QUASILINEAR LIMIT FOR PARTICLE MOTION IN A PRESCRIBED SPECTRUM OF RANDOM WAVES

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In memoriam R. Balescu

Résumé. The one-dimensional motion of N particles in the field of many incoherent waves is revisited with nonperturbative techniques of stochastic differential equations. When the wavefield has a single wavenumber κ and white noise time dependence, it is represented as $(q/m)\mathrm{d}E(\xi, t) = \alpha[\cos(\kappa\xi)\,\mathrm{d}W_t' + \sin(\kappa\xi)\,\mathrm{d}W_t'']$. For each particle the velocity is shown to be a Wiener process with the quasilinear diffusion coefficient $\sim \alpha^2$. The joint N velocity processes define a martingale, the components of which are conjectured to become independent in the strong noise limit $\alpha \to \infty$, ensuring propagation of chaos in this system. The connection with the concept of resonance box is discussed. Full nonlinear dynamics results are compared with the linearization around particle ballistic motions. The key quantity in the analysis is the relative velocity between two particles.

The motion of particles in a stochastic force field is a fundamental problem in statistical dynamics, investigated already from many viewpoints [2]. In a first, elementary idealization one may describe the force field by a random field, so that the appropriate mathematical setting is the theory of stochastic differential equations. The aim of the present work is to establish that in an appropriate scaling limit the motions of N particles in the same field approach N independent brownian motions in velocity space, though the force acting on them is spatially correlated.

The physical context for this model is the quasilinear theory for weak turbulence in plasma physics [6, 20]. From a more general perspective, this work also relates to the issue of "propagation of chaos" in statistical physics [12, 13] : how does chaotic dynamics enable a system, in which initial data are genuinely random but the evolution may generate correlations, to behave as if the evolution regenerated randomness or destroyed correlations ? Here, how do two Wiener processes, fully

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describing a prescribed "turbulent" environment, generate N independent brownian motions for particles?

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Our model is introduced in section 1. In section 2 we discuss its quasilinear treatment and show that the resulting N-particle problem has a brownian limit. In section 3 we conjecture the brownian limit for an auxiliary problem with one degree of freedom; this is the difficult part in our analysis. We formulate an approximation scheme hinting to the conjecture. The conjecture is shown in section 4 to imply the brownian limit for the full nonlinear N-particle system in the wavefield.

1. The force field

Denote by (W'_t, W''_t) the standard Wiener process on $\mathbb{R}^2$. Recall that $W'_0 = 0$, $\mathbb{E}(\mathrm{d}W'_t) = 0$, $\mathbb{E}(\mathrm{d}W'_t \mathrm{d}W''_s) = 0$ and $\mathbb{E}(\mathrm{d}W'_t \mathrm{d}W'_s) = \delta(t-s)\mathrm{d}s\mathrm{d}t$ formally.

We consider the motion of particles in one dimension on the microscopic scale with position ξ_τ and velocity η_τ as functions of time τ in a wave field which is the superposition of a wide spectrum of time harmonics with a single wavenumber κ. The equations of motion read formally

$$\mathrm{d}\xi_t = \eta_t \mathrm{d}t \tag{1.1}$$

$$\mathrm{d}\eta_t = \alpha \sum_{m=-\infty}^{\infty} c_m \cos(\kappa\xi - m\tilde{\omega}t - \phi_m)\,\mathrm{d}t \tag{1.2}$$

where $\tilde{\omega}$ is the relative detuning of harmonics (and $\tilde{\omega}/\kappa$ is the relative phase velocity) in the spectrum. The randomness of the wave field is characterized by the distribution of c_m and ϕ_m. For independent gaussian distributions[1] for $(c_m \cos\phi_m, c_m \sin\phi_m)$ the formal equation (1.2) is interpreted in terms of Wiener processes as representing a "temporally white noisy wave"

$$\mathrm{d}\eta_t = \alpha\cos(\kappa\xi)\,\mathrm{d}W'_t + \alpha\sin(\kappa\xi)\,\mathrm{d}W''_t \tag{1.3}$$

[1] Phases being uniformly distributed on the circle and intensities c_m^2 being exponentially distributed with $\mathbb{E}c_m^2 = \tilde{\omega}/\pi$.

where we denote by α the strength of the noise. For the system (1.1)-(1.3) Itô and Stratonovich stochastic integrals are equivalent.

One expects that in the limit $\alpha \to \infty$ the chaotic motion which the force field imposes to the particle will destroy correlations in the particle motion, in spite of the strong spatial correlation associated with the single wavenumber κ. To show this we turn to macroscopic variables $(X, V) = (\kappa\xi, \alpha^{-1}\eta)$, so that the position may be considered on the 2π-periodic circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and the equations of motion read

$$\mathrm{d}X_t = AV_t \,\mathrm{d}t \tag{1.4}$$

$$\mathrm{d}V_t = \cos X_t \,\mathrm{d}W'_t + \sin X_t \,\mathrm{d}W''_t \tag{1.5}$$

with parameter $A = \kappa\alpha$ scaling now the evolution for X but not directly for V. Solving this stochastic nonlinear dynamics for (X_t, V_t) for a single particle is elementary thanks to the identity $\cos^2 x + \sin^2 x = 1$: the velocity V_t is a standard Wiener process.

However the dynamics of several particles in the same wave field is nontrivial, as we discuss below. Indeed, the single wavenumber is expected to generate correlations in the particle motions. Yet, in the limit $\alpha \to \infty$, there is good evidence that such correlations disappear [3, 4, 9, 10, 11]. A prominent example of single wavenumber dynamics is the standard map (with all $c_m = c$, $\phi_m = 0$), the dynamics of which is being compared to random dynamics (with $c_m = c$ and uniform distribution of ϕ_m) in [5].

