& -\frac{\Gamma_{1,1}^7}{5040} - \frac{1}{144}\Gamma_{1,3}\Gamma_{1,1}^4 - \frac{1}{240}\Gamma_{1,5}\Gamma_{1,1}^2 - \frac{1}{72}\Gamma_{1,3}^2\Gamma_{1,1} - \frac{\Gamma_{1,7}}{5040}\Bigg) \\
& + m_1^2 m_2^6 \Bigg(-\frac{2}{15}\Gamma_{1,1}^7 - \Gamma_{1,3}\Gamma_{1,1}^4 - \frac{1}{6}\Gamma_{2,2}\Gamma_{1,1}^4 - \frac{11}{120}\Gamma_{1,5}\Gamma_{1,1}^2 - \frac{1}{24}\Gamma_{2,4}\Gamma_{1,1}^2 - \frac{5}{12}\Gamma_{1,3}^2\Gamma_{1,1} - \frac{1}{6}\Gamma_{1,3}\Gamma_{2,2}\Gamma_{1,1} - \frac{\Gamma_{2,6}}{1440}\Bigg) \\
& + m_1^3 m_2^5 \Bigg(-\frac{45}{16}\Gamma_{1,1}^7 - \frac{119}{16}\Gamma_{1,3}\Gamma_{1,1}^4 - \frac{9}{2}\Gamma_{2,2}\Gamma_{1,1}^4 - \frac{5}{48}\Gamma_{1,5}\Gamma_{1,1}^2 - \frac{7}{24}\Gamma_{2,4}\Gamma_{1,1}^2 - \frac{1}{8}\Gamma_{3,3}\Gamma_{1,1}^2 - \frac{7}{8}\Gamma_{1,3}^2\Gamma_{1,1} \\
& \qquad - \frac{3}{8}\Gamma_{2,2}^2\Gamma_{1,1} - \frac{3}{2}\Gamma_{1,3}\Gamma_{2,2}\Gamma_{1,1} - \frac{1}{720}\Gamma_{3,5}\Bigg) \\
& + \frac{m_1^4 m_2^4}{2} \Bigg(-\frac{64}{9}\Gamma_{1,1}^7 - \frac{104}{9}\Gamma_{1,3}\Gamma_{1,1}^4 - 12\Gamma_{2,2}\Gamma_{1,1}^4 - \frac{1}{6}\Gamma_{2,4}\Gamma_{1,1}^2 - \frac{5}{12}\Gamma_{3,3}\Gamma_{1,1}^2 - \frac{1}{6}\Gamma_{2,4}\Gamma_{1,1}^2 - \frac{5}{4}\Gamma_{2,2}^2\Gamma_{1,1} \\
& \qquad - 2\Gamma_{1,3}\Gamma_{2,2}\Gamma_{1,1} - \frac{17}{18}\Gamma_{1,3}^2\Gamma_{1,1} - \frac{1}{576}\Gamma_{4,4}\Bigg) \\
& + (m_1 \leftrightarrow m_2)\,,
\end{aligned}
\tag{A7}
$$

