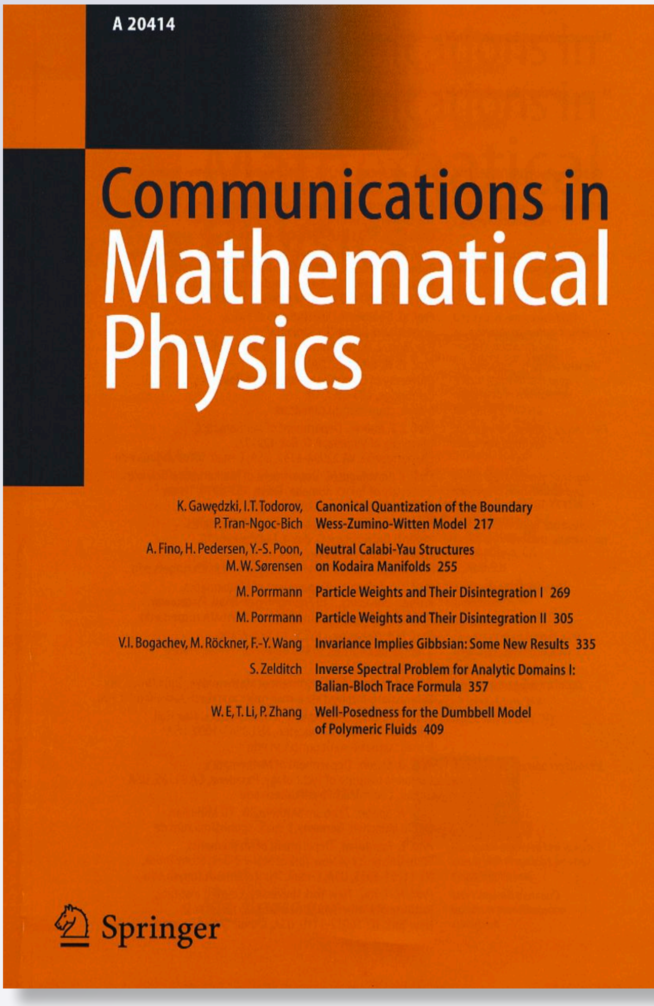
# *On the Recurrence and Robust Properties of Lorenz'63 Model*

**M. Gianfelice, F. Maimone, V. Pelino & S. Vaienti**

**Communications in Mathematical Physics**

ISSN 0010-3616

Commun. Math. Phys.
DOI 10.1007/s00220-012-1438-7



Commun. Math. Phys.
Digital Object Identifier (DOI) 10.1007/s00220-012-1438-7

Communications in Mathematical Physics

# On the Recurrence and Robust Properties of Lorenz'63 Model

**M. Gianfelice[1,⋆], F. Maimone[2,⋆⋆], V. Pelino[2,⋆⋆], S. Vaienti[3,⋆]**

[1] Dipartimento di Matematica, Università della Calabria, Campus di Arcavacata, Ponte P. Bucci - Cubo 30B, 87036 Arcavacata di Rende (CS), Italy. E-mail: gianfelice@mat.unical.it
[2] Italian Air Force, CNMCA, Aeroporto "De Bernardi", Via di Pratica di Mare, 00040 Roma, Italy. E-mail: maimone@meteoam.it; pelino@meteoam.it
[3] UMR-6207 Centre de Physique Théorique, CNRS, Université d'Aix-Marseille I, II, Université du Sud, Toulon-Var and FRUMAM, Fédération de Recherche des Unités des Mathématiques de Marseille, CPT Luminy, Case 907, 13288 Marseille Cedex 9, France. E-mail: vaienti@cpt.univ-mrs.fr

Received: 20 June 2011 / Accepted: 22 September 2011


**Abstract:** Lie-Poisson structure of the Lorenz'63 system gives a physical insight on its dynamical and statistical behavior considering the evolution of the associated Casimir functions. We study the invariant density and other recurrence features of a Markov expanding Lorenz-like map of the interval arising in the analysis of the predictability of the extreme values reached by particular physical observables evolving in time under the Lorenz'63 dynamics with the classical set of parameters. Moreover, we prove the statistical stability of such an invariant measure. This will allow us to further characterize the SRB measure of the system.

## Contents



⋆ Partially supported by GREFI-MEFI and PEPS *Mathematical Methods of Climate Models*.
⋆⋆ Partially supported by PEPS *Mathematical Methods of Climate Models*.

Published online: 11 February 2012

## 1. Introduction

In 1963 E. Lorenz by a drastic truncation of fluid-dynamics equations governing the atmospheric motion obtained a system of ODE which he proposed as a crude yet non trivial model of thermal convection of the atmosphere [L]. As a matter of fact, the Lorenz model is today understood as a basic toy-model for the evolution of Earth atmosphere's regimes, like zonal or blocked circulation or climate regimes (e.g. warm and cold), in which dynamics is described by equilibrium states [CMP, Se]. In his work Lorenz showed such a system to exhibit, for a large set of parameter values, a peculiar chaotic behavior, that is exponentially sensitive to perturbations of initial conditions and the existence of a global attracting set for the flow nowadays called the generalized nontrivial hyperbolic attractor. Although there exists an extensive literature on the subject, we refer the reader to [Sp] for a rather comprehensive overview on this problem and to [V] for a recent account on the progress made on the rigorous analysis of the Lorenz'63 ODE system and the relationship between this and its more abstract counterpart, the geometric Lorenz model, introduced in the second half of the seventies ([G, ABS and GW]) to describe the geometrical features a dynamical system should possess in order to exhibit the same asymptotic behavior as the Lorenz one. An affirmative answer to the long-standing question whether the original Lorenz'63 flow fits or not the description of the one modeled by the geometric Lorenz, which means it supports a robust singular hyperbolic (strange) attractor, has been given by W. Tucker in [T] by means of a computer assisted proof. As a byproduct, Tucker also proved that Lorenz flow admits a unique SRB measure supported on the strange attractor. Further results about the characterization of the set of geometric Lorenz-like maps are given in [LM].

In 2000 it has been emphasized that the Lorenz'63 model and the Kolmogorov one, considered as a low-order approximation of the Navier-Stokes equations, belong to a particular class of dynamical systems, named Kolmogorov-Lorenz systems [PP1], whose vector field admits a representation as a sum of a Hamiltonian $SO(3)$-invariant field, a dissipative linear field and constant forcing field (see also [PP2] for an extension of this analysis to the Lorenz'84 model). Moreover, they proved that the chaotic behavior of these models relies on the interplay between dissipation and forcing.

