

Computed Results

Contents

1	R_2	2
2	R_4	3
2.1	$R_{4,2}$	3
2.1.1	$R_{4,2}, N^\nu \neq 0$	3
2.1.2	$R_{4,2}, N^\nu = 0$	3
2.1.3	$R_{4,2}$, non simplified, $N^\nu \neq 0$	4
2.1.4	$R_{4,2}$, non simplified, $N^\nu = 0$	4
2.2	$R_{4,3}$	6
2.2.1	$R_{4,3}, N^\nu \neq 0$	6
2.2.2	$R_{4,3}, N^\nu = 0$	7
2.2.3	$R_{4,3}$, non simplified, $N^\nu \neq 0$	7
2.2.4	$R_{4,3}$, non simplified, $N^\nu = 0$	8
2.3	$R_{4,4}$	9
2.4	$R_{4,0}$	10
2.4.1	$R_{4,0}, N^\nu \neq 0$	10
2.4.2	$R_{4,0}, N^\nu = 0$	41
2.4.3	$R_{4,0}$, non simplified, $N^\nu \neq 0$	50
2.4.4	$R_{4,0}$, non simplified, $N^\nu = 0$	80
3	R_4 for u parallel	89
3.1	Case $N^\mu \neq 0$	89
3.2	Case $N^\mu = 0$	92
4	R_2 for Non Commutative Torus	93
4.1	R_2 for P_1	93
4.2	R_2 for P_2	93
4.3	R_2 for $P_1 + P_2$	95
4.4	R_2 for $P_1 + P_2$ with $d = 2$	97

1 R_2

The following result has been checked from expression computed by hand in terms of p^μ .

$$\begin{aligned}
R_2 &= \frac{1}{6} \Re X_1[u] \quad (\#1) \\
&\quad - (d+2) g^{\nu_1 \nu_2} X_3[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#2) \\
&\quad + \frac{1}{2} (d+2)(d+4) g^{\nu_1 \nu_2} X_4[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#3) \\
&\quad + X_2[u \otimes (\widehat{\Delta} u)] \quad (\#4) \\
&\quad - \frac{1}{2} (d+2) X_3[u \otimes (\widehat{\Delta} u) \otimes u] \quad (\#5) \\
&\quad + X_1[q] \quad (\#6) \\
&\quad + X_2[N^{\nu_1} \otimes (\widehat{\nabla}_{\nu_1} u)] \quad (\#7) \\
&\quad - \frac{1}{2} (d+2) X_3[N^{\nu_1} \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u] \quad (\#8) \\
&\quad + \frac{1}{2} (d+2) X_3[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes N^{\nu_1}] \quad (\#9) \\
&\quad - \frac{1}{2} g_{\nu_1 \nu_2} X_2[N^{\nu_1} \otimes N^{\nu_2}] \quad (\#10) \\
&\quad - X_2[u \otimes (\widehat{\nabla}_{\nu_1} N^{\nu_1})] \quad (\#11)
\end{aligned}$$

$$\begin{aligned}
R_2(N=0) &= \frac{1}{6} \Re X_1[u] \quad (\#1) \\
&\quad - (d+2) g^{\nu_1 \nu_2} X_3[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#2) \\
&\quad + \frac{1}{2} (d+2)(d+4) g^{\nu_1 \nu_2} X_4[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#3) \\
&\quad + X_2[u \otimes (\widehat{\Delta} u)] \quad (\#4) \\
&\quad - \frac{1}{2} (d+2) X_3[u \otimes (\widehat{\Delta} u) \otimes u] \quad (\#5) \\
&\quad + X_1[q] \quad (\#6)
\end{aligned}$$

2 R_4

2.1 $R_{4,2}$

2.1.1 $R_{4,2}, N^\nu \neq 0$

$$\begin{aligned}
R_{4,2} &= 2g^{\nu_2\nu_3} X_4[N^{\nu_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_3} u)] F_{\nu_1\nu_2} \quad (\#1) \\
&\quad - \frac{1}{2} d g^{\nu_2\nu_3} X_4[N^{\nu_1} \otimes u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes u] F_{\nu_1\nu_2} \quad (\#2) \\
&\quad - 2g^{\nu_2\nu_3} X_4[u \otimes u \otimes N^{\nu_1} \otimes (\widehat{\nabla}_{\nu_3} u)] F_{\nu_1\nu_2} \quad (\#3) \\
&\quad + \frac{1}{2} d g^{\nu_2\nu_3} X_4[u \otimes N^{\nu_1} \otimes (\widehat{\nabla}_{\nu_3} u) \otimes u] F_{\nu_1\nu_2} \quad (\#4) \\
&\quad + 2g^{\nu_2\nu_3} X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes N^{\nu_1}] F_{\nu_1\nu_2} \quad (\#5) \\
&\quad - \frac{1}{2} d g^{\nu_2\nu_3} X_4[u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes u \otimes N^{\nu_1}] F_{\nu_1\nu_2} \quad (\#6) \\
&\quad + \frac{1}{2} (d+4) g^{\nu_2\nu_3} X_4[u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes N^{\nu_1} \otimes u] F_{\nu_1\nu_2} \quad (\#7) \\
&\quad + \frac{1}{2} X_3[u \otimes N^{\nu_1} \otimes N^{\nu_2}] F_{\nu_1\nu_2} \quad (\#8) \\
&\quad - \frac{1}{2} X_3[N^{\nu_1} \otimes u \otimes N^{\nu_2}] F_{\nu_1\nu_2} \quad (\#9) \\
&\quad + \frac{1}{2} X_3[N^{\nu_1} \otimes N^{\nu_2} \otimes u] F_{\nu_1\nu_2} \quad (\#10) \\
&\quad - g^{\nu_2\nu_3} X_3[u \otimes (\widehat{\nabla}_{\nu_3} N^{\nu_1}) \otimes u] F_{\nu_1\nu_2} \quad (\#11) \\
&\quad - 8(d-2) X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#12) \\
&\quad + (d^2 - 4d + 16) X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#13) \\
&\quad - 8(d-2) X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] F^{\nu_1\nu_2} \quad (\#14) \\
&\quad + (d^2 - 2d + 16) X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#15) \\
&\quad - 6d X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#16) \\
&\quad + (d^2 + 8) X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] F^{\nu_1\nu_2} \quad (\#17) \\
&\quad + (d^2 - 4d + 8) X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] F^{\nu_1\nu_2} \quad (\#18) \\
&\quad - 8(d-1) X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes u] F^{\nu_1\nu_2} \quad (\#19) \\
&\quad + (d^2 + 8) X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes u] F^{\nu_1\nu_2} \quad (\#20) \\
&\quad - 6d X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes u \otimes u] F^{\nu_1\nu_2} \quad (\#21) \\
&\quad - 4(d-1) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1\nu_2}^2 u) \otimes u] F^{\nu_1\nu_2} \quad (\#22) \\
&\quad - 2d X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_1\nu_2}^2 u) \otimes u \otimes u] F^{\nu_1\nu_2} \quad (\#23) \\
&\quad - 2(3d-4) X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1\nu_2}^2 u)] F^{\nu_1\nu_2} \quad (\#24)
\end{aligned}$$

2.1.2 $R_{4,2}, N^\nu = 0$

$$\begin{aligned}
R_{4,2}(N=0) &= -8(d-2) X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#1) \\
&\quad + (d^2 - 4d + 16) X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#2) \\
&\quad - 8(d-2) X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] F^{\nu_1\nu_2} \quad (\#3) \\
&\quad + (d^2 - 2d + 16) X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#4) \\
&\quad - 6d X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] F^{\nu_1\nu_2} \quad (\#5) \\
&\quad + (d^2 + 8) X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] F^{\nu_1\nu_2} \quad (\#6) \\
&\quad + (d^2 - 4d + 8) X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] F^{\nu_1\nu_2} \quad (\#7) \\
&\quad - 8(d-1) X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes u] F^{\nu_1\nu_2} \quad (\#8) \\
&\quad + (d^2 + 8) X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes u] F^{\nu_1\nu_2} \quad (\#9)
\end{aligned}$$

$$-6d X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u \otimes u] F^{v_1 v_2} \quad (\#10)$$

$$-4(d-1) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u] F^{v_1 v_2} \quad (\#11)$$

$$-2d X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes u] F^{v_1 v_2} \quad (\#12)$$

$$-2(3d-4) X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u)] F^{v_1 v_2} \quad (\#13)$$

2.1.3 $R_{4,2}$, non simplified, $N^v \neq 0$

$$R_{4,2}(\text{n.s.}) = \frac{1}{2}(d+4) g^{v_2 v_3} X_3[u \otimes (\widehat{\nabla}_{v_3} u) \otimes N^{v_1}] F_{v_1 v_2} \quad (\#1)$$

$$+ \frac{1}{2} d g^{v_2 v_3} X_3[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_3} u)] F_{v_1 v_2} \quad (\#2)$$

$$- \frac{1}{2} d g^{v_2 v_3} X_3[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_3} u)] F_{v_1 v_2} \quad (\#3)$$

$$+ (d+2) g^{v_2 v_3} X_4[N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_3} u)] F_{v_1 v_2} \quad (\#4)$$

$$- (d+2) g^{v_2 v_3} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_3} u) \otimes N^{v_1}] F_{v_1 v_2} \quad (\#5)$$

$$- (d+2) g^{v_2 v_3} X_4[u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_3} u)] F_{v_1 v_2} \quad (\#6)$$

$$- (d+2) g^{v_2 v_3} X_4[u \otimes (\widehat{\nabla}_{v_3} u) \otimes u \otimes N^{v_1}] F_{v_1 v_2} \quad (\#7)$$

$$+ \frac{1}{2} X_2[N^{v_1} \otimes N^{v_2}] F_{v_1 v_2} \quad (\#8)$$

$$- X_3[N^{v_1} \otimes u \otimes N^{v_2}] F_{v_1 v_2} \quad (\#9)$$

$$- g^{v_2 v_3} X_2[u \otimes (\widehat{\nabla}_{v_3} N^{v_1})] F_{v_1 v_2} \quad (\#10)$$

$$+ 2 g^{v_2 v_3} X_3[u \otimes u \otimes (\widehat{\nabla}_{v_3} N^{v_1})] F_{v_1 v_2} \quad (\#11)$$

$$- d X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#12)$$

$$+ \frac{1}{2}(d+2)(d+4) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#13)$$

$$+ 2(d+2) X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#14)$$

$$- (d+4)^2 X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#15)$$

$$- 2(d+4) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#16)$$

$$- (d+2)(d+4) X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#17)$$

$$+ (d+4)(d+6) X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#18)$$

$$+ (d+4)(d+6) X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#19)$$

$$- d X_3[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u)] F^{v_1 v_2} \quad (\#20)$$

$$+ 2(d+2) X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u)] F^{v_1 v_2} \quad (\#21)$$

$$- 2(d+4) X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u)] F^{v_1 v_2} \quad (\#22)$$

2.1.4 $R_{4,2}$, non simplified, $N^v = 0$

$$R_{4,2}(\text{n.s.}, N=0) = -d X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#1)$$

$$+ \frac{1}{2}(d+2)(d+4) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#2)$$

$$+ 2(d+2) X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#3)$$

$$- (d+4)^2 X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#4)$$

$$- 2(d+4) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#5)$$

$$- (d+2)(d+4) X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#6)$$

$$+ (d+4)(d+6) X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#7)$$

$$+ (d+4)(d+6) X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] F^{v_1 v_2} \quad (\#8)$$

$$- d X_3[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u)] F^{v_1 v_2} \quad (\#9)$$

$$+ 2(d+2) X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1 \nu_2}^2 u)] F^{\nu_1 \nu_2} \quad (\#10)$$

$$- 2(d+4) X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1 \nu_2}^2 u)] F^{\nu_1 \nu_2} \quad (\#11)$$

2.2 $R_{4,3}$

2.2.1 $R_{4,3}, N^\nu \neq 0$

$$\begin{aligned}
R_{4,3} &= 8g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#1) \\
&\quad - (d+2)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#2) \\
&\quad - 4g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#3) \\
&\quad + (d-2)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#4) \\
&\quad + 8g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#5) \\
&\quad - (d+2)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#6) \\
&\quad - 4g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#7) \\
&\quad + (d-2)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#8) \\
&\quad + 8g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#9) \\
&\quad - 2(d-2)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#10) \\
&\quad + 2(d+2)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#11) \\
&\quad - 16g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#12) \\
&\quad + g^{\nu_1\nu_2}X_4[u \otimes N^{\nu_3} \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#13) \\
&\quad - g^{\nu_1\nu_2}X_4[N^{\nu_3} \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#14) \\
&\quad - g^{\nu_1\nu_2}X_4[u \otimes u \otimes u \otimes N^{\nu_3}](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#15) \\
&\quad + g^{\nu_1\nu_2}X_4[u \otimes u \otimes N^{\nu_3} \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#16) \\
&\quad - g^{\nu_2\nu_3}X_4[u \otimes u \otimes u \otimes N^{\nu_1}](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#17) \\
&\quad + g^{\nu_2\nu_3}X_4[u \otimes u \otimes N^{\nu_1} \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#18) \\
&\quad + g^{\nu_2\nu_3}X_4[u \otimes N^{\nu_1} \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#19) \\
&\quad - g^{\nu_2\nu_3}X_4[N^{\nu_1} \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#20) \\
&\quad - 2g^{\nu_1\nu_3}X_4[u \otimes u \otimes N^{\nu_2} \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#21) \\
&\quad - 2g^{\nu_1\nu_3}X_4[u \otimes N^{\nu_2} \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#22) \\
&\quad + 2g^{\nu_1\nu_3}X_4[N^{\nu_2} \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#23) \\
&\quad + 2g^{\nu_1\nu_3}X_4[u \otimes u \otimes u \otimes N^{\nu_2}](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#24)
\end{aligned}$$

2.2.2 $R_{4,3}, N^\nu = 0$

$$\begin{aligned}
R_{4,3}(N=0) &= 8g^{v_1 v_4} g^{v_2 v_3} X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_4} u)](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#1) \\
&\quad - (d+2)g^{v_1 v_4} g^{v_2 v_3} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_4} u) \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#2) \\
&\quad - 4g^{v_1 v_4} g^{v_2 v_3} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_4} u) \otimes u \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#3) \\
&\quad + (d-2)g^{v_1 v_4} g^{v_2 v_3} X_5[u \otimes (\widehat{\nabla}_{v_4} u) \otimes u \otimes u \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#4) \\
&\quad + 8g^{v_1 v_2} g^{v_3 v_4} X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_4} u)](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#5) \\
&\quad - (d+2)g^{v_1 v_2} g^{v_3 v_4} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_4} u) \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#6) \\
&\quad - 4g^{v_1 v_2} g^{v_3 v_4} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_4} u) \otimes u \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#7) \\
&\quad + (d-2)g^{v_1 v_2} g^{v_3 v_4} X_5[u \otimes (\widehat{\nabla}_{v_4} u) \otimes u \otimes u \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#8) \\
&\quad + 8g^{v_1 v_3} g^{v_2 v_4} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_4} u) \otimes u \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#9) \\
&\quad - 2(d-2)g^{v_1 v_3} g^{v_2 v_4} X_5[u \otimes (\widehat{\nabla}_{v_4} u) \otimes u \otimes u \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#10) \\
&\quad + 2(d+2)g^{v_1 v_3} g^{v_2 v_4} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_4} u) \otimes u](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#11) \\
&\quad - 16g^{v_1 v_3} g^{v_2 v_4} X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_4} u)](\nabla_{v_1 v_2 v_3}^3 \mathbf{1}) \quad (\#12)
\end{aligned}$$

