Introduction to asymptotic symmetries

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Abstract

Support material for lectures at the Mai '25 Galileo Galilei Institute school on asymptotic symmetries and flat holography. Contains an introduction to Noether theorem for gauge theories and gravity, covariant phase space formalism, boundary and asymptotic symmetries, future null infinity in Bondi-Sachs coordinates and in Penrose conformal compactification, BMS symmetries and their extensions.

Contents

1 Introduction

Gauge symmetries capture a redundancy in the field equations, and map solutions to solutions but which are typically undistinguishable: the same electric field described by a different potential, or the same spacetime geometry described in different coordinates. The situation can however change in the presence of boundaries. Boundaries, and more in particular the boundary conditions one chooses, can turn gauge symmetries to genuine symmetries such as isometries of the boundary conditions, and more in general making the mapped solutions physically distinguishable. When boundaries are at infinity, one speaks about asymptotic symmetries. This is the topic of this class. A prominent example is the BMS symmetry group of gravitational waves at future null infinity.

When talking about symmetries, a prominent role is taken by Noether's theorem, which identifies conserved currents that can be used to study conservation laws or flux-balance laws. This theorem is particularly useful to treat gauge symmetries, however it is the most delicate case to cover. There are two reasons for this. The first, is that Noether currents are only defined up to exact forms. But in gauge theories, the Noether current is itself an exact form, on-shell. Therefore the whole current is ambiguous. Second, one cannot always fix such ambiguities looking at the canonical generator, because in the presence of radiation, some symmetry generators correspond to vector fields which are not Hamiltonian, hence don't admit a canonical generator in the standard sense. For these reasons for instance, charges for the gravitational BMS asymptotic symmetries where identified first using physical arguments [1, 2, 3, 4, 5], and only later it was shown how to derive them in a consistent and unambiguous way from Noether's theorem [6] (see also discussion in [7]).

In the last few years the interest in boundary and asymptotic symmetries has increased enormously. We remark the connection between BMS symmetries and soft theorems in perturbative quantum gravity championed by Strominger, the relation between corner symmetries and entanglement, the experimental and theoretical work around memory effects, the relation between asymptotic symmetries and perturbation theory, the new explorations proposed by celestial holography, flat holography and Carollian geometry. The current ongoing research has motivated the program of the GGI workshop and the series of lectures we have proposed: introduction to asymptotic symmetries, to celestial holography, to twistor methods for amplitudes, to amplitude methods for gravitational waves, to Carollian geometry. I have the pleasure to propose you the first of these.

Disclaimer: these lecture notes cover only a small part of the large amount of interesting work that has been done in this topic. They furthermore present a rather personal viewpoint, built on my own perspective and work, and limited by it. I hope in due time to have the opportunity to complete them with at least a more extensive bibliography. Any feedback, corrections and comments appreciated.

2 Boundary symmetries

In the context of gauge theories and gravity, we will talk about boundary symmetries in the sense of residual gauge transformations allowed by the boundary conditions. What makes boundary gauge transformations special, if they are allowed by the boundary conditions, is that it can happen that the symplectic 2-form is non-degenerate along these directions, suggesting that they may not be a redundancy of the description. Rather, they could change the way the boundary data influence the physical interpretation of the solution. The simplest, and possibly oldest application of this idea, is an asymptotic diffeomorphism at spatial infinity. Assuming fall-off conditions to a flat metric leaves as residual diffeomorphisms the isometries of the flat metric, namely Poincaré transformations, and their interpretation is to describe the same physical spacetime as it would look like from the perspective of observers that can be translated, rotated or boosted with respect to one another. This idea can be applied also to null infinity, to finite boundaries, and to other gauge theories than gravity. In all cases, one needs first a study of boundary conditions to identify the residual gauge transformations, and then an analysis of Noether's theorem and canonical generators in order to determine the dynamical properties that charges for the symmetries capture. A useful setup to have in mind when thinking about boundary conditions is a finite region of spacetime, bounded by two space-like hypersurfaces, as in Fig. 1. The codimension-1 boundary connecting the two hypersurfaces could be time-like, or null. If the spatial hypersurfaces extend all the way to infinity and data on them captures all the solutions of the physical theory under consideration, we refer to them as Cauchy hypersurfaces. Otherwise we will generically refer to them as partial Cauchy hypersurfaces.

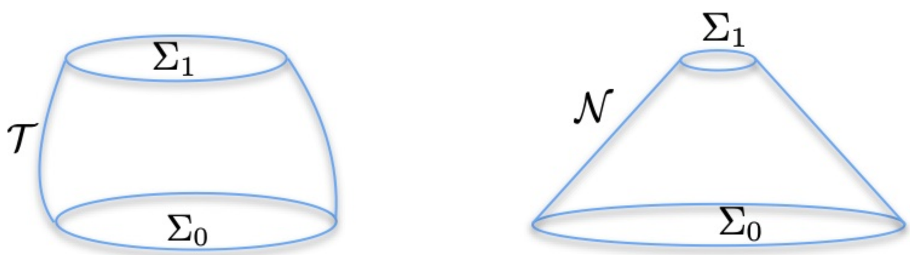

Figure 1: *Two space-like hypersurfaces Σ_1 and Σ_2 joined by a time-like boundary (left panel, $\mathcal{T}$) or a null boundary (right panel, $\mathcal{N}$).*

2.1 Covariant phase space

The covariant phase space is a very convenient tool to discuss boundary symmetries. Before introducing it, let us briefly recall the more conventional construction of phase space through the canonical formalism. Roughly speaking, for a simple mechanical system with second order equations of motion, one intersects the space of trajectories with an 'initial time surface', whose position and velocity can be taken as initial conditions identifying each trajectory. The space of such initial conditions can be equipped with a symplectic structure induced from an action principle in Hamiltonian form:

$$S = \int dt\,(p\dot{q} - H) \quad \Rightarrow \quad \theta := pdq, \quad \omega := d\theta = dp \wedge dq. \tag{2.1}$$

We call θ the symplectic potential, and ω the symplectic 2-form. It is closed ($d\omega = 0$), non-degenerate ($\det\omega \neq 0$), and it is furthermore conserved in that sense that

$$\dot{\omega} \hat{=} 0. \tag{2.2}$$

Here and in the following the short-hand notation $\hat{=}$ means on-shell of the equations of motion.

Box 1. Some details and conventions. Poisson brackets:

$$\{q,p\} = 1, \qquad \{F,G\} = \partial_q F \partial_p G - \partial_p F \partial_q G \tag{2.3}$$

Hamiltonian vector fields:

$$\hat{F} := \{\cdot, F\}, \qquad \hat{p} = \partial_q, \qquad \hat{q} = -\partial_p, \qquad \hat{H} = \partial_t. \tag{2.4}$$

Realization of a Lie algebra: given a canonical transformation or symmetry that belongs to a Lie algebra

$$[\xi_1, \xi_2]^c = f_{ab}{}^c \xi_1^a \xi_2^b, \tag{2.5}$$

we can seek its phase space realization via generators

$$\{G_a, G_b\} = f_{ab}{}^c G_c, \qquad \delta_2 G_1 := \delta_{\hat{\xi}_2} G(\xi_1) = \{G(\xi_1), G(\xi_2)\} = G_{[\xi_1,\xi_2]} \tag{2.6}$$

The sign of f is conventional, hence also on the RHS of the last equality. For instance for rotations, it depends on whether the phase space configuration q are to transform covariantly or contravariantly. Then from the Jacobi identity it follows that

$$\widehat{\{G_1, G_2\}} = -[\hat{G}_1, \hat{G}_2]. \tag{2.7}$$

Symplectic structure:

$$\omega = dp \wedge dq, \qquad \theta = p dq, \qquad d\omega = 0. \tag{2.8}$$

A vector field v is called Hamiltonian if it preserves the symplectic structure, and this guarantees (up to non-trivial cohomology) that its flow is generated by a scalar in field space h_v, called Hamiltonian of the vector field:

$$\pounds_v \omega = i_v d\omega + d i_v \omega = d i_v \omega = 0 \quad \Rightarrow \quad -i_v \omega = dh_v. \tag{2.9}$$

For example, time evolution and the energy Hamiltonian:

$$\partial_t = \dot{q}\partial_q + \dot{p}\partial_p, \qquad -i_{\partial_t}\omega = -\dot{p}dq + \dot{q}dp \hat{=} dH = \partial_q H dq + \partial_p H dp \tag{2.10}$$

and

$$\dot{\omega} = \pounds_{\partial_t}\omega = d i_{\partial_t}\omega \hat{=} -d^2 H = 0. \tag{2.11}$$

In general,

$$-i_{\hat{F}}\omega = dF, \qquad \{F,G\} = i_{\hat{F}} i_{\hat{G}} \omega = \omega(\hat{G}, \hat{F}). \tag{2.12}$$

The key step of the canonical formalism are a choice of time, of initial value surface, and of momenta identified by the chosen time. These steps hide covariance, an issue that becomes more significative in relativistic field theory, where the initial data are associated with a choice of Cauchy slice, and even more so in general relativistic field theories, where there is no preferred simultaneity surface to be chosen. The idea of the covariant phase space (whose germ actually goes back to Lagrange himself and pre-dates the canonical formalism) is to associate a symplectic structure to the trajectories themselves, as opposed to the initial data identifying them. Such a construction does not require any choice of time or momenta, and manifestly preserves covariance. To realize this idea, we first define the *field space* as the ensemble of all trajectories $q(t)$ (not necessarily solutions). We can think of this functional space as an uncountable infinite-dimensional space, for which we can take coordinates $q(t)$ that are labelled by a continuous index t, and define the functional derivative $\frac{\delta q(t)}{\delta q(t')} = \delta(t,t')$. We view the infinitesimal variations $\delta q(t)$ as coordinate differentials, namely δ now denotes the exterior derivative for differential forms on the field space. We denote a generic 1-form $F = F[q(t)]\delta q(t)$, and the wedge product $\curlywedge$. Notice that the 2-form $\delta q(t) \curlywedge \delta q(t')$ is not zero as long as $t \neq t'$, just like $dx^\mu \wedge dx^\nu$ is not zero as long as $\mu \neq \nu$. Vector fields have functionals for components, and can be represented in the coordinate basis as $V := \int dt V[q(t)] \frac{\delta}{\delta q(t)}$. If these notions feel initially too abstract, it is useful to ground them as a standard vector space whose coordinate label has been made continuous:

$$x^\mu \to q(t), \qquad \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \to \frac{\delta q(t)}{\delta q(t')} = \delta(t,t'), \qquad \partial_\mu \to \frac{\delta}{\delta q(t)}, \qquad dx^\mu = \delta q(t). \tag{2.13}$$

One then moves on to build a complete differential calculus in the field space, with an interior product I_V pairing forms and vectors,

$$I_V F = \int dt' V[q(t')] F[q(t)] \frac{\delta q(t)}{\delta q(t')} = \int dt V[q(t)] F[q(t)], \tag{2.14}$$

just like $i_v \alpha = v^\mu \alpha_\mu$, and a field space Lie derivative $\delta_V = I_V \delta + \delta I_V$ satisfying Cartan's formula, just like $\pounds_v = i_v d + d i_v$. Notice that when acting on field-space scalars like the trajectories themselves or the Lagrangian, the field-space Lie derivative has a single term, $\delta_V = I_V \delta$, and acquires the connotation of a variation specialized to the direction identified by the vector field V. In other words, we can recover a functional variation from a 1-form in field space acting on it with a vector field whose components are the desired variation.

A difference however is that in the functional case we can also consider derivatives of the function. Consider for instance the case of the Lagrangian, which is a functional of the trajectories and its time derivatives. Their variations should be treated as independent quantities, as we know from the variational principle. Accordingly, $\delta q(t)$ and $\delta \dot{q}(t)$ are two different 1-forms, and $\delta/\delta q(t)$ and $\delta/\delta \dot{q}(t)$ two different vector fields. As a result, while the finite-dimensional exterior calculus (namely the quantities that in (2.13) are in the left of the arrows) makes reference to the notions of tangent and cotangent bundles over a manifold, the exterior calculus on field space makes reference to a bundle whose base is the argument of the function, or of the fields in the field theory case, and whose fibers are the function and all its derivatives. In mathematical terms, this is called *jet bundle*. Just like a vector field is a section of the tangent bundle, a field and its derivatives seen as functions of the coordinates is a section of the jet bundle. A convenient aspect of this formalism is that we can formally treat the fields and all their derivatives as independent variations (each a different *jet*), with their dependence restored when we look at specific solutions. Other than this, we will not need the mathematical properties of the jet bundle in the following.

This notation has the advantage of scaling up immediately from trajectories of a finite-dimensional system to field theories, whose trajectories are the spacetime field configuration. We simply replace $q(t)$ by $\phi(x^\mu)$, where ϕ is the dynamical field under consideration, and instead of a single continuous label t we have n continuous labels $x^0, \ldots x^{n-1}$. From this perspective, the finite-dimensional case can be thought of as a special case of field theory in $1+0$ dimensions.

Since each field $\phi(x^\mu)$ has now a double differential structure, with respect to the spacetime manifold and with respect to the field space, one can define a *variational bi-complex*, where both operations can be performed consistently. To do so, one has to keep track of the two gradings of each object, say (p, P) for a quantity that is a p form in spacetime and a P form in field space, and choose a convention for the total differential. In most mathematical literature on the subject [8, 9, 10], the total differential is defined to be $d + \delta$. This implies that d and δ anti-commute, in order to guarantee that the differential square to zero. We prefer instead to define the total differential as $d + (-1)^{p+P}\delta$, so that d and δ commute, which simplifies one's life when doing calculations. We then define the graded commutator $[F^{(p,P)}, G^{(q,Q)}] = FG - (-1)^{pq+PQ} GF$. The notation for the variational bi-complex is summarized in Table 1, and some useful basic commutators are

$$\begin{array}{lll}
[d, i_v] = \pounds_v & [d, \delta] = 0 & [\delta, I_X] = \delta_X \\
[\pounds_\xi, d] = 0 & [d, I_\chi] = 0 = [\delta, i_\xi] & [\delta_\xi, \delta] = 0 \\
[\pounds_\xi, i_\chi] = i_{[\xi,\chi]} & [i_\xi, I_\chi] = 0 & [\delta_\xi, I_\chi] = -I_{[\xi,\chi]} \\
[\pounds_\xi, \pounds_\chi] = \pounds_{[\xi,\chi]} & [\delta_\xi, \pounds_\chi] = 0 & [\delta_\xi, \delta_\chi] = -\delta_{[\xi,\chi]}
\end{array}$$

The notions of jet bundle and variational bi-complex may seem like unnecessary mathematical sophistications, for a subject like symmetries and Noether's theorem that after all have been at the hearth of physics for more than a century, and can be a priori described using just functional differentiation. And to be fair, I resisted it myself for a while. But in the end it amounts to a small set of additional notions, and it really pays off in the long term: A powerful notation can do a lot of good to simplify and sharpen one's understanding.

Variational bi-complex

spacetime		**field space**
x^μ	coordinates	$\phi(x)$
$v = v^\mu \partial_\mu$	vector field	$X = \int X[\phi] \frac{\delta}{\delta\phi}$
d	exterior derivative	δ
i_v	interior product	I_X
$\pounds_v$	Lie derivative	δ_X
$\wedge$	wedge product	$\curlywedge$

Table 1: *Notation for the exterior calculus in spacetime and field space.*

To equip the field space with a symplectic structure, we look at the variational principle, and the boundary term induced when we derive the Euler-Lagrange equations

$$\delta L = E + d\theta \mathrel{\hat{=}} d\theta, \qquad \omega := \delta\theta. \tag{2.15}$$

In doing so, it is convenient to think of the Lagrangian as a top-form, as opposed as to a scalar. In other words, we define

$$S = \int L, \qquad L = \mathcal{L}\epsilon, \tag{2.16}$$

where $\mathcal{L}$ is the Lagrangian scalar, and $\epsilon = \sqrt{-g} d^n x$ the volume form in n spacetime dimensions. In background-independent theories g is a dynamical variable, it is then also convenient to introduce the Lagrangian density $\tilde{\mathcal{L}} = \sqrt{-g}\mathcal{L}$ so that $\delta L = \delta\tilde{\mathcal{L}}\, d^n x$. Having done so, the short-hand notation for the Euler-Lagrange equations used in (2.15) is a 1-form in field space,

$$E = \frac{\delta L}{\delta\phi}\delta\phi = \left(\frac{\partial L}{\partial\phi} - \partial_\mu \frac{\partial L}{\partial\partial_\mu\phi}\right)\delta\phi, \tag{2.17}$$

where we assumed that the Lagrangian is first order in derivatives.