More specifically, and more recently, in [PM] it has been shown that the effect of the dissipative and forcing terms appearing in the previously described decomposition of the Lorenz'63 vector field, with the classical set of parameters, is to induce chaotic oscillations in the time evolution of the first integrals of the Hamiltonian system associated to the Lorenz'63 model, namely the Hamiltonian and the Casimir function for the (+) Lie-Poisson brackets associated the $so(3)$ algebra [MR], which represents the angular momentum of a free rigid body in the Kolmogorov-Lorenz representation of geofluid dynamics introduced in [PP1]. In particular, it has been shown that two subsequent oscillation peaks in the plot of the Casimir function $C$ as a function of time are related by a map $\Phi$ of the interval similar to the one originally computed by Lorenz in [L] depicting the functional dependence of two subsequent maximum values assumed by the third coordinate of the flow as a function of time. We remark that, $C$ being the square norm of the flow, the similarity of the plots of these two maps is therefore not surprising. In [PM], the recurrence properties of $\Phi$ are also studied, which allows to characterize the trajectories of the system through the number of revolution they perform around the unstable point lying on one side of the plane $x + y = 0$ when the initial condition is chosen on the opposite side ([PM], Figs. 2, 10 and 11).

In our paper we clarify what is stated in [PM] by giving an account in the first section of the rigid body formulation of the Lorenz'63 model and constructing, in the next section, a Markov expanding Lorenz-like map $T$ of the interval being the reduction to $[0, 1]$ of $\Phi$. Both maps are in fact derived throughout the Poincaré map associated to the surface in the configuration space of the system corresponding to the set of maxima reached by the Casimir function during its time evolution. Hence, we will study the invariant measure under the dynamics defined by $T$ characterizing its density and consequently the SRB measure of the system. Furthermore, we analyse the recurrence properties of the dynamics induced by $T$ clarifying more rigorously what is stated in Sect. IV B of [PM].

We will also perturb the system by adding an extra forcing term which will eventually cause the system to lose its symmetry under the involution $R : (x, y, z) \to (-x, -y, z)$ of $\mathbb{R}^3$. Due to the robustness of the attractor, i.e. persistence under perturbations of the parameters, proved in [T], maps analogous to $T$ can be defined and studied, and their statistical properties analysed as in the unperturbed case. Therefore, such perturbation of the Lorenz'63 field will only have the effect to induce a change in the statistics of the invariant measure for the system, the SRB measure. We will prove that this change could be detected by looking at the deviation of the invariant density of the perturbed map with respect to the unperturbed one. Such a result would confirm what has been empirically shown in [CMP] about the impact of anthropogenic forcing to climate dynamics of the northern hemisphere.

We also believe that this analysis could also be pursued in the case of more general $N$-dimensional models such as those introduced by Zeitlin in [Z] to approximate in the limit of $N$ tending to infinity the dynamics of the atmosphere in absence of dissipation and forcing.

We will present elsewhere our contributions in these directions; here we prove the first non-trivial result about the statistical stability of the invariant measure for $T$. The technique we propose is new and we believe it could be applied as well for other maps with some sort of criticalities.

We remark that, in particular, the distribution of the return times of a measurable subset of $[0, 1]$, which can be derived directly from the invariant measure of $T$, could be useful in studying the statistics of extreme meteorological events.

## 2. Return Lorenz-like Maps

*2.1. Rigid body formulation of the Lorenz'63 model.* It can be shown ([PP1]) that the Lorenz'63 ODE system [L],

$$\begin{cases} \dot{x}_1 = -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 = -x_1 x_3 + \rho x_1 - x_2 \\ \dot{x}_3 = x_1 x_2 - \beta x_3 \end{cases} \tag{1}$$

can be mapped, through the change of variables

$$\begin{cases} u_1 = x_1 \\ u_2 = x_2 \\ u_3 = x_3 - (\rho + \sigma) \end{cases}, \tag{2}$$

to the ODE system

$$\begin{cases} \dot{u}_1 = -\sigma u_1 + \sigma u_2 \\ \dot{u}_2 = -u_1 u_3 - \sigma u_1 - u_2 \\ \dot{u}_3 = u_1 u_2 - \beta u_3 - \beta (\rho + \sigma) \end{cases} \tag{3}$$

representing the evolution of a Hamiltonian system whose configuration space is the $SO(3)$ group, subject to dissipation and to a constant forcing. That is, denoting by

$$\{F, G\} := \omega_+^2 \left(ad^*_{\nabla F}x, ad^*_{\nabla G}x\right) = x \cdot \nabla F \times \nabla G \tag{4}$$

the Lie-Poisson brackets associated to the symplectic 2-form $\omega_+^2$ defined on the cotangent bundle of $SO(3)$ [MR], (3) reads

$$\dot{u}_i = \{u_i, H\} - (\Lambda u)_i + f_i, \quad i = 1, 2, 3, \tag{5}$$

where:

- $$H(u) := \frac{1}{2} u \cdot \Omega u + h \cdot u \tag{6}$$

  is the Hamiltonian of a rigid body whose kinetical term is given by the matrix $\Omega := diag(2, 1, 1)$, while $h := (0, 0, -\sigma)$ is an axial torque;
- $\Lambda := diag(\sigma, 1, \beta)$ is the dissipation matrix;
- $f := (0, 0, -\beta(\rho + \sigma))$ is a forcing term.

This representation allows to study the Lorenz system as a perturbation of the Hamiltonian system

$$v_i(u) := \{u_i, H\}, \qquad i = 1, 2, 3, \tag{7}$$

admitting, as in the case of a rigid body with a fixed point, two independent first integrals $H$ and the Casimir function $C$ for the Poisson brackets (4) [MR]. In fact, when rewritten in this form, it follows straightforwardly that the system is non chaotic for $\sigma = 0$ [PP1] while, for $\sigma \neq 0$, the values of $C$ and $H$ undergo chaotic oscillations [PM].

Moreover, when passing to the representation (5), the symmetries of the system are preserved as well as other features such as the invariance of the $x_3$ $(u_3)$ axis and the direction of rotation of the trajectories about this axis. The critical points of the velocity field of the system are then

$$c_1 := \left(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, -(\sigma + 1)\right), \; c_2 := (-x_1(c_1), -x_2(c_1), x_3(c_1)), \tag{8}$$

and $c_0 := (0, 0, -(\rho + \sigma))$.

We also remark that (5) can be rewritten in the form

$$\dot{u} = v - w, \tag{9}$$

where $v$ is the divergence free field (7) and

$$\mathbb{R}^3 \ni u \longmapsto w(u) := \Lambda u - f = \nabla K(u) \in \mathbb{R}^3 \tag{10}$$

with

$$K(u) := \frac{1}{2} u \cdot \Lambda u - f \cdot u \tag{11}$$

a convex function on $\mathbb{R}^3$. Notice that the fields $v$ nd $w$ are orthogonal in $L^2\left(r\mathcal{B};\mathbb{R}^3\right)$, where $B := \{u \in \mathbb{R}^3 : \|u\| \leq 1\}$ is the unitary ball in $\mathbb{R}^3$ and $rB$ denotes the ball of radius $r$.