2.2.3 $R_{4,3}$, non simplified, $N^v \neq 0$

$$\begin{aligned}
R_{4,3}(\text{n.s.}) &= 2g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_2[u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#1) \\
&- (d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_3[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#2) \\
&- (d+6)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_3[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#3) \\
&- (d+8)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_3[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#4) \\
&- 4g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_3[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#5) \\
&- 2g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_3[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#6) \\
&+ (5d+22)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#7) \\
&+ 4(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#8) \\
&+ (5d+22)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#9) \\
&+ (3d+10)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_4[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#10) \\
&+ 2(d+6)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#11) \\
&+ 2(d+8)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#12) \\
&+ (d+6)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_4[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#13) \\
&+ 2(d+6)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#14) \\
&+ 4g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_4[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#15) \\
&- 8(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#16) \\
&- 8(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#17) \\
&- 8(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#18) \\
&- 6(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#19) \\
&- 6(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#20) \\
&- 6(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#21) \\
&- 4(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbf{1}) \quad (\#22)
\end{aligned}$$

$$-4(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#23)$$

$$-4(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#24)$$

$$-2(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#25)$$

$$-2(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#26)$$

$$-2(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#27)$$

$$+g^{\nu_1\nu_2}X_2[u\otimes N^{\nu_3}](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#28)$$

$$+g^{\nu_2\nu_3}X_2[N^{\nu_1}\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#29)$$

$$-3g^{\nu_2\nu_3}X_3[N^{\nu_1}\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#30)$$

$$-g^{\nu_2\nu_3}X_3[u\otimes N^{\nu_1}\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#31)$$

$$-3g^{\nu_1\nu_2}X_3[u\otimes u\otimes N^{\nu_3}](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#32)$$

$$-g^{\nu_1\nu_2}X_3[N^{\nu_3}\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#33)$$

$$-g^{\nu_2\nu_3}X_3[u\otimes u\otimes N^{\nu_1}](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#34)$$

$$-2g^{\nu_1\nu_3}X_3[u\otimes N^{\nu_2}\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#35)$$

$$-g^{\nu_1\nu_2}X_3[u\otimes N^{\nu_3}\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#36)$$

$$+2g^{\nu_1\nu_2}X_4[N^{\nu_3}\otimes u\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#37)$$

$$+2g^{\nu_1\nu_3}X_4[N^{\nu_2}\otimes u\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#38)$$

$$+2g^{\nu_2\nu_3}X_4[N^{\nu_1}\otimes u\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#39)$$

$$+2g^{\nu_1\nu_2}X_4[u\otimes N^{\nu_3}\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#40)$$

$$+2g^{\nu_1\nu_3}X_4[u\otimes N^{\nu_2}\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#41)$$

$$+2g^{\nu_2\nu_3}X_4[u\otimes N^{\nu_1}\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#42)$$

$$+2g^{\nu_1\nu_2}X_4[u\otimes u\otimes N^{\nu_3}\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#43)$$

$$+2g^{\nu_1\nu_3}X_4[u\otimes u\otimes N^{\nu_2}\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#44)$$

$$+2g^{\nu_2\nu_3}X_4[u\otimes u\otimes N^{\nu_1}\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#45)$$

$$+2g^{\nu_1\nu_2}X_4[u\otimes u\otimes u\otimes N^{\nu_3}](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#46)$$

$$+2g^{\nu_1\nu_3}X_4[u\otimes u\otimes u\otimes N^{\nu_2}](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#47)$$

$$+2g^{\nu_2\nu_3}X_4[u\otimes u\otimes u\otimes N^{\nu_1}](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#48)$$

2.2.4 $R_{4,3}$, non simplified, $N^\nu = 0$

$$R_{4,3}(\text{n.s.}, N=0) = 2g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_2[u\otimes(\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#1)$$

$$-(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_3[u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#2)$$

$$-(d+6)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_3[u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#3)$$

$$-(d+8)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_3[u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#4)$$

$$-4g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_3[u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#5)$$

$$-2g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_3[u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#6)$$

$$+(5d+22)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_4[u\otimes u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#7)$$

$$+4(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_4[u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#8)$$

$$+(5d+22)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_4[u\otimes u\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#9)$$

$$+(3d+10)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_4[u\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u\otimes u](\nabla_{\nu_1\nu_2\nu_3}^3\mathbf{1}) \quad (\#10)$$

$$\begin{aligned}
& + 2(d+6)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#11) \\
& + 2(d+8)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#12) \\
& + (d+6)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_4[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#13) \\
& + 2(d+6)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#14) \\
& + 4g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_4[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#15) \\
& - 8(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#16) \\
& - 8(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#17) \\
& - 8(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u)](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#18) \\
& - 6(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#19) \\
& - 6(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#20) \\
& - 6(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#21) \\
& - 4(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#22) \\
& - 4(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#23) \\
& - 4(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#24) \\
& - 2(d+4)g^{\nu_1\nu_2}g^{\nu_3\nu_4}X_5[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#25) \\
& - 2(d+4)g^{\nu_1\nu_3}g^{\nu_2\nu_4}X_5[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#26) \\
& - 2(d+4)g^{\nu_1\nu_4}g^{\nu_2\nu_3}X_5[u \otimes (\widehat{\nabla}_{\nu_4}u) \otimes u \otimes u \otimes u](\nabla_{\nu_1\nu_2\nu_3}^3 \mathbb{1}) \quad (\#27)
\end{aligned}$$

2.3 $R_{4,4}$

$$R_{4,4} = \frac{1}{12} X_1[u] F^{\nu_1\nu_2} F_{\nu_1\nu_2} \quad (\#1)$$

2.4.1 $R_{4,0}$, $N^v \neq 0$

10

[illegible]

[illegible]

[illegible]

[illegible]

$$-\frac{1}{2}(d+2)(d+4)^2 g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3 v_4} u) \otimes u \otimes (\widehat{\nabla}_{v_4} u) \otimes u] \quad (\#222)$$

$$-(d+4)^3 g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3} u) \otimes (\widehat{\nabla}_{v_4} u) \otimes u \otimes u] \quad (\#225)$$

$$+3(d+2)(d+4) g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3} u) \otimes u \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#226)$$

$$+(d+2)(d+4) g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes (\widehat{\nabla}_{v_3} u) \otimes u \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#227)$$

$$+2(d+4)^2 g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#228)$$

$$-12(d+4) g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes (\widehat{\nabla}_{v_3} u) \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#229)$$

$$-4(d+4) g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_3} u) \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#230)$$

$$-24(d+4) g^{v_1 v_4} g^{v_2 v_3} X_7[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3} u) \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#231)$$

$$+2(d^2 - 3d - 16) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3 v_4}^2 u) \otimes u] \quad (\#232)$$

$$+(d^2 - 8) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes (\widehat{\nabla}_{v_3 v_4}^2 u) \otimes u] \quad (\#233)$$

$$+2(d+2)^2 g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3 v_4}^2 u) \otimes u \otimes u] \quad (\#234)$$

$$+2(d-2)(d+2) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes (\widehat{\nabla}_{v_3 v_4}^2 u)] \quad (\#235)$$

$$+d(d+2) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_3 v_4}^2 u)] \quad (\#236)$$

$$+2(d+2)(d-8) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3 v_4}^2 u)] \quad (\#237)$$

$$-2(d^2 + 3d + 8) g^{v_1 v_4} g^{v_2 v_3} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3 v_4}^2 u) \otimes u] \quad (\#238)$$

$$-(d^2 + 4d + 8) g^{v_1 v_4} g^{v_2 v_3} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_3 v_4}^2 u) \otimes u] \quad (\#239)$$

$$+4(d+2) g^{v_1 v_4} g^{v_2 v_3} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3 v_4}^2 u) \otimes u \otimes u] \quad (\#240)$$

$$-2(d-2)(d+2) g^{v_1 v_4} g^{v_2 v_3} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes (\widehat{\nabla}_{v_3 v_4}^2 u)] \quad (\#241)$$

$$-d(d+2) g^{v_1 v_4} g^{v_2 v_3} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_3 v_4}^2 u)] \quad (\#242)$$

$$-2(d+2)(d-8) g^{v_1 v_4} g^{v_2 v_3} X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 u) \otimes (\widehat{\nabla}_{v_3 v_4}^2 u)] \quad (\#243)$$

$$-8 g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u)] \quad (\#244)$$

$$+d(d+4) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u)] \quad (\#245)$$

$$-2(d+4) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u) \otimes u] \quad (\#246)$$

$$-(d^2 + 8) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u) \otimes u] \quad (\#247)$$

$$-(d^2 + 6d + 16) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u) \otimes u] \quad (\#248)$$

$$+4(d+2) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u) \otimes u \otimes u] \quad (\#249)$$

$$-(d^2 + 8) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u)] \quad (\#250)$$

$$+16(d+1) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3 v_4}^3 u)] \quad (\#251)$$

$$+8(d+4) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3}^3 u) \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#252)$$

$$+2(d+4) g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3}^3 u) \otimes u \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#253)$$

$$-(d+4)^2 g^{v_1 v_3} g^{v_2 v_4} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3}^3 u) \otimes (\widehat{\nabla}_{v_4} u) \otimes u] \quad (\#254)$$

$$-2(d+4) g^{v_1 v_3} g^{v_2 v_4} X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3 v_4}^4 u)] \quad (\#255)$$

$$+4\Re X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\Delta} u)] \quad (\#256)$$

$$-\frac{1}{2}(d-6)\Re X_5[u \otimes u \otimes u \otimes (\widehat{\Delta} u) \otimes u] \quad (\#257)$$

$$-\frac{1}{3}2(d-3)\Re X_5[u \otimes u \otimes (\widehat{\Delta} u) \otimes u \otimes u] \quad (\#258)$$

$$-\frac{1}{2}(d-2)\Re X_5[u \otimes (\widehat{\Delta} u) \otimes u \otimes u \otimes u] \quad (\#259)$$

$$+2d g^{v_1 v_2} X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta}(\widehat{\nabla}_{v_2} u)))] \quad (\#260)$$

$$+(d+2) g^{v_1 v_2} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta}(\widehat{\nabla}_{v_2} u))) \otimes u] \quad (\#261)$$

$$-8(d-2) g^{v_1 v_2} X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2}(\widehat{\Delta} u))] \quad (\#262)$$

$$+\frac{1}{2}(d^2 - 6d + 32) g^{v_1 v_2} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2}(\widehat{\Delta} u))] \quad (\#263)$$

$$+\frac{1}{2}(d^2 - 18d + 8) g^{v_1 v_2} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \$$

[illegible]