The fact that the boundary term in (2.15) is a good definition of symplectic potential should be natural from the way it is defined in the canonical formalism with the Legendre transform, but can be also verified explicitly. For instance for a non-relativistic point particle in a conservative potential,

$$L = \left(\frac{1}{2}m\dot{q}^2 - V(q)\right) dt, \qquad \delta L = (m\dot{q}\delta\dot{q} - \partial_q V \delta q)dt = -(m\ddot{q} + \partial_q V)\delta q\, dt + d(m\dot{q}\delta q), \tag{2.18}$$

hence

$$\theta = m\dot{q}\delta q, \qquad \omega = m\delta\dot{q} \curlywedge \delta q. \tag{2.19}$$

The field-space 2-form ω so defined is closed and conserved on-shell,

$$\delta\omega = 0, \qquad d\omega \mathrel{\hat{=}} 0. \tag{2.20}$$

The first property follows by construction since ω is field-space exact, and the second from

$$d\omega = \delta d\theta = \delta E + \delta^2 L = \delta E. \tag{2.21}$$

It is also non-degenerate since as we have said different jets are formally treated as independent (this will change in the presence of gauge symmetries, as we will see shortly).

We can also check that we recover the symplectic structure of the canonical formulation if we introduce a constant time slice $t = t_0$, and project the trajectories there. The functions become their values at t_0, the variations become standard variations of the function's values at that point, and we recover the canonical formulation:

$$q(t)|_{t_0} = q, \quad \delta q(t)|_{t_0} = dq, \quad m\dot{q}(t)|_{t_0} = p, \quad \delta p(t)|_{t_0} = dp, \quad \theta|_{t_0} = p dq, \quad \omega|_{t_0} = dp \wedge dq. \tag{2.22}$$

The space of fields equipped with the symplectic structure (2.15) is the *covariant phase space*. From the viewpoint of the variational bi-complex, θ has grading $(n-1, 1)$ and ω has $(n-1, 2)$. Namely, they are both co-dimension 1 forms in the base manifold, and respectively a 1-form and a 2-form in field space. In the finite-dimensional case, $n = 1$, and ω is directly the symplectic 2-form, as (2.22) shows. In field theory, this is the symplectic 2-form *current*. The actually symplectic structure is its integral over a Cauchy hypersurface Σ. By Cauchy hypersurface we mean in the canonical sense that knowledge of initial data on it determines the solutions everywhere. One can also consider 'smaller' hypersurfaces that contain only part of the full data, and we will see examples below. In this case one can talk of a partial Cauchy slice, and partial phase space associated with it.

In these lectures we will restrict attention to $n = 4$, so the currents are 3-forms. Their Hodge dual is a vector, and we will use the following conventions:

$$\theta_{\mu\nu\rho} = \theta^\alpha \epsilon_{\alpha\mu\nu\rho}, \qquad \theta^\mu := -\frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}\theta_{\nu\rho\sigma}, \qquad d\theta = \partial_\mu \tilde{\theta}^\mu d^4x. \tag{2.23}$$

Notice also that the Lagrangian only defines $d\theta$, hence θ is defined only up to adding a closed 3-form, which in fact would be necessarily exact thanks to a theorem by Wald [11]. We will refer to the choice of θ corresponding to simply removing d as the 'standard' choice. The freedom to change the standard choice plays an important role in the realization of asymptotic symmetries, and we will discuss it at length below.

CPS symplectic structure			
θ	symplectic potential current;	$\Theta_\Sigma = \int_\Sigma \theta$	symplectic potential
ω	symplectic 2-form current;	$\Omega_\Sigma = \int_\Sigma \omega$	symplectic 2-form

Table 2: *Components of the symplectic structure of the covariant phase space. Here Σ can be a complete or partial Cauchy slice, it can be space-like, or null.*

2.2 Examples

Let us work out the standard CPS symplectic structure for a few field theories of interest.

- Klein-Gordon
$$L = \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) d^4x, \qquad \theta^\mu = -\partial^\mu\phi\delta\phi. \tag{2.24}$$
Projecting on a space-like slice we recover the usual canonical formalism,
$$\theta^t = \pi\delta\phi, \qquad \pi = \dot{\phi} \tag{2.25}$$
Details.
$$\delta L = (-\partial_\mu\delta\phi\partial^\mu\phi - \partial_\phi V\delta\phi)\, d^4x = (\Box\phi - \partial_\phi V)\delta\phi d^4x - \partial_\mu(\delta\phi\partial^\mu\phi)d^4x$$
$$d\theta = -\partial_\mu(\delta\phi\partial^\mu\phi)d^4x, \qquad \theta^\mu = -\delta\phi\partial^\mu\phi$$

- Maxwell.
$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}d^4x, \qquad \theta^\mu = -F^{\mu\nu}\delta A_\nu \tag{2.26}$$
Projecting on a space-like slice we recover the usual canonical formalism,
$$\theta^t = \pi^\mu\delta A_\mu, \qquad \pi^\mu = -F^{0\mu} = \dot{A}^\mu - \partial^\mu A^0 \equiv E^\mu. \tag{2.27}$$
This polarization is associated to conservative boundary conditions of the Dirichlet type, $\delta A_a = 0$, and a notion of 'stationarity' as solutions with vanishing electric field.
Details.
$$\delta L = -\frac{1}{2}\delta F_{\mu\nu}F^{\mu\nu}d^4x = -\partial_\mu\delta A_\nu F^{\mu\nu}d^4x = \partial_\mu F^{\mu\nu}\delta A_\nu d^4x - \partial_\mu(\delta A_\nu F^{\mu\nu})d^4x$$
$$d\theta = -\partial_\mu(\delta A_\nu F^{\mu\nu})d^4x, \qquad \theta^\mu = -F^{\mu\nu}\delta A_\nu$$

- Yang-Mills.
$$L = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})d^4x, \qquad \theta^\mu = -\mathrm{Tr}(F^{\mu\nu}\delta A_\nu) \tag{2.28}$$

- Chern-Simons
$$L = \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \qquad \theta = -\mathrm{Tr}(A\wedge\delta A) \tag{2.29}$$

- General Relativity
$$L = \frac{1}{16\pi}(R-2\Lambda)\epsilon, \qquad \theta^\mu = \frac{1}{8\pi}g^{\rho[\sigma}\delta\Gamma^{\mu]}_{\rho\sigma} = \frac{1}{8\pi}g^{\mu[\rho}g^{\nu]\sigma}\nabla_\nu\delta g_{\rho\sigma}. \tag{2.30}$$

Details. The variation of the EH Lagrangian gives the field equations plus an exact 4-form, induced by the identity $g^{\mu\nu}\delta R_{\mu\nu} = 2\nabla_\mu(g^{\rho[\sigma}\delta\Gamma^{\mu]}_{\rho\sigma})$,

$$\delta L = \frac{1}{16\pi}\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\epsilon + d\theta \tag{2.31}$$

- General Relativity in tetrad variables

$$L_e = \frac{1}{2}\epsilon_{IJKL}\, e^I \wedge e^J \wedge \Big(F^{KL} - \frac{\Lambda}{6}\, e^K \wedge e^L\Big). \tag{2.32}$$

$$E_I = \epsilon_{IJKL}\, e^J \wedge \left(F^{KL} - \frac{2}{3}\Lambda\, e^K \wedge e^L\right), \tag{2.33}$$

$$\theta_e(\delta) = \frac{1}{2}\epsilon_{IJKL}\, e^I \wedge e^J \wedge \delta\omega^{KL}, \qquad \theta^\mu = \frac{1}{3!}\,(\theta_e)_{\nu\rho\sigma}\,\epsilon^{\nu\rho\sigma\mu} = 2e^{[\mu}_I e^{\nu]}_J \delta\omega^{IJ}_\nu. \tag{2.34}$$

2.3 Boundary and corner terms: CPS ambiguities

The symplectic potentials and 2-forms so constructed are not unique. First, recall that adding a boundary term to the Lagrangian does not change the field equations. It changes however the symplectic potential,

$$L' = L + d\ell, \qquad \theta' = \theta + \delta\ell, \qquad \omega' = \omega. \tag{2.35}$$

This transformation does not affect the symplectic 2-form. It plays nonetheless an important role in the phase space, because of its relevance in the study of boundary conditions: adding a boundary Lagrangian ℓ can change the boundary conditions needed in the variational principle. Indeed, the change (2.35) in symplectic potential is akin to a change of polarization, like $p\delta q - \delta(pq) = -q\delta p$. For instance in our basic example (2.18), we can take

$$\ell = -mq\dot{q}, \qquad \theta' = \theta + \delta\ell = -mq\delta\dot{q}, \tag{2.36}$$

which changes the boundary conditions from Dirichlet (fixed position) to Neumann (fixed velocity). An analogue boundary term to switch from Dirichlet to Neumann in the Klein-Gordon example is $\ell = i_{\phi\partial^\mu\phi}\epsilon$, and in the Maxwell example $\ell = i_{F^{\mu\nu}A_\nu}\epsilon$, the effect being to hold fixed the electric field as opposed to the magnetic vector potential. For the gravitational case, we will give more detailed examples in Sec. 2.5 below.

The second source of ambiguities is that even at fixed Lagrangian, the symplectic potential is defined by (2.15) only up to an exact form, as mentioned earlier. Modifying the symplectic potential in this way *does* change the symplectic 2-form:

$$L' = L, \qquad \theta' = \theta - d\vartheta, \qquad \omega' = \omega - d\delta\vartheta. \tag{2.37}$$

We will refer to ϑ as to a *corner term* modification to the symplectic potential. Notice that this modification of the standard symplectic potential cannot be engineered adding a corner term to the boundary Lagrangian, as this would have no effect on the symplectic 2-form:

$$\ell' = \ell + dc, \qquad \theta' = \theta + \delta\ell + \delta dc, \qquad \omega' = \omega. \tag{2.38}$$

The modification (2.37) of the symplectic structure by a corner term is compatible with the field equations, and plays a very important role in the recent developments of asymptotic symmetries.

It occurs naturally in different formulations of the same theory: For instance, the Einstein-Hilbert symplectic potential (2.30) differs by such an exact form from the tetrad symplectic potential (2.34)[12], as shown above, and from the ADM symplectic potential [13]. It can also occur within the same formulation if one derives the symplectic structure not from (2.15) but using homotopy methods as in [9, 10], see e.g. discussion in [14]. Its importance on general grounds was brought to the foreground by [15], which prompted a more systematic analysis (see e.g. [16, 17, 18, 19, 20, 21]). Among the applications of corner terms that will be most relevant to us, they allow to remove divergences in the case of asymptotic symmetries (a procedure sometimes called 'symplectic renormalization') [], and to achieve covariance and select the right phase space realization of the asymptotic symmetries [].

Summarizing, the general equivalence class of symplectic structures is

$$\theta \sim \theta' = \theta + \delta\ell - d\vartheta. \tag{2.39}$$

The freedom is the possibility to add field space or spacetime exact terms, associated respectively with boundary terms of the Lagrangian, and corner terms of the symplectic potential. It is possible to get rid of these ambiguities and select a unique representative with a mathematical prescription, for instance one could choose a specific boundary Lagrangian and a unique symplectic potential associated to it via Anderson's homotopy operator [], see discussion in []. We will see that it is on the other hand more convenient to work with the full equivalence class, and use instead a physical prescription to select a representative adequate to the problem under consideration, in a similar way as in thermodynamics one does not have a universal choice of state functions, but the most suitable ones are chosen only after the physical system and its boundary conditions are specified.

In many cases, we are only interested in the symplectic potential evaluated on a specific hypersurface, for instance the boundary $\mathcal{B}$, or the initial data surface Σ, and in the (partial) phase space there defined. It is then possible to use the ambiguities with a slightly different perspective. Namely, we start from *any* given θ, and we use the freedom to add exact terms in field space and spacetime only *after* pull-back. Namely, we consider the possibility of rearranging the pull-back $\underleftarrow{\theta}$ as follows,

$$\underleftarrow{\theta} = \theta_{\mathcal{B}} - \delta\ell + d\vartheta, \qquad \omega_{\mathcal{B}} = \delta\theta_{\mathcal{B}} = \underleftarrow{\omega} - d\delta\vartheta. \tag{2.40}$$

Any θ' so defined is a good symplectic potential for the phase space at $\mathcal{B}$. Notice that in doing so all three quantities on the RHS may be only defined at $\mathcal{B}$. This occurs specifically if the pull-back involves extra fields defined only at $\mathcal{B}$. Extra care is then needed over whether the extra field should or should not affect the dynamics, and we will talk about this later. In this perspective, the initial θ may well be taken with a mathematical prescription, for instance the 'standard' one, or the homotopy one, because the viewpoint is that it does not matter very much in the end which one one starts from, but only the chosen θ'.

2.4 Background-independence and anomaly operator

A split like (2.40) plays a prominent role in the analysis of gravitational radiation. In a general curved spacetime, one cannot rely on the usual tools granted by a flat background in order to identify radiative degrees of freedom. However this brings in the risk of background dependence in the split. This is the reason why for instance in the original formulation of BMS charges and fluxes one has to carefully check conformal and foliation invariance [1]. To treat this systematically using the variational bi-complex tools, it is very convenient to introduce the anomaly operator, defined by

$$\Delta_\xi F = (\delta_\xi - \pounds_\xi)F, \tag{2.41}$$

see [22, 23, 18, 24].[1] For instance, it allows to compute the field space transformation at null infinity locally on $\mathscr{I}$ from geometric considerations alone, without knowing nothing about asymptotic expansion or postulated fall-off conditions [24].

2.5 Variational principle and polarizations in general relativity

Let us see some examples of this discussion in the gravitational case.

2.5.1 Space-like and time-like boundaries

Consider a hypersurface Σ located at $\Phi = 0$, with n_μ its unit normal, and boundary $\partial\Sigma = S$, with u_μ its unit normal within $T^*\Sigma$, so that $u_\mu n^\mu = 0$. The corresponding volume forms are $\epsilon_\Sigma = i_n\epsilon$ and $\epsilon_S = i_u\epsilon_\Sigma$. The normal is not necessarily geodetic, with $k = 2\pounds_n \ln N$ and $a^\perp_\mu = -q^\nu_\mu\partial_\nu \ln N$. The extrinsic geometry is automatically symmetric thanks to the normalization of n.

Geometric elements of a (non-null) hypersurface

Boundary normal	$\Phi = 0, \quad n_\mu = sN\partial_\mu\Phi, \quad N = (sg^{\Phi\Phi})^{-1/2}$
	$n^2 = s, \quad s = \pm 1, \quad n^\nu\nabla_\nu n_\mu = kn_\mu + a^\perp_\mu$
Induced geometry	$q_{ab} = \underleftarrow{g}_{ab}, \quad \det q = -s, \quad \epsilon_\Sigma = i_n\epsilon$
Projector	$q_{\mu\nu} := g_{\mu\nu} - sn_\mu n_\nu$
Extrinsic geometry	$K_\mu{}^\nu = \underleftarrow{\nabla}_\mu n^\nu = q^\rho_\mu\nabla_\rho n^\nu$

Taking the pull-back of (2.30) one finds (see e.g. [27, 28, 14])

$$\underleftarrow{\theta}^{\text{EH}} = s\,(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K)\,\epsilon_\Sigma + d\vartheta^{\text{EH}}, \qquad \vartheta^{\text{EH}} = -u_\mu\delta n^\mu\epsilon_S = u^\mu n^\nu\delta g_{\mu\nu}\epsilon_S. \tag{2.42}$$

Let us compare different choices of (2.40) and their corresponding polarizations. First, we introduce the gravitational momentum

$$\tilde\Pi^{\mu\nu} := \sqrt{q}(K^{\mu\nu} - q^{\mu\nu}K), \qquad \tilde\Pi := g_{\mu\nu}\tilde\Pi^{\mu\nu} = -2\sqrt{q}K, \tag{2.43}$$

familiar from the ADM analysis, here written as a spacetime tensor. It is then easy to see that

$$\underleftarrow{\theta}^{\text{EH}} = s\tilde\Pi_{\mu\nu}\delta q^{\mu\nu}d^3x - \delta\ell^{\text{GHY}} + d\vartheta^{\text{EH}} = s\,q_{\mu\nu}\delta\tilde\Pi^{\mu\nu}d^3x + d\vartheta^{\text{EH}}, \tag{2.44}$$

where

$$\ell^{\text{GHY}} := 2sK\epsilon_\Sigma \tag{2.45}$$

is the Gibbons-Hawking-York boundary Lagrangian. We see from the second equality in (2.40) that the Einstein-Hilbert Lagrangian has a well-posed variational principle with Neumann boundary conditions (as should be obvious since it contains second derivatives of the fundamental field, the metric), and that to switch to Dirichlet boundary conditions we need to add a boundary term, given by (2.45).

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[1] In the presence of field dependent gauge transformations one has to also include a term $I_{\delta\xi}$ in the definition of the anomaly operator. In that case however the definition of covariance should be kept as the matching of the field-space and spacetime Lie derivative, and not the vanishing of the anomalies, see discussions in [25, 26].

Notice that the sign of the boundary term and of the symplectic structure depends on the signature of the boundary. Of course, boundary conditions on the time-like and space-like boundaries have different meanings: the former determine the nature of the system, whereas the latter determines how one is specifying the states of the system. Nonetheless, both are relevant to the covariant phase space, as discussed in Sec. 2.6.