6-loop coefficients

$$
\begin{aligned}
\mathcal{V}^{G_N^8}_{7\mathrm{PN}} = m_1 m_2^8 \Bigg(& \frac{\Gamma_{1,1}^8}{40320} + \frac{1}{720}\Gamma_{1,3}\Gamma_{1,1}^5 + \frac{1}{720}\Gamma_{1,5}\Gamma_{1,1}^3 + \frac{1}{144}\Gamma_{1,3}^2\Gamma_{1,1}^2 + \frac{\Gamma_{1,7}\Gamma_{1,1}}{5040} + \frac{1}{720}\Gamma_{1,3}\Gamma_{1,5}\Bigg) \\
& + m_1^2 m_2^7 \Bigg(\frac{2}{45}\Gamma_{1,1}^8 + \frac{5}{9}\Gamma_{1,3}\Gamma_{1,1}^5 + \frac{1}{15}\Gamma_{2,2}\Gamma_{1,1}^5 + \frac{1}{10}\Gamma_{1,5}\Gamma_{1,1}^3 + \frac{1}{36}\Gamma_{2,4}\Gamma_{1,1}^3 + \frac{11}{18}\Gamma_{1,3}^2\Gamma_{1,1}^2 + \frac{1}{6}\Gamma_{1,3}\Gamma_{2,2}\Gamma_{1,1}^2 \\
& \qquad + \frac{1}{720}\Gamma_{1,7}\Gamma_{1,1} + \frac{1}{720}\Gamma_{2,6}\Gamma_{1,1} + \frac{1}{48}\Gamma_{1,3}\Gamma_{1,5} + \frac{1}{240}\Gamma_{1,5}\Gamma_{2,2} + \frac{1}{144}\Gamma_{1,3}\Gamma_{2,4}\Bigg) \\
& + m_1^3 m_2^6 \Bigg(\frac{81}{40}\Gamma_{1,1}^8 + \frac{63}{16}\Gamma_{2,2}\Gamma_{1,1}^5 + \frac{747}{80}\Gamma_{1,3}\Gamma_{1,1}^5 + \frac{2}{5}\Gamma_{1,5}\Gamma_{1,1}^3 + \frac{1}{2}\Gamma_{2,4}\Gamma_{1,1}^3 + \frac{1}{8}\Gamma_{3,3}\Gamma_{1,1}^3 \\
& \qquad + \frac{9}{16}\Gamma_{2,2}^2\Gamma_{1,1}^2 + \frac{29}{8}\Gamma_{1,3}\Gamma_{2,2}\Gamma_{1,1}^2 + \frac{27}{8}\Gamma_{1,3}^2\Gamma_{1,1}^2 + \frac{1}{240}\Gamma_{2,6}\Gamma_{1,1} + \frac{1}{240}\Gamma_{3,5}\Gamma_{1,1} \\
& \qquad + \frac{1}{48}\Gamma_{1,5}\Gamma_{2,2} + \frac{1}{24}\Gamma_{1,3}\Gamma_{2,4} + \frac{1}{48}\Gamma_{2,2}\Gamma_{2,4} + \frac{1}{240}\Gamma_{1,5}\Gamma_{1,3} + \frac{1}{72}\Gamma_{1,3}\Gamma_{3,3}\Bigg) \\
& + m_1^4 m_2^5 \Bigg(\frac{100}{9}\Gamma_{1,1}^8 + \frac{239}{9}\Gamma_{1,3}\Gamma_{1,1}^5 + \frac{70}{3}\Gamma_{2,2}\Gamma_{1,1}^5 + \frac{25}{144}\Gamma_{1,5}\Gamma_{1,1}^3 + \frac{167}{144}\Gamma_{2,4}\Gamma_{1,1}^3 \\
& \qquad + \Gamma_{3,3}\Gamma_{1,1}^3 + \frac{721}{144}\Gamma_{1,3}^2\Gamma_{1,1}^2 + \frac{9}{2}\Gamma_{2,2}^2\Gamma_{1,1}^2 + \frac{233}{24}\Gamma_{1,3}\Gamma_{2,2}\Gamma_{1,1}^2 \\
& \qquad + \frac{1}{144}\Gamma_{3,5}\Gamma_{1,1} + \frac{1}{144}\Gamma_{4,4}\Gamma_{1,1} + \frac{1}{24}\Gamma_{2,2}\Gamma_{2,4} + \frac{1}{144}\Gamma_{1,5}\Gamma_{1,3} + \frac{5}{144}\Gamma_{2,4}\Gamma_{1,3} \\
& \qquad + \frac{1}{24}\Gamma_{1,3}\Gamma_{3,3} + \frac{1}{24}\Gamma_{2,2}\Gamma_{3,3}\Bigg) \\
& + (m_1 \leftrightarrow m_2)\,.
\end{aligned}
\tag{A8}
$$

…8PN & 9PN… static corrections to the GR-Effective Potential

Unknown 8-loop coefficients

[*explicit calculation required*]

$$\nu^{G_N^9}_{\rm 8PN} = -\frac{G_N^9}{r^9}\left(\frac{35}{128}m_1m_2^9 + \left(\frac{\bar\Gamma_{2,8}}{80640}+\frac{266}{9}\right)m_1^2m_2^8 + \left(\frac{\bar\Gamma_{3,7}}{30240}+\frac{250705}{288}\right)m_1^3m_2^7 + \left(\frac{\bar\Gamma_{4,6}}{17280}+\frac{20084}{3}\right)m_1^4m_2^6 + \left(\frac{\bar\Gamma_{5,5}}{14400}+\frac{7656577}{576}\right)\frac{m_1^5m_2^5}{2}\right) + (m_1 \leftrightarrow m_2)\,,$$

$$\nu^{G_N^{10}}_{\rm 9PN} = \frac{G_N^{10}}{r^{10}}\left(\frac{63}{256}m_1m_2^{10} + \left(\frac{\bar\Gamma_{2,8}}{40320}+\frac{577}{18}\right)m_1^2m_2^9 + \left(\frac{\bar\Gamma_{2,8}}{10080}+\frac{\bar\Gamma_{3,7}}{10080}+\frac{186233}{144}\right)m_1^3m_2^8 + \left(\frac{\bar\Gamma_{3,7}}{4320}+\frac{\bar\Gamma_{4,6}}{4320}+\frac{270781}{18}\right)m_1^4m_2^7 + \left(\frac{\bar\Gamma_{4,6}}{2880}+\frac{\bar\Gamma_{5,5}}{2880}+\frac{2501525}{48}\right)m_1^5m_2^6\right) + (m_1 \leftrightarrow m_2)\,,$$