The decomposition of the velocity field as the sum of a divergence free field ad a gradient one, together with the appearance in the Hamiltonian description of the flow of the Lie-Poisson brackets (4) in the space reference frame of the rigid body, i.e. right translation on $SO(3)$, is standard in fluid dynamics [A,MR] and can be seen as another source of analogy between the Lorenz'63 model and Navier-Stokes equations [PP1,FJKTV].

*2.2. The return map on the set of maxima of the Casimir function.* If $u_0$ is any non stationary point for the field (3) such that $\|u_0\| \leq \frac{\|f\|}{\sqrt{\lambda_\Lambda}}$, with $\lambda_\Lambda := \min\{t \in spec\Lambda\}$, and $C(t) := \|u(t,u_0)\|^2$, let

$$m := \inf_{t>0} C(t)\,; \quad M := \sup_{t>0} C(t)\,. \tag{12}$$

Clearly, $m \geq 0$ since $C \geq 0$. Moreover, $M < \infty$ since it has been shown in [PP1] that $C(t) \leq \frac{\|f\|}{\sqrt{\lambda_\Lambda}}$ where, for our choice of parameters,

$$\frac{\|f\|}{\sqrt{\lambda_\Lambda}} = \|f\| = \beta(\rho+\sigma)\,. \tag{13}$$

To construct the function which links two subsequent relative maximum values of $C(t)$ we proceed as follows:

- first we identify the manifold $\Sigma$ in the configuration space of the system corresponding to the relative maxima of $C(t)$,
- then we construct a map of the interval $[0,1]$ to itself as a function of the map of the interval $[m,M]$ of the possible values of $C(t)$ in itself, which can be defined through the Poincaré map of this manifold.

The existence of the aforementioned Poincaré map follows from the existence of the return map computed by Tucker in [T].

Throughout the paper we will assume $\sigma = 10,\ \rho = 28,\ \beta = \frac{8}{3}$.

*2.2.1. Identification of* $\Sigma$. By (9) we get

$$\dot{C}(u) = -2\left[E(u) - \frac{\beta(\rho+\sigma)^2}{4}\right], \tag{14}$$

with

$$E(u) := \sigma u_1^2 + u_2^2 + \beta\left(u_3 + \frac{(\rho+\sigma)}{2}\right)^2. \tag{15}$$

Therefore,

$$\mathcal{E} := \left\{u \in \mathbb{R}^3 : \dot{C}(u) = 0\right\} = \left\{u \in \mathbb{R}^3 : E(u) = \frac{\beta(\rho+\sigma)^2}{4}\right\}, \tag{16}$$

as already noticed in [PM], is an ellipsoid intersecting the vertical axis ($u_3$) in the origin and in $c_0$. This also implies, $M = \rho + \sigma$. Clearly, $c_1, c_2 \in \mathcal{E}$.

Moreover, by (14) and (4),

$$\begin{aligned}\ddot{C}(u) &= 2\nabla E \cdot [u \times \nabla H + \nabla K](u) \\ &= 4\left\{\sigma^2 u_1^2 + u_2^2 - \left[\sigma(\sigma-1) + (\beta-1)\left(u_3 + \frac{\rho+\sigma}{2}\right) + \frac{\rho+\sigma}{2}\right] u_1 u_2 \right. \\ &\quad \left. + \beta^2 \left(u_3 + \frac{\rho+\sigma}{2}\right)^2 + \beta^2 \frac{\rho+\sigma}{2}\left(u_3 + \frac{\rho+\sigma}{2}\right)\right\}. \end{aligned} \tag{17}$$

Let us set $z := u_3 + \frac{\rho+\sigma}{2}$, then

$$\begin{aligned}\mathcal{E}' &:= \left\{u \in \mathbb{R}^3 : \ddot{C}(u) = 0\right\} \\ &= \left\{u \in \mathbb{R}^3 : \sigma^2 u_1^2 + u_2^2 - \left[\sigma(\sigma-1) + (\beta-1) z + \frac{\rho+\sigma}{2}\right] u_1 u_2 \right. \\ &\quad \left. + \beta^2 z \left(z + \frac{\rho+\sigma}{2}\right) = 0\right\}. \end{aligned} \tag{18}$$

We remark that, denoting by $R$ the involution

$$\mathbb{R}^3 \ni u = (u_1, u_2, u_3) \longmapsto Ru := (-u_1, -u_2, u_3) \in \mathbb{R}^3, \tag{19}$$

leaving invariant the field $\dot{u}$, $R\mathcal{E} = \mathcal{E}$ and $R\mathcal{E}' = \mathcal{E}'$.

Consider the diffeomorphism

$$q_i = O(z) u_i, \quad i = 1, 2; \; q_3 = z \tag{20}$$

such that for any fixed value of $z$, $O(z)$ is an orthogonal matrix diagonalizing the symmetric quadratic form

$$\zeta \cdot A(z) \zeta := \sigma^2 \zeta_1^2 + \zeta_2^2 - \left[\sigma(\sigma-1) + (\beta-1) z + \frac{\rho+\sigma}{2}\right] \zeta_1 \zeta_2, \quad \zeta \in \mathbb{R}^2, \tag{21}$$

namely, setting $A(z) = O^t(z)\, diag\, (\lambda_1(z), \lambda_2(z))\, O(z)$,

$$u \cdot A(z) u = q \cdot O(z) A(z) O^t(z) q = \lambda_1(z) q_1^2 + \lambda_2(z) q_2^2, \tag{22}$$

with

$$\lambda_1(z) = \frac{\sigma^2 + 1 + \sqrt{\left(\sigma^2 - 1\right)^2 + \left[\frac{\rho+\sigma}{2} + \sigma(\sigma-1) + (\beta-1) z\right]^2}}{2}, \tag{23}$$

$$\lambda_2(z) = \frac{\sigma^2 + 1 - \sqrt{\left(\sigma^2 - 1\right)^2 + \left[\frac{\rho+\sigma}{2} + \sigma(\sigma-1) + (\beta-1) z\right]^2}}{2}. \tag{24}$$

Under this change of variables

$$\mathcal{E}' = \left\{q \in \mathbb{R}^3 : \lambda_1(q_3) q_1^2 + \lambda_2(q_3) q_2^2 + \beta^2 q_3 \left(q_3 + \frac{\rho+\sigma}{2}\right) = 0\right\}. \tag{25}$$

Since $\lambda_1(q_3)$ is positive for any choice of the parameters $\beta, \rho, \sigma$ and $q_3$, the equation giving the intersection of $\mathcal{E}'$ with the planes parallel to $q_3 = 0$ ($u_3 = -\frac{(\rho+\sigma)}{2}$) can have a solution only if $\lambda_2(q_3)$ is negative, that is for

$$q_3 > -\frac{\sigma(\sigma-3) + \frac{\rho+\sigma}{2}}{\beta - 1} \Rightarrow u_3 > -\frac{1}{\beta-1}\left[\sigma(\sigma-3) + \beta\frac{(\rho+\sigma)}{2}\right]; \tag{26}$$

$$q_3 < -\frac{\sigma(\sigma+1) + \frac{\rho+\sigma}{2}}{\beta - 1} \Rightarrow u_3 < -\frac{1}{\beta-1}\left[\sigma(\sigma+1) + \beta\frac{(\rho+\sigma)}{2}\right]. \tag{27}$$