There is a third interesting choice of polarization, given by mixed boundary conditions proposed by York where one uses the conformal equivalence class of boundary metrics, and the trace of the extrinsic curvature []. The corresponding symplectic potential is obtained via

$$\underleftarrow{\theta}^{\mathrm{EH}} = -s\left(\tilde{P}^{\mu\nu}\delta\hat{q}_{\mu\nu} + \tilde{P}_K\delta K\right)d^3x - \delta\ell^{\mathrm{Y}} + d\vartheta^{\mathrm{EH}}, \tag{2.46}$$

where

$$\tilde{P}^{\mu\nu} := q^{1/3}(\tilde{\Pi}^{\mu\nu} - \frac{1}{3}q^{\mu\nu}\tilde{\Pi}), \qquad \tilde{P}_K = \frac{4}{3}\sqrt{q}, \tag{2.47}$$

and

$$\ell^{\mathrm{Y}} = s\frac{2}{3}K\epsilon_\Sigma \tag{2.48}$$

is the York boundary Lagrangian. These three choices can thus be parametrized as

$$\theta^b = \underleftarrow{\theta} + \delta\ell^b - d\vartheta^{\mathrm{EH}}, \tag{2.49}$$

where the corner term is always the same given by (2.42), and

$$\ell^{\mathrm{Y}} = sbK\epsilon_\Sigma \tag{2.50}$$

with $b = 2, 2/3, 0$.

The correspondence boundary Lagrangian/polarization are reported in the table below.

boundary conditions	*quantity fixed on boundary*	*boundary Lagrangian*	symplectic potential
Dirichlet	$q_{\mu\nu}$	$2K\epsilon_\Sigma$	$\tilde{\Pi}_{\mu\nu}\delta q^{\mu\nu}$
York	$(\hat{q}_{\mu\nu}, K)$	$\frac{2}{3}K\epsilon_\Sigma$	$-\tilde{P}^{\mu\nu}\delta\hat{q}_{\mu\nu} - \tilde{P}_K\delta K$
Neumann	$\tilde{\Pi}^{\mu\nu}$	0	$q_{\mu\nu}\delta\tilde{\Pi}^{\mu\nu}$

Table 3: *Different boundary conditions for a time-like boundary,* $s = 1$.

For more details, and in particular the case of codimension-2 corner Lagrangians required when $n_\mu\bar{n}^\mu \neq 0$, namely non-orthogonal corners, see [15, 29]. As for anomalies, the boundary field is background: $\delta\Phi = 0$. However if we restrict to diffeos tangent to the boundary, *and* we use a unit-norm normal, there are no anomalies [29].

2.5.2 Null boundaries

The main difference of a null boundary is that its normal 1-form defines a vector that is *tangent* to the hypersurface, and not orthogonal to it. And furthermore, there is no canonical normalization for the normal, the induced metric is degenerate (with null direction the null tangent vector itself), and there is no projector on the hypersurface, nor unique induced Levi-Civita connection. A very convenient way to deal with a null boundary is to use the Newman-Penrose formalism. One introduces a doubly null tetrad $(l, n, m, \bar{m})$, of which one real vector is tangent to the null hypersurface (say l), and the second real null vector (then n) acts as a 'rigging vector', or its 1-form as 'rigging 1-form'. It provides a 2d space-like projector via $2m_{(\mu}\bar{m}_{n)} = \gamma_{\mu\nu} := g_{\mu\nu} + 2l_{(\mu}n_{\nu)}$ and, in the case in which it is hypersurface orthogonal, a $2+1$ foliation of $\mathcal{N}$ determined by n and to which $(m, \bar{m})$ are tangent.

Geometric elements of a null hypersurface

Boundary normal	$\Phi = 0, \quad l_\mu = -f\partial_\mu\Phi$
	$l^2 = 0, \quad l^\nu\nabla_\nu l_\mu = k l_\mu$
Induced geometry	$q_{ab} = \underset{\leftarrow}{g}_{ab}, \quad \det q = 0, \quad q_{ab}l^b = 0, \quad \epsilon_{\mathcal{N}} = i_n\epsilon, \quad \epsilon_S = i_l\epsilon_{\mathcal{N}}$
2d projector	$\gamma_{\mu\nu} = g_{\mu\nu} + 2l_{(\mu}n_{\nu)} = 2m_{(\mu}\bar{m}_{n)}$
Extrinsic geometry	$W_\mu{}^\nu := \underset{\leftarrow}{\nabla}_\mu l^\nu = \omega_{\underset{\leftarrow}{\mu}} l^\nu + \gamma^\nu_\rho B_{\underset{\leftarrow}{\mu}}{}^\rho$

Being null and hypersurface orthogonal, l is automatically geodesic. It is however not necessarily affinely parameterized, and an explicit calculation shows that

$$k = \pounds_l \ln f - \frac{f}{2}\partial_\Phi g^{\Phi\Phi}. \tag{2.51}$$

While there is no extrinsic curvature in the usual sense, one can still define the Weingarten map, and with the help of a choice of rigging vector, split it into a vertical and a horizontal component. The horizontal component is purely intrinsic, and features the deformation tensor B occurring in the standard analysis of null congruences. Its antisymmetric part vanishes because l is hypersurface orthogonal, and the rest can be decomposed in terms of shear σ and expansion θ:[2]

$$B_{\mu\nu} := \gamma^\rho_\mu\gamma^\sigma_\nu\nabla_\rho l_\sigma = \frac{1}{2}\gamma^\rho_\mu\gamma^\sigma_\nu\pounds_l\gamma_{\rho\sigma} \stackrel{\mathcal{N}}{=} \sigma_{\mu\nu} + \frac{1}{2}\gamma_{\mu\nu}\theta, \tag{2.52}$$

$$\sigma_{\mu\nu} := \gamma^\rho_{\langle\mu}\gamma^\sigma_{\nu\rangle}\nabla_\rho l_\sigma = -\bar{m}_\mu\bar{m}_\nu\sigma + cc, \qquad \theta := 2m^{(\mu}\bar{m}^{\nu)}\nabla_\mu l_\nu = -2\rho. \tag{2.53}$$

The vertical part is extrinsic, since it depends on the first derivatives of the metric off the hypersurface, and can be conveniently decomposed as follows,

$$\omega_\mu := -\eta_\mu - kn_\mu, \qquad \eta_\mu := \gamma^\rho_\mu n^\sigma\nabla_\rho l_\sigma = -(\alpha + \bar{\beta})m_\mu + cc, \qquad l^\mu\omega_\mu = k = 2\mathrm{Re}(\epsilon). \tag{2.54}$$

Here ω is the rotational 1-form of isolated and non-expanding horizons [30, 31], satisfying $\omega \cdot l = k$; η is the connection 1-form on the normal time-like planes spanned by (l, n), sometimes called Hajicek 1-form [32], or twist. In these formulas, the complex scalars $\alpha, \beta, \epsilon, \rho$ and σ make reference to the NP formalism.[3]

The lack of 3d projector means also that there is no canonical Levi-Civita connection on a null hypersurface. In fact, even the pull-back of the ambient connection does not define a connection on the hypersurface. To see this, we can take two tangent vectors X and Y and compute:

$$l_\mu X^\nu\nabla_\nu Y^\mu = -X^\nu Y^\mu\nabla_\nu l_\mu = -X^\nu Y^\mu(\sigma_{\mu\nu} + \frac{\theta}{2}\gamma_{\mu\nu}). \tag{2.55}$$

[2] Hopefully there should be no confusion between the scalar θ used for the expansion, and the 3-form or vector θ used for the symplectic potential current. When both occur in the same equation, we will put a label to distinguish them.

[3] With mostly-plus signature, we use the conventions of [30]. The twist should not be confused with the 2-sphere connection of the covariant derivative $\eth$ used in NP calculus, which is given by $\alpha - \bar{\beta}$ [33, 34].

For a general null hypersurface the right-hand side does not vanish, hence the pull-back of ambient covariant derivative takes them outside of the hypersurface. Only special hypersurface that are shear-free and expansion-free admit a canonical connection, given by the pull-back of the ambient connection. These hypersurfaces play indeed an important role in the study of non-expanding horizons [35] and future null infinity, see below. For more general hypersurfaces, there is no Levi-Civita connection, but one can take advantage of the rigging vector and introduce a family of *rigging connections*, defined by

$$\mathcal{D}_\mu v^\nu := \Pi^\rho{}_\mu \Pi^\nu{}_\sigma \nabla_\rho v^\sigma, \tag{2.56}$$

where $\Pi^\mu{}_\nu = \delta^\mu_\nu + l^\mu n_\nu$ is a 'half-projector'. The pull-back of (2.56) gives a well-defined 3d connection acting on hypersurface tensors and forms. There is however no canonical choice, and we have a different connection for each choice of rigging.[4] If the hypersurface is shear and expansion free, all rigging connections become rigging-independent and match the canonical, induced connection.

Taking the pull-back of (2.30) one finds [36, 14] (see also [])

$$\underset{\leftarrow}{\theta}^{\text{EH}} = \big[\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + 2\delta(\theta + k)\big]\epsilon_{\mathcal{N}} + \theta\delta\epsilon_{\mathcal{N}} + d\vartheta^{\text{EH}}, \tag{2.57}$$

where

$$\pi_\mu := -2\left(\omega_\mu + \frac{\theta}{2} n_\mu\right) = 2\left(\eta_\mu + \left(k - \frac{\theta}{2}\right) n_\mu\right), \tag{2.58}$$

and

$$\vartheta^{\text{EH}} = n^\mu \delta l_\mu \epsilon_S - i_{\delta l}\epsilon_{\mathcal{N}} = (n^\mu \delta l_\mu + n_\mu \delta l^\mu)\,\epsilon_S - n \wedge i_{\delta l}\epsilon_S. \tag{2.59}$$

In the null case there is less room for changes of polarization, because the spin-2 pair already captures twice the same information,[5] and $\gamma^{\mu\nu}\sigma_{\mu\nu} = 0$. As for the spin-1 pair, the issue is the (lack of) independence of η from the induced metric, to which is related by the field equations. It remains the spin-0 sector, where one can consider changes of polarization in both inaffinity k and expansion θ. This leads to the 2-parameter family of polarizations [26, 38]

$$\underset{\leftarrow}{\theta}^{\text{EH}} = \theta^{(b,c)} - \delta\ell^{(b,c)} + d\vartheta^{\text{EH}}, \qquad \ell^{(b,c)} = -(bk + c\theta)\epsilon_{\mathcal{N}}, \tag{2.60}$$

$$\theta^{(b,c)} = \big[\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + (2-b)\delta k + (2-c)\delta\theta\big]\epsilon_{\mathcal{N}} - (bk + (c-1)\theta)\delta\epsilon_{\mathcal{N}}. \tag{2.61}$$

The relation to boundary conditions is also more delicate than in the non-null case, because of potential loss of covariance. Notice in fact that while θ^{EH} is general covariant by construction, since it only depends on the dynamical metric and no background fields, the split in three different terms (2.60) makes explicit reference to background fields. Hence one has to check that the split does not introduce background-dependence and anomalies. This background dependence can be conveniently studied using the Newman-Penrose formalism, because changes in the background fields (the scale of the normal and the choice of rigging vector) can be generated using the so called class-III and class-I internal Lorentz transformations. This is studied in [26, 38] (see also [39] for a different approach to this question).

[4] One can reduce the freedom choosing for instance the rigging 1-form to be hypersurface orthogonal and Lie dragged by the null tangent vector, which leaves a super-translation residual freedom. There is also some gauge freedom, for instance changing the rigging by a global translation does not change the connection, so given a rigging connection, there is not a unique rigging vector associated to it.

[5] The shear is the Lie derivative of the induced metric, see (2.53). The dependence of momentum on position is a general property of null hypersurfaces, occurring also in the canonical formalism, and due to the presence of second class constraints, see e.g. [37].

boundary conditions	*cons. b.c.*	(b, c)	symplectic potential
Dirichlet	$\gamma_{\mu\nu}, l^\mu$	(2,2)	$(\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu)\epsilon_{\mathcal{N}} - (2k+\theta)\delta\epsilon_{\mathcal{N}}$
CFP	$\gamma_{\mu\nu}, l^\mu, k$	(0,2)	$(\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + 2\delta k)\epsilon_{\mathcal{N}} - \theta\delta\epsilon_{\mathcal{N}}$
ORBS	$\sigma_{\mu\nu}, l^\mu, \theta$	(0,1)	$(\sigma^{\mu\nu}\delta\gamma_{\mu\nu} + \pi_\mu \delta l^\mu + 2\delta k + \delta\theta)\epsilon_{\mathcal{N}}$

Table 4: *Different boundary conditions for a null boundary.*

Relation to Carollian geometry.

From the point of view of holography, it is useful to consider a purely intrinsic description of a null surface, that makes no reference to its embedding. An elegant approach to this problem is to exploit the natural fibration by null geodesics, and replace the (pull-back of the) rigging 1-form with a choice of *Ehresmann connection.* The intrinsic description of null hypersurfaces is the subject of *Carollian geometry*, which you will also learn about in this school. The rigging connections considered here are related to the Carollian connections.

2.6 Dissipative boundary conditions

We can frame the relation between polarizations and the variational principle of the previous section in the more general context of conservative and dissipative boundary conditions in the phase space. As we have seen, the field theory equivalent of $\dot{\omega} \hat{=} 0$ is $d\omega \hat{=} 0$. While at first sight similar, this equation does not immediately imply that the symplectic 2-form is conserved in time. To understand why, let us integrate it over a bounded region of spacetime. By Stokes' theorem,

$$\Omega_{\Sigma_1} \hat{=} \Omega_{\Sigma_2} + \Omega_{\mathcal{B}}, \tag{2.62}$$

where the boundary $\mathcal{B}$ can be for instance time-like or null, as in Fig 1. If the fields vanish at the boundary $\mathcal{B}$, then the symplectic 2-form is the same at the two space-like hypersurfaces, and indeed it is constant in time. This is the situation we are most familiar with in field theory, implemented taking the boundary at infinite distance, and the fields falling off sufficiently fast.

More in general, it is the boundary conditions at $\mathcal{B}$ that determine whether $\Omega_{\mathcal{B}}$ vanishes or not, be it at finite or infinite distance. We can distinguish two general classes of boundary conditions: Conservative boundary conditions, for which $\Omega_{\mathcal{B}} = 0$, and leaky boundary conditions, for which $\Omega_{\mathcal{B}} \neq 0$. In this case there is 'symplectic flux' through the boundary, and the data specified on Σ_1 are not sufficient to reconstruct the data on Σ_2 (intuitively, one can visualize information lost or added through the radiation outgoing $\mathcal{B}$ or incoming). The leaky boundary conditions are also known as radiative, or dissipative/absorbing. In the following we will be mostly interested in the dissipative case, hence we will use this term, but all considerations apply also in the case of absorption.[6] A finer characterization can be obtained if we look at the symplectic potential. Since conservative boundary conditions are typically related to the variational principle, it is convenient to characterize them as $\delta q \stackrel{\mathcal{B}}{=} 0$ where q here represents a complete set of independent configuration variables, in other words $p\delta q$ is an admissible choice of polarization. In the case of dissipative boundary conditions, it is crucial that one be able to identify the degrees of freedom responsible for the dissipation. In other words, to

[6]This is an aspect for which having a null boundary simplifies the analysis with respect to a time-like boundary, because causality then neatly separates the two cases.

be able to identify a special class of solutions for which these degrees of freedom are not excited, and dissipation does not occur. We refer to these special solutions as non-dissipative, or non-radiative, or 'stationary'.[7] Having identified a relevant class of 'stationary' solutions, we seek a polarization for the symplectic potential such that p vanishes on them. As we will see, this formulation is very important because it allows one to study dynamics in dissipative situation with the solid benchmark that conservation automatically occurs when the systems undergoes a non-dissipative epoch.

Summarizing:

- Conservative boundary conditions:

$$\omega_{\mathcal{B}} = 0, \qquad \theta_{\mathcal{B}} = p\delta q, \qquad \delta q = 0 \quad \text{everywhere in phase space} \tag{2.63}$$

- Dissipative boundary conditions:

$$\omega_{\mathcal{B}} \neq 0, \qquad \theta_{\mathcal{B}} = p\delta q, \qquad p = 0 \quad \text{on 'stationary" solutions} \tag{2.64}$$

These considerations give a physical perspective to (2.40): use the freedom in order to choose a symplectic potential realizing the conditions (2.63) or (2.64), according to the situation of interest. If this is possible using only the ℓ freedom, then the change is akin to a change of polarization. If a corner term is needed as well, then the change is more subtle and means that corner degrees of freedom play a role.

3 Noether's theorem for gauge symmetries and gravity

Noether exposed a profound relation between conservation laws and differentiable symmetries (continuous and connected to the identity), and which will be at the heart of our lectures.

