ENS Lyon
Master de Science de la Matiere 2018-2019
Année M2 – Relativité Generale

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LECTURES PROGRAM

1. **Introduction and basics of differential geometry**

   Equivalence and relativity principles; general covariance and diffeomorphisms; tensors, Lie and covariant derivatives, Riemann tensor.

2. **Geodesics**

   As autoparallel curves, as trajectories extremizing the proper length; auxiliary Lagrangian formalism and conserved quantities. Newtonian limit. Time delay. Geodesic deviation and Raychaudhuri equations.

3. **Einstein's equations**

   Bianchi identities. Cosmological constant and energy-momentum tensor. Linearized theory and gravitational waves. Analogy with electromagnetism, spin 2 waves, gauge vs physical degrees of freedom.

4. **Schwarzschild solution**

   External and internal solution; gravitational redshift; circular orbits. Light-cones and adapted null coordinates; Physics of the event horizon. Coordinate singularities versus curvature singularities. Rindler horizon. Birkhoff and area theorems (without proofs).

5. **Global structure of spacetimes**

   Compactification, null infinity, Penrose-Carter diagrams.

6. **Killing vectors**

   Isometries and conservation laws (along geodetic and of the matter-energy momentum tensor). Maximally isometric spaces.

7. **Kerr solution**

   Event horizon and ergosphere; Penrose energy extraction mechanism. Laws of black hole mechanics. Brief discussion of Unruh and Hawking effects.

8. **Action principle and Hamiltonian framework**

   Brief discussion of yypersurfaces, extrinsic geometry and Gauss-Codazzi equation. Noether theorem and conserved quantities. Brief mention of ADM and Bondi energies.

SUGGESTED BIBLIOGRAPHY

- E. Poisson, *A Relativist's Toolkit* (Concise, fairly advanced)
- R. Wald, *General Relativity* (Rather complete, mathematically rigorous)
- Misner, Thorne and Wheeler, *Gravitation* (A classic reference, very broad but valuable even though a bit old, especially for the many intuitive descriptions)
- M. Blau, *Lectures on General Relativity* (Very explicit and pedagogical, but it can be dispersive)
- M. Nakahara, *Geometry, Topology and Physics* (Differential geometry for physicists)
- T. Frankel, *The Geometry of Physics* (Differential geometry for physicists, more advanced)
- C. Rovelli, *Quantum Gravity* (Initial chapters: For a nice and modern description of GR's deepest messages on the relational nature of the laws of physics, and for the tetrad formalism)

TRAVAUX DIRIGEES 1

8 Novembre 2018

1. Geodesics

- Compute the geodesics on the Euclidean plane in Cartesian and polar coordinates, and on the two sphere S^2.
- Consider a 2-dimensional toy model of Minkowski spacetime with one time and one space. Show that there are three types of geodesics: time-like, null and space-like. Write coordinates u and v such that u =constant and v =constant label null geodesics.
- Show that if the metric is flat up to a diffeomorphism, the geodesics equation coincides with motion in the presence of inertial forces.
- An independent definition of geodesic is that of an 'autoparallel curve', namely one whose tangent vector v^μ is transported parallel to itself:

  $$v^\nu \nabla_\nu v^\mu = k v^\mu. \tag{1}$$

  Show that k can always be put to zero rescaling v^μ.

  Show that if the covariant derivative ∇ is defined by the Christoffel symbols, this definition of geodesics coincides with the metric one, namely the curve extremizing the spacetime distance.

2. Motion around a spherically symmetric gravitational source

- Consider the following static, spherically symmetric line element

  $$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2}$$

  Write the geodesic equation for time-like trajectories, and the associated Lagrangian.
- Observe that t and ϕ are cyclic variables in the Lagrangian, and find the associated conserved quantities.
- Using the conserved quantities and the conservation of the Lagrangian, derive an equation for the orbits in the equatorial plane. Compute the radius of the innermost stable circular orbit (ISCO) in the massive case.
- A real black hole is often surrounded by matter falling into it along an accretion disk. Assuming the black hole is spherically symmetric and described by a metric like (2), and approximating the inspiral trajectories with a sequence of circular orbits from infinity to the ISCO, what is the maximal energy that can be in principle extracted from the change in gravitational potential energy?