Therefore, for $q_3 \neq 0$ ($u_3 \neq -\frac{(\rho+\sigma)}{2}$), these intersections are hyperbolas while, if $q_3 = 0$ or $q_3 = -\frac{(\rho+\sigma)}{2}$ ($u_3 = -(\rho+\sigma)$), from the definition of $\mathcal{E}'$ we get the equations

- if $q_3 = 0$,

$$\sigma^2 u_1^2 + u_2^2 - \left[\frac{(\rho+\sigma)}{2} + \sigma(\sigma-1)\right] u_1 u_2 = 0\,; \tag{28}$$

- if $q_3 = -\frac{(\rho+\sigma)}{2}$,

$$\sigma^2 u_1^2 + u_2^2 - \left[\sigma(\sigma-1) - (\beta-2)\frac{\rho+\sigma}{2}\right] u_1 u_2 = 0. \tag{29}$$

Since for our choice of the values of the parameters of the model,

$$\lambda_2(0) = \frac{\sigma^2 + 1 - \sqrt{(\sigma^2-1)^2 + \left[\frac{(\rho+\sigma)}{2} + \sigma(\sigma-1)\right]^2}}{2} < 0, \tag{30}$$

$$\lambda_2\left(-\frac{(\rho+\sigma)}{2}\right) = \frac{\sigma^2 + 1 - \sqrt{(\sigma^2-1)^2 + \left[\sigma(\sigma-1) - (\beta-2)\frac{(\rho+\sigma)}{2}\right]^2}}{2} < 0, \tag{31}$$

the intersection of $\mathcal{E}'$ with the planes $z = q_3 = 0$ and $z = q_3 = \frac{(\rho+\sigma)}{2}$ are the straight lines. The manifold in $R^3$ corresponding to the relative maxima of $C(t)$ is

$$\begin{aligned}\Sigma &:= \left\{u \in \mathbb{R}^3 : \dot{C}(u) = 0,\ \ddot{C}(u) \le 0\right\} \\ &= \left\{ u \in \mathbb{R}^3 : \begin{cases} \sigma u_1^2 + u_2^2 + \beta\left(u_3 + \frac{(\rho+\sigma)}{2}\right)^2 = \frac{\beta(\rho+\sigma)^2}{4} \\ \sigma^2 u_1^2 + u_2^2 - \left[\sigma(\sigma-1) + (\beta-1)\left(u_3 + \frac{\rho+\sigma}{2}\right) + \frac{\rho+\sigma}{2}\right] u_1 u_2 + \\ +\beta^2\left(u_3 + \frac{\rho+\sigma}{2}\right)(u_3 + \rho + \sigma) \le 0 \end{cases} \right\}.\end{aligned} \tag{32}$$

Since for our choice of the parameters

$$\frac{1}{\beta-1}\left[\sigma(\sigma-3) + \beta\frac{(\rho+\sigma)}{2}\right] > \rho + \sigma, \tag{33}$$

$\Sigma$ is composed by two closed surfaces in $R^3$, $\Sigma_+$ and $\Sigma_-$, such that $R\Sigma_+ = \Sigma_-$ and intersecting only in the critical point $c_0$.

By definition, $\forall u \in \mathcal{E}$, the vector $\dot{u}(u)$ is orthogonal to the vector $\nabla C(u)$, hence it belongs to the plane spanned by $\nabla E(u) - (\nabla E \cdot \nabla C)(u) \nabla C(u)$ and $(\nabla C \times \nabla E)(u)$, where

$$\nabla C \times \nabla E = 4 \begin{pmatrix} u_2 \left[ (\beta - 1) u_3 + \beta \frac{\rho+\sigma}{2} \right] \\ -u_1 \left[ (\beta - \sigma) u_3 + \beta \frac{\rho+\sigma}{2} \right] \\ -(\sigma - 1) u_1 u_2 \end{pmatrix} \tag{34}$$

and, since $C$ is a constant of motion for the Hamiltonian field $v$, $\forall u \in \mathcal{E}$, $(w \cdot \nabla C)(u) = 0$.

Moreover:

- $|\nabla E| \upharpoonright_{\mathcal{E}}$, $|\nabla C| \upharpoonright_{\mathcal{E}}$ and $|\nabla C \times \nabla E| \upharpoonright_{\mathcal{E}}$ are always different from zero;
- from (14) and (32)

$$\ddot{C}(t) = \left(\dot{u} \cdot \nabla \dot{C}\right)(t) = \left(\dot{u} \cdot \nabla \left[ -2 \left( E - \beta \frac{(\rho+\sigma)^2}{4} \right) \right] \right)(t), \tag{35}$$

hence, $\forall u \in \partial \Sigma$, $\dot{u}(u)$ is parallel to $\nabla C \times \nabla E$, that is tangent to $\Sigma$.

Therefore, $\dot{u}$ is transverse to $\Sigma \backslash \partial \Sigma$ and since $\forall u \in \Sigma \backslash \partial \Sigma$,

$$\ddot{C}(u) = \left(\dot{u} \cdot \nabla \dot{C}\right)(u) = -2(\dot{u} \cdot \nabla E) < 0 \implies (\dot{u} \cdot \nabla E) > 0, \tag{36}$$

the direction of $\dot{u}(u)$ points outward to the bounded subset of $\mathbb{R}^3$,

$$\left\{ u \in \mathbb{R}^3 : E(u) \leq \beta \frac{(\rho+\sigma)^2}{4} \right\}. \tag{37}$$

*2.2.2. Parametrization of* $\Sigma$. If $r \in (0, \rho + \sigma)$, $\gamma := rB \cap \mathcal{E}$ is a regular closed curve. Therefore, we can parametrize $\Sigma_+$ by choosing an appropriate arc of $\gamma$ as a coordinate curve of the parametrization, that is there exist an open regular subset $\Omega$ of $\mathbb{R}^2$ and a map $b^+ \in C^1\left(\Omega, \mathbb{R}^3\right) \cap C\left(\overline{\Omega}, \mathbb{R}^3\right)$ such that

$$\begin{cases} u_1 = b_1^+(y, z) \\ u_2 = b_2^+(y, z) \\ u_3 = b_3^+(y, z) \end{cases}, \quad (y, z) \in \Omega. \tag{38}$$

Moreover, we can choose the parametrization such that $z = r^2$, therefore the coordinate curves $b_z^+(y)$ satisfy the equations

$$\begin{cases} C\left(b_z^+(y)\right) = z \\ \dot{C}\left(b_z^+(y)\right) = 0 \end{cases}. \tag{39}$$

We remark that the tangent field to $\gamma$ is parallel to $\nabla C \times \nabla E$, while, if $\zeta$ denotes the coordinate curve $b_y^+(z)$, the tangent field to $\zeta$ is parallel to $\nabla E \times (\nabla C \times \nabla E)$.