A physicist's usual approach to such dynamics focuses on propagators, i.e. fundamental solutions to the associated diffusion problem [2]. The issue is then to start from the "Liouville" equation

$$\partial_t f + Av\partial_x f + (\dot{W}'_t \cos x + \dot{W}''_t \sin x)\partial_v f = 0 \tag{1.6}$$

where $\dot{W}_t \equiv \mathrm{d}W_t/\mathrm{d}t$ formally denotes a white noise. If the latter were a smooth function, the weak solution to (1.6) for initial data $N^{-1}\sum_{n=1}^{N}\delta(x - x_0^n)\delta(v - v_0^n)$ would follow the evolution of N particles released in the wavefield. In the propagator approach, one obtains a formal representation for the evolution of the reduced distribution $\bar{f}(.) = (2\pi)^{-1}\int f(x, .)\mathrm{d}x$ as a power series in the free evolution operator $(v\partial_x)^{-1}$ and interaction operator $(\dot{W}'_t \cos x + \dot{W}''_t \sin x)\partial_v$. Ordering terms as $A \to \infty$ indicates that the limit yields a diffusion equation

$$\partial_t \bar{f} = \frac{1}{2}\partial_v D \partial_v \bar{f} \tag{1.7}$$

where $D = 1$ is the quasilinear diffusion coefficient. The role of the limit $A \to \infty$ is to ensure a strong microscopic chaos in each particle evolution, additional assumptions ensure the gaussian law of the wavefield [1, 19].

We focus here on individual processes to keep track of their whole time evolution – so that we obtain more information than the mere conditional distributions $f(x, v, t|x_0, v_0, t_0)$. This appears as a minor issue for the noisy force in (1.5), which defines a Markov process (ξ_t, η_t), but it is central in the case where the wave field also exhibit time correlations.

2. First correction to ballistic motion

We consider N particles released simultaneously in the force field. In quasilinear theory, one approximate the position of a particle in the force term (1.5) by a ballistic motion, to get rid of the nonlinear feedback from X_t into this term. Calculations below show that the analysis is straightforward.

Let $N \in \mathbb{N}_0$, $A \in \mathbb{R}^+$, $y_0 = (y_0^n) \in \mathbb{T}^N$ and $u_0 = (u_0^n) \in \mathbb{R}^N$. We consider the velocity process U_t^A in $\mathbb{R}^N$ solution of the stochastic differential equation

$$U_0^{A,n} = u_0^n \tag{2.1}$$
$$\mathrm{d}U_t^{A,n} = \cos(y_0^n + A u_0^n t)\,\mathrm{d}W_t' + \sin(y_0^n + A u_0^n t)\,\mathrm{d}W_t''\,. \tag{2.2}$$

This simplification to the system (1.4)-(1.5) is drastic. Itô integrals reduce to Wiener integrals, and the model is scalar for U instead of being two-dimensional, so that it can be integrated directly.

Then U_t^A is a diffusion process, with infinitesimal generator

$$\sum_{n,r} \frac{\mathcal{D}_{nr}^A}{2} \partial_{u^n} \partial_{u^r} \tag{2.3}$$

where the diffusion matrix $\mathcal{D}^A = (\mathcal{D}_{nr}^A)$ has elements

$$\mathcal{D}_{nr}^A = \cos(y_0^n - y_0^r + A(u_0^n - u_0^r)t)\,. \tag{2.4}$$

This implies that, for any $1 \leq n \leq N$, the component $U_t^{A,n} - u_0^n$ is a standard Wiener process. Moreover, U_t^A is gaussian but its components are not independent for $s, t \geq 0$ as

(2.5)
$$\mathbb{E}(U_t^{A,n} - u_0^n)(U_s^{A,r} - u_0^r) = \begin{cases} \frac{\sin(y_0^n - y_0^r + A(u_0^n - u_0^r)\min(s,t)) - \sin(y_0^n - y_0^r)}{A(u_0^n - u_0^r)} & \text{if } u_0^n \neq u_0^r\,, \\ \cos(y_0^n - y_0^r)\min(s,t) & \text{if } u_0^n = u_0^r\,. \end{cases}$$

Components with large relative velocity ($|u_0^n - u_0^r| \gg 1/A$) are thus weakly correlated, a result reminiscent of locality in velocity for wave-particle interaction [3, 4].

Proposition 2.1. *Assume either (i) that all components of u_0 are different, or (ii) that y_0 is random with uniform distribution on $\mathbb{T}^N$ and u_0 is arbitrary in $\mathbb{R}^N$. For any $T > 0$ and $N > 0$, the velocity process $U_t^A - u_0$ converges in law for $A \to \infty$ to the N-dimensional standard Wiener process for $0 \le t \le T$.*

Proof : $U_t^A - u_0$ is a gaussian process, and each of its components is a standard Wiener process. It suffices then to find its covariance.

(i) If all components of u_0 are different, the covariance of U_t^A for $n \neq r$ converges to zero by (2.5).

(ii) If the initial position y_0 is random, uniformly distributed on $\mathbb{T}^N$, then the covariance (2.5) vanishes for $n \neq r$.

For $n = r$, (2.5) reduces to

$$\mathbb{E}(U_t^{A,n} - u_0^n)(U_s^{A,n} - u_0^n) = \min(s,t) \tag{2.6}$$

for any $A > 0$. □

Remark 1 : the proof still holds if u_0 depends on A, provided that $\lim_{A\to\infty} A|u_0^n - u_0^r| = \infty$ for all $n \neq r$.

Remark 2 : the proof still holds for $N' = 2N$ particles if, given the initial data $(y_0, u_0) \in \mathbb{T}^N \times \mathbb{R}^N$, we consider initial data $({y'_0}^n, {u'_0}^n) = (y_0^n, u_0^n)$ and $({y'_0}^{n+N}, {u'_0}^{n+N}) = (y_0^n \pm \pi/2, u_0^n)$ for $1 \le n \le N$. In particular, the data $(y_0^1, u_0^1) = (0,0)$ and $(y_0^2, u_0^2) = (\pi/2, 0)$ generate $U^1 = W'$ and $U^2 = W''$ respectively : the limit process for N particles is found independent from these two specific processes (which fully describe the force field in space-time (x,t)).

Remark 3 : The ballistic approximation in (2.2) for (1.4)-(1.5) is valid for short times. It breaks down when $A \int_0^t (t-s)\,\mathrm{d}V_s$ becomes of order unity, i.e. when the nonlinear aspects of the dynamics show up : the timescale for this is $\tau_{\mathrm{NL}} = A^{-2/3}$. On the other hand, particles are more or less decorrelated when $A(U_t - u_0)$ becomes of the order unity : the timescale for this is $\tau_{U\mathrm{corr}} = A^{-2}$. As $A \to \infty$ the velocity decorrelation occurs faster than the nonlinearity gets into play, which supports the predictions of the quasilinear approximation while the ballistic approximation breaks down. For long times, the independence of the successive increments to W' and W'' ensures the validity of the approximation too, while the velocity returns to earlier values.