Vector Space Structure of Twisted Period Integrals

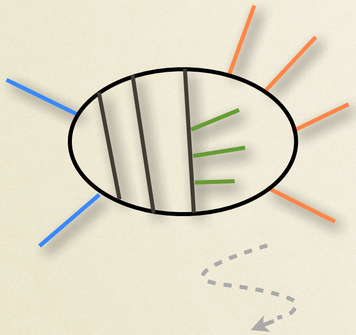
Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019, 2020)

Denominators as integration variables

Baikov (1996)

$$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$$



N-denominator
generic Integral

$$I^{[d]}_{a_1,\ldots,a_N} = \int_{\mathcal{C}} B(\mathbf{z})^{\gamma} \frac{d^n \mathbf{z}}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}} \qquad \gamma \equiv (d - E - L - 1)/2$$

$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \ldots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \ldots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z})$$

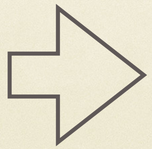
Gram Determinant

What does a (twisted) integral represent?

$$I(n, \mathbf{x}) \equiv \int_{\Gamma} f(\mathbf{x}, \mathbf{z})$$

vanishing condition
at the boundary of the integration domain

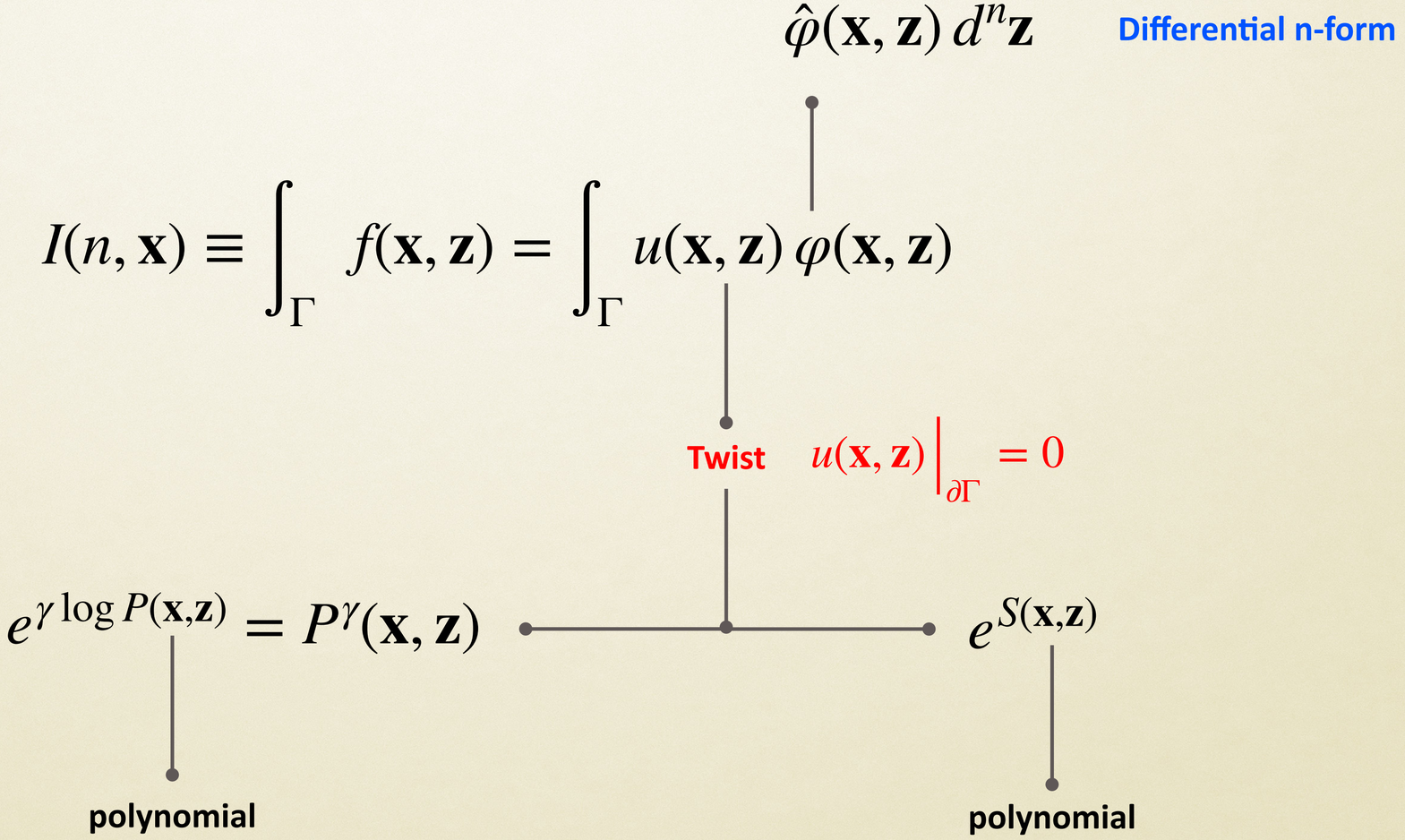
$$f(\mathbf{x}, \mathbf{z})\Big|_{\partial\Gamma} = 0$$



[Stokes' theorem]

$$\int_{\Gamma} df(\mathbf{x}, \mathbf{z}) = 0 = \int_{\partial\Gamma} f(\mathbf{x}, \mathbf{z})$$

What does a (twisted) integral represent?