Similar arguments also hold for $\Sigma_-$. Furthermore,

$$\overline{\Omega} \ni (y, z) \longmapsto b^-(y, z) := Rb^+(y, z) \in \mathbb{R}^3 \tag{40}$$

is easily seen to be a parametrization of $\Sigma_-$ sharing the same properties of $b^+$.

*2.2.3. Return maps on* $\Sigma$. The evolution of the system maps in itself the ball $\beta(\rho+\sigma)B$, $\mathcal{E}\subset\beta(\rho+\sigma)B$ and the velocity field is transverse to $\Sigma\backslash\partial\Sigma$ and $\mathcal{E}\backslash\Sigma$.

For our choice of parameters $\rho,\beta$ and $\sigma$, it has been shown in [T] that there exist periodic orbits crossing a two-dimensional compact domain $\Delta$ contained in the plane $\pi:=\left\{u\in\mathbb{R}^3 : u_3=1-(\rho+\sigma)\right\}$, which is also intersected by the stable manifold of the system $W_o^s$ along some curve $\Gamma_0$. Furthermore, the first eight shortest periodic orbits have been rigorously found in [GT]. Notice that, by symmetry, if $\varphi(t,u)$, $u\in\Sigma_+$, is a periodic orbit, $R\varphi(t,u)$ is also a periodic orbit and either $R\varphi(t,u)=\varphi(t,u)$ or $R\varphi(t,u)=\varphi(t,Ru)$, where $Ru\in\Sigma_-$; that is periodic orbits are either symmetric or appear in couples whose elements are mapped one into another by $R$, as already remarked in [Sp].

Since $\Delta$ is easily seen to be contained in

$$\left\{u\in\mathbb{R}^3 : \dot{C}(u)\leq 0\right\}\cap\pi, \tag{41}$$

these periodic orbits then necessarily cross $\Sigma$ which is also possibly intersected by $W_o^s$ along some curve $\Gamma$ lying in the half-space

$$\left\{u\in\mathbb{R}^3 : u_3\geq 1-(\rho+\sigma)\right\}. \tag{42}$$

Therefore, if $u_0\in\Sigma\backslash\Gamma$ lies on a periodic orbit of period $t_0$, there exists an open neighborhood $N\ni u_0$ and a $C^1(N,\mathbb{R})$ map $\tau$ such that $\tau(u_0)=t_0$ and $\varphi_{\tau(u)}(u)\in\Sigma$ for any $u\in N$. Then,

$$N\cap\Sigma\backslash\Gamma\ni u\longmapsto P_\Sigma(u):=\varphi_{\tau(u)}(u)\in\Sigma. \tag{43}$$

Moreover, it has been proved in [T] that $\Delta\backslash\Gamma_0$ is forward invariant under the return map on $\pi$ and that on $\Delta$ there exists a forward invariant unstable cone field. These properties are also shared by a compact subset $\Delta'\subset\Sigma$ such that any open subset of $\Delta'$ is diffeomorphic to a open subset of $\Delta$. Hence, $P_\Sigma$ admits an invariant stable foliation with $C^{1+\iota}$, $\iota\in(0,1)$ leaves.

*2.2.4. Construction of the map* $T$. Let $P^{(\pm)}:=P_{\Sigma_\pm}$. By the parametrization previously introduced for $\Sigma_+$, there exists an open subset $\Omega'\subset\Omega\backslash\Gamma'$, with $\Gamma':=\left(b^+\right)^{-1}(\Gamma)$, and a $C^1\left(\Omega',\mathbb{R}^2\right)$ map

$$\Omega'\ni(y,z)\longmapsto S(y,z)\in\Omega', \tag{44}$$

such that, $\forall(y,z)\in\Omega'$,

$$\left(b^+\circ S\right)(y,z):=\left(P^{(+)}\circ b^+\right)(y,z). \tag{45}$$

Furthermore,

$$G(y,z):=\left(\dot{C}\circ b^+\circ S\right)(y,z)=0. \tag{46}$$

Let $S_1,S_2$ be respectively the first and the second component of $S$. Since $b^+$ is a diffeomorphism and the components of $\nabla E=\nabla\dot{C}$ are different from zero on $\Sigma_+$, $\forall(y,z)\in\Omega'$, $\partial_y G(y,z)\neq 0$. Thus, by the implicit function theorem, $\forall(y_0,z_0)\in\Omega'$, there exist two open intervals $(y_1, y_2)$, $(z_1, z_2)$ such that $(y_0, z_0) \in (y_1, y_2) \times (z_1, z_2) \subseteq \Omega'$ and a unique $C^1((z_1, z_2))$ map

$$(z_1, z_2) \ni z \longmapsto y := U(z) \in (y_1, y_2) \tag{47}$$

such that, $\forall z \in (z_1, z_2)$, $G(U(z), z) = 0$ and, $\forall (y, z) \in (y_1, y_2) \times (z_1, z_2)$ such that $y \neq U(z)$, $G(y, z) \neq 0$.

Therefore, let

$$(z_1, z_2) \ni z \longmapsto V(z) := S_2(U(z), z) \in (z_1, z_2). \tag{48}$$

Notice that, since $b^+ \in C^1(\Omega)$, $S = \left(b^+\right)^{-1} \circ P^{(+)} \circ b^+$ is $C^1\left(\Omega'\right)$ if and only if $P^{(+)}$ is, and so are $U$ and $V$.

Moreover, by symmetry,

$$b^- \circ S = Rb^+ \circ S = RP^{(+)} \circ b^+ = RP^{(+)} \circ Rb^- = P^{(-)} \circ b^-. \tag{49}$$

Hence $S = \left(b^-\right)^{-1} \circ P^{(-)} \circ b^-$.

Clearly, $[z_1, z_2] \subseteq \left[m \vee (r^*)^2, \rho + \sigma\right]$, with $r^* := \inf\{r > 0 : rB \cap \Sigma \neq \varnothing\}$.

Let $u_+ \in \Sigma_+$, $(y_+, z_+) \in \overline{\Omega}$ be such that $P^{(+)}(u_+) = c_0$ and $b^+(y_+, z_+) = u_+$. Setting

$$[z_1, z_2] \ni z \longmapsto X(z) := \frac{z - z_1}{z_2 - z_1} \in [0, 1], \tag{50}$$

we define

$$[0, 1] \ni s \longmapsto T(s) := X \circ V \circ X^{-1}(s) \in [0, 1]. \tag{51}$$

Hence, by construction, $T$ is a $C^1((0, 1) \setminus \{x_0\})$ map, where $x_0 := X(z_+)$, and, since there exists $\iota \in (0, 1)$ such that $P_\Sigma$ admits an invariant stable foliation with $C^{1+\iota}$ leaves, then $T$ is also $C^{1+\iota}((0, 1) \setminus \{x_0\})$.