3. An auxiliary nonlinear dynamics

Given $(x_0, v_0) \in \mathbb{T} \times \mathbb{R}$ and $A > 0$, consider the solution (R_t^A, S_t^A) to the stochastic differential system

$$R_0^A = x_0 \tag{3.1}$$

$$S_0^A = v_0 \tag{3.2}$$

$$\mathrm{d}R_t^A = AS_t^A \mathrm{d}t \tag{3.3}$$

$$\mathrm{d}S_t^A = 2(\sin \frac{R_t^A}{2}) \, \mathrm{d}W_t \, . \tag{3.4}$$

where W_t is a standard Wiener process. We denote by $\mathcal{R}_t^A$ the lift on $\mathbb{R}$ of R_t^A, so that $\mathcal{R}_t^A$ is continuous and $\mathcal{R}_t^A - R_t^A \equiv 0 \bmod (2\pi)$ with $\mathcal{R}_0^A = R_0^A$.

For any $A > 0$, this equation has a unique strong solution. For initial data $(x_0, v_0) = (0, 0)$, the solution is readily found to be $(R_t^A, S_t^A) = (0, 0)$ for all t : the origin is a trap [18]. From here to the end of the section, let $(x_0, v_0) \neq (0, 0)$ on $\mathbb{T} \times \mathbb{R}$.

Conjecture 3.1. *For $A \to \infty$, if $(x_0, v_0) \neq (0, 0)$ on $\mathbb{T} \times \mathbb{R}$, the process $(S_t^A - v_0)/\sqrt{2}$ approaches a standard Wiener process, and for any $0 < t_1 < \ldots < t_\nu$ the law of the sequence $(R_{t_n}^A)_{1 \leq n \leq \nu}$ approaches a uniform distribution on $\mathbb{T}^\nu$ independent of $S_\cdot^A$.*

Remark : The process (R_t, S_t) is an inhomogeneous diffusion, but S_t alone is not a diffusion. This is why the limit $A \to \infty$ is interesting, as it defines an "autonomous" diffusion.

As a first step towards proving this claim, we prove

Proposition 3.2. *For any A, the process S_t^A is a square-integrable continuous martingale adapted to the same filtration as W_t. Its quadratic variation is*

$$\langle S^A - v_0 \rangle_t = 2t - 2C^A(t) \tag{3.5}$$

where $C^A(t) = \int_0^t \cos R_s^A \mathrm{d}s$.

Proof : The martingale property follows from the independence of $\mathrm{d}W_t$ with respect to the process R_s^A for $s \in [0, t]$. It is continuous and square-integrable since the coefficient $2(\sin \frac{R_t^A}{2})$ is bounded. The quadratic variation is estimated using

$$\int_0^t 4 \sin^2 \frac{R_s^A}{2} \mathrm{d}s = 2t - 2 \int_0^t \cos R_s^A \mathrm{d}s \, . \tag{3.6}$$

□

To investigate the limit $A \to \infty$, we note that the unique solution of the system for finite A depends on A, and that the limit $A \to \infty$ is singular for equation (3.3). The properties of this solution will be considered in a forthcoming paper. Here we introduce the iteration scheme

$$R_t^{A,m} = x_0 + Av_0 t + A\int_0^t (t-s)\,\mathrm{d}S_s^{A,m} \tag{3.7}$$

$$S_t^{A,m+1} = v_0 + \int_0^t 2(\sin\frac{R_t^{A,m}}{2})\,\mathrm{d}W_t\,. \tag{3.8}$$

with a "seed" $S_t^{A,0}$. For finite A the process (R_t^A, S_t^A) is the fixed point of this iteration scheme. One easily proves the following

Proposition 3.3. *If the seed is a continuous martingale, the process $S_t^{A,m}$ is a square-integrable continuous martingale for each $m > 0$, and $\langle S^{A,m} - v_0\rangle_t = 2t - 2C^{A,m}(t)$ with $C^{A,m}(t) = \int_0^t \cos R_s^{A,m-1}\,\mathrm{d}s$.*

To show that S_t^A converges for $A \to \infty$ to a brownian motion, we try to show that $C^A(t)$ vanishes in the limit. Here are some results in this direction.

Proposition 3.4. *If $S_t^{A,0} = v_0 + \sqrt{2}W_t^*$ where W_t^* is a standard Wiener process independent of W_t, the process $(S_t^{A,1}, W_t)$ converges for $A \to \infty$ to a brownian motion originating from $(v_0, 0)$ with diffusion matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, for any $0 < t_1 < \ldots < t_\nu$ the law of the sequence $(R_{t_n}^{A,1})_{1\le n\le\nu}$ approaches the uniform distribution on $\mathbb{T}^\nu$ independent of $S_\cdot^{A,1}$.*

Remark : This shows that the brownian motion in $\mathbb{R}^2$ is a kind of probabilistic fixed point for the process map $((S_\cdot^{A,m} - v_0)/\sqrt{2}, W_\cdot) \mapsto ((S_\cdot^{A,m+1} - v_0)/\sqrt{2}, W_\cdot)$. However, the S process is not invariant under the map – only the distributions are invariant. One should not expect convergence of processes in probability but only their convergence in law.

Proof : For any A, $S_t^{A,m}$ is a martingale adapted to the same filtration as (W_t, W_t^*). By Lévy's theorem [14, 18], this process is a brownian motion with diffusion coefficient 2 iff $C^{A,m}$ vanishes.