Definition: An infinitesimal transformation $\delta_\varepsilon \phi$ with continuous parameter ε is a symmetry if it leaves the field equations invariant, namely the variation of the Lagrangian is at most a boundary term:

$$\delta_\varepsilon L = dY_\varepsilon. \tag{3.1}$$

Noether theorem: For every differentiable symmetry of the Lagrangian there exists a current conserved on-shell, given by

$$j_\varepsilon := I_\varepsilon \theta - Y_\varepsilon, \qquad dj_\varepsilon \,\hat{=}\, 0. \tag{3.2}$$

If the symmetry transformation depends on derivatives of the symmetry parameters, then the field equations are not independent, and the conserved current is on-shell exact:

$$j_\varepsilon \,\hat{=}\, dq_\varepsilon. \tag{3.3}$$

[7]While both non-radiative and stationary have a useful intuitive meaning, they do not characterize the general case, because one could have non-radiative dissipation, or because the notion of stationarity as lack of radiation may not coincide with other uses of the word stationarity. This is for instance the case in general relativity, where stationarity typically refers to the presence of a time-translational Killing vector, and one can have spacetimes with no radiation neither a time-like Killing vector, hence the quotation marks in 'stationary'.

The 3-form j_ε, or its Hodge dual vector via (2.23), is the Noether current of the symmetry ε. Its integral $Q_\varepsilon[\Sigma] := \int_\Sigma j_\varepsilon$ is the Noether charge, and we will refer to q_ε as surface charge aspect. The proof of the first statement follows immediately from $\delta_\varepsilon L = I_\varepsilon E + dI_\varepsilon = dY_\varepsilon$. The proof of the second is only slightly longer, and we give it in Box 2. These two statements are also known separately as first and second Noether's theorems. Notice the power and elegance of the covariant phase space methods: compact and transparent formulas, and straightforward proofs.

Box 2. Proof of Noether's second theorem. Suppose that the infinitesimal transformation of the fields contains *derivatives* of the symmetry parameters, namely $\delta_\epsilon\phi = \epsilon\phi + \phi d\epsilon$, schematically. For instance in Maxwell and GR, we have (3.16) and (**??**). In this case we can write

$$I_\epsilon E = \frac{\delta L}{\delta\phi}\delta_\epsilon\phi = \frac{\delta L}{\delta\phi}(\epsilon\phi + \phi d\epsilon) = \epsilon\left(\frac{\delta L}{\delta\phi}\phi - d(\frac{\delta L}{\delta\phi}\phi)\right) + d\left(\frac{\delta L}{\delta\phi}\epsilon\phi\right). \tag{3.4}$$

Then

$$d(j_\epsilon - \frac{\delta L}{\delta\phi}\epsilon\phi) = \epsilon\left(\frac{\delta L}{\delta\phi}\phi - d(\frac{\delta L}{\delta\phi}\phi)\right). \tag{3.5}$$

The round bracket on the RHS must vanish in the bulk, because ϵ is an arbitrary parameter, and the LHS is a boundary term only. By continuity, it has to vanish on the boundary as well. We conclude two things: that the round bracket on the RHS is an off-shell identity (these are called Noether identities, or generalized Bianchi identity), and that the round bracket in the LHS is a closed form, hence exact by Wald's theorem. It follows that

$$j_\epsilon = dq_\epsilon + \frac{\delta L}{\delta\phi}\epsilon\phi \mathrel{\hat{=}} dq_\epsilon. \tag{3.6}$$

Dependence on derivatives of the symmetry parameters is precisely what happens in gauge theories and gravity. In this case the Noether identities are also called generalized Bianchi identities, and the part of the field equations entering (3.6) are the canonical constraints.

Let us explore the consequences of these conservation laws. By Stokes theorem,

$$Q_\varepsilon[\Sigma_1] \mathrel{\hat{=}} Q_\varepsilon[\Sigma_2] + Q_\varepsilon[\mathcal{B}]. \tag{3.7}$$

If the boundary conditions at $\mathcal{B}$ make $Q_\varepsilon[\mathcal{B}]$ vanish, then the Noether charges are conserved between one hypersurface and the next, namely they are constant in time. Observe the difference between a mechanical system and a field theory: in the first case the Noether charges are automatically conserved in time on solutions, whereas in the latter this requires specific boundary conditions.

On top of this codimension-1 conservation laws, in gauge theories and gravity we also have codimension-2 conservation laws relating the Noether current on Σ to the boundary of Σ via (3.3). Upon integration of this equation, we find

$$Q_\varepsilon[\Sigma] = \int_\Sigma j_\varepsilon \mathrel{\hat{=}} \oint_{\partial\Sigma} q_\varepsilon = Q_\varepsilon[\partial\Sigma]. \tag{3.8}$$

We refer to the term on the right as *surface charges*, because it has support on surfaces (or codimension-2 space in general dimensions). The 2-form integrand q_ϵ is the surface charge aspect. This is the heart of a gauge symmetry: For a gauge symmetry, the Noether charge is itself a surface charge, if Σ has a single boundary, or a difference of surface charges, if for instance Σ has two disconnected boundaries. The simplest example of such codimension-2 conservation law is Gauss's theorem relating the total electric charge in a region of space to the flux of the electric field, and we will cover this example in details shortly.

When a gauge theory is coupled to matter, there is also a special case that can occur: gauge transformations that leave the gauge fields invariant, but affect the matter. These are usually referred to as 'global' gauge transformations, because in electromagnetism this occurs for constant gauge transformations, and are otherwise typically globally defined. They can also be referred to as isometries, and I will favour this term in the following. When this happens, it is possible to consider the gauge fields as background and non-dynamical, and one recovers a symmetry for the matter fields alone, whose Noether current is 3d and not a surface term.

The fact that the Noether current is exact is not the only special feature of gauge transformations. As we will see shortly, another important one is that they correspond to degenerate directions of the symplectic 2-form.

3.1 Example 1: 'global' vs. local U(1) gauge symmetry

Consider a complex scalar field, with Lagrangian

$$\mathcal{L} = -\partial_\mu\phi\partial^\mu\bar{\phi} - V(|\phi|). \tag{3.9}$$

From the variation one obtains the field equations and symplectic potential current,

$$\partial^2\phi - \partial_{\bar{\phi}}V = 0, \qquad \theta^\mu = -\partial^\mu\bar{\phi}\delta\phi - \partial^\mu\phi\delta\bar{\phi}. \tag{3.10}$$

It is easy to see that the Lagrangian is invariant under the transformation $\phi \to e^{i\lambda\phi}$ with $\lambda \in \mathbb{R}$, whose infinitesimal version is $\delta_\lambda\phi = i\lambda\phi$. Its Noether current is

$$j^\mu_\lambda = I_\lambda\theta^\mu = i\lambda\bar{\phi}\overset{\leftrightarrow}{\partial}{}^\mu\phi. \tag{3.11}$$

It conservation can be easily checked, and it gives rise to a Noether charge

$$Q_\lambda = \int_\Sigma I_\lambda\theta^0 d^3x = -i\lambda\int_\Sigma(\bar{\phi}\dot{\phi} - \phi\dot{\bar{\phi}})d^3x, \tag{3.12}$$

that is constant in time, if the fields satisfy conservative boundary conditions at the lateral boundary.

Now let us 'gauge' this symmetry, by coupling the complex scalar field to the Maxwell Lagrangian ('scalar electro-dynamics')

$$\mathcal{L} = -\frac{1}{4}F^2 - D_\mu\phi\overline{D^\mu\phi} - V(|\phi|), \tag{3.13}$$

where $D_\mu\phi = (\partial_\mu + iA_\mu)\phi$ is the covariant derivative.[8] From the variation one obtains the field equations

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad J^\mu = -i\bar{\phi}\overset{\leftrightarrow}{D}{}^\mu\phi = -i\bar{\phi}\overset{\leftrightarrow}{\partial}{}^\mu\phi + 2A^\mu|\phi|^2, \qquad D^2\phi - \partial_{\bar{\phi}}V = 0, \tag{3.14}$$

and symplectic potential

$$\theta^\mu = -F^{\mu\nu}\delta A_\nu - \overline{D^\mu\phi}\delta\phi - D^\mu\phi\delta\bar{\phi}. \tag{3.15}$$

The Lagrangian is invariant under

$$\delta_\lambda\phi = i\lambda\phi, \qquad \delta_\lambda A_\mu = -\partial_\mu\lambda, \qquad \delta_\lambda\mathcal{L} = 0, \tag{3.16}$$

where λ is a real field. There is no boundary term, that is $Y_\lambda = 0$, hence the Noether current is

$$j^\mu_\lambda = I_\lambda\theta^\mu = F^{\mu\nu}\partial_\nu\lambda - \lambda J^\mu = \partial_\nu(\lambda F^{\mu\nu}) + \lambda(\partial_\nu F^{\nu\mu} - J^\mu) \mathrel{\hat{=}} \partial_\nu(\lambda F^{\mu\nu}). \tag{3.17}$$

[8] In the mathematical literature, one often describes ϕ and $\bar{\phi}$ has duals in a complex line bundle, with covariant derivative $D_\mu\phi = (\partial_\mu + A_\mu)\phi$ and $D_\mu\bar{\phi} = (\partial_\mu - A_\mu)\bar{\phi}$. Both conventions give the same results.

It is straightforward to verify that it is conserved on-shell, first statement of Noether's theorem, and the third equality shows the second statement explicitly. This is Noether's theorem for gauge symmetries: we have a conserved current on-shell that is itself vanishing, up to a corner term.

We can now distinguish two cases, depending on whether we take λ to be a constant or not. In the first case the symmetry is global, and it is an isometry in the sense that the gauge field is left invariant. We can then just set $\lambda = 1$. Integrating the current over a space-like hypersurface of constant time we find

$$Q = \int_\Sigma j^0 d^3x = -\int_\Sigma J^0 d^3x \mathrel{\hat{=}} \oint_{\partial\Sigma} E^a dS_a, \tag{3.18}$$

where we used that $F^{0a} = E^a$ the electric field. We recognize this as the electric charge, conserved in time (consequence of the first statement, with conservative boundary conditions), and related to the electric flux by Gauss' theorem (here seen as a direct consequence of the second statement). Notice in particular that the total electric charge must vanish on a spatially compact manifold, since the Noether current is still on-shell exact, also for constant λ.

Since the Noether current is a surface term also for global gauge transformations, when is it that we recover the statement that the Noether current is a 3d integral alone, non-vanishing also in the absence of boundaries? Since the electromagnetic field is left invariant by global transformations, we could then treat as a background, non dynamical field. In this case we recover a global U(1) symmetry for the complex scalar field, with Noether charge non-trivial even in the absence of boundaries. But we have changed the dynamics. In other words, there is a residual symmetry also if the gauge field is non dynamical, but given only by the smaller set of constant λ. This assumes that the electric charge is a manifestation of the electromagnetic field, described by a gauge theory. A global U(1) symmetry of a complex scalar field not related to any gauge field needs not vanish on a spatially compact manifold.

In the general, 'local' case, the Noether charge on the same hypersurface is

$$Q_\lambda[\Sigma] = \int_\Sigma (F^{0\nu}\partial_\nu\lambda - \lambda J^0) d^3x \mathrel{\hat{=}} \oint_{\partial\Sigma} \lambda E^a dS_a = Q_\lambda[\partial\Sigma]. \tag{3.19}$$

We see that the 3d integral has two contributions. These can be referred to as 'soft' and 'hard', in reference to their photonic and matter origin respectively. These names particularly used in applications at null infinity.

where Q_λ now is a *surface* charge. If there is a single boundary, then the 3d and 2d charges coincide. But if there are two boundaries, then they don't. On could refer to the first as Noether charge of flux, and to the second as surface charge. One often also considers both Σ and $\mathcal{B}$ then even more care with language needed.

This application of Gauss's law becomes however particularly useful in the case of dissipative boundary conditions. If these permit residula gauge transformations, then we we apply it to a laterla boundary $\mathcal{B}$, and derive a flux-balance law for each allowed λ that tells us how the surface charge changes under the dissipation.

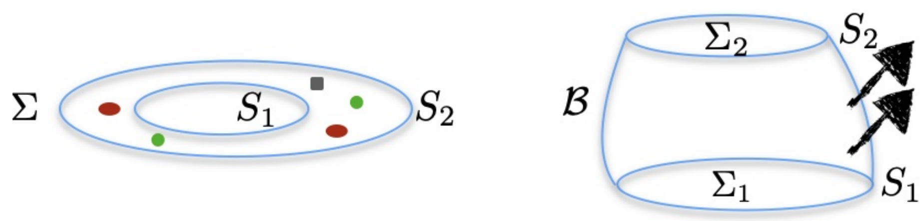

Figure 2: *Different applications of the co-dimension 2 flux-balance laws in electromagnetism.* Left panel: *On a single space-like hypersurface Σ with two boundaries, the surface charge difference is determine by the matter content in between.* Right panel: *Between different times, with dissipative boundary conditions – allowing residual gauge transformations at the boundary, to which λ must belong – the surface charge difference is determined by the flux.*

Now an important caveat about language. If Σ has a single boundary, the Noether charge is a surface integral, and in this example given by the electric flux. If Σ has two boundaries, then the Noether charge is the difference between two surface charges. This difference can also be referred to as a 'flux', in the sense that it captures the dynamical field content inside the boundaries, a name that makes even more sense if Σ is chosen to be time-like or null, as opposed to space-like. You will notice that depending on context, what one calls charge and flux can easily be swapped. So it is important to keep your eyes open and not just your ears, to avoid misunderstandings.

Using forms, the Noether current (3.17) reads

$$j_\lambda = \star F \wedge d\lambda - \lambda J \hat{=} d(\lambda \star F). \tag{3.20}$$

3.2 Example 2: 'global' vs. local diffeomorphisms

Consider a matter Lagrangian $\mathcal{L}_{\mathrm{m}}$ that depends on dynamical matter fields ϕ and a non-dynamical, spacetime metric g. Under a diffeomorphism, we have

$$\delta_\xi L_{\mathrm{m}} = \delta_\phi L_{\mathrm{m}}\, \delta_\xi \phi = \delta_\phi L_{\mathrm{m}}\, \pounds_\xi \phi = \pounds_\xi L_{\mathrm{m}} - \delta_g L_{\mathrm{m}}\, \pounds_\xi g = d i_\xi L_{\mathrm{m}} + \Delta_\xi L_{\mathrm{m}}. \tag{3.21}$$

Two possibilities:

- the metric g admits isometries, then its Killing vectors ξ are symmetries of the matter Lagrangian alone. E.g. in flat spacetime Poincarè transformations, and a standard calculation shows that the associated Noether charges are the energy-momentum and angular momentum tensors.

  add details

- make the metric dynamical. Then all diffeos are symmetries.

With the second option, we arrive to Einstein's principle of general covariance: if every field in the Lagrangian is dynamical, including the metric, diffeomorphisms are a symmetry:

$$\delta_\xi L = \delta_g L\, \delta_\xi g + \delta_\phi L\, \delta_\xi \phi = \delta_g L\, \pounds_\xi g + \delta_\phi L\, \pounds_\xi \phi = \pounds_\xi L = d i_\xi L. \tag{3.22}$$

In this case $Y_\xi = i_\xi L$, and the Noether current is $j_\xi = I_\xi \theta - i_\xi L$.

Let's consider the Einstein-Hilbert Lagrangian coupled to matter fields ϕ, and let's assume for simplicity that the matter Lagrangian $L_{\rm m}$ couples to the metric but not its derivatives. Then

$$j^\mu_\xi = 2\Big((G^\mu{}_\nu + \Lambda\delta^\mu_\nu - \frac{1}{2}T^\mu_\nu)\xi^\nu - \nabla_\nu\nabla^{[\mu}\xi^{\nu]}\Big), \tag{3.23}$$

where

$$T_{\mu\nu} = -\frac{2c}{\sqrt{-g}}\frac{\delta\mathcal{L}_{\rm M}}{\delta g^{\mu\nu}} \tag{3.24}$$

is the energy-momentum tensor of the matter Lagrangian. Taking the divergence of (3.23) we verify it vanishes on-shell of both the Einstein's and matter's field equations:

$$\nabla_\mu j^\mu_\xi = \frac{1}{8\pi G}E^{\mu\nu}\nabla_\mu\xi_\nu - \nabla_\mu T^{\mu\nu}\xi_\nu \,\hat{=}\, 0. \tag{3.25}$$

And the second statement of Noether's theorem is already manifest in (3.23) because the first term vanishes on shell, and the second is a boundary term. This is known as Komar integrand, and can be written as a 2-form as

$$\kappa_\xi := -\frac{1}{32\pi}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\xi^\sigma dx^\mu\wedge dx^\nu. \tag{3.26}$$

Add Details.

To turn (3.23) into a flux-balance law like (3.17), we can use the identity

$$\nabla_\nu\nabla^{[\mu}\xi^{\nu]} = \frac{1}{2}(R^\mu{}_\nu\xi^\nu - \Box\xi^\mu + \nabla^\mu\nabla_\nu\xi^\nu), \tag{3.27}$$

which follows from the definition of the Riemann tensor as the commutator of two covariant derivatives.