$$\begin{aligned}
& + \frac{1}{2}(d^2 + 8) X_6[u \otimes (\widehat{\Delta}u) \otimes u \otimes u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#361) \\
& + \frac{1}{2}d(d-6) X_6[u \otimes u \otimes (\widehat{\Delta}u) \otimes u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#362) \\
& + (d^2 + 2d + 12) X_6[u \otimes u \otimes (\widehat{\Delta}u) \otimes (\widehat{\Delta}u) \otimes u \otimes u] \quad (\#363) \\
& + \frac{1}{2}(d^2 + 8) X_6[u \otimes (\widehat{\Delta}u) \otimes u \otimes (\widehat{\Delta}u) \otimes u \otimes u] \quad (\#364) \\
& - 3d X_6[u \otimes (\widehat{\Delta}u) \otimes (\widehat{\Delta}u) \otimes u \otimes u \otimes u] \quad (\#365) \\
& + \frac{1}{3} \Re X_3[u \otimes u \otimes q] \quad (\#366) \\
& + \frac{1}{3} \Re X_3[u \otimes q \otimes u] \quad (\#367) \\
& + \frac{1}{3} \Re X_3[q \otimes u \otimes u] \quad (\#368) \\
& - 2(d+2) g^{\nu_1 \nu_2} X_5[u \otimes q \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#369) \\
& - 2(d+2) g^{\nu_1 \nu_2} X_5[q \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#370) \\
& - (d+2) g^{\nu_1 \nu_2} X_5[q \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#371) \\
& + \frac{1}{2}(d+2)^2 g^{\nu_1 \nu_2} X_5[q \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#372) \\
& - 2(d+2) g^{\nu_1 \nu_2} X_5[u \otimes u \otimes q \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#373) \\
& - (d+2) g^{\nu_1 \nu_2} X_5[u \otimes q \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#374) \\
& + \frac{1}{2}(d+2)^2 g^{\nu_1 \nu_2} X_5[u \otimes q \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#375) \\
& - 2(d+2) g^{\nu_1 \nu_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes q] \quad (\#376) \\
& - (d+2) g^{\nu_1 \nu_2} X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes q] \quad (\#377) \\
& + \frac{1}{2}(d+2)^2 g^{\nu_1 \nu_2} X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes q] \quad (\#378) \\
& + \frac{1}{2}(d+2)^2 g^{\nu_1 \nu_2} X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes q \otimes u] \quad (\#379) \\
& - 2(d+2) g^{\nu_1 \nu_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes q \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#380) \\
& - (d+2) g^{\nu_1 \nu_2} X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes q \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#381) \\
& - (d+2) g^{\nu_1 \nu_2} X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes q \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#382) \\
& + \frac{1}{2}(d+2)^2 g^{\nu_1 \nu_2} X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes q \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#383) \\
& + 2 X_4[u \otimes q \otimes u \otimes (\widehat{\Delta}u)] \quad (\#384) \\
& + 2 X_4[q \otimes u \otimes u \otimes (\widehat{\Delta}u)] \quad (\#385) \\
& - \frac{1}{2}d X_4[q \otimes u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#386) \\
& + 2 X_4[u \otimes u \otimes q \otimes (\widehat{\Delta}u)] \quad (\#387) \\
& - \frac{1}{2}d X_4[u \otimes q \otimes (\widehat{\Delta}u) \otimes u] \quad (\#388) \\
& + 2 X_4[u \otimes u \otimes (\widehat{\Delta}u) \otimes q] \quad (\#389) \\
& - \frac{1}{2}d X_4[u \otimes (\widehat{\Delta}u) \otimes u \otimes q] \quad (\#390) \\
& - \frac{1}{2}d X_4[u \otimes (\widehat{\Delta}u) \otimes q \otimes u] \quad (\#391) \\
& + X_2[q \otimes q] \quad (\#392) \\
& - (d+2) g^{\nu_1 \nu_2} X_4[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} q) \otimes u] \quad (\#393) \\
& + 4 g^{\nu_1 \nu_2} X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} q) \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#394) \\
& + 2 g^{\nu_1 \nu_2} X_4[u \otimes (\widehat{\nabla}_{\nu_1} q) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#395) \\
& - d g^{\nu_1 \nu_2} X_4[u \otimes (\widehat{\nabla}_{\nu_1} q) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#396) \\
& + X_3[u \otimes (\widehat{\Delta}q) \otimes u] \quad (\#397) \\
& + \frac{1}{2} (\widehat{\nabla}_{\nu_1} \Re) X_4[u \otimes N^{\nu_1} \otimes u \otimes u] \quad (\#398) \\
& + \frac{1}{2} (\widehat{\nabla}_{\nu_1} \Re) X_4[N^{\nu_1} \otimes u \otimes u \otimes u] \quad (\#399) \\
& - (\widehat{\nabla}_{\nu_1} \Re) X_4[u \otimes u \otimes u \otimes N^{\nu_1}] \quad (\#400) \\
& - \frac{1}{6}(d-6) g^{\nu_2 \nu_3} \text{Ric}_{\nu_1 \nu_3} X_5[u \otimes u \otimes N^{\nu_1} \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#401) \\
& - \frac{1}{3} 2d g^{\nu_2 \nu_3} \text{Ric}_{\nu_1 \nu_3} X_5[u \otimes N^{\nu_1} \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#402) \\
& - \frac{1}{3} d g^{\nu_2 \nu_3} \text{Ric}_{\nu_1 \nu_3} X_5[u \otimes N^{\nu_1} \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes u] \quad (\#403) \\
& + \frac{1}{2}(d+6) g^{\nu_2 \nu_3} \text{Ric}_{\nu_1 \nu_3} X_5[u \otimes u \otimes u \otimes N^{\nu_1} \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#404) \\
& - \frac{1}{2}(d-2) g^{\nu_2 \nu_3} \text{Ric}_{\nu_1 \nu_3} X_5[u \otimes u \otimes N^{\nu_1} \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)] \quad (\#405) \\
& - \frac{1}{6}(d-6) g^{\nu_2 \nu_3} \text{Ric}_{\nu_1 \nu_3} X_5[N^{\nu_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u] \quad (\#406) \\
& - \frac{1}{3} d g^{\nu_2 \nu_3} \text{Ric}_{\nu_1 \nu_3} X_5[N^{\nu_1} \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes u] \quad (\#407)
\end{aligned}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& + \frac{1}{2} g_{v_1 v_2} X_4 [N^{v_1} \otimes u \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_3} N^{v_3})] \quad (\#1321) \\
& + \frac{1}{2} g_{v_1 v_2} X_4 [u \otimes N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_3} N^{v_3})] \quad (\#1322) \\
& + \frac{1}{2} g_{v_1 v_3} X_4 [N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_2} N^{v_2}) \otimes N^{v_3}] \quad (\#1323) \\
& + \frac{1}{2} g_{v_1 v_3} X_4 [u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2} N^{v_2}) \otimes N^{v_3}] \quad (\#1324) \\
& + \frac{1}{2} g_{v_1 v_3} X_4 [N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_2} N^{v_3}) \otimes N^{v_2}] \quad (\#1325) \\
& + \frac{1}{2} g_{v_1 v_3} X_4 [u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2} N^{v_3}) \otimes N^{v_2}] \quad (\#1326) \\
& + \frac{1}{2} g_{v_2 v_3} X_4 [u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes N^{v_2} \otimes N^{v_3}] \quad (\#1327) \\
& + \frac{1}{2} g_{v_2 v_3} X_4 [u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes N^{v_1} \otimes N^{v_3}] \quad (\#1328) \\
& + \frac{1}{2} g_{v_2 v_3} X_4 [u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes N^{v_3} \otimes N^{v_1}] \quad (\#1329) \\
& - g_{v_2 v_4} g^{v_1 v_3} X_4 [u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes (\widehat{\nabla}_{v_3} N^{v_4}) \otimes u] \quad (\#1330) \\
& - X_4 [u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes (\widehat{\nabla}_{v_2} N^{v_1}) \otimes u] \quad (\#1331) \\
& + X_4 [u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes u \otimes (\widehat{\nabla}_{v_2} N^{v_2})] \quad (\#1332) \\
& + 2 X_4 [u \otimes u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes (\widehat{\nabla}_{v_2} N^{v_2})] \quad (\#1333) \\
& + (d-2) g^{v_1 v_2} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_3}) \otimes u] \quad (\#1334) \\
& + d g^{v_1 v_2} X_5 [u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_3}) \otimes u] \quad (\#1335) \\
& - 4 g^{v_1 v_2} X_5 [u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_3}) \otimes u \otimes u] \quad (\#1336) \\
& + (d-2) g^{v_1 v_2} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_3})] \quad (\#1337) \\
& + d g^{v_1 v_2} X_5 [u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_3})] \quad (\#1338) \\
& - 6 g^{v_1 v_2} X_5 [u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_3})] \quad (\#1339) \\
& - (d-2) g^{v_1 v_3} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_2})] \quad (\#1340) \\
& - d g^{v_1 v_3} X_5 [u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_2})] \quad (\#1341) \\
& + 2 g^{v_1 v_3} X_5 [u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_2}) \otimes u] \quad (\#1342) \\
& + 6 g^{v_1 v_3} X_5 [u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_2})] \quad (\#1343) \\
& + (d+6) g^{v_1 v_3} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2 v_3}^2 N^{v_2}) \otimes u] \quad (\#1344) \\
& - 6 g^{v_2 v_3} X_5 [u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_1}) \otimes (\widehat{\nabla}_{v_3} u)] \quad (\#1345) \\
& - 2 g^{v_2 v_3} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_1}) \otimes u \otimes (\widehat{\nabla}_{v_3} u)] \quad (\#1346) \\
& + (d+2) g^{v_2 v_3} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_1}) \otimes (\widehat{\nabla}_{v_3} u) \otimes u] \quad (\#1347) \\
& - 6 g^{v_1 v_3} X_5 [u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2}) \otimes (\widehat{\nabla}_{v_3} u)] \quad (\#1348) \\
& - 2 g^{v_1 v_3} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2}) \otimes u \otimes (\widehat{\nabla}_{v_3} u)] \quad (\#1349) \\
& + (d+2) g^{v_1 v_3} X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2}) \otimes (\widehat{\nabla}_{v_3} u) \otimes u] \quad (\#1350) \\
& - X_4 [u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2 v_1}^2 N^{v_2})] \quad (\#1351) \\
& - X_4 [u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2 v_1}^2 N^{v_2}) \otimes u] \quad (\#1352) \\
& + X_4 [N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_1}^2 N^{v_2})] \quad (\#1353) \\
& - X_4 [N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2})] \quad (\#1354) \\
& - X_4 [N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2}) \otimes u] \quad (\#1355) \\
& + X_4 [u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2})] \quad (\#1356) \\
& + X_4 [u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_1}) \otimes N^{v_2}] \quad (\#1357) \\
& + X_4 [u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2}) \otimes N^{v_1}] \quad (\#1358) \\
& + 2 g^{v_1 v_3} X_4 [u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3}^3 N^{v_2})] \quad (\#1359) \\
& + 4 X_5 [u \otimes u \otimes (\widehat{\Delta} N^{v_1}) \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#1360) \\
& - \frac{1}{2} (d-2) X_5 [u \otimes (\widehat{\Delta} N^{v_1}) \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#1361) \\
& - d X_5 [u \otimes (\widehat{\Delta} N^{v_1}) \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u] \quad (\#1362) \\
& + 2 X_5 [u \otimes u \otimes (\widehat{\Delta} N^{v_1}) \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#1363) \\
& + 2 X_5 [u \otimes (\widehat{\Delta} N^{v_1}) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#1364) \\
& + (d+2) X_5 [u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} N^{v_1}) \otimes u] \quad (\#1365)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(d+2) X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\Delta} N^{\nu_1}) \otimes u] \quad (\#1366) \\
& - 2 X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\Delta} N^{\nu_1}) \otimes u \otimes u] \quad (\#1367) \\
& - \frac{1}{2} g_{\nu_1 \nu_2} X_4[u \otimes N^{\nu_1} \otimes (\widehat{\Delta} N^{\nu_2}) \otimes u] \quad (\#1368) \\
& - \frac{1}{2} g_{\nu_1 \nu_2} X_4[N^{\nu_1} \otimes u \otimes (\widehat{\Delta} N^{\nu_2}) \otimes u] \quad (\#1369) \\
& - \frac{1}{2} g_{\nu_1 \nu_2} X_4[u \otimes (\widehat{\Delta} N^{\nu_1}) \otimes u \otimes N^{\nu_2}] \quad (\#1370) \\
& - \frac{1}{2} g_{\nu_1 \nu_2} X_4[u \otimes (\widehat{\Delta} N^{\nu_1}) \otimes N^{\nu_2} \otimes u] \quad (\#1371) \\
& - X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} (\widehat{\Delta} N^{\nu_1}))] \quad (\#1372) \\
& - X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} (\widehat{\Delta} N^{\nu_1})) \otimes u] \quad (\#1373) \\
& - X_4[u \otimes u \otimes u \otimes (\widehat{\Delta} (\widehat{\nabla}_{\nu_1} N^{\nu_1}))] \quad (\#1374) \\
& - X_4[u \otimes u \otimes (\widehat{\Delta} (\widehat{\nabla}_{\nu_1} N^{\nu_1})) \otimes u] \quad (\#1375)
\end{aligned}$$

2.4.2 $R_{4,0}, N^v = 0$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& + \frac{1}{2}d(d-6) X_6[u \otimes u \otimes (\widehat{\Delta}u) \otimes u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#362) \\
& + (d^2 + 2d + 12) X_6[u \otimes u \otimes (\widehat{\Delta}u) \otimes (\widehat{\Delta}u) \otimes u \otimes u] \quad (\#363) \\
& + \frac{1}{2}(d^2 + 8) X_6[u \otimes (\widehat{\Delta}u) \otimes u \otimes (\widehat{\Delta}u) \otimes u \otimes u] \quad (\#364) \\
& - 3d X_6[u \otimes (\widehat{\Delta}u) \otimes (\widehat{\Delta}u) \otimes u \otimes u \otimes u] \quad (\#365) \\
& + \frac{1}{3} \Re X_3[u \otimes u \otimes q] \quad (\#366) \\
& + \frac{1}{3} \Re X_3[u \otimes q \otimes u] \quad (\#367) \\
& + \frac{1}{3} \Re X_3[q \otimes u \otimes u] \quad (\#368) \\
& - 2(d+2) g^{v_1 v_2} X_5[u \otimes q \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#369) \\
& - 2(d+2) g^{v_1 v_2} X_5[q \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#370) \\
& - (d+2) g^{v_1 v_2} X_5[q \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#371) \\
& + \frac{1}{2}(d+2)^2 g^{v_1 v_2} X_5[q \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#372) \\
& - 2(d+2) g^{v_1 v_2} X_5[u \otimes u \otimes q \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#373) \\
& - (d+2) g^{v_1 v_2} X_5[u \otimes q \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#374) \\
& + \frac{1}{2}(d+2)^2 g^{v_1 v_2} X_5[u \otimes q \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#375) \\
& - 2(d+2) g^{v_1 v_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes q] \quad (\#376) \\
& - (d+2) g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes q] \quad (\#377) \\
& + \frac{1}{2}(d+2)^2 g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes q] \quad (\#378) \\
& + \frac{1}{2}(d+2)^2 g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes q \otimes u] \quad (\#379) \\
& - 2(d+2) g^{v_1 v_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes q \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#380) \\
& - (d+2) g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes q \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#381) \\
& - (d+2) g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes q \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#382) \\
& + \frac{1}{2}(d+2)^2 g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes q \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#383) \\
& + 2 X_4[u \otimes q \otimes u \otimes (\widehat{\Delta}u)] \quad (\#384) \\
& + 2 X_4[q \otimes u \otimes u \otimes (\widehat{\Delta}u)] \quad (\#385) \\
& - \frac{1}{2}d X_4[q \otimes u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#386) \\
& + 2 X_4[u \otimes u \otimes q \otimes (\widehat{\Delta}u)] \quad (\#387) \\
& - \frac{1}{2}d X_4[u \otimes q \otimes (\widehat{\Delta}u) \otimes u] \quad (\#388) \\
& + 2 X_4[u \otimes u \otimes (\widehat{\Delta}u) \otimes q] \quad (\#389) \\
& - \frac{1}{2}d X_4[u \otimes (\widehat{\Delta}u) \otimes u \otimes q] \quad (\#390) \\
& - \frac{1}{2}d X_4[u \otimes (\widehat{\Delta}u) \otimes q \otimes u] \quad (\#391) \\
& + X_2[q \otimes q] \quad (\#392) \\
& - (d+2) g^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} q) \otimes u] \quad (\#393) \\
& + 4 g^{v_1 v_2} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} q) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#394) \\
& + 2 g^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} q) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#395) \\
& - d g^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} q) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#396) \\
& + X_3[u \otimes (\widehat{\Delta}q) \otimes u] \quad (\#397)
\end{aligned}$$