To estimate $C^{A,1}$, note that the lift $\mathcal{R}_t^{A,0}$ is a gaussian process like $S_t^{A,0}$, with

$$\mathbb{E}\mathcal{R}_t^{A,0} = x_0 + Av_0 t\,, \tag{3.9}$$

$$\mathbb{E}\mathcal{R}_s^{A,0}\mathcal{R}_t^{A,0} - (x_0 + Av_0 s)(x_0 + Av_0 t)$$

$$= A^2 \int_0^s \int_0^t (s-s')(t-t') 2\delta(s'-t')\, \mathrm{d}s' \mathrm{d}t'$$

$$= \frac{2A^2}{3} \min(s^3, t^3) \tag{3.10}$$

Therefore

$$\mathbb{E} \cos R_t^{A,0} = \mathbb{E} \Re \mathrm{e}^{\mathrm{i} R_t^{A,0}} = \Re \mathrm{e}^{\mathrm{i}(x_0 + A v_0 t) - A^2 t^3/3}$$

$$= \cos(x_0 + A v_0 t) \mathrm{e}^{-A^2 t^3/3} \tag{3.11}$$

so that

$$\mathbb{E} C^{A,1}(t) = \int_0^t \cos(x_0 + A v_0 s) \mathrm{e}^{-A^2 s^3/3}\, \mathrm{d}s \tag{3.12}$$

and $\lim_{A \to \infty} \mathbb{E} C^{A,1}(t) = 0$ for any t.

Moreover,

$$\mathbb{E} C^{A,1}(t)^2 = \int_0^t \int_0^t \mathbb{E} \cos R_s^{A,0} \cos R_{s'}^{A,0}\, \mathrm{d}s' \mathrm{d}s$$

$$= \frac{1}{2} (a_+^{A,1}(t) + a_-^{A,1}(t)) \tag{3.13}$$

where

$$a_\pm^{A,1}(t) = \int_0^t \int_0^t \mathbb{E} \cos(R_s^{A,0} \pm R_{s'}^{A,0}) \mathrm{d}s' \mathrm{d}s\,. \tag{3.14}$$

Now, for $0 \leq s' \leq s$, the gaussian law of $\mathcal{R}_t^{A,0}$ yields

$$\mathbb{E} \cos(R_s^{A,0} - R_{s'}^{A,0}) = \cos(A v_0 (s - s')) \mathrm{e}^{-A^2 (s-s')^2 (s + 2s')/3} \tag{3.15}$$

$$\mathbb{E} \cos(R_s^{A,0} + R_{s'}^{A,0}) = \cos(2x_0 + A v_0 (s + s')) \mathrm{e}^{-A^2 s (s + 3 s'^2)/3} \tag{3.16}$$

so that $\lim_{A \to \infty} a_\pm^{A,1}(t) = 0$ for any $t > 0$. Therefore $\lim_{A \to \infty} C^{A,1}(t) = 0$ for any $t > 0$ with probability 1.

As to the joint process $(S_t^{A,1}, W_t)$, it is a martingale too thanks to the independence of increments $\mathrm{d}W_t$. The quadratic variation of W_t is t, and one need only prove that the cross-variation

$$\langle S^{A,1}, W \rangle_t = 2 \int_0^t \sin \frac{R_s^{A,0}}{2}\, \mathrm{d}s \tag{3.17}$$

vanishes in the limit $A \to \infty$. This is again a straightforward calculation thanks to the gaussian law of $\mathcal{R}_s^{A,0}$.

Finally, given $(k_1, \ldots, k_\nu) \in \mathbb{Z}^\nu$, $0 < t_1 < \ldots < t_\nu$, and $\varphi, \psi \in L^2(\mathbb{R})$, we shall prove that

$$a = \mathbb{E} \mathrm{e}^{\mathrm{i} \sum_n k_n R_{t_n}^{A,1} + \mathrm{i} \int_0^\infty \varphi(t)\, \mathrm{d}S_t^{A,1} + \mathrm{i} \int_0^\infty \psi(t)\, \mathrm{d}W_t} \tag{3.18}$$

vanishes if at least one k_n is nonzero. As zero entries in $(k_1, \dots, k_\nu)$ do not contribute to the exponent at all, we need only prove that a vanishes for any $\nu > 0$ and $(k_1, \dots, k_\nu) \in \mathbb{Z}_0^\nu$, and we estimate

$$\begin{aligned} a &= e^{\mathrm{i}\sum_n k_n (x_0 + A v_0 t_n)} \mathbb{E} e^{\mathrm{i}\int_0^\infty (\varphi(t) + A\sum_n k_n (t_n - t)^+)\,\mathrm{d}S_t^{A,1} + \mathrm{i}\int_0^\infty \psi(t)\,\mathrm{d}W_t} \\ &= e^{\mathrm{i}\sum_n k_n (x_0 + A v_0 t_n)} \mathbb{E} e^{-(1/2)\int_0^\infty \left((\varphi(t) + A\sum_n k_n (t_n - t)^+) 2\sin(R_t^{A,0}/2) + \psi(t)\right)^2 \mathrm{d}t} \end{aligned} \tag{3.19}$$

where we denote by f^+ the positive part of the function f. The first equality follows from (3.7). The second equality follows from the gaussian law of the brownian motion W_t which is independent of $S_\cdot^{A,0}$ and from (3.8). Then,

$$|a| = \mathbb{E} e^{-\frac{A^2}{2}\int_0^\infty \left(\psi(t)/A + (\varphi(t)/A + \sum_n k_n (t_n - t)^+) 2\sin(R_t^{A,0}/2)\right)^2 \mathrm{d}t} \tag{3.20}$$

where almost surely the integrand in the exponent is positive. Letting $\Theta = [t_{\nu-1}, t_\nu]$ we truncate the integral and expand the exponential, so that

$$|a| \le \sum_{P=0}^\infty \frac{(-A^2)^P}{P! 2^P} \int_{\Theta^P} \mathbb{E} \prod_{p=1}^P \Big(\frac{\psi(t'_p)}{A} + (\frac{\varphi(t'_p)}{A} + k_\nu (t_\nu - t'_p)) 2\sin(\frac{R_{t'_p}^{A,0}}{2})\Big)^2 \mathrm{d}t'_p \tag{3.21}$$

and in expanding the square in the product we note that the expectation of terms linear in the sine yields a vanishing contribution in the limit $A \to \infty$ after integration, for the same reasons as for $a_\pm^{A,1}$ above. Moreover, the quadratic term yields

$$4\sin^2(\frac{R_{t'_p}^{A,0}}{2}) = 2 - 2\cos R_{t'_p}^{A,0} \tag{3.22}$$

which will also contribute only via the constant 2. This yields the estimate

$$\begin{aligned} |a| &\le \sum_{P=0}^\infty \frac{(-A^2)^P}{P! 2^P} \int_{\Theta^P} \mathbb{E} \prod_{p=1}^P (\psi^2(t'_p)/A^2 + 2(\varphi(t'_p)/A + k_\nu (t_\nu - t'_p))^2)\,\mathrm{d}t'_p \\ &= e^{-\frac{A^2}{2}\int_{t_{\nu-1}}^{t_\nu} (\psi^2(t)/A^2 + 2(\varphi(t)/A + k_\nu (t_\nu - t))^2)\,\mathrm{d}t} \end{aligned} \tag{3.23}$$

which tends to 0 as $A \to \infty$ for $k_\nu \neq 0$. Note that if $k_\nu = 0$, we let $\Theta = [0, \infty[$ and are left with the usual expectation for the brownian motion in the plane for $(S_\cdot^{A,1}, W_\cdot)$.