- Write the equation for radial null geodesics.
- Find adapted coordinates u and v describing respectively ingoing and outgoing null geodesics.
- Change coordinates to (v, r, θ, ϕ) and show that in these coordinates the metric is non-singular even if F has zeroes.
- Time allowing, we can at this point derive together in class one of the two historical first tests of GR, either the correction to the precession of Mercury's perihelion, or the bending of light by gravity.

3. Tetrads and local almost flatness

- At each spacetime point, the metric is a symmetric matrix, and can thus be locally (i.e. point by point) diagonalized. The equivalence principle imposes that the local diagonal form of the metric is the Minkowskian one, $\eta_{IJ} = \mathrm{diag}(-1, 1, 1, 1)$,
$$g_{\mu\nu}(x) = e^I_\mu(x) e^J_\nu(x) \eta_{IJ}. \tag{3}$$
The reason to use a different type of index (I vs μ) is to stress that there is an 'internal' gauge freedom: How many tetrads describe the same metric?
- At any given point P, we can diagonalize the metric as described above so that in some coordinates x^μ it looks like $\eta_{\mu\nu}$. In the neighbourhood of P, we can write an arbitrary coordinate transformation in power series
$$x'^\mu = a^\mu{}_\nu x^\nu + b^\mu_{\rho\sigma} x^\rho x^\sigma + O(x^3) \tag{4}$$
with constant coefficients. Recalling the transformation property of $\Gamma^\mu_{\nu\rho}$ under change of coordinates, show that it is always possible to set $\Gamma^\mu{}_{\nu\rho}(P) = 0$.
- Observe that it does not imply that also $\partial_\lambda \Gamma^\mu_{\nu\rho}(P) = 0$. This is very the local almost flatness stops: we can always find coordinates such that the metric is flat and its first derivatives vanish, but not the second derivatives.
- Write the Riemann tensor at P using $g = \eta_{\mu\nu}$ and $\Gamma^\mu_{\nu\rho}(P) = 0$, and deduce its symmetries. Argue that it would have the same symmetries in any other coordinate system.

TRAVAUX DIRIGEES 2

15 Novembre 2018

1. Rindler metric

- Consider an observer with a uniform acceleration, $a^\mu := \frac{d^2 x^\mu}{d\tau^2} = (0, a, 0, 0)$ in the (instantaneous) rest frame.[1]
- Compute the 4-velocity and the trajectory. Confirm that it is not a geodesic.
- Introduce coordinates (η, ρ) adapted to the rest frame of the observer. Show that the coordinates span only a quadrant of Minkowski, and that the boundary of the domain is an horizon.
- What is the topology of the horizon?

2. Penrose-Carter diagrams: Minkowski

- Start from Minkowski spacetime in spherical coordinates. Write the metric in the null coordinates
$$u = \frac{1}{\sqrt{2}}(t - r), \qquad v = \frac{1}{\sqrt{2}}(t + r) \tag{5}$$
Observe that if we draw only the (u, v) plane, each point describes a sphere of radius $r(u, v)$ with center at $r = 0$. Show that in this coordinate system the ingoing and outgoing directions of the null cones are at 90 degrees,

[1] Why the need to specify that it is the instantaneous rest frame?

- Perform a further change of coordinates to

$$U = \arctan u, \qquad V = \arctan v, \tag{6}$$

and draw the boundaries of spacetime corresponding to spatial infinity, future and past time infinity, and future and past null infinity.

- Even though the whole of Minkowski has been 'compactified' to a finite region, show that the physical distance between points can still be infinite.

3. Penrose-Carter diagrams: Schwarzschild

- Start with coordinates (u, v, θ, ϕ), for which the metric is still degenerate at $r = 2M$. Change coordinates to

$$U = -e^{-u/4M}, \qquad V = e^{v/4M} \tag{7}$$

and show that the metric is now regular, as it is with the Eddington-Finkelstein coordinates (v, r, θ, ϕ).