What does a (twisted) integral represent?

Space of **Integrals**
and their (linear and quadratic) relations



$$I(n, \mathbf{x}) \equiv \int_{\Gamma} f(\mathbf{x}, \mathbf{z})$$

What does a (twisted) integral represent?

Space of **Integrands**
that differ by a total differential
$g = f + df$

Space of **Integrals**
and their (linear and quadratic) relations

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Space of **Integration Contours**
that differ by boundary terms
$\Sigma = \Gamma + \partial\Gamma$

What does a (twisted) integral represent?

Space of **Integrands**
that differ by a total differential
$g = f + df$

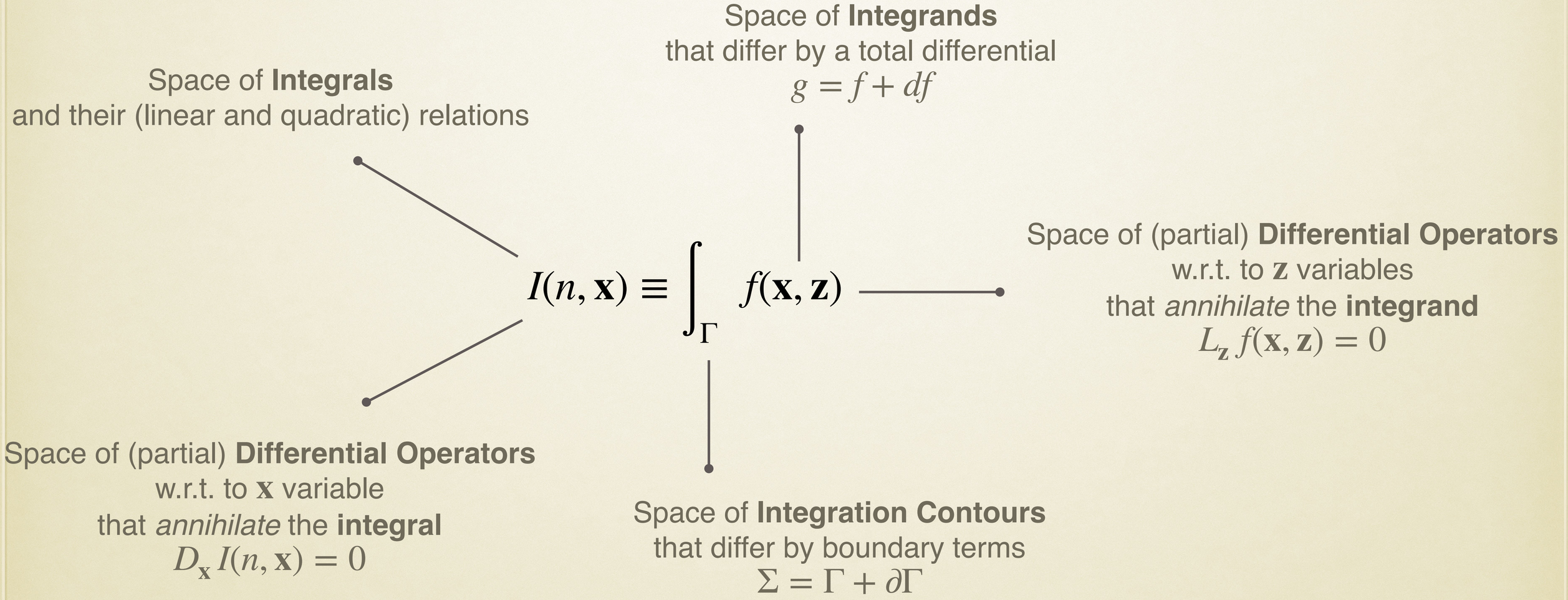
Space of **Integrals**
and their (linear and quadratic) relations

$$I(n, \mathbf{x}) \equiv \int_{\Gamma} f(\mathbf{x}, \mathbf{z})$$

Space of (partial) **Differential Operators**
w.r.t. to $\mathbf{z}$ variables
that *annihilate* the **integrand**
$L_{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) = 0$

Space of **Integration Contours**
that differ by boundary terms
$\Sigma = \Gamma + \partial\Gamma$

What does a (twisted) integral represent?