□

The drawback with this proposition is that it involves expectations with respect to an auxiliary process W_t^*. Though the solution to (3.1)-(3.2)-(3.3)-(3.4) is unique, we must show how the sequence of processes defined from our stochastic seed $S_t^{A,0}$ converges to our process, which is independent of W_t^* – or show that the limit $A \to \infty$ is legitimate here though the iteration (3.7)-(3.8) with respect to m is performed for $A < \infty$. Comparing (3.10) with its analogue[2] for the next process shows that the presence of the auxiliary brownian motion makes a difference in the laws of our processes for finite A.

Proposition 3.5. *If $S_t^{A,0} = v_0 \neq 0$, the process $S_t^{A,1}$ converges for $A \to \infty$ to a brownian motion originating from v_0 with diffusion coefficient 2. This process is independent of $W_.$. Any sequence $(R_{t_1}^{A,1}, \dots, R_{t_\nu}^{A,1})$ converges to a random sequence with uniform distribution on $\mathbb{T}^\nu$ independent of $(S_.^{A,1}, W_.)$.*

Proof : Here $\mathcal{R}_t^{A,0} = x_0 + Av_0 t$. Then, $C^{A,1}(t) = t \cos x_0$ if $v_0 = 0$, and

$$C^{A,1}(t) = \frac{\sin(x_0 + Av_0 t) - \sin x_0}{Av_0} \tag{3.24}$$

if $v_0 \neq 0$. The latter expression vanishes for $A \to \infty$, proving the first claim.

For the second claim, we compute the cross-variation $\langle S_.^{A,1}, W_. \rangle_t = 2 \int_0^t \sin \frac{x_0 + Av_0 t}{2} \, \mathrm{d}t$ which vanishes for $A \to \infty$ unless $v_0 = 0$. For $v_0 = 0$, one finds $\langle S_.^{A,1}, W_. \rangle_t = 2t \sin(x_0/2)$.

For the third claim, we compute the characteristic function for any $(k_1, \dots, k_\nu) \in \mathbb{Z}_0^\nu$ and $\varphi, \psi \in L^2(\mathbb{R})$,

$$\begin{aligned} a &= \mathbb{E} \mathrm{e}^{\mathrm{i} \sum_n k_n R_{t_n}^{A,1} + \mathrm{i} \int_0^\infty \varphi(t) \mathrm{d} S_t^{A,1} + \mathrm{i} \int_0^\infty \psi(t) \, \mathrm{d} W_t} \\ &= \mathrm{e}^{\mathrm{i} \sum_n k_n (x_0 + Av_0 t_n)} \mathbb{E} \mathrm{e}^{\mathrm{i} A \int_0^\infty \left(\psi(t) + (\varphi(t)/A + \sum_n k_n (t_n - t)^+) 2 \sin \frac{x_0 + Av_0 t}{2} \right) \mathrm{d} W_t} \\ &= \mathrm{e}^{\mathrm{i} \sum_n k_n (x_0 + Av_0 t_n)} \mathrm{e}^{-\frac{A^2}{2} \int_0^\infty \left(\psi(t)/A + (\varphi(t)/A + \sum_n k_n (t_n - t)^+) 2 \sin((x_0 + Av_0 t)/2) \right)^2 \mathrm{d} t} . \end{aligned} \tag{3.25}$$

In the last exponential, we restrict the integral to $[t_{\nu-1}, t_\nu]$ and expand the square, so that

$$\begin{aligned} |a| &\leq \mathrm{e}^{-\frac{A^2}{2} \int_{t_{\nu-1}}^{t_\nu} \left((\varphi(t)/A + k_\nu (t_\nu - s)) 2 \sin((x_0 + Av_0 (t_\nu - s))/2) + \psi(t)/A \right)^2 \mathrm{d} s} \\ &\leq \mathrm{e}^{-A^2 g(t_{\nu-1}, t_\nu)/2} \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} g(t_{\nu-1}, t_\nu) &= \int_{t_{\nu-1}}^{t_\nu} \left((A^{-1} \varphi(t) + k_\nu (t_\nu - s)) 2 \sin \frac{x_0 + Av_0 (t_\nu - s)}{2} + A^{-1} \psi(t) \right)^2 \mathrm{d} s \\ &\geq \int_{t_{\nu-1}}^{t_\nu} k_\nu^2 (t_\nu - t)^2 2 (1 - \cos(x_0 + Av_0 t)) \, \mathrm{d} t \end{aligned}$$

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[2] The latter reads explicitly $A^2 \int_0^{\min(s,t)} 2(s - s')(t - s')(1 - \cos(x_0 + Av_0 s')) \, \mathrm{d} s'$.

$$+\frac{2}{A}\int_{t_{\nu-1}}^{t_\nu} k_\nu(t_\nu - s)\big(\varphi(t) 2\sin\frac{x_0 + Av_0(t_\nu - s)}{2} + \psi(t)\big)\,\mathrm{d}s\,. \tag{3.27}$$

In the last expression, the second integral is bounded by the Schwarz inequality. In the first integral (which can be computed explicitly) the trigonometric part tends to zero by the Riemann-Lebesgue lemma, so that

$$\lim_{A\to\infty} g(t_{\nu-1}, t_\nu) = \int_{t_{\nu-1}}^{t_\nu} k_\nu^2(t_\nu - t)^2\, 2\,\mathrm{d}t = 2k_\nu^2(t_\nu - t_{\nu-1})^3/3 \tag{3.28}$$

and $\lim_{A\to\infty}|a| = 0$. Therefore, $(R^{A,1}_{t_1}, \dots, R^{A,1}_{t_\nu})$ converges to a random sequence with uniform distribution on $\mathbb{T}^\nu$ independent of $(S^{A,1}_{\cdot}, W_{\cdot})$ $\square$

Proposition 3.6. *If $S^{A,0}_t = v_0 \neq 0$, the process $S^{A,2}_t$ converges for $A \to \infty$ to a brownian motion originating from v_0 with diffusion coefficient* 2.