- Draw cartesian axes associated with $T = (U + V)/2$ and $R = (V - U)/2$, and show that in these cooridnates, radial null lines are at 90 degrees.

- Draw the lines of constant t and constant r. Where is the event horizon? How much of $\mathbb{R}^2$ is covered by these coordinates?

- Extend your coordinates so that they span the whole of $\mathbb{R}^2$. This is known as Kruskal's maximal extension, and it is still a vacuum solution of GR.

- Compactify using coordinates

$$\tilde{U} = 2 \arctan U, \qquad \tilde{V} = \arctan V \tag{8}$$

Find the boundaries of the spacetime and draw the Penrose-Carter diagram of the Kruskal extension.

4. The doomed astronaut

- Consider the equation for radial time-like geodesics in the Schwarschild metric, $L = 0$, and $E^2 - 1 = -1$, *inside* the event horizon $r = 2M$. Show that on its solution the proper time and the radius satisfy

$$d\tau = (\frac{2M}{r} - 1)^{-1/2} dr. \tag{9}$$

- Integrating this equation gives

$$\tau = -M \arctan\left(\frac{M - r}{\sqrt{r(r - 2M)}}\right) - \sqrt{r(2M - r)} + constant \tag{10}$$

Show that an astronaut starting at rest from just inside the event horizon reaches the central singularity in a finite proper time

$$\Delta\tau = \pi M \tag{11}$$

How much is this time for a solar mass black hole?

5. Friedmann-Lemaitre-Robertson-Walker Consider the FLRW metric in co-moving coordinates,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right) \tag{12}$$

- Show that it is conformally flat, namely that there exist coordinates (η, y^a), $a = 1, 2, 3$, such that

$$ds^2 = \Omega^2(\eta, y^a)\Big(- d\eta^2 + \sum_a (dy^a)^2\Big) \tag{13}$$

- What is the general form of the Levi-Civita connection for a conformally flat metric?
- Back to the original coordinates, show that the vector field $u^\mu = (1,0,0,0)$ is geodesic
- Compute the shear, twist and expansion of its geodesic congruence
- Use the Raychaudhuri and the Einstein equations with perfect fluid source to show that

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) \tag{14}$$

6. Killing vectors

- Show that for a Killing vector,

$$\nabla_\mu \nabla_\nu \xi_\rho = R^\sigma{}_{\mu\nu\rho}\xi_\sigma \tag{15}$$

- Show that Killing vectors form a Lie algebra: if ξ_1 and ξ_2 are Killing, also $[\xi_1, \xi_2]$ is.
- A *Killing horizon* is a null hypersurface whose normal is a Killing vector. Show that the event horizon of Schwarzschild is a Killing horizon for the time translation Killing vector.
- Show that the event horizon of Kerr is not the Killing horizon for the time translation Killing vector, but for a certain linear combination of the time translation and axial symmetries.

7. Gravitational waves

Show that the linearized conservation of the energy-momentum tensor implies that gravitational radiation has no dipole moment, and that the leading order term is the quadrupole one.

TRAVAUX DIRIGEES 3

22 Novembre 2018

1. Kerr black holes

- The area of the event horizon of a Kerr black hole is given by $A = 4\pi(r_+^2 + a^2)$, and the location given by $r_+ = M + \sqrt{M^2 - a^2}$. Prove that under a small perturbation $(M, J) \mapsto (M + \delta M, J + \delta J)$,

$$\delta A = \frac{8\pi}{k}\Big(\delta M - \Omega_{\rm H}\delta J\Big) \tag{16}$$

  where $k = \sqrt{M^2 - a^2}/(r_+^2 + a^2)$ is the surface gravity, and $\Omega_{\rm H} = a/(r_+^2 + a^2)$ the angular velocity of the horizon.