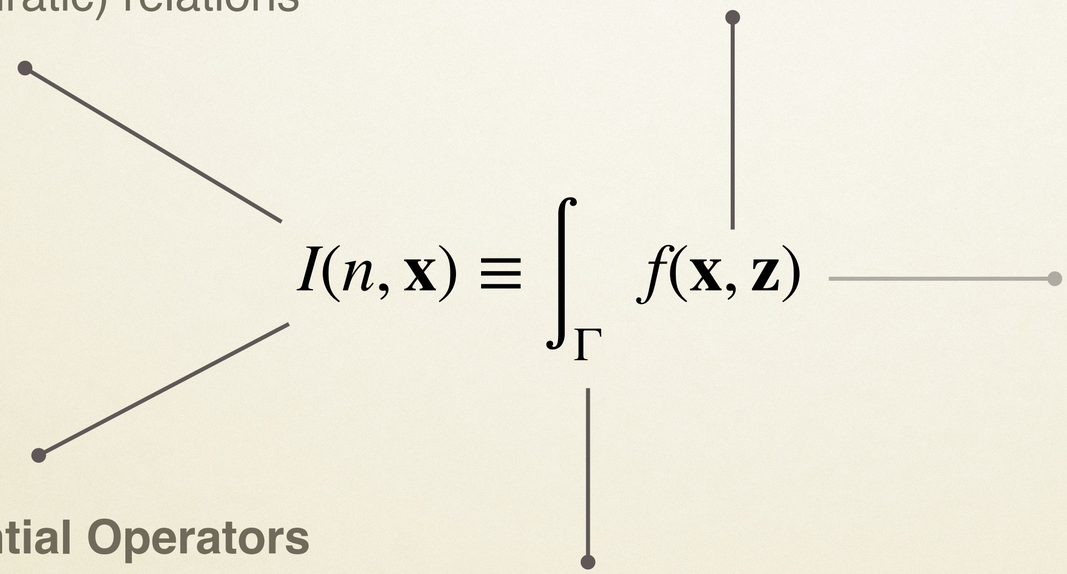
What does a (twisted) integral represent?

Integrals Vector Space

Space of **Integrals**
and their (linear and quadratic) relations

De-Rham Cohomology Group

Space of **Integrands**
that differ by a total differential
$g = f + df$



Space of (partial) **Differential Operators**
w.r.t. to $\mathbf{z}$ variables
that *annihilate* the **integrand**
$L_{\mathbf{z}}\, f(\mathbf{x}, \mathbf{z}) = 0$

D-Module
(internal variables)

Space of (partial) **Differential Operators**
w.r.t. to $\mathbf{x}$ variable
that *annihilate* the **integral**
$D_{\mathbf{x}}\, I(n, \mathbf{x}) = 0$

D-Module
(external variables)

Space of **Integration Contours**
that differ by boundary terms
$\Sigma = \Gamma + \partial\Gamma$

De-Rham Homology Group

De Rham Thm & Vector Spaces *Isomorphism/Correspondence*

generators / basis

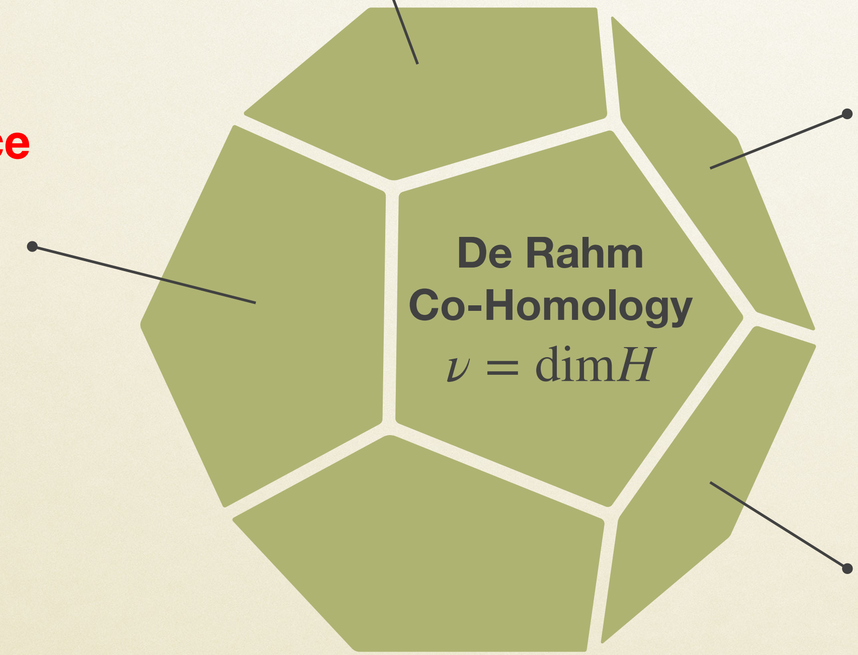
dimensions = # of basis elements

De-Rham Cohomology Group

Vector Space of differential n-forms (twisted cocycles)