Proof : By proposition 3.5 we have for $s, t > 0$, with $s \neq t$,

$$\lim_{A\to\infty} \mathbb{E}\cos R^{A,1}_t = \lim_{A\to\infty} \mathbb{E}\cos R^{A,1}_s \cos R^{A,1}_t = 0 \tag{3.29}$$

so that $\lim_{A\to\infty} \mathbb{E}C^{A,2}(t) = 0$ and $\lim_{A\to\infty} \mathbb{E}C^{A,2}(t)^2 = 0$. Hence $C^{A,2}(t) \to 0$ with probability 1. $\square$

To iterate the scheme, we see that as long as the law of any pair $(R^{A,m}_{t_1}, R^{A,m}_{t_2})$ with $t_2 > t_1 > 0$ is uniform on $\mathbb{T}^2$ and independent of $(S^{A,m}_{\cdot}, W_{\cdot})$ as $A \to \infty$ we also have $\lim_{A\to\infty} C^{A,m+1}(t) = 0$ with probability 1. The harder part is to prove that $a \to 0$ for $(k_1, \dots, k_\nu) \in \mathbb{Z}^\nu_0$ and $\psi, \varphi \in L^2(\mathbb{R})$, where

$$\begin{aligned} a &= \mathbb{E}\mathrm{e}^{\mathrm{i}\sum_n k_n R^{A,m+1}_{t_n} + \mathrm{i}\int_0^\infty \varphi(t)\,\mathrm{d}S^{A,m+1}_t + \mathrm{i}\int_0^\infty \psi(t)\,\mathrm{d}W_t} \\ &= \mathrm{e}^{\mathrm{i}\sum_n k_n(x_0 + Av_0 t_n)}\mathbb{E}\mathrm{e}^{\mathrm{i}A\int_0^\infty (\psi(t)/A + (\varphi(t)/A + \sum_n k_n(t_n - t)^+)2\sin\frac{R^{A,m}_t}{2})\mathrm{d}W_t} \end{aligned} \tag{3.30}$$

so that

$$|a| = |\mathbb{E}\mathrm{e}^{\mathrm{i}A\int_0^\infty (\psi(t)/A + (\varphi(t)/A + \sum_n k_n(t_n - t)^+)2\sin\frac{R^{A,m}_t}{2})\mathrm{d}W_t}|\,. \tag{3.31}$$

Assuming the limit $A \to \infty$ commutes with the stochastic integral with respect to W_t, the (asymptotic) independence of $R^{A,m}_t$ simplifies this estimate to

$$\begin{aligned} |a| &= \mathbb{E}\mathrm{e}^{-\frac{A^2}{2}\int_0^\infty (\psi(t)/A + (\varphi(t)/A + \sum_n k_n(t_n - t)^+)2\sin\frac{R^{A,m}_t}{2})^2\,\mathrm{d}t} \\ &= \sum_{P=0}^{\infty}\frac{(-A^2)^P}{P!2^P}\int_0^\infty \mathbb{E}\prod_{p=1}^{P}\Big(\frac{\psi(t'_p)}{A} + (\frac{\varphi(t'_p)}{A} + \sum_n k_n(t_n - t'_p)^+)2\sin(\frac{R^{A,m}_{t'_p}}{2})\Big)^2\,\mathrm{d}t'_p\,. \end{aligned} \tag{3.32}$$

For each finite P, the expectation would leave only the contribution $1/2$ from each square sine,

$$|a| = \sum_{P=0}^{\infty} \frac{(-A^2)^P}{P! 2^P} \int_0^\infty \prod_{p=1}^{P} \Big(\frac{\psi^2(t'_p)}{A^2} + 2(\frac{\varphi(t'_p)}{A} + \sum_n k_n (t_n - t'_p)^+)^2 \Big)^2 \mathrm{d}t'_p$$

$$(3.33)\ \mathrm{e}^{-\frac{A^2}{2} \int_0^\infty \big(\frac{\psi^2(t'_p)}{A^2} + 2(\frac{\varphi(t)}{A} + \sum_n k_n (t_n - t)^+)^2 \big)^2 \mathrm{d}t} .$$

Again, if $k_\nu \neq 0$, restricting the integral to $[t_{\nu-1}, t_\nu]$ yields a vanishing estimate ; and if all k_n vanish, one recovers the characteristic functional for the brownian motion $(S_., W_.)$. This argument is heuristic but does not prove the conjecture, because of its nonrigorous exchange of limits.

4. Nonlinear dynamics of N particles

Given $(x_0, v_0) \in \mathbb{T}^N \times \mathbb{R}^N$ (with all components (x_0^n, v_0^n) distinct in $\mathbb{T} \times \mathbb{R}$), consider now the solution (X_t^A, V_t^A) to the stochastic differential system

$$X_0^{A,n} = x_0^n \tag{4.1}$$

$$V_0^{A,n} = v_0^n \tag{4.2}$$

$$\mathrm{d}X_t^{A,n} = A V_t^{A,n} \, \mathrm{d}t \tag{4.3}$$

$$\mathrm{d}V_t^{A,n} = (\cos X_t^{A,n}) \, \mathrm{d}W'_t + (\sin X_t^{A,n}) \, \mathrm{d}W''_t . \tag{4.4}$$

We denote by $\mathcal{X}_t^A$ the continuous lift of X_t^A on $\mathbb{R}^N$ with $\mathcal{X}_0^A = X_0^A$.

Proposition 4.1. *The process (X_t^A, V_t^A) is a diffusion, with infinitesimal generator*

$$\sum_n A v^n \partial_{x^n} + \sum_{n,r} \frac{D_{nr}}{2} \partial_{v^n} \partial_{v^r} \tag{4.5}$$

where the elements of the diffusion matrix $D = (D_{nr})$ are

$$D_{nr} = \cos(x^n - x^r) . \tag{4.6}$$

Moreover, V_t^A is a continuous, square-integrable martingale, adapted to the same filtration as (W'_t, W''_t).