Again, we can distinguish two cases, isometries or not. Isometries exists if there are solutions of the Killing equation

$$\pounds_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} = 0. \tag{3.28}$$

In this case it is possible to ignore the dynamics of g, and consider it as a background field, and we go back to option 1 above. Furthermore if ξ^ν is a Killing vector, the last two terms in (3.27) vanish. Then integrating both sides of the equation over a 3d portion of space V delimited by two boundaries S_1 and S_2, and using Stokes' theorem, we find

$$Q_\xi[S] = \oint_S \nabla_\nu\nabla^{[\mu}\xi^{\nu]}dS_\mu, \tag{3.29}$$

$$Q_\xi[S_2] - Q_\xi[S_1] = -\frac{1}{2}\int_V R^\mu{}_\nu\xi^\nu dV_\mu \,\hat{=}\, -\frac{1}{2}\int_V\left(T^{\mu\nu}\xi_\nu - (\Lambda + \frac{T}{2})\xi^\mu\right)dV_\mu. \tag{3.30}$$

The Noether charge (3.29) obtained in this way is known as Komar charge. If the right-hand side of (3.27) vanishes, the Komar charge is conserved in the sense that it has the same value independently of the surface S used, and its value changes only when the deformations of S include some source terms. If the right-hand side does not vanish, the Noether charge varies by an amount determined by the matter energy-momentum in the enclosed region. The simplest application of the Komar formulas is the Kerr spacetime, which possesses Killing vectors corresponding to stationarity and axial symmetry, and whose Komar integrals on an arbitrary 2-sphere S encompassing the singularity give respectively the mass and angular momentum. The identification however requires different choices of overall normalization, an issue which is known as the factor of 2 problem of the Komar mass. As we will see below this issue is elegantly solved looking at the canonical generators in phase space. Another perplexing aspect of the flux-balance law (3.30) is that we have the trace-reversed energy-momentum appearing as source, as opposed to the expected one. But a much more severe problem is that the flux-balance law derived from (3.27) is not very useful in a generic spacetime without Killing vectors. If we apply it for instance to BMS transformations, we would get a non-zero flux also in the absence of radiation.

The idea then is to use the ambiguities (2.39) in the covariant phase space to look for a different Noether current whose flux balance law has a wider range of applicability. Let us first see how the ambiguities affect the charges, and then how we can prescribe a preferred choice.

3.3 Improved Noether charges

When changing the symplectic potential within the equivalence class (2.39) one gets [15] (see also [16, 18, 29, 20, 21])

$$\theta' = \theta + \delta\ell - d\vartheta, \qquad q'_\xi = q_\xi + i_\xi\ell - I_\xi\vartheta. \tag{3.31}$$

In general, q'_ξ is not only determined by the choice of θ' but also from the specific choice of ℓ, in the sense that adding a corner term dc to ℓ can change the charge [20, 21]. The perspective in changing the charges is similar to the one used in thermodynamics, where one looks at different state functions such as internal energy or free energy, depending on the problem. In thermodynamics the different choices are typically related to changes of polarizations, which is controlled by ℓ here, but now we also have the additional possibility of corner term changes, given by ϑ. These play an important role in many situations.

3.4 (Generalized) Wald-Zoupas prescription

In general, characterizing the preferred symplectic potential requires the use of a background, and of background structures that can be associated with the boundary. For instance in general relativity we would like to select preferred charges that are conserved in the absence of gravitational radiation. But this is hard to characterize in a diffeomorphism-invariant way. What is easier is to use a reference background to distinguish radiation from the other modes in the field. So in general there is no general preferred symplectic potential that applies to all situations. One has to first specify the system by specifying a boundary and the boundary conditions, and then look for the preferred symplectic potential at that boundary and with variations restricted to preserving the boundary conditions. In other words, we look only at the equivalence class (2.40). This introduces the caveat discussed earlier that one has to make sure that the split chosen does not introduce dependence on the background structures of the boundary. Namely one has to make sure that the chosen preferred potential is covariant. It may feel ironical that one should worry about covariance in a theory that is general covariant, but the problem is that to do physics it is typically more convenient to introduce a reference frame rather then to look only at gauge invariant quantities, and this reference frame has to be handled in a way that does not break covariance.

Accordingly, we give the following criteria for selecting the preferred symplectic potential [21]:

1. Covariance: $\delta_\xi\bar\theta = \pounds_\xi\bar\theta$, namely independence of any background structure
2. Stationarity: $\bar\theta = p\delta q$ where $p = 0$ for solutions satisfying a notion of stationarity

In the case of asymptotic symmetries, one has to make sure also that the preferred symplectic potential is well-defined, in other words any divergences should also be removed using the equivalence class freedom. The notion of stationarity should be prescribed based on the physical problem at hand. For instance, it could be all solutions without radiation, all solutions without dissipation, all solutions with a time-translation Killing vector, etc. These criteria are based on the seminal Wald-Zoupas paper [6], where the prescription was applied at $\mathscr{I}$, with the stationarity condition defined by the vanishing of the news function. The main generalization is the inclusion of corner terms in the equivalence class (2.40). These are important in situations where it is not possible to realize the two conditions otherwise, or because one wants to introduce corner degrees of freedom in the phase space. I have for instance seen an example of the former when full-filling this construction for the extended BMS symmetry in [24], and an example of the latter when constructing a purely hard flux for BMS transformations in [40].

A very important consequence of these conditions is that they guarantee that the Noether currents realize the symmetry algebra in the covariant phase space in terms of the Barnich-Troessaert bracket [41], without field-dependent cocycles [42].

3.5 Canonical generators and dissipation

On top of being conserved, the Noether charges also provide canonical generators for the symmetry transformations in phase space. To see this, we use (3.2) t derive

$$-I_\epsilon \omega = -\delta_\epsilon \theta + \delta I_\epsilon \theta = -\delta_\epsilon \theta + \delta j_\epsilon + \delta Y_\epsilon. \tag{3.32}$$

Here we have taken the short-hand notation of representing the vector field representing the symmetry transformation with its symmetry parameter. Using repeatedly $[d, \delta] = 0$, and assuming $\delta\varepsilon = 0$ for the time being, we have

$$d\delta_\epsilon \theta \mathrel{\hat{=}} \delta_\epsilon \delta L = d\delta Y_\epsilon \qquad \Rightarrow \qquad \delta_\epsilon \theta \mathrel{\hat{=}} \delta Y_\epsilon + dX_\varepsilon, \tag{3.33}$$

hence

$$-I_\epsilon \omega \mathrel{\hat{=}} \delta j_\epsilon - dX_\varepsilon. \tag{3.34}$$

The form of X_ε depends on the specific theory and symmetry considered.

From this general result we can draw two important conclusions:

- If the symmetry is gauge, j_ε is exact, hence $I_\varepsilon \omega$ only has support on the corners: bulk gauge transformations are degenerate directions of the symplectic 2-form, boundary ones generically not.

- If $X_\varepsilon = \delta b_\varepsilon$ is field-space exact, so is $I_\varepsilon \omega$: the transformation corresponds to a Hamiltonian vector field, with Hamiltonian aspect
$$h_\epsilon \mathrel{\hat{=}} j_\epsilon + b_\epsilon. \tag{3.35}$$

This discussion is completely general. Once we choose the boundary, and the boundary conditions, we can refine the analysis and draw more specific conclusions. In this respect, it is useful to work at the level of the 3-form ω so that we are not committing to a specific pull-back on a given boundary, and we can make statements that are general to the whole CPS.

We can now distinguish three cases.

Case 1: $b_\varepsilon = 0$. The simplest, and indeed most common case, is when $b_\epsilon = 0$, then the Noether charge already computed is the canonical generator of the symmetry on a given phase space. This occurs for instance in electromagnetism, where

$$-I_\lambda \omega = \delta_\lambda F^{\mu\nu} \delta A_\nu - \delta F^{\mu\nu} \delta_\lambda A_\nu = \delta F^{\mu\nu} \partial_\nu \lambda = \delta(F^{\mu\nu}\partial_\nu \lambda) - F^{\mu\nu}\partial_\nu \delta\lambda \,\hat{=}\, \delta j_\lambda \,\hat{=}\, d\delta q_\lambda, \tag{3.36}$$

for $\delta\lambda = 0$ and in vacuum. One can proceed similarly when the complex scalar field is included, the intermediate steps are longer, but using (3.15) and (3.17) we arrive at the same end result. The last equality shows that bulk gauge transformations are degenerate directions of the symplectic 2-form, hence they don't affect the nature of the solution. On the other hand gauge transformations with support on the boundary can be non-trivial in general.[9] The question is whether boundary conditions allow any gauge transformations at the boundary. A prominent example in the Maxwell case are the fall-off conditions at future null infinity, where a residual gauge symmetry of time-independent boundary λ's is allowed, and gives rise to the conservation laws that have remarkably been related to the soft photon theorems of the quantum theory [43].

Case 2: $b_\varepsilon \neq 0$. In this case the canonical generator is shifted with respect to the initial Noether charge computed. It may still be possible to interpret the shift as an *improved* Noether charge, namely from the Noether charge obtained adding a boundary Lagrangian and/or a corner term. This situation occurs for instance in the gravitational case. A famous result by Iyer and Wald is that

$$-I_\xi \omega = -\delta_\xi \theta + \delta I_\xi \theta = -\pounds_\xi \theta + \delta(j_\xi + i_\xi L) \,\hat{=}\, \delta j_\xi - d i_\xi \theta \,\hat{=}\, d(\delta q_\xi - i_\xi \theta). \tag{3.37}$$

If at the boundary $\underleftarrow{\theta} = \delta b$ then we are in case 2, so we have a canonical generator, the charge is integrable, and it is given by $q'_\xi = q_\xi + i_\xi b$. An example of this occurs at spatial infinity, with standard ADM fall-off conditions. These boundary conditions allow residual diffeomorphisms which are isometries of the asymptotic flat metric and are given by the Poincaré group. In this way one can reconstruct the ADM charges from covariant phase space methods, and in particular the b shift solves the famous issue of the missing factor of 2 in the Komar mass [44].

The generator obtained in this way can be interpreted as an improved Noether charge with boundary Lagrangian b [15, 29]. If we restrict the freedom to only changes of polarizations, then the symplectic 2-form is invariant, but the split between integrable and non-integrable is not:

$$\theta' = \theta + \delta\ell, \qquad q'_\xi = q_\xi + i_\xi \ell, \qquad -I_\xi \omega \,\hat{=}\, d(\delta q_\xi - i_\xi \theta) \,\hat{=}\, d(\delta q'_\xi - i_\xi \theta'). \tag{3.38}$$

Another example of case 2 is a finite time-like boundary with conservative boundary conditions. In this case one gets the Brown-York charges [15, 16] or the alternatives with Neumann and York boundary conditions [29],

$$Q^b_\xi = \oint_S q_\xi + i_\xi \ell^b - I_\xi \vartheta^{\text{EH}} = -2 \oint_S n^\mu \xi^\nu (\bar{K}_{\mu\nu} - \frac{b}{2} \bar{q}_{\mu\nu} \bar{K}) \epsilon_S. \tag{3.39}$$

It is also possible to show that the counter-term needed in order to recover the ADM charges in the limit to spatial infinity can be written as a suitable boundary Lagrangian [29].

Case 3: $X_\varepsilon \neq \delta b_\varepsilon$. In this case the canonical generator is not integrable. This occurs typically in the presence of radiation. For instance in the gravitational case, we see from the Iyer-Wald result

[9] Sometimes the name *large* is also used, but this term is also used for the completely unrelated notion of gauge transformations not connected to the identity. For this reason I prefer to avoid it and use instead boundary, or asymptotic gauge transformations.

$X_\xi = i_\xi\theta$ that this would occur for ξ not tangent to the boundary of Σ hyperbolic space-like manifold intersecting $\mathscr{I}$, namely those boundary transformations that are not Hamiltonian vector fields because they deform Ω_Σ in a direction that 'sees' the dissipation. In this case one needs a more careful prescription. The idea is to use the preferred symplectic potential defined by the (generalized) Wald-Zoupas prescription, and define the charges as those that would be canonical generators in the stationary subset of the phase space.

An alternative approach, but which ends up giving the same result, is to look at the flux, namely at the pull-back of $I_\xi\omega$ on $\mathscr{I}$ (or a portion thereof). In that case one still has the problem of non-integrability, but can be dealt with introducing a norm in field space which roughly speaking makes the non-integrable terms measure zero, and then defining the generator as a completion of the integrable one in that dense subset [3, 7]. This is specifically the case with the asymptotic symmetries at future null infinity, and what we will focus on in the rest of the class.

4 BMS symmetry

4.1 Future null infinity

To study the asymptotic symmetries of gravitational waves, we are interested in the behaviour of asymptotically flat spacetimes along null directions. To gain some intuition about these asymptotics, let us first consider the case of flat spacetime. If we use spherical coordinates and retarded time $u := t - r$, the metric reads

$$ds^2 = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B, \tag{4.1}$$

where q_{AB} is the standard round sphere metric. Hypersurfaces of constant u describe outgoing null cones, ruled by null geodesics, and r is an affine parameter along them. Taking the limit $r \to \infty$ at constant u is thus a way to reach future null infinity. A difficulty with this limit is that the metric becomes ill-defined, since the sphere part diverges, and dr is no longer defined. A way to improve the mathematical control is to use Penrose's idea of conformal compactification. To do that, we change the radial coordinate r to

$$\Omega = \frac{1}{r}. \tag{4.2}$$

We have

$$d\Omega = -r^{-2}dr, \qquad \partial_\Omega = -r^2\partial_r, \qquad dr = -\Omega^{-2}d\Omega, \qquad \partial_r = -\Omega^2\partial_\Omega. \tag{4.3}$$

It follows that vector and form components change as follows,

$$v^\Omega = -\Omega^2 v^r, \qquad v_\Omega = -\Omega^{-2}v_r. \tag{4.4}$$

This has a strong impact into the study of limits. Something that looks divergent in r coordinates may not be so, once a well-defined coordinate system is used. The vector field $r\partial_r = -\Omega\partial_\Omega$ may look divergent as $r \to \infty$, but it is in fact well-defined, and actually vanishing, if we use a good coordinate system and $\Omega \to 0$. This opens up the possibility of adding points corresponding to $\Omega = 0$ to the spacetime manifold. We thus obtain a new manifold $\hat{M}$, which we refer to as the conformally completed manifold. The hypersurface $\Omega = 0$ is the boundary of $\hat{M}$.

In the coordinate chart $x^\mu = (u, \Omega, x^A)$ the Minkowski metric reads

$$ds^2 = -du^2 + \Omega^{-2}(2dud\Omega + q_{AB}dx^A dx^B). \tag{4.5}$$

Or as a matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & \frac{1}{\Omega^2} & 0 \\ & 0 & 0 \\ & & \frac{1}{\Omega^2} q_{AB} \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} 0 & \Omega^2 & 0 \\ & \Omega^4 & 0 \\ & & \Omega^2 q^{AB} \end{pmatrix}. \tag{4.6}$$

The metric blows up at the boundary of $\hat{M}$ in a way that can now be controlled. We define the conformally rescaled metric (aka 'unphysical metric')

$$d\hat{s}^2 = 2dud\Omega + q_{AB}dx^A dx^B - \Omega^2 du^2. \tag{4.7}$$

That is,

$$\hat{\eta}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}, \qquad \hat{\eta}_{\mu\nu} = \begin{pmatrix} -\Omega^2 & 1 & 0 \\ & 0 & 0 \\ & & q_{AB} \end{pmatrix}, \qquad \hat{\eta}^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ & \Omega^2 & 0 \\ & & q^{AB} \end{pmatrix}. \tag{4.8}$$

The unphysical metric is well defined at $\Omega = 0$, and given by

$$d\hat{s}^2 \stackrel{\mathscr{I}}{=} 2dud\Omega + q_{AB}dx^A dx^B. \tag{4.9}$$

The pair $(\hat{M}, \hat{g})$ is the conformally rescaled spacetime, and its boundary $\Omega = 0$ is the hypersurface we refer to as future null infinity $\mathscr{I}^+$, or $\mathscr{I}$ in short, since we will talk mostly about future null infinity alone. While $\mathscr{I}$ does not 'exist' in the physical spacetime, it is the boundary of the conformally completed spacetime. Since $\mathscr{I}$ is a hypersurface of $\hat{M}$, its properties can be studied using local differential geometry. Its induced metric is degenerate, and in the (u, x^A) coordinates induced from the bulk coordinates, reads

$$q_{ab} = \begin{pmatrix} 0 & 0 \\ & q_{AB} \end{pmatrix}. \tag{4.10}$$

The null directions are along ∂_u, which provides a an affinely parameterized tangent vector to the null geodesics of $\mathscr{I}$. In particular, we can take as normal to $\mathscr{I}$

$$\mathbb{n} := d\Omega, \tag{4.11}$$

and observe that its norm given by the unphysical metric vanishes, hence $\mathscr{I}$ is a null hypersurface. Furthermore as a vector,

$$\mathbb{n} = \partial_u + \Omega^2 \partial_\Omega \stackrel{\mathscr{I}}{=} \partial_u. \tag{4.12}$$

The retarded time vector ∂_u is thus time-like everywhere in the bulk, and becomes null at $\mathscr{I}$.