Differential forms (closed / exact)

$\dim H^n_{\mathrm{dR}} =$ # of independent forms



Integrals Vector Space

Vector Space of **Feynman** (Twisted Period) Integrals

Master integrals

# of Master Integrals

De-Rham Homology Group

Vector Space of integration contours (twisted cycles)

Integration contours (closed / boundary)

$\dim H_{n,\mathrm{dR}} =$ # of independent forms

D-Module

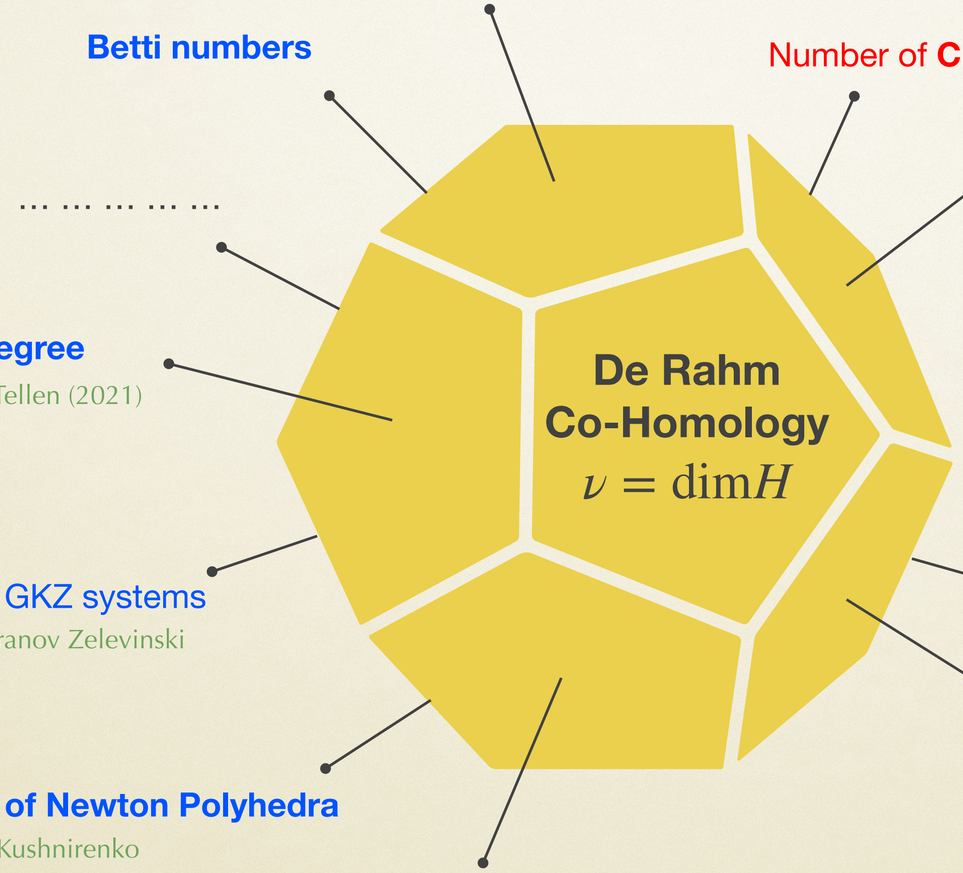
Ring of Differential operators (w.r.t. external variables)

Standard (Std) Monomials

Rank = # of independent Std's

- Annihilator ideal
- Macaulay matrix
- D-module
- Pfaffian systems
- Restrictions
- **Picard-Fuchs eq's**
- **Special Functions**

Vector Space Dimensions / counting “holes”



Chetyrkin, Tkachov (1981); Remiddi, Laporta (1996); Laporta (2000)

Number of **Master Integrals**

Betti numbers

Number of **Critical Points** Lee, Pomeranski (2013)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

… … … … …

Relation with χ_E

Aluffi, Marcolli (2008)

Bitoun, Bogner, Klausen, Panzer (2018)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Maximum likelihood degree

Agostini, Brysiewicz, Fevola, Sturmfels, Tellen (2021)

Bosma, Sogaard, Zhang (2017) Primo, Tancredi (2017)

Number of **Independent Contours**

Holonomic rank of GKZ systems

Gelfand Kapranov Zelevinski

Number of **Independent Forms**

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Mixed volume of Newton Polyhedra

Bernstein-Khobaskii-Kushnirenko

Saito Sturmfels Takayama

Ideal saturation / **dimension of quotient space**

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2020)

Vector Space Structure of *Feynman* [- *Euler-Mellin - GKZ - A-hypegeometric*] Integrals