Proof : The diffusion property follows immediately from the coefficients in the differential equation. The martingale property too. □

Proposition 4.2. *For any $1 \leq n \leq N$, the single component $V_t^{A,n} - v_0^n$ is a standard Wiener process, and $(\mathcal{X}_t^{A,n}, V_t^{A,n})$ is gaussian with*

$$\mathbb{E}\mathcal{X}_t^{A,n} = x_0^n + A v_0^n t \tag{4.7}$$

$$\mathbb{E}V_t^{A,n} = v_0^n \tag{4.8}$$

$$\begin{aligned} & \mathbb{E}(\mathcal{X}_t^{A,n} - x_0^n - Av_0^n t)(\mathcal{X}_s^{A,n} - x_0^n - Av_0^n s) \\ (4.9) \qquad = & \ \frac{A^2}{6}\min(s^2, t^2)(s+t+2|s-t|) \end{aligned}$$

$$\begin{aligned} & \mathbb{E}(\mathcal{X}_t^{A,n} - x_0^n - Av_0^n t)(V_s^{A,n} - v_0^n) \\ (4.10) \qquad = & \ \frac{A}{2}(t^2 - (t-s)^2 1_{s\leq t}) \end{aligned}$$

$$(4.11) \qquad \mathbb{E}(V_t^{A,n} - v_0^n)(V_s^{A,n} - v_0^n) \ = \ \min(s,t)$$

for $0 \leq s < \infty$, $0 \leq t < \infty$.

Proof : For $A > 0$ and any n, the process $(\mathcal{X}_t^{A,n}, V_t^{A,n})$ is a diffusion on $\mathbb{R}^{2N}$ with generator $Av^n\partial_{x^n} + \frac{1}{2}\partial_{v^n}^2$, which proves the claim. □

Remark : This implies that $\mathbb{E}\cos k(X_t^{A,n} - x_0^n - Av_0^n t) = \mathrm{e}^{-k^2A^2t^3/6}$ for any $k \in \mathbb{R}$, $t > 0$.

Proposition 4.3. *For any* $N > 0$*, the velocity process* $V_t^A - v_0$ *converges for* $A \to \infty$ *to the* N*-dimensional standard Wiener process for* $0 \leq t < \infty$ *if conjecture 3.1 holds.*

Proof : We use Lévy's characterization of the brownian motion [14]. By the above propositions, $V_t^A - v_0$ is a continuous, square-integrable martingale, and each of its components is a standard Wiener process on $\mathbb{R}$. This property carries over in the limit $A \to \infty$.

Thus it suffices to show that the cross-variation process $\langle V^{A,1}, V^{A,2}\rangle_t$ vanishes in the limit $A \to \infty$, i.e. that in this limit $V_t^{A,1}V_t^{A,2}$ is also a martingale. Denoting by $\mathcal{F}_s$ the filtration adapted to (W_s', W_s'') we now show that

$$(4.12) \qquad \mathbb{E}(V_t^{A,1}V_t^{A,2} - V_s^{A,1}V_s^{A,2}|\mathcal{F}_s) = \mathbb{E}(V_t^{A,1}V_t^{A,2}|\mathcal{F}_s) - V_s^{A,1}V_s^{A,2}$$

vanishes in the limit $A \to \infty$. First,

$$\begin{aligned} & \mathbb{E}(V_t^{A,1}V_t^{A,2} - V_s^{A,1}V_s^{A,2}|\mathcal{F}_s) \\ = & \int_s^t\int_s^t \mathbb{E}(\cos X_\tau^{A,1}\cos X_{\tau'}^{A,2}|\mathcal{F}_s)\,\mathrm{d}W_\tau'\mathrm{d}W_{\tau'}' \\ & + \int_s^t\int_s^t \mathbb{E}(\sin X_\tau^{A,1}\sin X_{\tau'}^{A,2}|\mathcal{F}_s)\,\mathrm{d}W_\tau''\mathrm{d}W_{\tau'}'' \\ = & \int_s^t \mathbb{E}(\cos(X_\tau^{A,2} - X_\tau^{A,1})|\mathcal{F}_s)\,\mathrm{d}\tau \\ (4.13) \qquad = & \int_s^t \mathbb{E}(\cos R_\tau^A|\mathcal{F}_s)\,\mathrm{d}\tau \end{aligned}$$

where the first equality uses the independence between W' and W''. Moreover, the increments of (W_t', W_t'') at time t are independent from the paths $X_s^{A,n}$ for $0 \le s \le t$. Last, we introduced the process $R_t^A = X_t^{A,2} - X_t^{A,1}$, with initial data $x_0 = x_0^2 - x_0^1$.

Given the process $V_t^{A,1}$ we define the process W_t^A by $W_0^A = 0$ and

$$\mathrm{d}W_t^A = -\sin(X_t^{A,1})\,\mathrm{d}W_t' + \cos(X_t^{A,1})\,\mathrm{d}W_t'' \tag{4.14}$$

so that $(V_t^{A,1} - v_0^1, W_t^A)$ is a two-dimensional standard Wiener process. It follows that

$$\mathrm{d}W_t' = \cos X_t^{A,1}\,\mathrm{d}V_t^{A,1} - \sin X_t^{A,1}\,\mathrm{d}W_t^A \tag{4.15}$$

$$\mathrm{d}W_t'' = \sin X_t^{A,1}\,\mathrm{d}V_t^{A,1} + \cos X_t^{A,1}\,\mathrm{d}W_t^A\,. \tag{4.16}$$

Then for $S_t^A = V_t^{A,2} - V_t^{A,1}$:

$$\mathrm{d}R_t^A = AS_t^A\,\mathrm{d}t \tag{4.17}$$

$$\mathrm{d}S_t^A = (\cos R_t^A - 1)\,\mathrm{d}V_t^{A,1} + \sin R_t^A\,\mathrm{d}W_t^A\,. \tag{4.18}$$

Clearly (R_t^A, S_t^A) is a diffusion process, such that $\mathbb{E}\mathrm{d}S_t^A = 0$ and

$$\mathbb{E}\mathrm{d}S_t^A\,\mathrm{d}S_s^A = 2(1 - \mathbb{E}\cos R_t^A)\delta(s-t)\,\mathrm{d}s\mathrm{d}t \tag{4.19}$$

so that (3.4) can be replaced with

$$\mathrm{d}S_t^A = \sqrt{2(1-\cos R_t^A)}\,\mathrm{d}\tilde{W}_t = 2\sin\frac{R_t^A}{2}\,\mathrm{d}\tilde{W}_t' \tag{4.20}$$

where $\tilde{W}_t$ and $\tilde{W}_t'$ are standard Wiener processes.