2.4.3 $R_{4,0}$, non simplified, $N^\nu \neq 0$

$$\begin{aligned}
R_{4,0}(\text{n.s.}) &= \frac{1}{180} |R|^2 X_1[u] \quad (\#1) \\
&- \frac{1}{180} |\text{Ric}|^2 X_1[u] \quad (\#2) \\
&+ \frac{1}{30} (\widehat{\Delta}\mathfrak{H})X_1[u] \quad (\#3) \\
&+ \frac{1}{72} \mathfrak{H}^2 X_1[u] \quad (\#4) \\
&+ g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#5) \\
&+ 2g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_3[u \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#6) \\
&- (d+8)g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#7) \\
&- (d+6)g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#8) \\
&- \frac{1}{2}(d+6)g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u] \quad (\#9) \\
&+ (d+4)g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes u] \quad (\#10) \\
&+ 2(d+4)g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u] \quad (\#11) \\
&+ 3(d+4)g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#12) \\
&+ 4(d+4)g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{H}) X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#13) \\
&+ \frac{1}{3}4\text{Ric}^{v_1 v_2} X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#14) \\
&+ \frac{1}{3}2g^{v_1 v_2} \mathfrak{H} X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#15) \\
&- \frac{1}{3}(9d+34)\text{Ric}^{v_1 v_2} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#16) \\
&- \frac{1}{3}(3d+14)\text{Ric}^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#17) \\
&- \frac{1}{3}2(d+4)\text{Ric}^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#18) \\
&- \frac{1}{3}4(d+4)g^{v_1 v_2} \mathfrak{H} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#19) \\
&- \frac{1}{3}2(d+4)g^{v_1 v_2} \mathfrak{H} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#20) \\
&- \frac{1}{3}2(d+3)g^{v_1 v_2} \mathfrak{H} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#21) \\
&+ \frac{1}{6}(d+4)(d+22)\text{Ric}^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#22) \\
&+ \frac{1}{6}5(d+4)(d+10)\text{Ric}^{v_1 v_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#23) \\
&+ (d+4)(d+12)\text{Ric}^{v_1 v_2} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#24) \\
&+ \frac{1}{3}(d+4)(d+8)\text{Ric}^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#25) \\
&+ \frac{1}{3}2(d+4)(d+8)\text{Ric}^{v_1 v_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#26) \\
&+ \frac{1}{3}2(d+4)\text{Ric}^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#27) \\
&+ \frac{1}{2}(d+4)(d+12)g^{v_1 v_2} \mathfrak{H} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#28) \\
&+ \frac{1}{3}(d+4)(d+12)g^{v_1 v_2} \mathfrak{H} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#29) \\
&+ \frac{1}{6}(d+4)(d+12)g^{v_1 v_2} \mathfrak{H} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#30) \\
&+ \frac{1}{6}(d+4)^2 g^{v_1 v_2} \mathfrak{H} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#31) \\
&+ \frac{1}{6}(d+4)(d+8)g^{v_1 v_2} \mathfrak{H} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#32) \\
&+ \frac{1}{3}(d+4)(d+8)g^{v_1 v_2} \mathfrak{H} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#33) \\
&- (d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#34) \\
&- 2(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#35) \\
&- 3(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#36) \\
&- 4(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#37) \\
&- \frac{1}{3}(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#38) \\
&- \frac{1}{3}2(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#39) \\
&- \frac{1}{3}4(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#40) \\
&- \frac{1}{3}2(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#41) \\
&- 2(d+4)(d+6)\text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#42) \\
&- \frac{1}{2}(d+4)(d+6)g^{v_1 v_2} \mathfrak{H} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#43) \\
&- (d+4)(d+6)g^{v_1 v_2} \mathfrak{H} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#44) \\
&- \frac{1}{2}3(d+4)(d+6)g^{v_1 v_2} \mathfrak{H} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#45)
\end{aligned}$$

[illegible]

[illegible]

[illegible]

[illegible]

$-\frac{1}{2}(d+4)(d+6)(d+8)g^{v_1v_3}g^{v_2v_4}X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}u)] \quad (\#224)$
 $-\frac{1}{2}(d+4)(d+6)(d+8)g^{v_1v_4}g^{v_2v_3}X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}u)] \quad (\#225)$
 $-(d+4)(d+6)(d+8)g^{v_1v_3}g^{v_2v_4}X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}u)] \quad (\#226)$
 $-(d+4)(d+6)(d+8)g^{v_1v_4}g^{v_2v_3}X_7[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}u)] \quad (\#227)$
 $+(d+2)^2g^{v_1v_3}g^{v_2v_4}X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#228)$
 $+2(d+2)g^{v_1v_4}g^{v_2v_3}X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#229)$
 $-2(d+4)(2d+7)g^{v_1v_3}g^{v_2v_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#230)$
 $-(d+4)^2g^{v_1v_4}g^{v_2v_3}X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#231)$
 $-2(d+4)(d+5)g^{v_1v_4}g^{v_2v_3}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#232)$
 $-(d+4)^2g^{v_1v_3}g^{v_2v_4}X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#233)$
 $+(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#234)$
 $+(d+4)(d+6)g^{v_1v_4}g^{v_2v_3}X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#235)$
 $+3(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#236)$
 $+3(d+4)(d+6)g^{v_1v_4}g^{v_2v_3}X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes u \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#237)$
 $+6(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#238)$
 $+6(d+4)(d+6)g^{v_1v_4}g^{v_2v_3}X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^2u) \otimes (\widehat{\nabla}_{v_3v_4}^2u)] \quad (\#239)$
 $+2(d+2)g^{v_1v_3}g^{v_2v_4}X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^3u) \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#240)$
 $-(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^3u) \otimes u \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#241)$
 $-(d+4)(d+8)g^{v_1v_3}g^{v_2v_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2}^3u) \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#242)$
 $-(d+4)^2g^{v_1v_3}g^{v_2v_4}X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2v_3}^3u) \otimes (\widehat{\nabla}_{v_4}^3u)] \quad (\#243)$
 $-2(d+4)g^{v_1v_3}g^{v_2v_4}X_5[u \otimes (\widehat{\nabla}_{v_1}^3u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#244)$
 $+(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes (\widehat{\nabla}_{v_1}^3u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#245)$
 $+2(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1}^3u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#246)$
 $+3(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}^3u) \otimes u \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#247)$
 $+4(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}^3u) \otimes (\widehat{\nabla}_{v_2v_3v_4}^3u)] \quad (\#248)$
 $+4(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2v_3}^3u) \otimes (\widehat{\nabla}_{v_4}^3u)] \quad (\#249)$
 $+(d+4)(d+6)g^{v_1v_3}g^{v_2v_4}X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2v_3}^3u) \otimes u \otimes (\widehat{\nabla}_{v_4}^3u)] \quad (\#250)$
 $-2(d+4)g^{v_1v_3}g^{v_2v_4}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1v_2v_3v_4}^4u)] \quad (\#251)$
 $+\frac{1}{3}2\Re X_3[u \otimes u \otimes (\widehat{\Delta}u)] \quad (\#252)$
 $+\frac{1}{3}\Re X_3[u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#253)$
 $-\frac{1}{2}(d+6)\Re X_4[u \otimes u \otimes u \otimes (\widehat{\Delta}u)] \quad (\#254)$
 $-\frac{1}{6}(d+2)\Re X_4[u \otimes (\widehat{\Delta}u) \otimes u \otimes u] \quad (\#255)$
 $-\frac{1}{3}(d+4)\Re X_4[u \otimes u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#256)$
 $+2(d+4)\Re X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\Delta}u)] \quad (\#257)$
 $+\frac{1}{3}(d+4)\Re X_5[u \otimes u \otimes (\widehat{\Delta}u) \otimes u \otimes u] \quad (\#258)$
 $+(d+4)\Re X_5[u \otimes u \otimes u \otimes (\widehat{\Delta}u) \otimes u] \quad (\#259)$
 $+(d+2)g^{v_1v_2}X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta}(\widehat{\nabla}_{v_2}u)))] \quad (\#260)$
 $-2(d+4)g^{v_1v_2}X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta}(\widehat{\nabla}_{v_2}u)))] \quad (\#261)$
 $-d g^{v_1v_2}X_3[u \otimes (\widehat{\nabla}_{v_1}u) \otimes (\widehat{\nabla}_{v_2}(\widehat{\Delta}u))] \quad (\#262)$
 $+\frac{1}{2}(d+2)^2g^{v_1v_2}X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta}u)) \otimes (\widehat{\nabla}_{v_2}u)] \quad (\#263)$
 $+\frac{1}{2}(d+2)^2g^{v_1v_2}X_4[u \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1}u)) \otimes (\widehat{\nabla}_{v_2}u)] \quad (\#264)$
 $+\frac{1}{2}(d+2)(d+8)g^{v_1v_2}X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1}u) \otimes (\widehat{\nabla}_{v_2}(\widehat{\Delta}u))] \quad (\#265)$
 $+\frac{1}{2}(d+2)(d+6)g^{v_1v_2}X_4[u \otimes (\widehat{\nabla}_{v_1}u) \otimes u \otimes (\widehat{\nabla}_{v_2}(\widehat{\Delta}u))] \quad (\#266)$
 $+(d+2)g^{v_1v_2}X_4[u \otimes (\widehat{\nabla}_{v_1}u) \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_2}u))] \quad (\#267)$
 $+\frac{1}{2}(d+2)(d+4)g^{v_1v_2}X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1}u) \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_2}u))] \quad (\#268)$

[illegible]

[illegible]

$$\begin{aligned}
& -\frac{1}{2}(d+2)(d+4)g^{\nu_1\nu_2}X_5[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u\otimes q\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#362) \\
& -(d+2)(d+4)g^{\nu_1\nu_2}X_5[u\otimes u\otimes q\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#363) \\
& -\frac{1}{2}(d+2)(d+4)g^{\nu_1\nu_2}X_5[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes q\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#364) \\
& -\frac{1}{2}(d+2)(d+4)g^{\nu_1\nu_2}X_5[u\otimes q\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#365) \\
& -(d+2)(d+4)g^{\nu_1\nu_2}X_5[u\otimes q\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#366) \\
& -\frac{1}{2}(d+2)(d+4)g^{\nu_1\nu_2}X_5[q\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#367) \\
& -(d+2)(d+4)g^{\nu_1\nu_2}X_5[q\otimes u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#368) \\
& -\frac{1}{2}dX_3[u\otimes(\widehat{\Delta}u)\otimes q] \quad (\#369) \\
& -\frac{1}{2}dX_3[u\otimes q\otimes(\widehat{\Delta}u)] \quad (\#370) \\
& -\frac{1}{2}dX_3[q\otimes u\otimes(\widehat{\Delta}u)] \quad (\#371) \\
& +(d+2)X_4[u\otimes u\otimes(\widehat{\Delta}u)\otimes q] \quad (\#372) \\
& +(d+2)X_4[u\otimes u\otimes q\otimes(\widehat{\Delta}u)] \quad (\#373) \\
& +(d+2)X_4[u\otimes q\otimes u\otimes(\widehat{\Delta}u)] \quad (\#374) \\
& +(d+2)X_4[q\otimes u\otimes u\otimes(\widehat{\Delta}u)] \quad (\#375) \\
& +X_2[q\otimes q] \quad (\#376) \\
& -(d+2)g^{\nu_1\nu_2}X_3[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}q)] \quad (\#377) \\
& -dg^{\nu_1\nu_2}X_3[u\otimes(\widehat{\nabla}_{\nu_1}q)\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#378) \\
& +(d+2)g^{\nu_1\nu_2}X_4[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u\otimes(\widehat{\nabla}_{\nu_2}q)] \quad (\#379) \\
& +2(d+2)g^{\nu_1\nu_2}X_4[u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}q)] \quad (\#380) \\
& +2(d+2)g^{\nu_1\nu_2}X_4[u\otimes u\otimes(\widehat{\nabla}_{\nu_1}q)\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#381) \\
& +(d+2)g^{\nu_1\nu_2}X_4[u\otimes(\widehat{\nabla}_{\nu_1}q)\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#382) \\
& +X_2[u\otimes(\widehat{\Delta}q)] \quad (\#383) \\
& -2X_3[u\otimes u\otimes(\widehat{\Delta}q)] \quad (\#384) \\
& +\frac{1}{2}(\widehat{\nabla}_{\nu_1}\mathfrak{R})X_3[u\otimes N^{\nu_1}\otimes u] \quad (\#385) \\
& +\frac{1}{2}(\widehat{\nabla}_{\nu_1}\mathfrak{R})X_3[N^{\nu_1}\otimes u\otimes u] \quad (\#386) \\
& -(\widehat{\nabla}_{\nu_1}\mathfrak{R})X_4[u\otimes u\otimes N^{\nu_1}\otimes u] \quad (\#387) \\
& -(\widehat{\nabla}_{\nu_1}\mathfrak{R})X_4[u\otimes N^{\nu_1}\otimes u\otimes u] \quad (\#388) \\
& -(\widehat{\nabla}_{\nu_1}\mathfrak{R})X_4[N^{\nu_1}\otimes u\otimes u\otimes u] \quad (\#389) \\
& -(\widehat{\nabla}_{\nu_1}\mathfrak{R})X_4[u\otimes u\otimes u\otimes N^{\nu_1}] \quad (\#390) \\
& +\frac{1}{3}2g^{\nu_2\nu_3}\text{Ric}_{\nu_1\nu_3}X_3[u\otimes N^{\nu_1}\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#391) \\
& -\frac{1}{3}g^{\nu_1\nu_3}\text{Ric}_{\nu_2\nu_3}X_3[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes N^{\nu_2}] \quad (\#392) \\
& +\frac{1}{3}2g^{\nu_2\nu_3}\text{Ric}_{\nu_1\nu_3}X_3[N^{\nu_1}\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#393) \\
& +\frac{1}{3}\mathfrak{R}X_3[u\otimes N^{\nu_1}\otimes(\widehat{\nabla}_{\nu_1}u)] \quad (\#394) \\
& +\frac{1}{3}\mathfrak{R}X_3[N^{\nu_1}\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)] \quad (\#395) \\
& +\frac{1}{3}\mathfrak{R}X_3[N^{\nu_1}\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u] \quad (\#396) \\
& -\frac{1}{6}(d+10)g^{\nu_2\nu_3}\text{Ric}_{\nu_1\nu_3}X_4[u\otimes u\otimes N^{\nu_1}\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#397) \\
& -\frac{1}{3}2(d+4)g^{\nu_2\nu_3}\text{Ric}_{\nu_1\nu_3}X_4[u\otimes N^{\nu_1}\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#398) \\
& -\frac{1}{6}(d+10)g^{\nu_2\nu_3}\text{Ric}_{\nu_1\nu_3}X_4[N^{\nu_1}\otimes u\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)] \quad (\#399) \\
& +\frac{1}{6}(d-2)g^{\nu_1\nu_3}\text{Ric}_{\nu_2\nu_3}X_4[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u\otimes N^{\nu_2}] \quad (\#400) \\
& -\frac{1}{6}(d+10)g^{\nu_1\nu_3}\text{Ric}_{\nu_2\nu_3}X_4[u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes N^{\nu_2}] \quad (\#401) \\
& -\frac{1}{3}(d+4)g^{\nu_1\nu_3}\text{Ric}_{\nu_2\nu_3}X_4[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes N^{\nu_2}\otimes u] \quad (\#402) \\
& -\frac{1}{3}(d+4)g^{\nu_2\nu_3}\text{Ric}_{\nu_1\nu_3}X_4[u\otimes N^{\nu_1}\otimes(\widehat{\nabla}_{\nu_2}u)\otimes u] \quad (\#403) \\
& -\frac{1}{3}(d+4)g^{\nu_2\nu_3}\text{Ric}_{\nu_1\nu_3}X_4[N^{\nu_1}\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)\otimes u] \quad (\#404) \\
& -\frac{1}{6}(d+6)\mathfrak{R}X_4[u\otimes u\otimes N^{\nu_1}\otimes(\widehat{\nabla}_{\nu_1}u)] \quad (\#405) \\
& -\frac{1}{6}(d+6)\mathfrak{R}X_4[u\otimes N^{\nu_1}\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)] \quad (\#406) \\
& -\frac{1}{6}(d+6)\mathfrak{R}X_4[N^{\nu_1}\otimes u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)] \quad (\#407) \\
& -\frac{1}{3}2\mathfrak{R}X_4[u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes N^{\nu_1}] \quad (\#408)
\end{aligned}$$