## 3. The Invariant Density for the Evolution Under $T$

In this section we compute the density of the unique (by ergodicity) absolutely continuous invariant measure for the map $T$ and we prove its statistical stability. For the construction of the density we use the techniques recently introduced in the paper [CHMV] (see also [BH] for results related to similar maps), which dealt with Lorenz maps admitting indifferent fixed points besides points with unbounded derivative. For the statistical stability we will follow the recent article [BV], but with some new substantial improvements. The techniques used in [CHMV] turned around Young's towers [LSY] and, substantiated by a careful analysis of the distortion, led to a detailed study of the density of the absolutely continuous invariant measure, of the recurrence properties of the dynamics and of limit theorems for Hölder continuous observables. This analysis could in particular be carried over when the map has a derivative larger than one at the fixed point, as in the case we are going to treat, but possibly smaller than one in some other point (see below). We recall that whenever the Lorenz map is a Markov expanding map with finite derivative, it could be investigated with the spectral techniques of Keller [K]. Young's towers are useful when the map loses the Markov property, but preserves points with unbounded derivative. This has been analysed in [KDO], see also [OHL] when there are critical points too. Our main effort will be in investigating the smoothness of the density. We

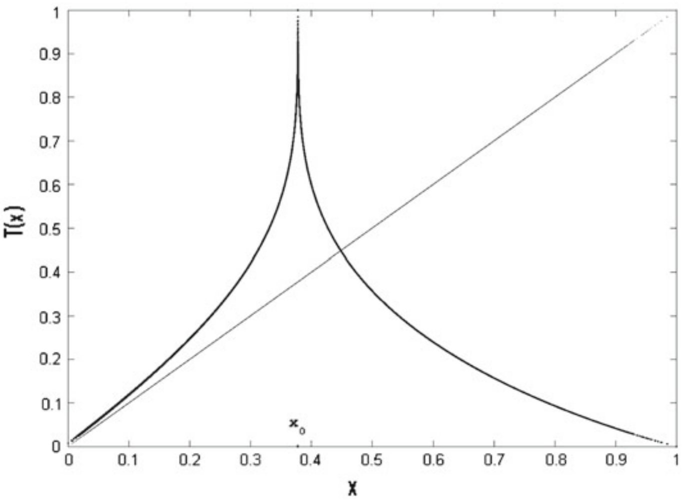



**Fig. 1.** Normalized Lorenz cusp map for the Casimir maxima

will show that such a density is Lipschitz continuous on the whole unit interval. The argument we produce is a strong improvement with respect to the result achieved in [CHMV] (and applies to it as well), where we solely proved the Lipschitz continuity on countably many intervals partitioning the unit interval. We stress that, as far as we know, this is the first result where the smoothness of the density for Lorenz like maps is explicitly exhibited.

*Notations.* With $a_n \approx b_n$ we mean that there exists a constant $C \geq 1$ such that $C^{-1}b_n \leq a_n \leq Cb_n$ for all $n \geq 1$; with $a_n \sim b_n$ we mean that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. We will also use the symbols "$O$" and "$o$" in the usual sense.

The analysis we perform in this section applies to a large class of Lorenz-like maps which includes in particular those whose behavior is given by the theoretical arguments of the preceding section and by the numerical investigations of the paper [PM]. The map $T$ (Fig. 1) has a left and a right convex branch around the point $0 < x_0 < 1$; the left branch is monotonically increasing and uniformly expanding even at the fixed point 0, while the right one is monotonically decreasing with the derivative bounded from below by a constant less than one; at the cusp, located at $x_0$, the left and right derivative blow up to infinity. Both branches are onto [0, 1] and this makes the map Markovian. Moreover, we recall that our map is $C^1$ on $[0, 1]\backslash\{x_0\}$ and $C^{1+\iota}$, $\iota \in (0, 1)$ , on $(0, x_0) \cup (x_0, 1)$.

The local behaviors are ($c$ will denote a positive constant which could take different values from one formula to another):

$$\begin{cases} T(x) = \alpha' x + \beta' x^{1+\psi} + o(x^{1+\psi}); \ x \to 0^+ \\ DT(x) = \alpha' + cx^{\psi} + o(x^{\psi}), \ \alpha' > 1; \ \beta' > 0; \ \psi > 1 \end{cases}, \tag{52}$$

$$\begin{cases} T(x) = \alpha(1-x) + \tilde{\beta}(1-x)^{1+\kappa} + o((1-x)^{1+\kappa}); \ x \to 1^- \\ DT(x) = -\alpha - c(1-x)^{\kappa} + o((1-x)^{\kappa}), \ 0 < \alpha < 1, \ \tilde{\beta} > 0, \ \kappa > 1 \end{cases}, \tag{53}$$

$$\begin{cases} T(x) = 1 - A'(x_0 - x)^{B'} + o((x_0 - x)^{B'}); \ x \to x_0^-, \ A' > 0 \\ DT(x) = c(x_0 - x)^{B'-1} + o((x_0 - x)^{B'-1}), \ 0 < B' < 1 \end{cases}, \tag{54}$$

$$\begin{cases} T(x) = 1 - A(x-x_0)^B + o((x-x_0)^B);\ x \to x_0^+,\ A > 0 \\ DT(x) = -c(x-x_0)^{B-1} + o((x-x_0)^{B-1}),\ 0 < B < 1 \end{cases}. \tag{55}$$

We set $B^* := \max(B, B')$; moreover we set $T_1$ (resp. $T_2$) the restriction of $T$ to $[0, x_0]$ (resp. to $[x_0, 1]$). A key role is played by the preimages of $x_0$ since they will give the sets where we will induce with the first return map; so we set: $a_0 := T_2^{-1}x_0$; $a'_0 := T_1^{-1}x_0$; $a'_p = T_1^{-p}a'_0$; $a_p = T_2^{-1}T_1^{-(p-1)}a'_0$, $p \geq 1$. We also define the sequences $\{b_p\}_{p\geq 1} \subset (x_0, a_0)$ and $\{b'_p\}_{p\geq 1} \subset (a'_0, x_0)$ as $Tb'_p = Tb_p = a_{p-1}$. The idea is now to induce on some domain $I$ and to replace the action of $T$ on $I$ with that of the first return map $T_I$ into $I$. We will see that the systems $(I, T_I)$ will admit an absolutely continuous invariant measure $\mu_I$ which is in particular equivalent to the Lebesgue measure with a density $\rho_I$ bounded from below and from above. There will be finally a link between the induced measure $\mu_I$ and the absolutely continuous invariant measure $\mu$ on the interval, which will allow us to get some information on the density $\rho$ of $\mu$. The principal set where we will choose to induce is the open interval $I = (a'_0, a_0)\backslash\{x_0\}$. The subsets $Z_p \subset I$ with first return time $p$ will have the form

$$Z_1 = (a'_0, b'_1) \cup (b_1, a_0), \tag{56}$$
$$Z_p = (b'_{p-1}, b'_p) \cup (b_p, b_{p-1}) \quad p > 1. \tag{57}$$