Recall that $(0,0)$ in $\mathbb{T}\times\mathbb{R}$ is a trap [18] for this process (R_t^A, S_t^A) : physically speaking, two particles released at the same point $(x_0^2 = x_0^1)$ with the same velocity $(v_0^2 = v_0^1)$ follow almost surely the same trajectory under the force field. Reversibility of the system (4.17)-(4.18) suggests however that almost surely a trajectory starting away from $(0,0)$ never visits this point; however, the subtleties of time reversal for Itô integrals call for a separate proof in a forthcoming paper.

Thus we restrict the discussion to the open set $\Lambda = \mathbb{T}\times\mathbb{R}\setminus\{(0,0)\}$ and invoke the results of the previous section. All cross-variations $\langle V^{A,r}, V^{A,n}\rangle_t$ are shown in the same way to vanish for $r \neq n$ as $A \to \infty$. □

There is no need to prove threefold or higher independence between processes $V_t^{A,n}$ as $A \to \infty$ because the convergence of the cross-variation matrix to t times the identity ensures the brownian limit of V_t^A.

Remark : there is a coupling between $V_t^{A,1}$ and $U_t^{A,1}$. Let $\mathrm{d}\bar{W}_t' + \mathrm{i}\mathrm{d}\bar{W}_t'' = \mathrm{e}^{\mathrm{i}(y_0^1 + Au_0^1 t - X_t^{A,1})}(\mathrm{d}W_t' + \mathrm{i}\mathrm{d}W_t'')$. This rotation of the two-dimensional brownian motion in the plane (W_t', W_t'') is well defined because $X_t^{A,1}$ depends only on the history of (W_t', W_t'') and on initial data (x_0^1, v_0^1), (y_0^1, u_0^1). Then $(\bar{W}_t', \bar{W}_t'')$ is also a two-dimensional brownian motion in the plane.

5. Perspectives

Early works on quasilinear transport focused on diffusion-type equations for distribution functions and the generation of correlations, dubbed "clumps" [7, 15, 16, 17]. Here we use the language of stochastic processes to describe full particle trajectories and take advantage of $A \to \infty$ to obtain the brownian limit, and we hope to produce the proof of the conjecture in the near future [8]. One may expect that this extension will enable a more complete discussion of the self-consistent field-particle dynamics too and a deeper understanding of transport in turbulent many-body systems.

Références

[1] Adam, J-C, Laval, G., and Pesme, D. *Effet des interactions résonnantes ondes-particules en turbulence faible des plasmas*, **Ann. Phys. Fr. 6** (1981) 319-420.

[2] Balescu, R. **Statistical dynamics – Matter out of equilibrium**, Imperial college press, London, 1997.

[3] Bénisti, D., and Escande, D.F. *Origin of diffusion in hamiltonian dynamics*, **Phys. Plasmas 4** (1997) 1576-1581.

[4] Bénisti, D., and Escande, D.F. *Finite range of large perturbations in hamiltonian dynamics*, **J. Stat. Phys. 92** (1998) 909-972.

[5] Bénisti, D., and Escande, D.F. *Nonstandard diffusion properties of the standard map*, **Phys. Rev. Lett. 80** (1998) 4871-4874.

[6] Drummond, W.E., and Pines, D. *Nonlinear stability of plasma oscillations*, **Nuclear Fusion Suppl. 3** (1962) 1049-1057.

[7] Dupree, T.H. *Theory of phase space density granulation in plasma*, **Phys. Fluids 15** (1972) 334-344.

[8] Elskens, Y., and Pardoux, E. to be published.

[9] Elskens, Y., and Escande, D. **Microscopic dynamics of plasmas and chaos**, IoP publishing, Bristol, 2003.

[10] Escande, D.F., and Elskens, Y. *Quasilinear diffusion for the chaotic motion of a particle in a set of longitudinal waves*, **Acta Phys. Pol. B 33** (2002) 1073-1084.

[11] Escande, D.F., and Elskens, Y. *Microscopic dynamics of plasmas and chaos : the wave-particle interaction paradigm*, **Plasma Phys. Control. Fusion 45** (2003) A115-A124.

[12] Kac, M. *Foundations of kinetic theory*, pp. 171-197 in **Proc. 3rd Berkely Symp. Math. Stat. Prob.** (ed. Neyman, J.) **3**, University of California, 1956.

[13] Kac, M. **Probability and related topics in physical sciences**, American mathematical society, Providence, 1959.

[14] Karatzas, I., and Schreve, S.E. **Brownian motion and stochastic calculus**, Springer, New York, 1988.

[15] Misguich, J.H, and Balescu, R. *"Clump" as enhanced correlations and turbulent Debye screening*, **Plasma Phys. 20** (1978) 781-823.

[16] Misguich, J.H, and Balescu, R. *The Ljapunov exponent of plasma clumps : Relative diffusion of charged particles correlated by turbulent electric fields*, **Annual Workshop on Nonlinear Dynamics, Dynamics Days Twente**, Enschede (NL) (May 6-7, 1981).

[17] Misguich, J.H, and Balescu, R. *On relative spatial diffusion in plasma and fluid turbulences : Clumps, Richardson's law and intrinsic stochasticity*, **Plasma Phys. 24** (1982) 289-318.

[18] Øksendal B. **Stochastic differential equations**, Springer, Berlin, 2003.

[19] Pesme, D. *The Fokker-Planck description in the context of quasilinear theory*, **Physica Scr. T50** (1994) 7-14.

[20] Vedenov, A.A., Velikhov, E.P., and Sagdeev, R.Z. *Quasilinear theory of plasma oscillations*, **Nuclear Fusion Suppl. 2** (1962) 465-475.

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