- **Twisted Period Integral**

$$I \equiv \int_\Gamma u\,\varphi$$

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- **Vector space dimensions and Master Integrals**

$$\nu = \dim H = \#\text{ solutions of } \partial_{z_i} \log u = 0$$

$$J_i \equiv \int_\Gamma u\, e_i \qquad i = 1,\ldots,\nu$$

Master Integrals — J_i

Master forms = basis of H^n_{dR} — e_i

- **Vector decomposition**

$$I = \sum_{i=1}^{\nu} c_i\, J_i$$

IBP decomposition

- **Coefficients as Intersection Numbers**

$$c_i = J_i^\vee \cdot I = \langle e_i^\vee \,|\, \varphi \rangle \equiv \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n - \mathbb{V}} (u^{-1}\, e_i^\vee) \wedge (u\,\varphi)$$

Grothendiek Residue formula if $u = e^S$

(Generalised) Cho-Matsumoto Residue formula if $u = P^\gamma = e^{\gamma \log P}$

To Conclude:

Underpinning Correspondence: Physics vs Math

Physics	(Quantum) Field Theory	Feynman Integrals	Integrals Vector Space	Master Integrals	Integration by Parts Identities	Differential Equations	Quadratic Integral Relations	Laporta Systems
Math	Differential & Algebraic Geometry	Twisted Period Integrals	Co-Homology Groups	Master Forms & Master Contours	Contiguity Relations	Pfaffian Systems	Riemann Twisted Period Relations	Macaulay Matrices

Underpinning Correspondence: Physics vs Math

Physics	(Quantum) Field Theory	Feynman Integrals	Integrals Vector Space	Master Integrals	Integration by Parts Identities	Differential Equations	Quadratic Integral Relations	Laporta Systems
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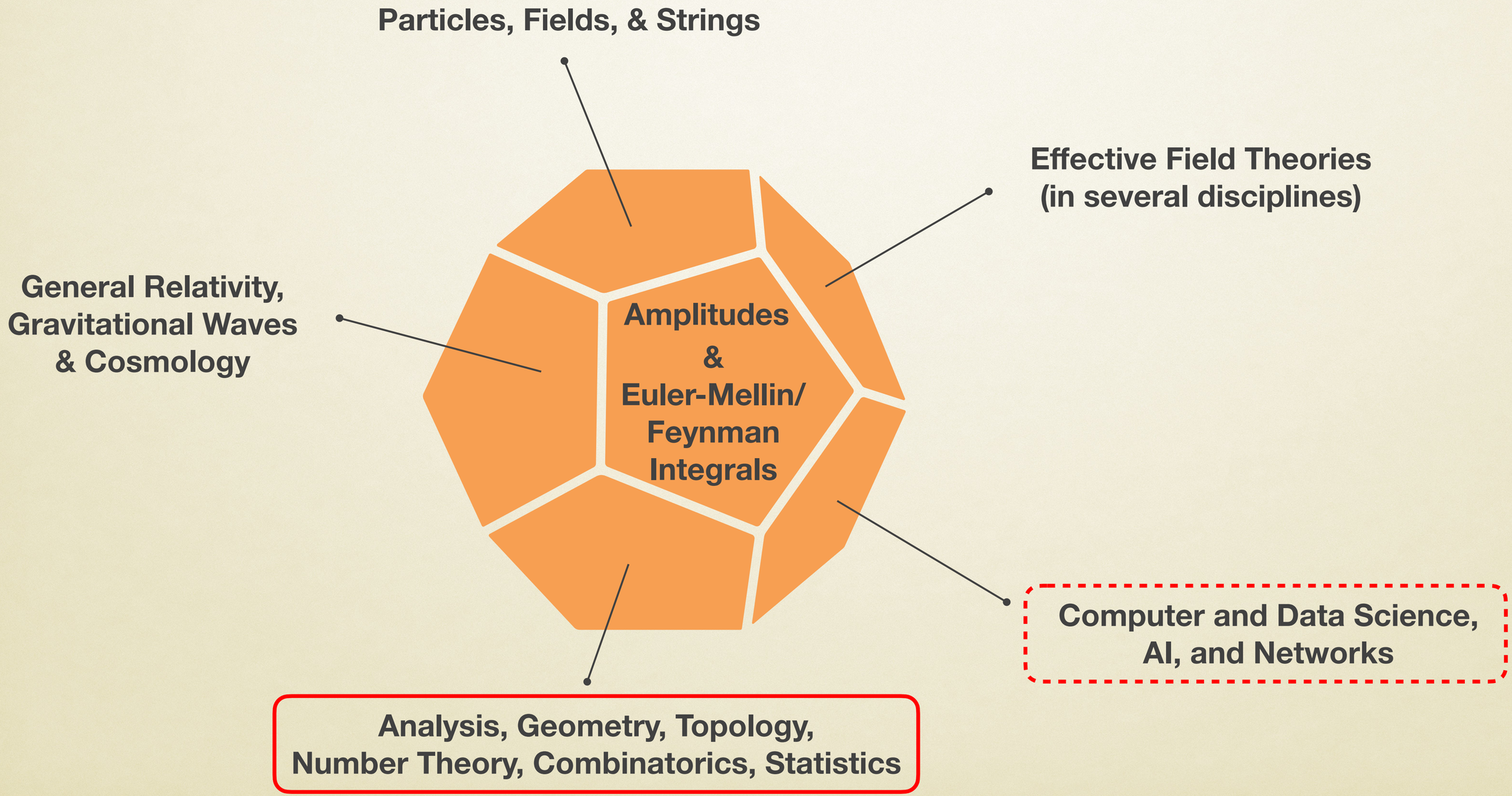
A unifying vision on Calculus in Fundamental Physics: Twisted Period Integrals