We will also induce over the open sets $(a'_n, a'_{n-1})$ and $(a_n, a_{n+1})$, $n > 1$, simply denoted in the following as the rectangles $I_n$. In order to apply the techniques of [CHMV], we have to show that the induced maps are aperiodic uniformly expanding Markov maps with bounded distortion on each set with prescribed return time. On the sets $I_n$ the first return map $T_{I_n}$ is Bernoulli, while the aperiodicity condition on $I$ follows easily by the inspection of the graph of the first return map $T_I : I \to I$ showing that it maps: $(a'_0, b'_1)$ onto $(x_0, a_0)$; the intervals $(b'_l, b'_{l+1})$, $l \geq 1$, onto the interval $(a'_0, x_0)$ and $(b_1, a_0)$ onto $(x_0, a_0)$. Finally, $T_I$ sends the intervals $(b_{l+1}, b_l)$, $l \geq 1$ onto $(a'_0, x_0)$. Bounds on the distortion of the first return map on $I$ and on the $I_n$ can be proved exactly in the same way as in Proposition 3 of [CHMV] (we defer to it for the details) provided we show that the first return maps are uniformly expanding[1]. The proof of this fact is given in the next lemma and it requires a few assumptions which can be checked numerically with a finite number of steps and by a direct inspection of the graph of $T$. With abuse of language we will say that the derivative is larger than 1 if its absolute value is larger than 1.

**Lemma 1.** *Let us suppose that in addition to (52)-(55) the map $T$ satisfies the assumptions:*

[1] The Lorenz-like map considered in [CHMV] was $C^2$ outside the boundary points and the cusp; our map is instead $C^{1+\iota}$. This will not change the proof of the distortion in [CHMV] and all the statistical properties which follow from it. As a matter of fact, in the initial formula (5) in [CHMV], we have to replace the term $\left|\frac{D^2T(\xi)}{DT(\xi)}\right| \left|T^q(x) - T^q(y)\right|$ with $\frac{C_h}{|DT(\xi)|}\left|T^q(x) - T^q(y)\right|^\iota$, where $C_h$ is the Hölder constant larger than 0 depending only on $T$ and $\xi$ is a point between the iterates $T^q(x)$, $T^q(y)$. The only delicate point where the $C^{1+\iota}$ assumption could give problems is the summability of the series at point (i) in the statement of Lemma 4 in [CHMV]. The general term of this series will be of the form (we adapt to our case): $(a_{n+1} - a_n)^\iota$. In [CHMV], due to the presence of the indifferent fixed point, the term $(a_{n+1} - a_n)$ decays polynomially like $n^{-\kappa}$, say, where $\kappa > 1$ depends on the map. In order to guarantee the aforecited summability property we have therefore to ask an additional assumption on $\kappa$, namely $\kappa > \iota^{-1}$. We do not have such a constraint in our case since the length $(a_{n+1} - a_n)$ decays exponentially fast.

*(i)* $d_{(1,0)} := \inf_{x\in(b_1,a_0)} |DT(x)| > 1$;
*(ii)* $|DT(b_1)| \geq DT(a_0')$;
*(iii)* $|DT(a_{p-1})|DT(a'_{p-2})\cdots DT(a'_1)DT(a'_0) > \alpha''$, $\forall p \leq p^* := \left\lfloor 1 + \frac{\log(\alpha''\alpha^{-1})}{\log \alpha'}\right\rfloor$, *for* $1 < \alpha'' \leq d_{(1,0)} \wedge \alpha'$.

*Then the first return time maps* $T_I$ *and* $T_{I_n}, n > 1$ *have the derivative uniformly bounded below away from* $\alpha''$.

*Proof.* We give the proof for $I$ and we generalize after to all the $I_n$. We represent with an arrow "$\rightarrow$" the evolution under $T$ of a subset $Z_p \subset I,\ p \geq 1$, given in (56) and (57). Consequently $(b_1, a_0) \rightarrow (x_0, a_0)$ and $(a_0', b_1') \rightarrow (x_0, a_0)$. In the latter case the derivative of the map coincides with that of $T$ and is larger than 1, since $T_1$ has derivative larger than $\alpha' > 1$. The former case follows by condition (i). For $p > 1$ we have:

$$\begin{cases} (b_2, b_1) \rightarrow (a_0, a_1) \rightarrow \left(a_0', x_0\right), \quad p = 2 \\ (b_p, b_{p-1}) \rightarrow (a_{p-2}, a_{p-1}) \rightarrow (a'_{p-2}, a'_{p-3}) \rightarrow (a'_{p-3}, a'_{p-4}) \rightarrow \cdots \\ \quad \rightarrow (a_1', a_0') \rightarrow (a_0', x_0) \quad p \geq 3 \end{cases} \tag{58}$$

and

$$\begin{cases} \left(b_2', b_1'\right) \rightarrow (a_0, a_1) \rightarrow \left(a_0', x_0\right), \quad p = 2 \\ (b'_{p-1}, b'_p) \rightarrow (a_{p-2}, a_{p-1}) \rightarrow (a'_{p-2}, a'_{p-3}) \rightarrow (a'_{p-3}, a'_{p-4}) \rightarrow \cdots \\ \quad \rightarrow (a_1', a_0') \rightarrow (a_0', x_0) \quad p \geq 3 \end{cases} \tag{59}$$

In order to get that the derivative of $T^p$ is larger than one we need:

- in the first case

$$\left|DT(b_{p-1})DT(a_{p-1})\right| DT(a'_{p-2})\cdots DT(a_2')DT(a_1') > 1\,; \tag{60}$$

- in the second case

$$DT(b'_{p-1})\left|DT(a_{p-1})\right| DT(a'_{p-2})\cdots DT(a_2')DT(a_1') > 1. \tag{61}$$

Let us suppose now that we have, for $p > 1$,

$$\left|DT(a_{p-1})\right| DT(a'_{p-2})\cdots DT(a_1')DT(a_0') > \alpha''. \tag{62}$$

If this condition holds, then the inequality (61) follows too with the same uniform (in $p$) bound $\alpha''$, since, by monotonicity of the first derivative, $DT(b'_{p-1}) > DT(a_0')$. In order to satisfy the inequality (60) with a lower bound given by $\alpha''$, by assuming again (62), it will be sufficient to show that $\left|DT(b_{p-1})\right| \geq DT(a_0')$, which, by monotonicity, is implied by $|DT(b_1)| \geq DT(a_0')$ and this follows from assumption (ii). Therefore, we are left with the proof of the validity of condition (62). By condition (iii) this holds true for $p \leq p^*$. Moreover, since the points $a'_{p-2}, a'_{p-3}, \ldots, a_0'$ lie on the left of $x_0$, all the $(p-1)$ derivatives in the block $DT(a'_{p-2})\cdots DT(a_2')DT(a_0'))$ are larger $\alpha'$. On the other hand, the derivative in $a_{p-1}$ is surely larger than $\alpha$. Hence, (62) holds for all $p$ such that $\alpha\left(\alpha'\right)^{p-1} > \alpha''$.