[illegible]

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& -(d+4)g^{v_2 v_3} X_5[N^{v_1} \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_1 v_3}^3 u)] \quad (\#724) \\
& -(d+4)g^{v_2 v_3} X_5[u \otimes N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_1 v_3}^3 u)] \quad (\#725) \\
& -(d+4)g^{v_2 v_3} X_5[u \otimes u \otimes N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_2 v_1 v_3}^3 u)] \quad (\#726) \\
& -(d+4)g^{v_1 v_3} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3}^3 u) \otimes N^{v_2}] \quad (\#727) \\
& -(d+4)g^{v_2 v_3} X_5[u \otimes u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2 v_1 v_3}^3 u)] \quad (\#728) \\
& -\frac{1}{2}d X_3[u \otimes N^{v_1} \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#729) \\
& -\frac{1}{2}d X_3[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#730) \\
& +\frac{1}{2}3(d+2) X_4[u \otimes u \otimes N^{v_1} \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#731) \\
& +\frac{1}{2}3(d+2) X_4[N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#732) \\
& +\frac{1}{2}(d+2) X_4[u \otimes N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#733) \\
& +\frac{1}{2}(d+2) X_4[N^{v_1} \otimes u \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#734) \\
& +\frac{1}{2}(d+2) X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u)) \otimes N^{v_1}] \quad (\#735) \\
& +\frac{1}{2}(d+2) X_4[u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#736) \\
& +\frac{1}{2}(d+2) X_4[u \otimes N^{v_1} \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#737) \\
& +\frac{1}{2}(d+2) X_4[u \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u)) \otimes N^{v_1}] \quad (\#738) \\
& -(d+4) X_5[N^{v_1} \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#739) \\
& -(d+4) X_5[N^{v_1} \otimes u \otimes u \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#740) \\
& -(d+4) X_5[u \otimes N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#741) \\
& -(d+4) X_5[u \otimes N^{v_1} \otimes u \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#742) \\
& -(d+4) X_5[u \otimes u \otimes N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#743) \\
& -(d+4) X_5[u \otimes u \otimes N^{v_1} \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#744) \\
& -(d+4) X_5[u \otimes u \otimes u \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u)) \otimes N^{v_1}] \quad (\#745) \\
& -(d+4) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u)) \otimes N^{v_1}] \quad (\#746) \\
& -(d+4) X_5[u \otimes u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1}(\widehat{\Delta} u))] \quad (\#747) \\
& -(d+4) X_5[u \otimes u \otimes u \otimes N^{v_1} \otimes (\widehat{\Delta}(\widehat{\nabla}_{v_1} u))] \quad (\#748) \\
& -\frac{1}{2}d X_3[N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} u)] \quad (\#749) \\
& +\frac{1}{4}(d+2)^2 X_4[u \otimes (\widehat{\Delta} u) \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#750) \\
& +\frac{1}{4}(d+2)^2 X_4[u \otimes N^{v_1} \otimes (\widehat{\Delta} u) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#751) \\
& +\frac{1}{4}(d+2)(d+6) X_4[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} u)] \quad (\#752) \\
& +\frac{1}{4}(d+2)(d+4) X_4[N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\Delta} u)] \quad (\#753) \\
& +\frac{1}{4}(d+2)(d+6) X_4[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} u)] \quad (\#754) \\
& +\frac{1}{4}(d+2)^2 X_4[N^{v_1} \otimes u \otimes (\widehat{\Delta} u) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#755) \\
& +\frac{1}{2}(d+2) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} u) \otimes N^{v_1}] \quad (\#756) \\
& +\frac{1}{2}(d+2) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_1} \otimes (\widehat{\Delta} u)] \quad (\#757) \\
& +\frac{1}{2}(d+2) X_4[u \otimes (\widehat{\Delta} u) \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_1}] \quad (\#758) \\
& -\frac{1}{4}(d+4)(3d+10) X_5[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\Delta} u)] \quad (\#759) \\
& -(d+4)^2 X_5[N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} u)] \quad (\#760) \\
& -(d+3)(d+4) X_5[N^{v_1} \otimes u \otimes u \otimes (\widehat{\Delta} u) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#761) \\
& -\frac{1}{4}(d+4)(d+6) X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_1} \otimes u \otimes (\widehat{\Delta} u)] \quad (\#762) \\
& -\frac{1}{4}(d+4)(3d+10) X_5[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\Delta} u)] \quad (\#763) \\
& -(d+4)^2 X_5[u \otimes N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} u)] \quad (\#764) \\
& -(d+3)(d+4) X_5[u \otimes N^{v_1} \otimes u \otimes (\widehat{\Delta} u) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#765) \\
& -(d+3)(d+4) X_5[u \otimes u \otimes (\widehat{\Delta} u) \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#766) \\
& -(d+3)(d+4) X_5[u \otimes u \otimes N^{v_1} \otimes (\widehat{\Delta} u) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#767) \\
& -(d+4)^2 X_5[u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} u)] \quad (\#768) \\
& -\frac{1}{4}(d+2)(d+4) X_5[N^{v_1} \otimes u \otimes (\widehat{\Delta} u) \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#769)
\end{aligned}$$

[illegible]

$$\begin{aligned}
& + \frac{1}{2}(d+2) X_4[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes q] \quad (\#816) \\
& + \frac{1}{2}(d+2) X_4[u \otimes N^{v_1} \otimes q \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#817) \\
& + \frac{1}{2}(d+2) X_4[N^{v_1} \otimes q \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#818) \\
& + \frac{1}{2}(d+2) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes q \otimes N^{v_1}] \quad (\#819) \\
& + \frac{1}{2}(d+2) X_4[u \otimes q \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_1}] \quad (\#820) \\
& + \frac{1}{2}(d+2) X_4[u \otimes q \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#821) \\
& + \frac{1}{2}(d+2) X_4[q \otimes N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#822) \\
& + \frac{1}{2}(d+2) X_4[q \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_1}] \quad (\#823) \\
& + \frac{1}{2}(d+2) X_4[q \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#824) \\
& + X_2[N^{v_1} \otimes (\widehat{\nabla}_{v_1} q)] \quad (\#825) \\
& - X_3[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1} q)] \quad (\#826) \\
& - X_3[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1} q)] \quad (\#827) \\
& - X_3[u \otimes (\widehat{\nabla}_{v_1} q) \otimes N^{v_1}] \quad (\#828) \\
& - \frac{1}{6} \text{Ric}_{v_1 v_2} X_3[N^{v_1} \otimes u \otimes N^{v_2}] \quad (\#829) \\
& + \frac{1}{3} \text{Ric}_{v_1 v_2} X_3[u \otimes N^{v_1} \otimes N^{v_2}] \quad (\#830) \\
& + \frac{1}{3} \text{Ric}_{v_1 v_2} X_3[N^{v_1} \otimes N^{v_2} \otimes u] \quad (\#831) \\
& - \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[N^{v_1} \otimes u \otimes u \otimes N^{v_2}] \quad (\#832) \\
& - \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[u \otimes N^{v_1} \otimes u \otimes N^{v_2}] \quad (\#833) \\
& - \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[u \otimes u \otimes N^{v_1} \otimes N^{v_2}] \quad (\#834) \\
& - \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[N^{v_1} \otimes N^{v_2} \otimes u \otimes u] \quad (\#835) \\
& - \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[N^{v_1} \otimes u \otimes N^{v_2} \otimes u] \quad (\#836) \\
& - \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[u \otimes N^{v_1} \otimes N^{v_2} \otimes u] \quad (\#837) \\
& - \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[N^{v_1} \otimes N^{v_2} \otimes u \otimes u] \quad (\#838) \\
& - \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[N^{v_1} \otimes u \otimes u \otimes N^{v_2}] \quad (\#839) \\
& - \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[N^{v_1} \otimes u \otimes N^{v_2} \otimes u] \quad (\#840) \\
& - \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[u \otimes u \otimes N^{v_1} \otimes N^{v_2}] \quad (\#841) \\
& - \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[u \otimes N^{v_1} \otimes u \otimes N^{v_2}] \quad (\#842) \\
& - \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[u \otimes N^{v_1} \otimes N^{v_2} \otimes u] \quad (\#843) \\
& + \frac{1}{4}(d+2)^2 X_4[N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_2} u) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#844) \\
& + \frac{1}{4}(d+2)^2 X_4[N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#845) \\
& + \frac{1}{2}(d+2) X_4[N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes N^{v_2}] \quad (\#846) \\
& + \frac{1}{4}(d+2)^2 X_4[N^{v_1} \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#847) \\
& - \frac{1}{4}(d+4)^2 g_{v_1 v_2} g^{v_3 v_4} X_5[N^{v_1} \otimes N^{v_2} \otimes u \otimes (\widehat{\nabla}_{v_3} u) \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#848) \\
& - \frac{1}{4}(d+4)^2 g_{v_1 v_4} g^{v_2 v_3} X_5[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes (\widehat{\nabla}_{v_3} u) \otimes N^{v_4}] \quad (\#849) \\
& - \frac{1}{4}(d+4)^2 g_{v_1 v_3} g^{v_2 v_4} X_5[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes N^{v_3} \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#850) \\
& - \frac{1}{4}(d+4)^2 g_{v_1 v_2} g^{v_3 v_4} X_5[N^{v_1} \otimes u \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_3} u) \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#851) \\
& - \frac{1}{4}(d+4)^2 g_{v_3 v_4} g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes N^{v_3} \otimes N^{v_4}] \quad (\#852) \\
& - \frac{1}{4}(d+4)^2 g_{v_2 v_4} g^{v_1 v_3} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_3} u) \otimes N^{v_4}] \quad (\#853) \\
& - \frac{1}{4}(d+4)^2 g_{v_2 v_3} g^{v_1 v_4} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_2} \otimes N^{v_3} \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#854) \\
& - \frac{1}{4}(d+4)^2 g_{v_1 v_4} g^{v_2 v_3} X_5[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2} u) \otimes (\widehat{\nabla}_{v_3} u) \otimes N^{v_4}] \quad (\#855) \\
& - \frac{1}{4}(d+4)^2 g_{v_1 v_3} g^{v_2 v_4} X_5[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2} u) \otimes N^{v_3} \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#856) \\
& - \frac{1}{4}(d+4)^2 g_{v_1 v_2} g^{v_3 v_4} X_5[u \otimes N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_3} u) \otimes (\widehat{\nabla}_{v_4} u)] \quad (\#857) \\
& - \frac{1}{2}(d+3)(d+4) X_5[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#858) \\
& - \frac{1}{2}(d+3)(d+4) X_5[N^{v_1} \otimes u \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#859) \\
& - \frac{1}{2}(d+3)(d+4) X_5[N^{v_1} \otimes u \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_2} u) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#860) \\
& - \frac{1}{4}(d+4)^2 X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#861)
\end{aligned}$$