Remark: We noticed in Section 2.5.2 that on a null hypersurfaces there is no canonical choice of normal. However in the case of $\mathscr{I}$ the situation is special, because the conformal compactification provides a preferred choice, given by (4.11). The existence of this choice tying up the normal to the conformal factor chosen is ultimately responsible for the extra generator of the symmetry group of physical non-expanding horizons with respect to the BMS group at $\mathscr{I}$ [45, 35].

4.2 Global and asymptotic symmetries

A metric possesses isometries if there are non-trivial solutions to the Killing equation (3.28). Minkowski spacetime is maximally symmetric and admits ten Killing vectors. They form an algebra under Lie bracket which is isomorphic to the Poincaré algebra. The description of the Killing vectors is simplest in Cartesian coordinates, where the metric is constant everywhere, and we get

$$\xi = a^\mu + b^\mu{}_\nu x_\nu, \qquad b_{(\mu\nu)} = 0, \qquad b^0{}_a = b^a, \qquad b^a{}_b = -\epsilon^a{}_{bc} r^c. \tag{4.13}$$

Changing coordinates to retarded time, we find

$$\xi = f\partial_u + Y^A\partial_A - \frac{r}{2}\mathscr{D}_A Y^A \partial_r - \frac{1}{r}\mathscr{D}^A f\partial_A + \frac{1}{2}\mathscr{D}^2 f\partial_r, \tag{4.14}$$

where $\mathscr{D}$ is the covariant derivative on the 2-sphere, and $f = T + \frac{u}{2}\mathscr{D}_A Y^A$. The function $T = T(x^A)$ is a linear combination of the lowest harmonics $l = 0, 1$ and encodes the translation parameters a. The vectors Y^A are conformal Killing vectors of the sphere, which span the Lorentz group and encode the parameters b. Using the inverse radius coordinate $\Omega = 1/r$ makes it manifest that the Killing vector are tangent to $\mathscr{I}$:

$$\xi = f\partial_u + Y^A\partial_A + \Omega(\dot{f}\partial_\Omega - \mathscr{D}^A f\partial_A) - \frac{1}{2}\Omega^2\mathscr{D}^2 f\partial_\Omega. \tag{4.15}$$

These expressions are exact to all orders in r, or Ω: these are the global Killing vectors. Notice that the Killing vectors are also conformal Killing vectors of the unphysical metric, since they satisfy

$$\pounds_\xi \hat{\eta}_{\mu\nu} = 2\pounds_\xi \ln\Omega\, \hat{\eta}_{\mu\nu} + \Omega^2 \pounds_\xi \eta = 2\alpha_\xi \hat{\eta}_{\mu\nu}, \qquad \alpha_\xi := \pounds_\xi \ln\Omega = \frac{\mathbb{n}\cdot\xi}{\Omega}. \tag{4.16}$$

Now let's look at (4.7). We can ask for a smaller condition, namely *asymptotic Killing vectors* that preserve only the leading order at $\mathscr{I}$ of (4.7). Namely, we seek ξ that solve the equation

$$\pounds_\xi \hat{\eta}_{\mu\nu} \stackrel{\mathscr{I}}{=} 2\alpha_\xi \hat{\eta}_{\mu\nu}, \tag{4.17}$$

as opposed to (4.16). This equation is much weaker, and has two strong effect on the global solution (4.14): first, only the $O(\Omega)$ is determined, all higher orders are left free. The reason why the $O(\Omega)$ is fixed is because we are requiring (4.47) for the spacetime metric. If we restrict the requirement to hold only for the pull-back, then only the tangent part of the vector field is determined. Second, T no longer needs to be in the lowest two harmonics, it can be an arbitrary function on the sphere. The resulting vector fields form a closed sub-algebra of the diffeomorphism algebra, given by

$$[\xi, \chi] = (T_\xi \dot{f}_\chi + Y_\xi[f_\chi] - (\xi \leftrightarrow \chi))\partial_u + [Y_\xi, Y_\chi]^A\partial_A. \tag{4.18}$$

This algebra exponentiates to a finite group action, a subgroup of the full diffeomorphism group of $\mathscr{I}$ that we call BMS group.

The lowest two harmonics of T, which correspond to global translations, are the solutions of the equation

$$\mathscr{D}_{\langle A}\mathscr{D}_{B\rangle} T = 0. \tag{4.19}$$

In general, we can distinguish modes using the Laplacian eigenvalue equation $\mathscr{D}^2 T = -l(l+1)T$. The new allowed diffeomorphisms are arbitrary angle-dependent time translations,

$$u' = u + T(x^A), \tag{4.20}$$

and are called super-translations. Notice that if we act with a boost, we change the round sphere to another round sphere, but the two sets of spherical harmonics get mixed up. Only the lowest sector is 'pure' in the sense that it gets no contribution from higher harmonics. What this means is that while global translations are characterized by (4.19) in any frame, non-global-super-translations are not: We first pick a frame, then we can talk about the $l \geq 2$ modes in that frame. But changing frame, these modes will mix, and get contributions from the $l = 0, 1$ modes as well. In more mathematical terms, we have that global translations, and global translations only, form a normal subgroup of the BMS group.

To gain intuition about super-translations, observe that acting with an asymptotic Killing vector on the Minkowski metric (4.1) 'spoils' its form by terms constructed from $\mathscr{D}_{\langle A}\mathscr{D}_{B\rangle}T$. The consequence is that while the codimension-2 leaves of constant (u, r) are round spheres, the new codimension-2 leaves of constant (u', r') are non-round spheres. It further follows that while for $r \to 0$ the constant u ingoing geodesics of different angles x^A all converge to a point, the constant u' ingoing geodesics of different angles start individually crossing before and don't focus to a point. From the point of view of $\mathscr{I}$, there is no difference between u and u', both are equally good coordinates, corresponding to different choice of cuts foliating $\mathscr{I}$. But from the bulk perspective, some cuts come from light emitted from a point, and are called good cuts. The rest are the bad cuts.

A tricky feature of the BMS group is that only the global translations are a canonical subgroup, and there is no canonical subgroup corresponding to the Lorentz group. The situation echoes that of the global Killing vectors, for which there is no unique Lorentz subgroup of the Poincaré group. But while to pick a Lorentz subgroup of the Poincaré group we need to pick an origin, and the freedom in doing so is spanned by the finite-dimensional group of translations, we have a different Lorentz subgroup for each choice of cut, and this is an infinite-dimensional freedom. Once we have chosen a Lorentz subgroup, we also get a Poincaré subgroup immediately, since global translations are anyways an ideal.

These properties can be deduced from the group Lie algebra (4.18). It implies that the group has the semi-direct product structure

$$G^{\mathrm{BMS}} = \mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{R}^S. \tag{4.21}$$

Two important special cases of (4.18) are that super-translations commute, $[\xi_T, \xi_{T'}] = 0$, and that Lorentz transformations do not commute with generic super-translations,

$$[\xi_Y, \chi_T] = \xi_{T'}, \qquad T' = Y^A \partial_A T - \dot{f} T. \tag{4.22}$$

This makes it manifest that the notion of Lorentz subgroup of the BMS group is cross-section dependent: acting with a super-translation changes the cross-section and the Lorentz symmetry vector is shifted by a super-momentum contribution. For the angular momentum piece $\dot{f} = 0$, and the shift is by $Y^A \partial_A T$ only. This vanishes for global translations, recovering the standard Poincaré result that rotations commute with translations, but they do not with super-translations.

Summarizing, the global Killing vectors preserve all of 4.7, including the $O(\Omega^{-2})$; the asymptotic Killing instead only the lowest order of (4.7).

4.3 Asymptotically flat spacetimes

In the literature one can find two different approaches to asymptotically flat spacetimes at null infinity. The Bondi-Sachs approach, based on a bulk gauge fixing and asymptotic expansion of the metric, and the Penrose-Geroch approach, based on the idea of conformal compactification used above for Minkowski. They are both useful and complementary, and I will try to show how one can take advantage of both perspectives.

We start from a coordinate patch (u, r, x^A), where $A = 1, 2$ are coordinates on topological 2-spheres, and require that the level sets of u are null, and that x^A are Lie dragged along the null geodesics at constant u. This implies that

$$g^{uu} = g^{uA} = 0 \quad \Leftrightarrow \quad g_{rr} = g_{rA} = 0. \tag{4.23}$$

These 3 gauge-fixing conditions can be referred to as partial Bondi gauge (e.g. [46, 47, 48]). The remaining gauge freedom can be used to fix r to be the area radius of the 2-spheres, as in the Bondi (aka Bondi-Sachs) coordinates:

$$\partial_r(\det{}^{(2)}g_{AB}/r^4) = 0; \tag{4.24}$$

or an affine parameter for the null geodesics at constant u, as in the Newman-Unti coordinates, where one fixes $g^{ur} = -1$.

We parametrize the gauge-fixed metric as

$$g_{\mu\nu} = \begin{pmatrix} -Ve^{2\mathcal{B}} + \gamma_{AB}V^AV^B & -e^{2\mathcal{B}} & -\gamma_{AB}V^B \\ & 0 & 0 \\ & & \gamma_{AB} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\mathcal{B}} & 0 \\ & Ve^{-2\mathcal{B}} & -e^{-2\mathcal{B}}V^A \\ & & \gamma^{AB} \end{pmatrix}, \tag{4.25}$$

with determinant

$$\sqrt{-g} = e^{2\mathcal{B}}\sqrt{\gamma}. \tag{4.26}$$

In the original formulation, the condition that the metric be asymptotically flat was imposed requiring that it matches (4.1) at leading order in $r \to \infty$. This is achieved taking

$$V = 1 + O(r^{-1}), \qquad \mathcal{B} = \frac{1}{r}\mathcal{B}^{(1)} + O(r^{-2}), \qquad V^A = O(r^{-1}), \qquad g_{AB} = r^2 q_{AB} + O(r). \tag{4.27}$$

Since the asymptotic background is non-dynamical, this corresponds to the boundary conditions

$$\delta g_{ur} = O(r^{-1}), \qquad \delta g_{uA} = O(r^{-1}), \qquad \delta g_{uu} = O(r^{-1}), \qquad \delta g_{AB} = O(r). \tag{4.28}$$

One unconvenient feature of this analysis is that the boundary conditions are manifestly coordinate-dependent, since different metric components scale differently. This can be fixed using the conformal completion picture, which also offers a broader perspective on the fall-off conditions. If we switch to $(u, \Omega = 1/r, x^A)$ coordinates, the conformal rescaling of (4.25) gives

$$\hat{g}_{\mu\nu} = \begin{pmatrix} -V\Omega^2 e^{2\mathcal{B}} + \hat{q}_{AB}V^AV^B & e^{2\mathcal{B}} & -\hat{q}_{AB}V^B \\ & 0 & 0 \\ & & \hat{q}_{AB} \end{pmatrix}, \qquad \hat{g}^{\mu\nu} = \begin{pmatrix} 0 & e^{-2\mathcal{B}} & 0 \\ & \Omega^2 Ve^{-2\mathcal{B}} & e^{-2\mathcal{B}}V^A \\ & & \hat{q}^{AB} \end{pmatrix}. \tag{4.29}$$

If we require the metric to be smooth at $\mathscr{I}$, we can posit the generic asymptotic behaviour[10]

$$V = \Omega^{-2}V^{(-2)} + \Omega^{-1}V^{(-1)} + V^{(0)} + \Omega V^{(1)} + O(\Omega^2), \tag{4.30a}$$

$$\mathcal{B} = \mathcal{B}^{(0)} + \Omega\mathcal{B}^{(1)} + \Omega^2\mathcal{B}^{(2)} + O(\Omega^3), \tag{4.30b}$$

$$V^A = V^{(0)A} + \Omega V^{(1)A}, \tag{4.30c}$$

$$\hat{q}_{AB} = q_{AB} + \Omega C_{AB} + \Omega^2 D_{AB} + O(\Omega^3). \tag{4.30d}$$

[10]Comment on log terms, cite Chrisuel, celine, marc

The determinant condition (4.24) imposes

$$q^{AB}C_{AB}=0,\qquad D_{AB}=\frac{1}{4}q_{AB}C_{CD}C^{CD}, \tag{4.31}$$

and similar conditions on the lower orders of the expansion. We can define as in the flat case the normal to $\mathscr{I}$ via (4.11). Then using conformal transformations, the Einstein equations for $\hat{g}$ can be written as

$$\hat{S}_{\mu\nu}:=\hat{R}_{\mu\nu}-\frac{1}{6}\hat{g}_{\mu\nu}\hat{R}\,\hat{=}\,-2\Omega^{-1}\hat{\nabla}_{\mu}\mathbb{n}_{\nu}+\Omega^{-2}\mathbb{n}^{2}\hat{g}_{\mu\nu}+O(\Omega^{2}), \tag{4.32}$$

where $\hat{S}$ is the unphysical Schouten tensor, and we assumed that $T_{\mu\nu}=O(\Omega^2)$. If we require that $\hat{S}$ is smooth at $\mathscr{I}$, it follows that (4.11) is shear-free and has expansion tied to its inaffinity:

$$\mathbb{n}^{2}\overset{\mathscr{I}}{=}0,\qquad \hat{\nabla}_{\langle a}\mathbb{n}_{b\rangle}\overset{\mathscr{I}}{=}0,\qquad \hat{\nabla}_{\mu}\mathbb{n}^{\mu}=\pounds_{\mathbb{n}}\ln\sqrt{-\hat{g}}\overset{\mathscr{I}}{=}\theta+2k\,\hat{=}\,2\theta. \tag{4.33}$$

For any given asymptotically flat spacetime, there is considerable freedom in choosing the conformal factor. Given an Ω, any other $\Omega'=\omega\Omega$ with non-vanishing ω at $\mathscr{I}$ is an equally valid choice. One class of such transformations is generated by the asymptotic boosts, since as we have seen already in flat spacetime, a boost in retarded time spherical coordinates rescales r. One can consider more general transformations, and if we see $\Omega=1/r$ as a coordinate, they can be interpreted as radial diffeomorphisms. This freedom can be exploited to always restrict attention to so-called *divergent-free frames*, namely to set $\hat{\nabla}_{\mu}\mathbb{n}^{\mu}\overset{\mathscr{I}}{=}0$. Upon doing so, $\mathbb{n}$ has also vanishing expansion and inaffinity. The residual conformal invariance is restricted to be time-independent, namely $\pounds_{\mathbb{n}}\omega=0$. Since vanishing shear and expansion are then properties of any normal to $\mathscr{I}$ within this residual class, we say that this choice of conformal compactifications makes $\mathscr{I}$ a non-expanding horizon, or more precisely a *weakly isolated horizon* [35].

In the parametrization (4.30),

$$\mathbb{n}^{\mu}:=\hat{g}^{\mu\nu}\mathbb{n}_{\nu}=e^{-2\mathcal{B}}(1,\Omega^{2}V,V^{A}), \tag{4.34}$$

hence $V^{(-2)}\,\hat{=}\,0$ from the null nature of $\mathscr{I}$. We can use the freedom of choosing coordinates on $\mathscr{I}$ to fix $n\overset{\mathscr{I}}{=}\partial_u$ namely $\mathcal{B}^{(0)}=1$ and $V^{(0)A}=0$. The asymptotic Einstein's equations then give (see e.g. [49, 48, 50])

$$\dot{q}_{AB}\,\hat{=}\,q_{AB}\partial_u\ln\sqrt{q},\qquad V^{(-1)}\,\hat{=}\,\partial_u\ln\sqrt{q},\qquad V^{(0)}\,\hat{=}\,\frac{\mathcal{R}}{2}, \tag{4.35}$$

$$\mathcal{B}^{(1)}\,\hat{=}\,0,\qquad \mathcal{B}^{(2)}\,\hat{=}\,\upbeta:=-\frac{1}{32}C_{AB}C^{AB},\qquad U^{A}:=V^{(1)A}=-\frac{1}{2}\mathscr{D}^{B}C_{AB}, \tag{4.36}$$

where $\mathcal{R}$ is the curvature of q_{AB}, and $\mathcal{R}=2$ if q_{AB} is round. The equations (4.35) show that it is possible to consider a more general leading order than the one defined by (4.7): we can take an arbitrary sphere metric, not necessary a round one, and furthermore we can make its determinant time-dependent. From the perspective of (4.7), this means allowing diffeomorphisms of the type $r'=r'(u,r,x^A)$ with a non-trivial dependence on time in the limit. Choosing a conformal completion with $\dot{q}=0$ is known as choosing the *Bondi condition*, and we have seen from the earlier discussion that it is on-shell equivalent to choosing a divergent-free conformal frame, namely making $\mathscr{I}$ a non-expanding horizon.[11]

[11] While this is always possible, it may not always be the most convenient option. For instance, the Robinson-Trautman solution can be naturally written in BS coordinates with $\dot{q}\neq 0$, and changing radial coordinate so that $\dot{q}=0$ makes it bulk expression much more complicated.