- Verify that the hypothesis of the area theorem are satisfied by the Kerr metric.
- Consider two Kerr black holes of same mass M and same angular momentum $J = aM$, with $a \ll M$. Assume that they merge into a final black hole with mass M_f and $a_f \ll M_f$. The energy radiated away to future null infinity is $E_{\rm rad} = 2M - M_f$. Use the area theorem to provide an upper bound at first order in $(a/M)^2$ and $(a_f/M_f)^2$.

2. Follow-ups from the Unruh effect

- The transformation of the Ricci scalar under a conformal rescaling $g_{\mu\nu} \mapsto g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$ is

$$R' = \Omega^{-2}\Big(R - 2(n-1)\Box \ln\Omega - (n-2)(n-1)\nabla_\mu \ln\Omega\, \nabla^\mu \ln\Omega\Big) \tag{17}$$

  Use this result to find the right numerical factor ξ so that the following wave equation,

$$(\Box - \xi R)\Phi = 0, \tag{18}$$

  is conformally invariant in n dimensions. Remark: You will need also to assume that $\Phi \mapsto \Phi' = \Omega^{1-n/2}\Phi$.

- Consider two quantum harmonic oscillators with same mass and same frequency, and denote their creation operators $a_{\rm L}^\dagger$ and $a_{\rm R}^\dagger$. Define the squeezing operator

$$\hat{S}(\zeta) := e^{\bar{\zeta} a_{\rm L} a_{\rm R} - \zeta a_{\rm L}^\dagger a_{\rm R}^\dagger} \tag{19}$$

Show that its action on the vacuum gives

$$\hat{S}(\zeta)|0,0\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (-e^{i\phi} \tanh r)^n |n,n\rangle \tag{20}$$

with $\zeta = re^{i\phi}$. *Hint: use the BCH formula to rewrite* $S(\zeta) = e^{\gamma_+ a_{\rm L}^\dagger a_{\rm R}^\dagger} e^{\gamma_0 (a_{\rm L}^\dagger a_{\rm L} + a_{\rm R}^\dagger a_{\rm R} + 1)} e^{\gamma_- a_{\rm L} a_{\rm R}}$ *with coefficients* $\gamma_{\pm,0}$ *to be determined.*

- Show that the reduced density matrix obtained tracing out one of the two oscillators is thermal, and find the relation between the temperature and the squeezing factor ζ.

- Hence, a squeezed state on a factorized Hilbert space gives naturally a thermal density matrix. What is the origin of the squeezing and of the factorization in the case of the Unruh effect?

3. Conserved quantities

- Using Stokes' theorem, show that

$$\int_M d^4x \sqrt{-g} \nabla_\mu (n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu) = \int_{\partial M} d^3x \sqrt{q} K \tag{21}$$

- Compute the Komar charge

$$Q = \frac{1}{16\pi} \int_{S^2} \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \xi^\sigma dS^{\mu\nu} \tag{22}$$

associated with the two Killing vectors of Kerr, using a sphere of coordinate radius r, and show that it is proportional to the mass and angular momentum. *Hint: since the result does not depend on the location of the sphere, you can simplify the calculation assuming the sphere to be at infinity and using the* $1/r$ *leading behaviour of the Kerr metric and its inverse.* Muse over the numerical mismatch of a factor of 2.

- Denote by $\theta^\mu(\delta)$ the boundary term obtained from the variation of the Einstein-Hilbert action. Compute the component $\theta^r(\delta)$ when the background is Minkowski, and $\delta g_{\mu\nu}$ the leading $1/r$ behaviour of the Schwarzschild metric. *Hint: Use asymptotic Cartesian coordinates* (t, x^a) *to write* $\delta g_{\mu\nu}$ *and compute* θ^a *first. Then prove* $\theta^r = (1/r) x_a \theta^a$.

- Use the large r behaviour of the Schwarzschild metric to prove that

$$\frac{1}{16\pi} \int_{S^2} \epsilon_{\mu\nu\rho\sigma} (\nabla^\rho (\partial_t)^\sigma - \theta^\rho (\partial_t)^\sigma) dS^{\mu\nu} = M, \tag{23}$$

without a mismatch of a factor of 2.