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& +3(d+4)(d+6)g^{v_2v_3}X_6[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}u)\otimes(\bar{\nabla}_{v_2}u)\otimes(\bar{\nabla}_{v_3}N^{v_1})] \quad (\#11136) \\
& +3(d+4)(d+6)g^{v_1v_2}X_6[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}u)\otimes(\bar{\nabla}_{v_2}N^{v_3})\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11137) \\
& +3(d+4)(d+6)g^{v_1v_3}X_6[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}u)\otimes(\bar{\nabla}_{v_2}N^{v_2})\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11138) \\
& +3(d+4)(d+6)g^{v_2v_3}X_6[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}u)\otimes(\bar{\nabla}_{v_2}N^{v_1})\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11139) \\
& +(d+4)(d+6)g^{v_1v_3}X_6[u\otimes(\bar{\nabla}_{v_1}u)\otimes u\otimes u\otimes(\bar{\nabla}_{v_2}N^{v_2})\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11140) \\
& +(d+4)(d+6)g^{v_1v_2}X_6[u\otimes(\bar{\nabla}_{v_1}u)\otimes u\otimes u\otimes(\bar{\nabla}_{v_2}N^{v_3})\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11141) \\
& +(d+4)(d+6)g^{v_2v_3}X_6[u\otimes(\bar{\nabla}_{v_1}u)\otimes u\otimes u\otimes(\bar{\nabla}_{v_2}N^{v_1})\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11142) \\
& +3(d+4)(d+6)g^{v_1v_3}X_6[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_3}u)\otimes(\bar{\nabla}_{v_2}u)] \quad (\#11143) \\
& +3(d+4)(d+6)g^{v_1v_3}X_6[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_2}u)\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11144) \\
& +3(d+4)(d+6)g^{v_2v_3}X_6[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_1})\otimes(\bar{\nabla}_{v_2}u)\otimes(\bar{\nabla}_{v_3}u)] \quad (\#11145) \\
& -d g^{v_1v_3}X_3[u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_3}^2u)] \quad (\#11146) \\
& +3(d+2)g^{v_1v_3}X_4[u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_3}^2u)] \quad (\#11147) \\
& +(d+4)g^{v_2v_3}X_4[u\otimes u\otimes(\bar{\nabla}_{v_1}^2u)\otimes(\bar{\nabla}_{v_3}N^{v_1})] \quad (\#11148) \\
& +(d+2)g^{v_1v_3}X_4[u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_2}^2u)] \quad (\#11149) \\
& +(d+2)g^{v_1v_3}X_4[u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes u\otimes(\bar{\nabla}_{v_2}^2u)] \quad (\#11150) \\
& +(d+2)g^{v_1v_3}X_4[u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes u\otimes(\bar{\nabla}_{v_3}^2u)] \quad (\#11151) \\
& +(d+4)g^{v_1v_3}X_4[u\otimes u\otimes(\bar{\nabla}_{v_1}^2u)\otimes(\bar{\nabla}_{v_3}N^{v_2})] \quad (\#11152) \\
& -(d+4)g^{v_1v_3}X_5[u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes u\otimes u\otimes(\bar{\nabla}_{v_3}^2u)] \quad (\#11153) \\
& -(d+4)g^{v_1v_3}X_5[u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes u\otimes u\otimes(\bar{\nabla}_{v_2}^2u)] \quad (\#11154) \\
& -2(d+4)g^{v_1v_3}X_5[u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes u\otimes(\bar{\nabla}_{v_3}^2u)] \quad (\#11155) \\
& -2(d+4)g^{v_1v_3}X_5[u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes u\otimes(\bar{\nabla}_{v_2}^2u)] \quad (\#11156) \\
& -(d+4)g^{v_1v_3}X_5[u\otimes u\otimes(\bar{\nabla}_{v_1}^2u)\otimes u\otimes(\bar{\nabla}_{v_3}N^{v_2})] \quad (\#11157) \\
& -(d+4)g^{v_2v_3}X_5[u\otimes u\otimes(\bar{\nabla}_{v_1}^2u)\otimes u\otimes(\bar{\nabla}_{v_3}N^{v_1})] \quad (\#11158) \\
& -3(d+4)g^{v_1v_3}X_5[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}^2u)\otimes(\bar{\nabla}_{v_3}N^{v_2})] \quad (\#11159) \\
& -3(d+4)g^{v_2v_3}X_5[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}^2u)\otimes(\bar{\nabla}_{v_3}N^{v_1})] \quad (\#11160) \\
& -3(d+4)g^{v_1v_3}X_5[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_3}^2u)] \quad (\#11161) \\
& -3(d+4)g^{v_1v_3}X_5[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_2}^2u)] \quad (\#11162) \\
& -X_3[u\otimes(\widehat{\Delta}u)\otimes(\bar{\nabla}_{v_1}N^{v_1})] \quad (\#11163) \\
& +\frac{1}{2}(d+2)X_4[u\otimes(\bar{\nabla}_{v_1}N^{v_1})\otimes u\otimes(\widehat{\Delta}u)] \quad (\#11164) \\
& +(d+4)X_4[u\otimes u\otimes(\widehat{\Delta}u)\otimes(\bar{\nabla}_{v_1}N^{v_1})] \quad (\#11165) \\
& +(d+2)X_4[u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_1})\otimes(\widehat{\Delta}u)] \quad (\#11166) \\
& +\frac{1}{2}(d+2)X_4[u\otimes(\widehat{\Delta}u)\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_1})] \quad (\#11167) \\
& -(d+4)X_5[u\otimes(\bar{\nabla}_{v_1}N^{v_1})\otimes u\otimes u\otimes(\widehat{\Delta}u)] \quad (\#11168) \\
& -2(d+4)X_5[u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_1})\otimes u\otimes(\widehat{\Delta}u)] \quad (\#11169) \\
& -(d+4)X_5[u\otimes u\otimes(\widehat{\Delta}u)\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_1})] \quad (\#11170) \\
& -3(d+4)X_5[u\otimes u\otimes u\otimes(\widehat{\Delta}u)\otimes(\bar{\nabla}_{v_1}N^{v_1})] \quad (\#11171) \\
& -3(d+4)X_5[u\otimes u\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_1})\otimes(\widehat{\Delta}u)] \quad (\#11172) \\
& -X_3[u\otimes(\bar{\nabla}_{v_1}N^{v_1})\otimes q] \quad (\#11173) \\
& -X_3[u\otimes q\otimes(\bar{\nabla}_{v_1}N^{v_1})] \quad (\#11174) \\
& -X_3[q\otimes u\otimes(\bar{\nabla}_{v_1}N^{v_1})] \quad (\#11175) \\
& -X_3[N^{v_1}\otimes(\bar{\nabla}_{v_1}u)\otimes(\bar{\nabla}_{v_2}N^{v_2})] \quad (\#11176) \\
& -\frac{1}{2}d X_3[N^{v_1}\otimes(\bar{\nabla}_{v_1}N^{v_2})\otimes(\bar{\nabla}_{v_2}u)] \quad (\#11177) \\
& +\frac{1}{2}(d+4)g_{v_1v_4}g^{v_2v_3}X_4[N^{v_1}\otimes u\otimes(\bar{\nabla}_{v_2}u)\otimes(\bar{\nabla}_{v_3}N^{v_4})] \quad (\#11178) \\
& +\frac{1}{2}(d+2)g_{v_1v_3}g^{v_2v_4}X_4[N^{v_1}\otimes u\otimes(\bar{\nabla}_{v_2}N^{v_3})\otimes(\bar{\nabla}_{v_4}u)] \quad (\#11179) \\
& +\frac{1}{2}(d+4)g_{v_3v_4}g^{v_1v_2}X_4[u\otimes(\bar{\nabla}_{v_1}u)\otimes(\bar{\nabla}_{v_2}N^{v_3})\otimes N^{v_4}] \quad (\#11180)
\end{aligned}$$

[illegible]

$$\begin{aligned}
& + X_4[u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2 v_1}^2 N^{v_2})] \quad (\#1318) \\
& + 2 g^{v_1 v_3} X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3}^3 N^{v_2})] \quad (\#1319) \\
& - \frac{1}{2} d X_3[u \otimes (\widehat{\Delta} N^{v_1}) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#1320) \\
& - X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} N^{v_1})] \quad (\#1321) \\
& + 2(d+2) X_4[u \otimes u \otimes (\widehat{\Delta} N^{v_1}) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#1322) \\
& + \frac{1}{2}(d+6) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\Delta} N^{v_1})] \quad (\#1323) \\
& + (d+6) X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} N^{v_1})] \quad (\#1324) \\
& + \frac{1}{2}(d+2) X_4[u \otimes (\widehat{\Delta} N^{v_1}) \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#1325) \\
& - (d+4) X_5[u \otimes u \otimes (\widehat{\Delta} N^{v_1}) \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#1326) \\
& - (d+4) X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\Delta} N^{v_1})] \quad (\#1327) \\
& - 2(d+4) X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\Delta} N^{v_1})] \quad (\#1328) \\
& - 3(d+4) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\Delta} N^{v_1})] \quad (\#1329) \\
& - 3(d+4) X_5[u \otimes u \otimes u \otimes (\widehat{\Delta} N^{v_1}) \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#1330) \\
& - \frac{1}{2} g_{v_1 v_2} X_3[N^{v_1} \otimes u \otimes (\widehat{\Delta} N^{v_2})] \quad (\#1331) \\
& - \frac{1}{2} g_{v_1 v_2} X_3[u \otimes (\widehat{\Delta} N^{v_1}) \otimes N^{v_2}] \quad (\#1332) \\
& - \frac{1}{2} g_{v_1 v_2} X_3[u \otimes N^{v_1} \otimes (\widehat{\Delta} N^{v_2})] \quad (\#1333) \\
& + g_{v_1 v_2} X_4[N^{v_1} \otimes u \otimes u \otimes (\widehat{\Delta} N^{v_2})] \quad (\#1334) \\
& + g_{v_1 v_2} X_4[u \otimes N^{v_1} \otimes u \otimes (\widehat{\Delta} N^{v_2})] \quad (\#1335) \\
& + g_{v_1 v_2} X_4[u \otimes u \otimes (\widehat{\Delta} N^{v_1}) \otimes N^{v_2}] \quad (\#1336) \\
& + g_{v_1 v_2} X_4[u \otimes u \otimes N^{v_1} \otimes (\widehat{\Delta} N^{v_2})] \quad (\#1337) \\
& - X_3[u \otimes u \otimes (\widehat{\nabla}_{v_1} (\widehat{\Delta} N^{v_1}))] \quad (\#1338) \\
& - X_3[u \otimes u \otimes (\widehat{\Delta} (\widehat{\nabla}_{v_1} N^{v_1}))] \quad (\#1339) \\
& + 2 X_4[u \otimes u \otimes u \otimes (\widehat{\Delta} (\widehat{\nabla}_{v_1} N^{v_1}))] \quad (\#1340) \\
& + 2 X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} (\widehat{\Delta} N^{v_1}))] \quad (\#1341)
\end{aligned}$$

2.4.4 $R_{4,0}$, non simplified, $N^\nu = 0$

$$\begin{aligned}
R_{4,0}(\text{n.s.}, N=0) &= \frac{1}{180} |R|^2 X_1[u] \quad (\#1) \\
&- \frac{1}{180} |\text{Ric}|^2 X_1[u] \quad (\#2) \\
&+ \frac{1}{30} (\widehat{\Delta} \mathfrak{R}) X_1[u] \quad (\#3) \\
&+ \frac{1}{72} \mathfrak{R}^2 X_1[u] \quad (\#4) \\
&+ g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#5) \\
&+ 2 g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_3[u \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#6) \\
&- (d+8) g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#7) \\
&- (d+6) g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#8) \\
&- \frac{1}{2} (d+6) g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u] \quad (\#9) \\
&+ (d+4) g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes u] \quad (\#10) \\
&+ 2(d+4) g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u] \quad (\#11) \\
&+ 3(d+4) g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u] \quad (\#12) \\
&+ 4(d+4) g^{v_1 v_2} (\widehat{\nabla}_{v_2} \mathfrak{R}) X_5[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u)] \quad (\#13) \\
&+ \frac{1}{3} 4 \text{Ric}^{v_1 v_2} X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#14) \\
&+ \frac{1}{3} 2 g^{v_1 v_2} \mathfrak{R} X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#15) \\
&- \frac{1}{3} (9d+34) \text{Ric}^{v_1 v_2} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#16) \\
&- \frac{1}{3} (3d+14) \text{Ric}^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#17) \\
&- \frac{1}{3} 2(d+4) \text{Ric}^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#18) \\
&- \frac{1}{3} 4(d+4) g^{v_1 v_2} \mathfrak{R} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#19) \\
&- \frac{1}{3} 2(d+4) g^{v_1 v_2} \mathfrak{R} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#20) \\
&- \frac{1}{3} 2(d+3) g^{v_1 v_2} \mathfrak{R} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#21) \\
&+ \frac{1}{6} (d+4)(d+22) \text{Ric}^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#22) \\
&+ \frac{1}{6} 5(d+4)(d+10) \text{Ric}^{v_1 v_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#23) \\
&+ (d+4)(d+12) \text{Ric}^{v_1 v_2} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#24) \\
&+ \frac{1}{3} (d+4)(d+8) \text{Ric}^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#25) \\
&+ \frac{1}{3} 2(d+4)(d+8) \text{Ric}^{v_1 v_2} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#26) \\
&+ \frac{1}{3} 2(d+4) \text{Ric}^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#27) \\
&+ \frac{1}{2} (d+4)(d+12) g^{v_1 v_2} \mathfrak{R} X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#28) \\
&+ \frac{1}{3} (d+4)(d+12) g^{v_1 v_2} \mathfrak{R} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#29) \\
&+ \frac{1}{6} (d+4)(d+12) g^{v_1 v_2} \mathfrak{R} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#30) \\
&+ \frac{1}{6} (d+4)^2 g^{v_1 v_2} \mathfrak{R} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#31) \\
&+ \frac{1}{6} (d+4)(d+8) g^{v_1 v_2} \mathfrak{R} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#32) \\
&+ \frac{1}{3} (d+4)(d+8) g^{v_1 v_2} \mathfrak{R} X_5[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#33) \\
&- (d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#34) \\
&- 2(d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#35) \\
&- 3(d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#36) \\
&- 4(d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#37) \\
&- \frac{1}{3} (d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#38) \\
&- \frac{1}{3} 2(d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u \otimes u] \quad (\#39) \\
&- \frac{1}{3} 4(d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#40) \\
&- \frac{1}{3} 2(d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#41) \\
&- 2(d+4)(d+6) \text{Ric}^{v_1 v_2} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u) \otimes u] \quad (\#42) \\
&- \frac{1}{2} (d+4)(d+6) g^{v_1 v_2} \mathfrak{R} X_6[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#43) \\
&- (d+4)(d+6) g^{v_1 v_2} \mathfrak{R} X_6[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#44) \\
&- \frac{1}{2} 3(d+4)(d+6) g^{v_1 v_2} \mathfrak{R} X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#45)
\end{aligned}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& -\frac{1}{2}(d+2)(d+4)g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes q \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#362) \\
& -(d+2)(d+4)g^{v_1 v_2} X_5[u \otimes u \otimes q \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#363) \\
& -\frac{1}{2}(d+2)(d+4)g^{v_1 v_2} X_5[u \otimes (\widehat{\nabla}_{v_1} u) \otimes q \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#364) \\
& -\frac{1}{2}(d+2)(d+4)g^{v_1 v_2} X_5[u \otimes q \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#365) \\
& -(d+2)(d+4)g^{v_1 v_2} X_5[u \otimes q \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#366) \\
& -\frac{1}{2}(d+2)(d+4)g^{v_1 v_2} X_5[q \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#367) \\
& -(d+2)(d+4)g^{v_1 v_2} X_5[q \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#368) \\
& -\frac{1}{2}d X_3[u \otimes (\widehat{\Delta} u) \otimes q] \quad (\#369) \\
& -\frac{1}{2}d X_3[u \otimes q \otimes (\widehat{\Delta} u)] \quad (\#370) \\
& -\frac{1}{2}d X_3[q \otimes u \otimes (\widehat{\Delta} u)] \quad (\#371) \\
& +(d+2) X_4[u \otimes u \otimes (\widehat{\Delta} u) \otimes q] \quad (\#372) \\
& +(d+2) X_4[u \otimes u \otimes q \otimes (\widehat{\Delta} u)] \quad (\#373) \\
& +(d+2) X_4[u \otimes q \otimes u \otimes (\widehat{\Delta} u)] \quad (\#374) \\
& +(d+2) X_4[q \otimes u \otimes u \otimes (\widehat{\Delta} u)] \quad (\#375) \\
& +X_2[q \otimes q] \quad (\#376) \\
& -(d+2)g^{v_1 v_2} X_3[u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} q)] \quad (\#377) \\
& -d g^{v_1 v_2} X_3[u \otimes (\widehat{\nabla}_{v_1} q) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#378) \\
& +(d+2)g^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} u) \otimes u \otimes (\widehat{\nabla}_{v_2} q)] \quad (\#379) \\
& +2(d+2)g^{v_1 v_2} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} u) \otimes (\widehat{\nabla}_{v_2} q)] \quad (\#380) \\
& +2(d+2)g^{v_1 v_2} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} q) \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#381) \\
& +(d+2)g^{v_1 v_2} X_4[u \otimes (\widehat{\nabla}_{v_1} q) \otimes u \otimes (\widehat{\nabla}_{v_2} u)] \quad (\#382) \\
& +X_2[u \otimes (\widehat{\Delta} q)] \quad (\#383) \\
& -2 X_3[u \otimes u \otimes (\widehat{\Delta} q)] \quad (\#384)
\end{aligned}$$