An important role is also played by the equations for $V^{(1)}$ and $V^{(2)A}$. To write them, it is convenient to parameterize

$$V^{(1)} = -2M, \qquad V^{(2)A} = -\frac{2}{3}(J^A + \partial^A \beta + C^{AB} U_C). \tag{4.37}$$

Then,

$$\dot{M} = -\frac{1}{8}\dot{C}^2 + \frac{1}{4}\mathscr{D}\mathscr{D}\dot{C} + \frac{1}{8}\mathscr{D}^2\mathcal{R}, \tag{4.38}$$

$$\begin{aligned}\dot{J}_A = {}& \frac{1}{4}\dot{C}^{BC}\mathscr{D}_B C_{AC} + \frac{1}{2}C_{AB}\mathscr{D}_C\dot{C}^{BC} - \frac{1}{4}\dot{C}_{AB}\mathscr{D}_C C^{BC} - \frac{1}{8}\partial_A(C\dot{C}) + \frac{1}{4}C_{AB}\partial^B\mathcal{R} \\ & + \partial_A M + \frac{1}{2}\mathscr{D}^B\mathscr{D}_{[A}\mathscr{D}_C C_{B]}{}^C. \end{aligned} \tag{4.39}$$

The reason for this parameterization is that J_A is chosen to match Dray-Streubel's Lorentz charge aspect. It corresponds to the choice $(1,1)$ in the parametrization of [51], and it is related to the common Barnich-Troessaert (BT) [41] and Flanagan-Nichols (FN) [52] choices by

$$J_A = N_A^{\mathrm{BT}} - \partial_A\beta = N_A^{\mathrm{FN}} + 2\partial_A\beta + \frac{1}{2}C_{AB}U^B. \tag{4.40}$$

After these choices, the general fall-off on an arbitrary frame satisfying the Bondi condition is

$$g_{uu} = -\frac{\mathcal{R}}{2} + \frac{2M}{r} + O(r^{-2}), \qquad g_{ur} = -1 - \frac{2\beta}{r^2} + O(r^{-3}), \tag{4.41a}$$

$$g_{uA} = -U_A + \frac{2}{3r}(J_A + \partial_A\beta - \frac{1}{2}C_{AB}U^B), \qquad g_{AB} = r^2 q_{AB} + rC_{AB} + O(1). \tag{4.41b}$$

Further restricting the background q_{AB} to be a round sphere, so that $\mathcal{R} = 2$, is referred to as picking a Bondi frame.

In the Bondi-Sachs framework, we define the asymptotic symmetries as the residual diffeomorphisms preserving the bulk coordinates and the boundary conditions. That is,

$$\pounds_\xi g_{rr} = 0, \qquad \pounds_\xi g_{rA} = 0, \qquad \partial_r(g^{AB}\pounds_\xi g_{AB}) = 0, \tag{4.42}$$

and

$$\pounds_\xi g_{ur} = O(r^{-2}), \qquad \pounds_\xi g_{uA} = O(1) \tag{4.43a}$$

$$\pounds_\xi g_{AB} = O(r), \qquad \pounds_\xi g_{uu} = O(r^{-1}). \tag{4.43b}$$

Solving these equations give

$$\xi = f\partial_u + Y^A\partial_A - \frac{r}{2}\mathscr{D}^A Y_A \partial_r - \frac{1}{r}\mathscr{D}^A f\partial_A + \frac{1}{2}\mathscr{D}^2 f\partial_r + \frac{1}{2r^2}C^{AB}\mathscr{D}_B f\partial_A + O(r^{-3}) \tag{4.44}$$

$$= f\partial_u + Y^A\partial_A + \Omega(\dot{f}\partial_\Omega - \mathscr{D}^A f\partial_A) - \frac{1}{2}\Omega^2(\mathscr{D}^2 f\partial_\Omega - C^{AB}\mathscr{D}_B f\partial_A) + O(\Omega^3), \tag{4.45}$$

where f and Y satisfy the same relations as found before,

$$f = T(x^A) + \frac{u}{2}\mathscr{D}_A Y^A(x^B), \qquad \mathscr{D}_{\langle A}Y_{B\rangle} = 0. \tag{4.46}$$

This is the BMS group. We thus find that the asymptotic symmetry group of the BS metrics is exactly given by the asymptotic Killing vectors of Minkowski's spacetime. This should not come as a surprise, because if we rewrite (4.43) in terms of the unphysical metric, we find

$$\pounds_\xi \hat{g}_{\mu\nu} \overset{\mathscr{I}}{=} 2\alpha_\xi \hat{g}_{\mu\nu}, \tag{4.47}$$

and since the metric on $\mathscr{I}$ is flat, we obtain the same result as before in (**??**). Notice also that

$$\pounds_\xi \mathbb{n}^\mu = \pounds_\xi \hat{g}^{\mu\nu}\partial_\mu\Omega + \hat{g}^{\mu\nu}\partial_\nu \pounds_\xi \Omega = (\xi^\Omega - 2\alpha_\xi)\mathbb{n}^\mu = -\alpha_\xi \mathbb{n}^\mu \tag{4.48}$$

where in the last step we used (4.16). Since $\mathbb{n}^\mu$ is tangent to $\mathscr{I}$, we can use hypersurface indices and write the equations that define the symmetry generators in terms of intrinsic quantities only, as

$$\pounds_\xi q_{ab} = 2\alpha_\xi q_{ab}, \qquad \pounds_\xi \mathbb{n}^a = -\alpha_\xi \mathbb{n}^a. \tag{4.49}$$

These equations allows us to interpret the BMS symmetries as diffeomorphism preserving the equivalence class of conformal transformations

$$(q_{ab}, \mathbb{n}^a) \sim (\omega^2 q_{ab}, \omega^{-1}\mathbb{n}^a). \tag{4.50}$$

This is the *universal structure* of asymptotically flat metrics. The upshot is that we have two equivalent ways of thinking about BMS asymptotic symmetries: as boundary diffeomorphisms that preserve the boundary conditions, or equivalently as isometries of the universal structure allowed by the boundary conditions.

This equivalence holds also for the weakened boundary conditions and larger symmetry groups that have been considered in the literature, such as gBMS, BMSW (details to appear), and also for null boundaries at finite distance [26].

From the same calculation we can also deduce the field space transformations of the asymptotic phase space $\Phi = (M, J_A, C_{AB})$, through the definition

$$\pounds_\xi g_{\mu\nu}[\Phi] \equiv g_{\mu\nu}[\Phi + \delta_\xi \Phi] - g_{\mu\nu}[\Phi]. \tag{4.51}$$

The result can be written as

$$\delta_\xi \Phi = \pounds_\xi \Phi + \Delta_\xi \Phi, \tag{4.52}$$

where the second term, called anomaly, captures the dependence of Φ on the background fields, which in this case are the conformal factor, and the foliation u used to define the different metric components being identified as mass, angular momentum and shear. This gives a more geometric and intuitive way of understanding the transformation laws, and which furthermore can be derived intrinsically at $\mathscr{I}$ without the need of any bulk expansion. See [24].

5 Fluxes and charges for the BMS symmetry

Charges and fluxes for the BMS symmetry in the radiative phase space were first identified in [1, 2, 3, 4, 5]. Using the Bondi-Sachs parametrization (**??**) and a Bondi frame with $\mathcal{R} = 2$, we can write the flux between two cuts of $\mathscr{I}$ as

$$F_\xi = \int j_\xi = -\frac{1}{32\pi}\int \dot{C}_{AB}\delta_\xi C^{AB}\epsilon_{\mathscr{I}} \,\hat{=}\, Q_\xi[S_2] - Q_\xi[S_1], \tag{5.1}$$

where

$$\delta_\xi C_{AB} = (f\partial_u + \pounds_Y - \dot{f})C_{AB} - 2\mathscr{D}_{\langle A}\mathscr{D}_{B\rangle}f, \tag{5.2}$$

and

$$Q_\xi = \frac{1}{8\pi}\oint_S (2fM + Y^A J_A)\epsilon_S. \tag{5.3}$$

For reference, these charge aspects read in Newman-Penrose formalism

$$M = -\left(\psi_2 + \sigma\dot{\bar\sigma} + \frac{1}{2}\left(\eth^2\bar\sigma - cc\right)\right) = -\mathrm{Re}(\psi_2 + \sigma\dot{\bar\sigma}), \qquad m^A J_A = -\left(\psi_1 + \sigma\eth\bar\sigma + \frac{1}{2}\eth(\sigma\bar\sigma)\right). \tag{5.4}$$

These expressions can be derived in three different ways: the Ashtekar-Streubel approach, the Wald-Zoupas approach, and the Barnich-Brandt approach. We will now briefly describe the methods and their equivalence. For simplicity we are considering only the case of Bondi frames, namely round spheres. However non-round spheres can be very useful also within the BMS symmetry, see [24]. We further restrict attention to the BMS case, for which there is no need of symplectic renormalization, nor of corner terms. We refer to the literature for the eBMS and gBMS cases where the situation is more complicated and requires these additional techniques.

5.1 Ashtekar-Streubel approach

- *Advantages:* Intrinsic at $\mathscr{I}$; independent of bulk coordinates and bulk extensions of the symmetry vector fields; independent of symplectic potential ambiguities that don't affect the symplectic 2-form.
- *Caveats:* Integration of the angular momentum charge complicated; closed-form ambiguities to be resolved in a second stage; no relation to canonical generators on Σ

In this approach one first evaluates the pull-back at $\mathscr{I}$ of the standard Einstein-Hilbert symplectic 2-form. The result can be described in terms of geometric quantities in an arbitrary coordinate system of $\mathscr{I}$, however for simplicity let us use the specific foliation induced by Bondi coordinates, and the parametrization (4.41). This gives

$$\Omega_{\mathcal{N}} = -\frac{1}{32\pi}\int \delta\dot{C}_{AB} \curlywedge \delta C^{AB}\epsilon_{\mathscr{I}}, \tag{5.5}$$

known as Ashtekar-Streubel symplectic form since it was first derived in [3]. Here $\mathcal{N}$ could be all of $\mathscr{I}$, or a portion of it. From this formula one can compute the canonical generator on $\mathscr{I}$, finding

$$-I_\xi\Omega - \mathcal{N} = \delta F_\xi + \frac{1}{32\pi}\oint_{S_1}^{S_2} f\dot{C}_{AB}\delta C^{AB}\epsilon_S, \tag{5.6}$$

where F_ξ is the flux given in (5.1), and $S_{1,2}$ two arbitrary cuts of $\mathscr{I}$. The boundary terms make the infinitesimal generator not integrable, however the obstruction is measure zero with respect to the natural measure that one can introduce in the field space. This makes it possible to define a genuine generator in the full radiative phase space starting from the integrable one defined in the dense subset [3, 7]. This procedure identifies the flux (5.1) as the canonical generator of BMS symmetries. Indeed one can prove using the Poisson bracket and the Barnich-Troessaert brackets that it reproduces the

BMS algebra correctly. On the other hand, it was pointed out in [53] that this flux fails to reproduce the correct BMS transformation of the shear at the level of the Dirac bracket, suggesting that the symplectic 2-form should be modified by a corner term. This is something I am currently working on, but I don't have a definite answer yet.

Next, in order to introduce the surface charges, one can use Einstein's equations to show that the flux is on-shell exact. Hence if we evaluate it between two arbitrary cuts of $\mathscr{I}$, it would provide the difference of two surface terms, each of which would coincide with the Noether current integrated on the hyperbolic Σ intersecting $\mathscr{I}$ at that cut, see Fig. 3.

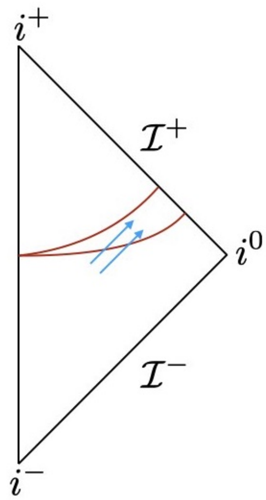

Figure 3: *Two space-like hypersurfaces intersecting $\mathscr{I}$. By the conservation law $d\omega \hat{=} 0$, the canonical generator on the phase space defined on the portion of $\mathscr{I}$ is equal to the difference of the two canonical generators on the space-like hypersurfaces.*

It is particularly simple and illuminating to do this for the super-translation part of the symmetry. Specializing (5.1) to a super-translation we have

$$\begin{aligned} j_T^{\rm BMS} &= -\frac{1}{32\pi}\dot{C}_{AB}\delta_T C^{AB}\epsilon_{\mathscr{I}} = -\frac{1}{32\pi}\dot{C}^{AB}(T\dot{C}_{AB} - 2\mathscr{D}_A\mathscr{D}_B T)\epsilon_{\mathscr{I}} \\ &= \frac{1}{32\pi}\Big[T(-\dot{C}^{AB}\dot{C}_{AB} + 2\mathscr{D}_A\mathscr{D}_B\dot{C}_{AB}) + 2\mathscr{D}_A(\dot{C}^{AB}\partial_B T - T\mathscr{D}_B\dot{C}^{AB})\Big]\epsilon_{\mathscr{I}} \\ &\hat{=} \frac{1}{4\pi}\partial_u\left(TM + \frac{1}{4}\mathscr{D}_A(C^{AB}\mathscr{D}_B T - T\mathscr{D}_B C^{AB})\right)\epsilon_{\mathscr{I}}. \end{aligned} \tag{5.7}$$

The result can be written as

$$j_T^{\rm BMS} \hat{=} \frac{1}{4\pi}D_a P_T^a \epsilon_{\mathscr{I}} = \frac{1}{4\pi}dP_T, \tag{5.8}$$

where

$$P_T^a = \left(TM_\rho,\ \frac{1}{4}\big(T\mathscr{D}_B N^{AB} - N^{AB}\mathscr{D}_B T\big)\right). \tag{5.9}$$

This quantity is with Geroch's super-momentum [1], in Bondi coordinates and with $l = -du$. Its Hodge dual defines the 2-form $P_T := \frac{1}{2}P_T^a \epsilon_{\mathscr{I}abc}dx^b \wedge dx^c$, whose pull-back on the cross sections gives

$P_T^u \epsilon_S = TM_\rho \epsilon_S$, hence we recover (5.3). This calculation shows explicitly the Ashtekar-Streubel strategy of obtaining the surface charges 'integrating the fluxes'.

Deriving the charges in this way obviously leaves the ambiguity of adding time-independent terms. One then has to prove that all time-independent terms that could be added would spoil covariance, in order to identify a unique charge. The identification is possible for the charge, but not for its integrand, or aspect, which remains partially ambiguous.

The same procedure for the angular momentum is significantly more complicated. It was carried out in [5], and one obtains the Dray-Streubel charge given in (5.3).

5.2 Wald-Zoupas approach

- *Advantages:* Closed-form ambiguity fixed (for fixed symplectic 2-form); possibility of bootstrapping the charge from the Komar formula, which makes it particularly simpler to derive the angular momentum charge;

- *Caveats:* Depends on choice of Σ, bulk coordinates and bulk extensions of the symmetry vector fields; subtlety with certain choices of field-dependent extensions due to extension-dependence of Komar formula; field-space constant ambiguity to be resolved after the procedure

In the Wald-Zoupas approach, we first select a preferred symplectic potential for (5.5). The condition of stationarity is identified with the vanishing of the news function, which for the special case of Bondi frames (and this case only!) can be identified with $\dot{C}_{AB}$. This leads to the choice

$$\theta^{\rm BMS} = -\frac{1}{32\pi}\int \dot{C}_{AB}\delta C^{AB}\epsilon_{\mathscr{I}}, \tag{5.10}$$

for which one can check covariance, namely conformal invariance and foliation independence. The Noether current is then immediately seen to match the Ashtekar-Streubel flux.

To extract the charges however, the Wald-Zoupas strategy is different. Instead of 'integrating' the flux, they propose to match the charge with the integrable part of the canonical generator at Σ obtained subtracting the preferred symplectic potential. This procedure has the advantage that the charge does not have the ambiguity of adding time-independent terms, as in the previous procedure. It has the ambiguity of adding a field-space independent term, but this can be easily get rid of requiring that the charges all vanishing in Minkowski, for instance.