Area	High-energy Physics & String theory	General Relativity	Cosmology	Algebraic and Differential Geometry
Target	Scattering Amplitudes	Interaction Hamiltonian	Correlator Functions	Euler Mellin Integrals

Summary

- **EFT Diagrammatic Approach to General Relativity**
  - Multi-loop Calculus from Quantum Field Theory to Effective Filed Theory of Gravity
  - Precision Gravitational Wave Physics: PM and PN corrections
- **(The road to) 7PN static corrections & beyond**
  - Crucial result for the complete 6PN potential
  - (Crucial result for QCD)
  - Correlation function based approach
- **Applied Mathematical & Computational Algebraic Geometry**
  - Using state-of-the-art public software for computational algebraic geometry
  - Development of a novel high-performance IBP decomposition code (pushing the state-of-the-art) of PN and PM calculations
- **Outlook**
  - Higher-order PN and PM corrections
  - QFT & Amplitudes approach to investigate the “beautiful and simple” mathematical structures of GR

Scattering Amplitudes & Multiloop Calculus: interdisciplinary toolbox



Based on:

- Brunello, Mandal, **PM**, Patil, Pegorin, Steinhoff,
  All-order structure of static gravitational interactions and the seventh post-Newtonian potential
  accepted by Phys. Rev. Lett.

- Brunello, Mandal, **PM**, Patil, Pegorin, Ronca, Steinhoff, Torres Bobadilla,
  Six-loop gravitational interactions at the sixth post-Newtonian order
  Phys.Rev.Lett. 137 (2026) 11

- Mandal, **PM**, Patil, Steinhoff,
  Gravitational spin-orbit Hamiltonian at NNNLO in the post-Newtonian framework
  JHEP 03 (2023) 130

- Mandal, **PM**, Patil, Steinhoff,
  Gravitational quadratic in spin Hamiltonian at NNNLO in the post-Newtonian framework
  JHEP 07 (2023) 128

- Mandal, **PM**, Patil, Silva, Steinhoff,
  Gravitoelctric dynamical tides at second post-Newtonian framework
  JHEP 11 (2023) 067

- Mandal, **PM**, Patil, Silva, Steinhoff,
  Renormalizing Love: tidal effects at third post-Newtonian framework
  JHEP 02 (2024) 188

- Mandal, **PM**, Patil, Silva, Steinhoff,
  Radiating Love: adiabatic tidal fluxes and modes up to NNL post-Newtonian order
  JHEP 05 (2025) 008

- Bini, Damour, Geralico, Laporta, **PM**,
  Gravitational dynamics at O(G^6): perturbative gravitational scattering meets experimental mathematics
  2008.09389 [gr-qc]

- Bini, Damour, Geralico, Laporta, **PM**,
  Gravitational scattering at seventh order in G: non-local contribution at the sixth Post-Newtonian order
  Pays Rev D 103 (2021) 4, 044038

- Foffa, **PM**, Sturani, Sturm, Torres Bobadilla
  Static two-body potential at fifth post-Newtonian order
  Phys.Rev.Lett. 122 (2019) 24, 241605

- Foffa, **PM**, Sturani, Sturm,
  Effective field theory approach to the gravitational two-body dynamics, at fourth post-Newtonian order and quintic in the Newton constant
  Phys.Rev.D 95 (2017) 10, 104009

Acknowledgements

For collaboration and discussions:

D. Bini, T. Damour, S. Laporta, E. Remiddi,
Foffa, Sturani, Sturm,
G. Brunello, M. Mandal, R. Patil, M. Pegorin, J. Ronca, J. Steinhoff, S. Smith, W. J. Torres
V. Chestnov, G. Crisanti, M. Giroux,
W. Flieger, S. Matsubara-Heo, N. Takayama,
S. Cacciatori, L. Bresciani.