3 R_4 for u parallel

3.1 Case $N^\mu \neq 0$

$$\begin{aligned}
R_{4,0}(u = \text{parallel}) &= \frac{1}{180} |R|^2 X_1[u] \quad (\#1) \\
&- \frac{1}{180} |\text{Ric}|^2 X_1[u] \quad (\#2) \\
&+ \frac{1}{30} (\widehat{\Delta} \mathfrak{R}) X_1[u] \quad (\#3) \\
&+ \frac{1}{72} \mathfrak{R}^2 X_1[u] \quad (\#4) \\
&+ \frac{1}{3} \mathfrak{R} X_3[u \otimes u \otimes q] \quad (\#5) \\
&+ \frac{1}{3} \mathfrak{R} X_3[u \otimes q \otimes u] \quad (\#6) \\
&+ \frac{1}{3} \mathfrak{R} X_3[q \otimes u \otimes u] \quad (\#7) \\
&+ X_2[q \otimes q] \quad (\#8) \\
&+ X_3[u \otimes (\widehat{\Delta} q) \otimes u] \quad (\#9) \\
&+ \frac{1}{2} (\widehat{\nabla}_{v_1} \mathfrak{R}) X_4[u \otimes N^{v_1} \otimes u \otimes u] \quad (\#10) \\
&+ \frac{1}{2} (\widehat{\nabla}_{v_1} \mathfrak{R}) X_4[N^{v_1} \otimes u \otimes u \otimes u] \quad (\#11) \\
&- (\widehat{\nabla}_{v_1} \mathfrak{R}) X_4[u \otimes u \otimes u \otimes N^{v_1}] \quad (\#12) \\
&+ X_3[N^{v_1} \otimes (\widehat{\nabla}_{v_1} q) \otimes u] \quad (\#13) \\
&- X_3[u \otimes (\widehat{\nabla}_{v_1} q) \otimes N^{v_1}] \quad (\#14) \\
&+ \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[u \otimes N^{v_1} \otimes N^{v_2} \otimes u] \quad (\#15) \\
&- \frac{1}{6} \text{Ric}_{v_1 v_2} X_4[N^{v_1} \otimes u \otimes N^{v_2} \otimes u] \quad (\#16) \\
&+ \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[N^{v_1} \otimes N^{v_2} \otimes u \otimes u] \quad (\#17) \\
&+ \frac{1}{3} \text{Ric}_{v_1 v_2} X_4[u \otimes u \otimes N^{v_1} \otimes N^{v_2}] \quad (\#18) \\
&- \frac{1}{6} \text{Ric}_{v_1 v_2} X_4[u \otimes N^{v_1} \otimes u \otimes N^{v_2}] \quad (\#19) \\
&- \frac{1}{3} 2 \text{Ric}_{v_1 v_2} X_4[N^{v_1} \otimes u \otimes u \otimes N^{v_2}] \quad (\#20) \\
&- \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[N^{v_1} \otimes N^{v_2} \otimes u \otimes u] \quad (\#21) \\
&- \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[N^{v_1} \otimes u \otimes u \otimes N^{v_2}] \quad (\#22) \\
&- \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[N^{v_1} \otimes u \otimes N^{v_2} \otimes u] \quad (\#23) \\
&- \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[u \otimes u \otimes N^{v_1} \otimes N^{v_2}] \quad (\#24) \\
&- \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[u \otimes N^{v_1} \otimes u \otimes N^{v_2}] \quad (\#25) \\
&- \frac{1}{6} g_{v_1 v_2} \mathfrak{R} X_4[u \otimes N^{v_1} \otimes N^{v_2} \otimes u] \quad (\#26) \\
&- \frac{1}{2} g_{v_1 v_2} X_3[N^{v_1} \otimes N^{v_2} \otimes q] \quad (\#27) \\
&- \frac{1}{2} g_{v_1 v_2} X_3[N^{v_1} \otimes q \otimes N^{v_2}] \quad (\#28) \\
&- \frac{1}{2} g_{v_1 v_2} X_3[q \otimes N^{v_1} \otimes N^{v_2}] \quad (\#29) \\
&+ \frac{1}{4} g_{v_1 v_2} g_{v_3 v_4} X_4[N^{v_1} \otimes N^{v_2} \otimes N^{v_3} \otimes N^{v_4}] \quad (\#30) \\
&+ \frac{1}{4} g_{v_1 v_3} g_{v_2 v_4} X_4[N^{v_1} \otimes N^{v_2} \otimes N^{v_3} \otimes N^{v_4}] \quad (\#31) \\
&+ \frac{1}{4} g_{v_1 v_4} g_{v_2 v_3} X_4[N^{v_1} \otimes N^{v_2} \otimes N^{v_3} \otimes N^{v_4}] \quad (\#32) \\
&+ \frac{1}{3} g^{v_1 v_3} \text{Ric}_{v_2 v_3} X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes u] \quad (\#33) \\
&+ \frac{1}{3} 2 g^{v_1 v_3} \text{Ric}_{v_2 v_3} X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes u \otimes u] \quad (\#34) \\
&- g^{v_1 v_3} \text{Ric}_{v_2 v_3} X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} N^{v_2})] \quad (\#35) \\
&- \frac{1}{3} \mathfrak{R} X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes u \otimes u] \quad (\#36) \\
&- \mathfrak{R} X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} N^{v_1})] \quad (\#37)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{3}2\Re X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes u] \quad (\#38) \\
& -X_3[u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes q] \quad (\#39) \\
& -X_3[u \otimes q \otimes (\widehat{\nabla}_{v_1} N^{v_1})] \quad (\#40) \\
& -X_3[q \otimes u \otimes (\widehat{\nabla}_{v_1} N^{v_1})] \quad (\#41) \\
& -\frac{1}{2}g_{v_1 v_3} X_4[N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_2} N^{v_3}) \otimes u] \quad (\#42) \\
& -\frac{1}{2}g_{v_2 v_3} X_4[N^{v_1} \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes u \otimes N^{v_3}] \quad (\#43) \\
& -\frac{1}{2}g_{v_2 v_3} X_4[N^{v_1} \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes N^{v_3} \otimes u] \quad (\#44) \\
& -\frac{1}{2}g_{v_2 v_3} X_4[N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_1} N^{v_3}) \otimes u] \quad (\#45) \\
& +\frac{1}{2}g_{v_1 v_2} X_4[N^{v_1} \otimes N^{v_2} \otimes u \otimes (\widehat{\nabla}_{v_3} N^{v_3})] \quad (\#46) \\
& +\frac{1}{2}g_{v_1 v_2} X_4[N^{v_1} \otimes u \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_3} N^{v_3})] \quad (\#47) \\
& +\frac{1}{2}g_{v_1 v_2} X_4[u \otimes N^{v_1} \otimes N^{v_2} \otimes (\widehat{\nabla}_{v_3} N^{v_3})] \quad (\#48) \\
& +\frac{1}{2}g_{v_1 v_3} X_4[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_2} N^{v_2}) \otimes N^{v_3}] \quad (\#49) \\
& +\frac{1}{2}g_{v_1 v_3} X_4[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2} N^{v_2}) \otimes N^{v_3}] \quad (\#50) \\
& +\frac{1}{2}g_{v_1 v_3} X_4[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_2} N^{v_3}) \otimes N^{v_2}] \quad (\#51) \\
& +\frac{1}{2}g_{v_1 v_3} X_4[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2} N^{v_3}) \otimes N^{v_2}] \quad (\#52) \\
& +\frac{1}{2}g_{v_2 v_3} X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes N^{v_2} \otimes N^{v_3}] \quad (\#53) \\
& +\frac{1}{2}g_{v_2 v_3} X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes N^{v_1} \otimes N^{v_3}] \quad (\#54) \\
& +\frac{1}{2}g_{v_2 v_3} X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes N^{v_3} \otimes N^{v_1}] \quad (\#55) \\
& -g_{v_2 v_4} g^{v_1 v_3} X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes (\widehat{\nabla}_{v_3} N^{v_4}) \otimes u] \quad (\#56) \\
& -X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_2}) \otimes (\widehat{\nabla}_{v_2} N^{v_1}) \otimes u] \quad (\#57) \\
& +X_4[u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes u \otimes (\widehat{\nabla}_{v_2} N^{v_2})] \quad (\#58) \\
& +2X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} N^{v_1}) \otimes (\widehat{\nabla}_{v_2} N^{v_2})] \quad (\#59) \\
& -X_4[u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2 v_1}^2 N^{v_2})] \quad (\#60) \\
& -X_4[u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_2 v_1}^2 N^{v_2}) \otimes u] \quad (\#61) \\
& +X_4[N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_2 v_1}^2 N^{v_2})] \quad (\#62) \\
& -X_4[N^{v_1} \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2})] \quad (\#63) \\
& -X_4[N^{v_1} \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2}) \otimes u] \quad (\#64) \\
& +X_4[u \otimes u \otimes N^{v_1} \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2})] \quad (\#65) \\
& +X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_1}) \otimes N^{v_2}] \quad (\#66) \\
& +X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2}^2 N^{v_2}) \otimes N^{v_1}] \quad (\#67) \\
& +2g^{v_1 v_3} X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1 v_2 v_3}^3 N^{v_2})] \quad (\#68) \\
& -\frac{1}{2}g_{v_1 v_2} X_4[u \otimes N^{v_1} \otimes (\widehat{\Delta} N^{v_2}) \otimes u] \quad (\#69) \\
& -\frac{1}{2}g_{v_1 v_2} X_4[N^{v_1} \otimes u \otimes (\widehat{\Delta} N^{v_2}) \otimes u] \quad (\#70) \\
& -\frac{1}{2}g_{v_1 v_2} X_4[u \otimes (\widehat{\Delta} N^{v_1}) \otimes u \otimes N^{v_2}] \quad (\#71) \\
& -\frac{1}{2}g_{v_1 v_2} X_4[u \otimes (\widehat{\Delta} N^{v_1}) \otimes N^{v_2} \otimes u] \quad (\#72) \\
& -X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{v_1} (\widehat{\Delta} N^{v_1}))] \quad (\#73) \\
& -X_4[u \otimes u \otimes (\widehat{\nabla}_{v_1} (\widehat{\Delta} N^{v_1})) \otimes u] \quad (\#74) \\
& -X_4[u \otimes u \otimes u \otimes (\widehat{\Delta} (\widehat{\nabla}_{v_1} N^{v_1}))] \quad (\#75) \\
& -X_4[u \otimes u \otimes (\widehat{\Delta} (\widehat{\nabla}_{v_1} N^{v_1})) \otimes u] \quad (\#76)
\end{aligned}$$

$$R_{4,2}(u = \text{parallel}) = \frac{1}{2} X_3[u \otimes N^{\nu_1} \otimes N^{\nu_2}] F_{\nu_1 \nu_2} \quad (\#1)$$

$$- \frac{1}{2} X_3[N^{\nu_1} \otimes u \otimes N^{\nu_2}] F_{\nu_1 \nu_2} \quad (\#2)$$

$$+ \frac{1}{2} X_3[N^{\nu_1} \otimes N^{\nu_2} \otimes u] F_{\nu_1 \nu_2} \quad (\#3)$$

$$- g^{\nu_2 \nu_3} X_3[u \otimes (\widehat{\nabla}_{\nu_3} N^{\nu_1}) \otimes u] F_{\nu_1 \nu_2} \quad (\#4)$$

$$R_{4,3}(u = \text{parallel}) = g^{\nu_1 \nu_2} X_4[u \otimes N^{\nu_3} \otimes u \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#1)$$

$$- g^{\nu_1 \nu_2} X_4[N^{\nu_3} \otimes u \otimes u \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#2)$$

$$- g^{\nu_1 \nu_2} X_4[u \otimes u \otimes u \otimes N^{\nu_3}] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#3)$$

$$+ g^{\nu_1 \nu_2} X_4[u \otimes u \otimes N^{\nu_3} \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#4)$$

$$- g^{\nu_2 \nu_3} X_4[u \otimes u \otimes u \otimes N^{\nu_1}] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#5)$$

$$+ g^{\nu_2 \nu_3} X_4[u \otimes u \otimes N^{\nu_1} \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#6)$$

$$+ g^{\nu_2 \nu_3} X_4[u \otimes N^{\nu_1} \otimes u \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#7)$$

$$- g^{\nu_2 \nu_3} X_4[N^{\nu_1} \otimes u \otimes u \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#8)$$

$$- 2 g^{\nu_1 \nu_3} X_4[u \otimes u \otimes N^{\nu_2} \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#9)$$

$$- 2 g^{\nu_1 \nu_3} X_4[u \otimes N^{\nu_2} \otimes u \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#10)$$

$$+ 2 g^{\nu_1 \nu_3} X_4[N^{\nu_2} \otimes u \otimes u \otimes u] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#11)$$

$$+ 2 g^{\nu_1 \nu_3} X_4[u \otimes u \otimes u \otimes N^{\nu_2}] (\nabla_{\nu_1 \nu_2 \nu_3}^3 \mathbf{1}) \quad (\#12)$$

$$R_{4,4}(u = \text{parallel}) = \frac{1}{12} X_1[u] F^{\nu_1 \nu_2} F_{\nu_1 \nu_2} \quad (\#1)$$

3.2 Case $N^\mu = 0$

$$R_{4,0}(u = \text{parallel}, N = 0) = \frac{1}{180} |R|^2 X_1[u] \quad (\#1)$$

$$- \frac{1}{180} |\text{Ric}|^2 X_1[u] \quad (\#2)$$

$$+ \frac{1}{30} (\widehat{\Delta} \mathfrak{R}) X_1[u] \quad (\#3)$$

$$+ \frac{1}{72} \mathfrak{R}^2 X_1[u] \quad (\#4)$$

$$+ \frac{1}{3} \mathfrak{R} X_3[u \otimes u \otimes q] \quad (\#5)$$

$$+ \frac{1}{3} \mathfrak{R} X_3[u \otimes q \otimes u] \quad (\#6)$$

$$+ \frac{1}{3} \mathfrak{R} X_3[q \otimes u \otimes u] \quad (\#7)$$

$$+ X_2[q \otimes q] \quad (\#8)$$

$$+ X_3[u \otimes (\widehat{\Delta} q) \otimes u] \quad (\#9)$$

$$R_{4,2}(u = \text{parallel}, N = 0) = 0$$

$$R_{4,3}(u = \text{parallel}, N = 0) = 0$$

$$R_{4,4}(u = \text{parallel}, N = 0) = \frac{1}{12} X_1[u] F^{\nu_1 \nu_2} F_{\nu_1 \nu_2} \quad (\#1)$$