Another potential advantage is that one can bootstrap the calculation from the Komar 2-form. The idea is that $\theta^{\rm BMS} = \underleftarrow{\theta} + \delta b$ for a certain b, hence one could be tempted to use the formula $q_\xi^{\rm BMS} \stackrel{?}{=} q_\xi + i_\xi b$. However this turns out to be delicate, because the Komar 2-form on the second and third order expansion of the symmetry vector field off $\mathscr{I}$. And these are not canonical. Worse, if one uses the common choice of extension (4.44), they are field dependent. In this case (3.37) is no longer valid, and has to be replaced by

$$-I_\xi\omega \mathrel{\hat{=}} d(\delta q_\xi - q_{\delta\xi} - i_\xi\theta). \tag{5.11}$$

The correction $q_{\delta\xi}$ gets rid of the spurious contribution coming from the field dependence of (4.44), and including it one recovers (5.3). Forgetting it leads to including a soft term in the charge that spoils conservation in Minkowski for all BMS generators. For more details on this see [21], and also discussion in [24, 7]. Remarkably, even the charge obtained from $\delta q_\xi - q_{\delta\xi}$ can be interpreted as an improved Noether charge, because it turns out that $q_{\delta\xi} = \delta s_\xi$, and this can be generated adding a corner term to the boundary Lagrangian,

$$c = -\frac{1}{8\pi}\beta\epsilon_S. \tag{5.12}$$

With this corner term one can indeed recover (5.3) starting from the limit of Komar q_ξ:

$$q_\xi^{\rm BMS} = q_\xi + i_\xi \ell^{\rm BT} - I_\xi \delta c. \tag{5.13}$$

5.3 Barnich-Brandt approach

- *Advantages:* Explicit formula directly in terms of the metric components; avoids the subtlety with field-dependent extensions
- *Caveats:* Hides the role of the symplectic structure; needs to be supplemented by Wald-Zoupas prescription in order to identify a covariant and stationary split

The Barnich-Brandt formula gives

$$-I_\xi\omega \mathrel{\hat=} -\frac{1}{32\pi}\epsilon_{\mu\nu\rho\sigma}\big[(\delta\ln\sqrt{-g})\nabla^\rho\xi^\sigma + \delta g^{\rho\alpha}\nabla_\alpha\xi^\sigma + \xi^\rho\left(\nabla_\alpha\delta g^{\alpha\sigma} + 2\nabla^\sigma\delta\ln\sqrt{-g}\right) \\ - \xi_\alpha\nabla^\rho\delta g^{\sigma\alpha}\big]dx^\mu\wedge dx^\nu, \tag{5.14}$$

where we have subtracted off the extra contribution that comes from the different corner term in the symplectic structure, and which anyways plays no role in the BMS symmetry because it vanishes in the limit. An immediate advantage is that this formula gives directly (5.11), so any spurious field dependence introduced by the choice of extension (4.44) is removed, and one gets [41]

$$-I_\xi\Omega_\Sigma \mathrel{\hat=} \delta Q_\xi - F_\xi, \tag{5.15}$$

with the candidate charges and fluxes given precisely by (5.1) and (5.3). So the only thing that remains to be done is to identify them in a canonical way, which can be done applying the Wald-Zoupas prescription. This leads to non-trivial insights, for instance the need for Geroch tensor correcting the formulas (5.1) and (5.3) for frames which are not round spheres. This correction removes the cocycle found in [41], leading to a centerless realization of the BMS algebra [24].

Because in the presence of dissipation some of the symmetry vector fields are not Hamiltonian, the algebra cannot be realized using Poisson brackets. In general, two symmetries ξ and χ give

$$I_\xi I_\chi\Omega_\Sigma = \delta_\chi Q_\xi - I_\chi F_\xi \neq \delta_\chi Q_\xi. \tag{5.16}$$

The key idea of Barnich and Troessaert was to define a bracket with the non-integrable, flux term subtracted off:

$$\{Q_\xi, Q_\chi\}_* := \delta_\chi Q_\xi - I_\xi F_\chi = I_\xi I_\chi\Omega_\Sigma + I_\chi F_\xi - I_\xi F_\chi. \tag{5.17}$$

The result of this calculation depends on the integrable/non-integrable split chosen, or in other words, on the choice of preferred symplectic potential. It was then proved in [42] that if the split satisfies the Wald-Zoupas conditions,

$$\{Q_\xi, Q_\chi\}_* \mathrel{\hat=} Q_{[\xi,\chi]} + K_{(\xi,\chi)}, \tag{5.18}$$

where the only possible cocyle comes from a closed 2-form, or more inuitively, cointains only time-independent terms. Furthermore, an analogue definition applied to the Noether current on $\mathscr{I}$ (namely the charge from the point of view of $\mathscr{I}$, or the flux from the point of view of Σ) realized the algebra *without any cocycle*. This is a remarkable consequence of insisting on covariance, the key requirement 1 of the generalized Wald-Zoupas prescription.

These results have motivated us also to apply the Wald-Zoupas prescription to larger symmetry groups. For eBMS, this is indeed possible, see [24] again, and the result provides a solid foundation for the symplectic structure used in [54, 55]. For gBMS this is harder, and after succeeding for an intermediate construction that we called the rest-frame Bondi sphere group (RBS), we now have a candidate and we hope to finish verifications soon.

6 Further reading

- BMS symmetries: [52, 56, 57]; larger symmetries at $\mathscr{I}$: [49, 58, 59, 60, 61, 54, 62, 63, 47, 50, 64]
- Horizons and more general null boundary symmetries: [65, 66, 67, 68, 69, 70, 71, 72, 39, 73]

References

[1] R. Geroch, *Asymptotic Structure of Space-Time*, ch. 1, pp. 1–106. Springer US, Boston, MA, 1977.

[2] A. Ashtekar, *Radiative Degrees of Freedom of the Gravitational Field in Exact General Relativity*, J. Math. Phys. **22** (1981) 2885–2895.

[3] A. Ashtekar and M. Streubel, *Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity*, Proc. Roy. Soc. Lond. A **376** (1981) 585–607.

[4] T. Dray and M. Streubel, *Angular momentum at null infinity*, Class. Quant. Grav. **1** (1984), no. 1 15–26.

[5] T. Dray, *Momentum Flux At Null Infinity*, Class. Quant. Grav. **2** (1985) L7–L10.

[6] R. M. Wald and A. Zoupas, *A General definition of 'conserved quantities' in general relativity and other theories of gravity*, Phys. Rev. D **61** (2000) 084027 [gr-qc/9911095].

[7] A. Ashtekar and S. Speziale, *Null infinity and horizons: A new approach to fluxes and charges*, Phys. Rev. D **110** (2024), no. 4 044049 [2407.03254].

[8] I. M. Anderson, *Introduction to the variational bicomplex*, bz Contemporary Mathematics **132** (1992).

[9] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in gauge theories*, Phys. Rept. **338** (2000) 439–569 [hep-th/0002245].

[10] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, Nucl. Phys. **B633** (2002) 3–82 [hep-th/0111246].

[11] R. M. Wald, *On identically closed forms locally constructed from a field*, J. Math. Phys. **31** (1990), no. 10 2378.

[12] E. De Paoli and S. Speziale, *A gauge-invariant symplectic potential for tetrad general relativity*, JHEP **07** (2018) 040 [1804.09685].

[13] G. A. Burnett and R. M. Wald, *A conserved current for perturbations of Einstein-Maxwell space-times*, Proc. Roy. Soc. Lond. A **430** (1990), no. 1878 57–67.

[14] R. Oliveri and S. Speziale, *Boundary effects in General Relativity with tetrad variables*, Gen. Rel. Grav. **52** (2020), no. 8 83 [1912.01016].

[15] D. Harlow and J.-Q. Wu, *Covariant phase space with boundaries*, JHEP **10** (2020) 146 [1906.08616].

[16] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part I. Corner potentials and charges*, JHEP **11** (2020) 026 [2006.12527].

[17] J. Margalef-Bentabol and E. J. S. Villaseñor, *Geometric formulation of the Covariant Phase Space methods with boundaries*, Phys. Rev. D **103** (2021), no. 2 025011 [2008.01842].

[18] L. Freidel, R. Oliveri, D. Pranzetti and S. Speziale, *Extended corner symmetry, charge bracket and Einstein's equations*, JHEP **09** (2021) 083 [2104.12881].

[19] L. Ciambelli, R. G. Leigh and P.-C. Pai, *Embeddings and Integrable Charges for Extended Corner Symmetry*, Phys. Rev. Lett. **128** (2022) [2111.13181].

[20] V. Chandrasekaran, E. E. Flanagan, I. Shehzad and A. J. Speranza, *A general framework for gravitational charges and holographic renormalization*, Int. J. Mod. Phys. A **37** (2022), no. 17 2250105 [2111.11974].

[21] G. Odak, A. Rignon-Bret and S. Speziale, *Wald-Zoupas prescription with soft anomalies*, Phys. Rev. D **107** (2023), no. 8 084028 [2212.07947].

[22] F. Hopfmüller and L. Freidel, *Null Conservation Laws for Gravity*, Phys. Rev. D **97** (2018), no. 12 124029 [1802.06135].

[23] V. Chandrasekaran and A. J. Speranza, *Anomalies in gravitational charge algebras of null boundaries and black hole entropy*, JHEP **01** (2021) 137 [2009.10739].

[24] A. Rignon-Bret and S. Speziale, *Centerless BMS charge algebra*, Phys. Rev. D **110** (2024), no. 4 044050 [2405.01526].

[25] S. Carrozza, S. Eccles and P. A. Hoehn, *Edge modes as dynamical frames: charges from post-selection in generally covariant theories*, SciPost Phys. **17** (2024), no. 2 048 [2205.00913].

[26] G. Odak, A. Rignon-Bret and S. Speziale, *General gravitational charges on null hypersurfaces*, JHEP **12** (2023) 038 [2309.03854].

[27] J. Brown, S. Lau and J. York, *Action and energy of the gravitational field*, Annals of Physics **297** (2002), no. 2 175–218 [gr–qc/0010024].

[28] L. Lehner, R. C. Myers, E. Poisson and R. D. Sorkin, *Gravitational action with null boundaries*, Phys. Rev. D **94** (2016), no. 8 084046 [1609.00207].

[29] G. Odak and S. Speziale, *Brown-York charges with mixed boundary conditions*, JHEP **11** (2021) 224 [2109.02883].

[30] A. Ashtekar, S. Fairhurst and B. Krishnan, *Isolated horizons: Hamiltonian evolution and the first law*, Phys. Rev. D **62** (2000) 104025 [gr-qc/0005083].

[31] A. Ashtekar and B. Krishnan, *Isolated and dynamical horizons and their applications*, Living Rev. Rel. **7** (2004) 10 [gr-qc/0407042].

[32] P. Hájiček, *Exact models of charged black holes*, Communications in Mathematical Physics **34** (1973), no. 1 53–76.

[33] E. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J.Math.Phys. **3** (1962) 566–578.

[34] I. Booth and S. Fairhurst, *Isolated, slowly evolving, and dynamical trapping horizons: Geometry and mechanics from surface deformations*, Phys. Rev. D **75** (2007) 084019 [gr-qc/0610032].

[35] A. Ashtekar and S. Speziale, *Null infinity as a weakly isolated horizon*, Phys. Rev. D **110** (2024), no. 4 044048 [2402.17977].

[36] F. Hopfmüller and L. Freidel, *Gravity Degrees of Freedom on a Null Surface*, Phys. Rev. D **95** (2017), no. 10 104006 [1611.03096].

[37] S. Alexandrov and S. Speziale, *First order gravity on the light front*, Phys. Rev. D **91** (2015), no. 6 064043 [1412.6057].

[38] V. Chandrasekaran and E. E. Flanagan, *Horizon phase spaces in general relativity*, JHEP **07** (2024) 017 [2309.03871].

[39] L. Ciambelli, L. Freidel and R. G. Leigh, *Null Raychaudhuri: canonical structure and the dressing time*, JHEP **01** (2024) 166 [2309.03932].

[40] A. Rignon-Bret and S. Speziale, *Spatially local energy density of gravitational waves*, JHEP **03** (2025) 048 [2405.08808].

[41] G. Barnich and C. Troessaert, *BMS charge algebra*, JHEP **12** (2011) 105 [1106.0213].

[42] A. Rignon-Bret and S. Speziale, *General covariance and boundary symmetry algebras*, EPL **149** (2025), no. 6 69002 [2403.00730].

[43] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, 1703.05448.

[44] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. D **50** (1994) 846–864 [gr-qc/9403028].

[45] A. Ashtekar, N. Khera, M. Kolanowski and J. Lewandowski, *Non-expanding horizons: multipoles and the symmetry group*, JHEP **01** (2022) 028 [2111.07873].

[46] E. De Paoli and S. Speziale, *Sachs free data in real connection variables*, JHEP **11** (2017) 205 [1707.00667].

[47] M. Geiller and C. Zwikel, *The partial Bondi gauge: Further enlarging the asymptotic structure of gravity*, SciPost Phys. **13** (2022) 108 [2205.11401].

[48] M. Geiller and C. Zwikel, *The partial Bondi gauge: Gauge fixings and asymptotic charges*, SciPost Phys. **16** (2024) 076 [2401.09540].

[49] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, JHEP **05** (2010) 062 [1001.1541].

[50] R. McNees and C. Zwikel, *The symplectic potential for leaky boundaries*, JHEP **01** (2025) 049 [2408.13203].

[51] G. Compère, R. Oliveri and A. Seraj, *The Poincaré and BMS flux-balance laws with application to binary systems*, JHEP **10** (2020) 116 [1912.03164].

[52] E. E. Flanagan and D. A. Nichols, *Conserved charges of the extended Bondi-Metzner-Sachs algebra*, Phys. Rev. D **95** (2017), no. 4 044002 [1510.03386].

[53] T. He, V. Lysov, P. Mitra and A. Strominger, *BMS supertranslations and Weinberg's soft graviton theorem*, JHEP **05** (2015) 151 [1401.7026].

[54] M. Campiglia and J. Peraza, *Generalized BMS charge algebra*, Phys. Rev. D **101** (2020), no. 10 104039 [2002.06691].

[55] L. Donnay, K. Nguyen and R. Ruzziconi, *Loop-corrected subleading soft theorem and the celestial stress tensor*, JHEP **09** (2022) 063 [2205.11477].

[56] A. M. Grant, K. Prabhu and I. Shehzad, *The Wald-Zoupas prescription for asymptotic charges at null infinity in general relativity*, Class. Quant. Grav. **39** (2022), no. 8 085002 [2105.05919].

[57] G. Barnich, P. Mao and R. Ruzziconi, *BMS current algebra in the context of the Newman-Penrose formalism*, Class. Quant. Grav. **37** (2020), no. 9 095010 [1910.14588].

[58] M. Campiglia and A. Laddha, *Asymptotic symmetries and subleading soft graviton theorem*, Phys. Rev. D **90** (2014), no. 12 124028 [1408.2228].

[59] G. Barnich and C. Troessaert, *Finite BMS transformations*, JHEP **03** (2016) 167 [1601.04090].

[60] G. Compère, A. Fiorucci and R. Ruzziconi, *Superboost transitions, refraction memory and super-Lorentz charge algebra*, JHEP **11** (2018) 200 [1810.00377]. [Erratum: JHEP 04, 172 (2020)].

[61] G. Compère, A. Fiorucci and R. Ruzziconi, *The* Λ*-BMS*$_4$ *charge algebra*, JHEP **10** (2020) 205 [2004.10769].

[62] L. Freidel, R. Oliveri, D. Pranzetti and S. Speziale, *The Weyl BMS group and Einstein's equations*, JHEP **07** (2021) 170 [2104.05793].

[63] G. Barnich and R. Ruzziconi, *Coadjoint representation of the BMS group on celestial Riemann surfaces*, JHEP **06** (2021) 079 [2103.11253].

[64] G. Satishchandran and R. M. Wald, *Asymptotic behavior of massless fields and the memory effect*, Phys. Rev. D **99** (2019), no. 8 084007 [1901.05942].

[65] A. Ashtekar, M. Campiglia and S. Shah, *Dynamical Black Holes: Approach to the Final State*, Phys. Rev. D **88** (2013), no. 6 064045 [1306.5697].

[66] L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, *Supertranslations and Superrotations at the Black Hole Horizon*, Phys. Rev. Lett. **116** (2016), no. 9 091101 [1511.08687].

[67] H. Adami, D. Grumiller, M. M. Sheikh-Jabbari, V. Taghiloo, H. Yavartanoo and C. Zwikel, *Null boundary phase space: slicings, news & memory*, JHEP **11** (2021) 155 [2110.04218].

[68] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos and K. Siampos, *Flat holography and Carrollian fluids*, JHEP **07** (2018) 165 [1802.06809].

[69] L. Ciambelli, R. G. Leigh, C. Marteau and P. M. Petropoulos, *Carroll Structures, Null Geometry and Conformal Isometries*, Phys. Rev. D **100** (2019), no. 4 046010 [1905.02221].

[70] L. Freidel, D. Pranzetti and A.-M. Raclariu, *Higher spin dynamics in gravity and w1+∞ celestial symmetries*, Phys. Rev. D **106** (2022), no. 8 086013 [2112.15573].

[71] L. Ciambelli and R. G. Leigh, *Universal corner symmetry and the orbit method for gravity*, Nucl. Phys. B **986** (2023) 116053 [2207.06441].

[72] L. Donnay, A. Fiorucci, Y. Herfray and R. Ruzziconi, *Carrollian Perspective on Celestial Holography*, Phys. Rev. Lett. **129** (2022), no. 7 071602 [2202.04702].

[73] M. Geiller, *Celestial $w_{1+\infty}$ charges and the subleading structure of asymptotically-flat spacetimes*, 2403.05195.