4 R_2 for Non Commutative Torus

4.1 R_2 for P_1

$$\begin{aligned}
R_2 = & -(d+2)g^{\nu_1\nu_2}X_3[u \otimes (\delta_{\nu_1}k)(\delta_{\nu_2}k) \otimes u] \quad (\#1) \\
& + \frac{1}{2}(d^2+2d+8)g^{\nu_1\nu_2}X_4[u \otimes k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#2) \\
& - (d-2)g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes u \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#3) \\
& - 2dg^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k \otimes u \otimes u] \quad (\#4) \\
& - (d-2)g^{\nu_1\nu_2}X_4[u \otimes k(\delta_{\nu_1}k) \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#5) \\
& + 4g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes u \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#6) \\
& - 2dg^{\nu_1\nu_2}X_4[u \otimes u \otimes k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k] \quad (\#7) \\
& + \frac{1}{2}(d+2)^2g^{\nu_1\nu_2}X_4[u \otimes k(\delta_{\nu_1}k) \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#8) \\
& - (d+2)g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes u \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#9) \\
& - 2(d+2)g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes k(\delta_{\nu_2}k) \otimes u \otimes u] \quad (\#10) \\
& - 2(d+2)g^{\nu_1\nu_2}X_4[u \otimes u \otimes (\delta_{\nu_1}k)k \otimes (\delta_{\nu_2}k)k] \quad (\#11) \\
& - (d+2)g^{\nu_1\nu_2}X_4[u \otimes (\delta_{\nu_1}k)k \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#12) \\
& + \frac{1}{2}(d+2)^2g^{\nu_1\nu_2}X_4[u \otimes (\delta_{\nu_1}k)k \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#13) \\
& + \frac{1}{2}(d+2)(d+4)g^{\nu_1\nu_2}X_4[u \otimes (\delta_{\nu_1}k)k \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#14) \\
& - \frac{1}{2}dg^{\nu_1\nu_2}X_3[u \otimes k(\delta_{\nu_1\nu_2}^2k) \otimes u] \quad (\#15) \\
& + 2g^{\nu_1\nu_2}X_3[k(\delta_{\nu_1\nu_2}^2k) \otimes u \otimes u] \quad (\#16) \\
& + 2g^{\nu_1\nu_2}X_3[u \otimes u \otimes (\delta_{\nu_1\nu_2}^2k)k] \quad (\#17) \\
& - \frac{1}{2}dg^{\nu_1\nu_2}X_3[u \otimes (\delta_{\nu_1\nu_2}^2k)k \otimes u] \quad (\#18)
\end{aligned}$$

4.2 R_2 for P_2

$$\begin{aligned}
R_2 = & \frac{1}{2}df^{\nu_1\nu_2}X_3[u \otimes k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k] \quad (\#1) \\
& - f^{\nu_1\nu_2}X_3[k(\delta_{\nu_1}k) \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#2) \\
& + \frac{1}{2}df^{\nu_1\nu_2}X_3[k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#3) \\
& + \frac{1}{2}df^{\nu_1\nu_2}X_3[u \otimes (\delta_{\nu_1}k)k \otimes (\delta_{\nu_2}k)k] \quad (\#4) \\
& - f^{\nu_1\nu_2}X_3[(\delta_{\nu_1}k)k \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#5) \\
& + \frac{1}{2}df^{\nu_1\nu_2}X_3[(\delta_{\nu_1}k)k \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#6) \\
& + \frac{1}{2}df^{\nu_1\nu_2}X_3[u \otimes k(\delta_{\nu_1}k) \otimes k(\delta_{\nu_2}k)] \quad (\#7) \\
& - f^{\nu_1\nu_2}X_3[k(\delta_{\nu_1}k) \otimes u \otimes k(\delta_{\nu_2}k)] \quad (\#8) \\
& + \frac{1}{2}df^{\nu_1\nu_2}X_3[k(\delta_{\nu_1}k) \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#9) \\
& + \frac{1}{2}df^{\nu_1\nu_2}X_3[u \otimes (\delta_{\nu_1}k)k \otimes k(\delta_{\nu_2}k)] \quad (\#10) \\
& - f^{\nu_1\nu_2}X_3[(\delta_{\nu_1}k)k \otimes u \otimes k(\delta_{\nu_2}k)] \quad (\#11) \\
& + \frac{1}{2}df^{\nu_1\nu_2}X_3[(\delta_{\nu_1}k)k \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#12) \\
& - (d+2)g^{\nu_1\nu_2}X_3[u \otimes (\delta_{\nu_1}k)(\delta_{\nu_2}k) \otimes u] \quad (\#13) \\
& + \frac{1}{2}(d^2+4d+8)g^{\nu_1\nu_2}X_4[u \otimes k(\delta_{\nu_1}k) \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#14) \\
& - \frac{1}{2}(d-2)g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes u \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#15) \\
& - dg^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes k(\delta_{\nu_2}k) \otimes u \otimes u] \quad (\#16) \\
& - \frac{1}{2}(d-2)g^{\nu_1\nu_2}X_4[u \otimes k(\delta_{\nu_1}k) \otimes u \otimes k(\delta_{\nu_2}k)] \quad (\#17)
\end{aligned}$$

$$\begin{aligned}
& + 2g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes u \otimes u \otimes k(\delta_{\nu_2}k)] \quad (\#18) \\
& - d g^{\nu_1\nu_2}X_4[u \otimes u \otimes k(\delta_{\nu_1}k) \otimes k(\delta_{\nu_2}k)] \quad (\#19) \\
& + \frac{1}{2}(d^2 + 4d + 8)g^{\nu_1\nu_2}X_4[u \otimes k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#20) \\
& - \frac{1}{2}(d - 2)g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes u \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#21) \\
& - d g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k \otimes u \otimes u] \quad (\#22) \\
& - \frac{1}{2}(d - 2)g^{\nu_1\nu_2}X_4[u \otimes k(\delta_{\nu_1}k) \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#23) \\
& + 2g^{\nu_1\nu_2}X_4[k(\delta_{\nu_1}k) \otimes u \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#24) \\
& - d g^{\nu_1\nu_2}X_4[u \otimes u \otimes k(\delta_{\nu_1}k) \otimes (\delta_{\nu_2}k)k] \quad (\#25) \\
& + \frac{1}{2}(d^2 + 4d + 8)g^{\nu_1\nu_2}X_4[u \otimes (\delta_{\nu_1}k)k \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#26) \\
& - \frac{1}{2}(d - 2)g^{\nu_1\nu_2}X_4[(\delta_{\nu_1}k)k \otimes u \otimes k(\delta_{\nu_2}k) \otimes u] \quad (\#27) \\
& - d g^{\nu_1\nu_2}X_4[(\delta_{\nu_1}k)k \otimes k(\delta_{\nu_2}k) \otimes u \otimes u] \quad (\#28) \\
& - \frac{1}{2}(d - 2)g^{\nu_1\nu_2}X_4[u \otimes (\delta_{\nu_1}k)k \otimes u \otimes k(\delta_{\nu_2}k)] \quad (\#29) \\
& + 2g^{\nu_1\nu_2}X_4[(\delta_{\nu_1}k)k \otimes u \otimes u \otimes k(\delta_{\nu_2}k)] \quad (\#30) \\
& - d g^{\nu_1\nu_2}X_4[u \otimes u \otimes (\delta_{\nu_1}k)k \otimes k(\delta_{\nu_2}k)] \quad (\#31) \\
& + \frac{1}{2}(d^2 + 4d + 8)g^{\nu_1\nu_2}X_4[u \otimes (\delta_{\nu_1}k)k \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#32) \\
& - \frac{1}{2}(d - 2)g^{\nu_1\nu_2}X_4[(\delta_{\nu_1}k)k \otimes u \otimes (\delta_{\nu_2}k)k \otimes u] \quad (\#33) \\
& - d g^{\nu_1\nu_2}X_4[(\delta_{\nu_1}k)k \otimes (\delta_{\nu_2}k)k \otimes u \otimes u] \quad (\#34) \\
& - \frac{1}{2}(d - 2)g^{\nu_1\nu_2}X_4[u \otimes (\delta_{\nu_1}k)k \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#35) \\
& + 2g^{\nu_1\nu_2}X_4[(\delta_{\nu_1}k)k \otimes u \otimes u \otimes (\delta_{\nu_2}k)k] \quad (\#36) \\
& - d g^{\nu_1\nu_2}X_4[u \otimes u \otimes (\delta_{\nu_1}k)k \otimes (\delta_{\nu_2}k)k] \quad (\#37) \\
& - \frac{1}{2}(d + 2)g^{\nu_1\nu_2}X_3[u \otimes (\delta_{\nu_1\nu_2}^2k)k \otimes u] \quad (\#38) \\
& - \frac{1}{2}(d + 2)g^{\nu_1\nu_2}X_3[u \otimes k(\delta_{\nu_1\nu_2}^2k) \otimes u] \quad (\#39)
\end{aligned}$$

4.3 R_2 for $P_1 + P_2$

$$\begin{aligned}
R_2 = & \frac{1}{2}d f^{\nu_1 \nu_2} X_3[u \otimes k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k] \quad (\#1) \\
& - f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#2) \\
& + \frac{1}{2}d f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#3) \\
& + \frac{1}{2}d f^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k] \quad (\#4) \\
& - f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#5) \\
& + \frac{1}{2}d f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#6) \\
& + \frac{1}{2}d f^{\nu_1 \nu_2} X_3[u \otimes k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k)] \quad (\#7) \\
& - f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#8) \\
& + \frac{1}{2}d f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#9) \\
& + \frac{1}{2}d f^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k)] \quad (\#10) \\
& - f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#11) \\
& + \frac{1}{2}d f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#12) \\
& - 2(d+2) g^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u] \quad (\#13) \\
& + (d^2 + 3d + 8) g^{\nu_1 \nu_2} X_4[u \otimes k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#14) \\
& - \frac{1}{2}3(d-2) g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes u \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#15) \\
& - 3d g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k \otimes u \otimes u] \quad (\#16) \\
& - \frac{1}{2}3(d-2) g^{\nu_1 \nu_2} X_4[u \otimes k(\delta_{\nu_1} k) \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#17) \\
& + 6 g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes u \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#18) \\
& - 3d g^{\nu_1 \nu_2} X_4[u \otimes u \otimes k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k] \quad (\#19) \\
& + (d^2 + 4d + 6) g^{\nu_1 \nu_2} X_4[u \otimes k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#20) \\
& - \frac{1}{2}(3d+2) g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes u \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#21) \\
& - (3d+4) g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k) \otimes u \otimes u] \quad (\#22) \\
& - \frac{1}{2}(d-2) g^{\nu_1 \nu_2} X_4[u \otimes k(\delta_{\nu_1} k) \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#23) \\
& + 2 g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes u \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#24) \\
& - d g^{\nu_1 \nu_2} X_4[u \otimes u \otimes k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k)] \quad (\#25) \\
& + (d^2 + 5d + 8) g^{\nu_1 \nu_2} X_4[u \otimes (\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#26) \\
& - \frac{1}{2}(d-2) g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes u \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#27) \\
& - d g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k) \otimes u \otimes u] \quad (\#28) \\
& - \frac{1}{2}(d-2) g^{\nu_1 \nu_2} X_4[u \otimes (\delta_{\nu_1} k)k \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#29) \\
& + 2 g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes u \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#30) \\
& - d g^{\nu_1 \nu_2} X_4[u \otimes u \otimes (\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k)] \quad (\#31) \\
& + (d^2 + 4d + 6) g^{\nu_1 \nu_2} X_4[u \otimes (\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#32) \\
& - \frac{1}{2}(d-2) g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes u \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#33) \\
& - d g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k \otimes u \otimes u] \quad (\#34) \\
& - \frac{1}{2}(3d+2) g^{\nu_1 \nu_2} X_4[u \otimes (\delta_{\nu_1} k)k \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#35) \\
& + 2 g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes u \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#36) \\
& - (3d+4) g^{\nu_1 \nu_2} X_4[u \otimes u \otimes (\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k] \quad (\#37) \\
& - (d+1) g^{\nu_1 \nu_2} X_3[u \otimes k(\delta_{\nu_1 \nu_2}^2 k) \otimes u] \quad (\#38) \\
& + 2 g^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1 \nu_2}^2 k) \otimes u \otimes u] \quad (\#39)
\end{aligned}$$

$$+ 2 g^{\nu_1 \nu_2} X_3[u \otimes u \otimes (\delta_{\nu_1 \nu_2}^2 k)k] \quad (\#40)$$

$$- (d+1) g^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1 \nu_2}^2 k)k \otimes u] \quad (\#41)$$

4.4 R_2 for $P_1 + P_2$ with $d = 2$

$$\begin{aligned}
R_2(d=2) = & f^{\nu_1 \nu_2} X_3[u \otimes k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k] \quad (\#1) \\
& - f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#2) \\
& + f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#3) \\
& + f^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k] \quad (\#4) \\
& - f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#5) \\
& + f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#6) \\
& + f^{\nu_1 \nu_2} X_3[u \otimes k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k)] \quad (\#7) \\
& - f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#8) \\
& + f^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#9) \\
& + f^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k)] \quad (\#10) \\
& - f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#11) \\
& + f^{\nu_1 \nu_2} X_3[(\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#12) \\
& - 8g^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u] \quad (\#13) \\
& + 18g^{\nu_1 \nu_2} X_4[u \otimes k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#14) \\
& - 6g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k \otimes u \otimes u] \quad (\#15) \\
& + 6g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes u \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#16) \\
& - 6g^{\nu_1 \nu_2} X_4[u \otimes u \otimes k(\delta_{\nu_1} k) \otimes (\delta_{\nu_2} k)k] \quad (\#17) \\
& + 18g^{\nu_1 \nu_2} X_4[u \otimes k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#18) \\
& - 4g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes u \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#19) \\
& - 10g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k) \otimes u \otimes u] \quad (\#20) \\
& + 2g^{\nu_1 \nu_2} X_4[k(\delta_{\nu_1} k) \otimes u \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#21) \\
& - 2g^{\nu_1 \nu_2} X_4[u \otimes u \otimes k(\delta_{\nu_1} k) \otimes k(\delta_{\nu_2} k)] \quad (\#22) \\
& + 22g^{\nu_1 \nu_2} X_4[u \otimes (\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k) \otimes u] \quad (\#23) \\
& - 2g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k) \otimes u \otimes u] \quad (\#24) \\
& + 2g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes u \otimes u \otimes k(\delta_{\nu_2} k)] \quad (\#25) \\
& - 2g^{\nu_1 \nu_2} X_4[u \otimes u \otimes (\delta_{\nu_1} k)k \otimes k(\delta_{\nu_2} k)] \quad (\#26) \\
& + 18g^{\nu_1 \nu_2} X_4[u \otimes (\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k \otimes u] \quad (\#27) \\
& - 2g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k \otimes u \otimes u] \quad (\#28) \\
& - 4g^{\nu_1 \nu_2} X_4[u \otimes (\delta_{\nu_1} k)k \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#29) \\
& + 2g^{\nu_1 \nu_2} X_4[(\delta_{\nu_1} k)k \otimes u \otimes u \otimes (\delta_{\nu_2} k)k] \quad (\#30) \\
& - 10g^{\nu_1 \nu_2} X_4[u \otimes u \otimes (\delta_{\nu_1} k)k \otimes (\delta_{\nu_2} k)k] \quad (\#31) \\
& - 3g^{\nu_1 \nu_2} X_3[u \otimes k(\delta_{\nu_1 \nu_2}^2 k) \otimes u] \quad (\#32) \\
& + 2g^{\nu_1 \nu_2} X_3[k(\delta_{\nu_1 \nu_2}^2 k) \otimes u \otimes u] \quad (\#33) \\
& + 2g^{\nu_1 \nu_2} X_3[u \otimes u \otimes (\delta_{\nu_1 \nu_2}^2 k)k] \quad (\#34) \\
& - 3g^{\nu_1 \nu_2} X_3[u \otimes (\delta_{\nu_1 \nu_2}^2 k)k \otimes u] \quad (\#35)
\end{aligned}$$