Now we return to the rectangles in $I_n$. Let us first call *complete path* the graphs given above starting respectively from $(a_{p-1}, a_p),\ p > 1$, and ending in $(a_0', x_0)$ and starting from $(a_p', a'_{p-1}),\ p > 1$ and ending in $(a_0', x_0)$. It is easy to check, by looking at the grammar[2] given by the arrows, that any subset of the rectangles in $I_n$ with first return time $q > n$ will contain points whose trajectory follows a complete path, or spends some time in $(x_0, a_0)$. In any case, and by the condition (62) whose validity has been checked above, the derivative $DT^q$ will be strictly larger than $\alpha''$. □

*Remark 2.* The assumption (i)–(iii) in the previous lemma are easily verified for the map investigated in [PM]. In particular, with the values $\alpha = 0.4603$ and $\alpha' = 1.113$ associated to the map and with $\alpha'' \sim 1.01$, the inequality (iii) is verified for $p \geq 9$; hence we have only to check (62) for $1 < p \leq 8$ and this has been done, and confirmed, by a direct easy numerical inspection.

As a consequence of the preceding results, we could apply, as in [CHMV], the L.-S. Young tower theory and conclude the following statements:

- On the induced set $I$, the tail of the Lebesgue measure of the set of points with first return bigger than $n$, to be more precise the quantity $\sum_{k>n} m\{x \in I\ ;\ \tau_I(x) \geq k\}$, where $\tau_I(x)$ denotes the first return of the point $x$ into $I$, decays exponentially fast with $n$. By using (56) and the asymptotic values for the $b_n$ and $b'_n$ given below, it is immediate to find that the previous rate of decay is $O((\alpha')^{-\frac{n}{B^*}})$. This implies the existence on the Borel $\sigma$-algebra $\mathcal{B}$ ([0, 1]) of an absolutely continuous invariant measure $\mu$ with exponential decay of correlations for Hölder observables evolving under $T$ w.r.t. $\mu$ (the rate of this decay will be of the type $\hat{\alpha}^{-n}$, where $\hat{\alpha}$ is possibly different from $\alpha'$).
- Since the first return maps $T_I$ and $T_{I_n}$ are aperiodic uniformly expanding Markov maps, they admit invariant measures $\mu_I$ and $\mu_{I_n}$ which turn out to be equivalent to Lebesgue on $I$ and $I_n$ with densities bounded away from 0 and $\infty$ [CHMV] and also Lipschitz continuous on the images of the rectangles of the their associated Markov partition [AD].[3] In the sequel we will show that such densities coincide, but a constant, with the restriction, on the inducing sets, of the density $\rho$ of the invariant measure $\mu$ for the map $T$. Now, the images of the rectangles of the Markov partitions are the (disjoint) sets $(a'_0, x_0)$ and $(x_0, a_0)$, when we induce over $I$, and the whole intervals $(a'_n, a'_{n-1})$ and $(a_n, a_{n+1})$, $n > 1$, when we induce over the rectangles in $I_n$. Therefore we could conclude that the density of the invariant measure $\mu$ is a

[2] Let us give the coding for the map $T$ with the grammar which we invoked above. To use a coherent notation we will redefine $a_{-0} \equiv a'_0$; $a_{-p} \equiv a'_p = T_1^{-p} a'_0$, $p \geq 1$. We associate with each point $x \in [0, 1]\backslash\chi$, where $\chi = \cup_{i \geq 0} T^{-i}\{x_0\}$, the unique coding $x = (\omega_0, \omega_1, \ldots, \omega_n, \ldots)$, $\omega_l \in \mathbb{Z}$, where (from now on $n$ will denote a positive integer larger than 1), $\omega_l = n$ iff $T^l x \in (a_{n-1}, a_n)$; $\omega_l = -n$ iff $T^l x \in (a_{-n}, a_{-(n-1)})$; $\omega_l = 0$ iff $T^l x \in I$. The grammar is the following (the formal symbol $-0$ must be intended as 0) :

$$\omega_i = n > 0 \Rightarrow \omega_{i+1} = -n; \quad \omega_i = -n \Rightarrow \omega_{i+1} = -(n-1),$$
$$\omega_i = 0 \Rightarrow \omega_{i+1} = n \geq 0 \text{ (any } n).$$

[3] It is argued in [AD] that if $\alpha$ is a Markov partition of the standard probability metric space $(X, \mathcal{B}, m, T)$ with distance $d$, then $T\alpha \subset \sigma(\alpha)$, where $\sigma(\alpha)$ denotes the sigma-algebra generated by the partition $\alpha$, and therefore it exists a (possibly countable) partition $\beta$ coarser than $\alpha$ such that $\sigma(T\alpha) = \sigma(\beta)$. Moreover, if the system is Gibbs-Markov, as in our case, then the space $Lip_{\infty,\beta}$ of functions $f : X \to \mathbb{R}$, $f \in L^\infty_m := L^\infty_m(X)$, which are Lipschitz continuous on each $Z \in \beta$, is a Banach space with the norm: $\|f\|_{Lip_{\infty,\beta}} = \|f\|_{L^\infty_m} + D_\beta f$, where $D_\beta f = \sup_{Z \in \beta} \sup_{x,y \in Z} \frac{|f(x)-f(y)|}{d(x,y)}$. The space $Lip_{\infty,\beta}$ is compactly injected into $L^1_m$, which gives the desired conclusions on the smoothness of the density as a consequence of the Lasota-Yorke inequality. Notice that in our case $m$ in just the Lebesgue measure. We denote by $B(I)$ the Banach space $Lip_{\infty,\beta}$ defined on $I$.

piecewise Lipschitz continuous functions with possible discontinuities at the points $a_p, a'_p,\ \ p > 1$, $a_0$, $a'_0$ and $x_0$.

We now improve this last result by showing that the density is Lipschitz continuous over the unit interval. We stress that this result will improve as well Proposition 13 in [CHMV].

**Proposition 3.** *The density* $\rho$ *of the invariant measure* $\mu$ *is Lipschitz continuous and bounded over the interval* [0, 1]. *Moreover,*

$$\lim_{x\to 0^+} \rho(x) = \lim_{x\to 1^-} \rho(x) = 0. \tag{63}$$

*Proof.* We work on the induced set $I$. The invariant measure $\mu_I$ for the induced map $T_I$ is related to the invariant measure $\mu$ over the whole interval thanks to the well-known formula due to Pianigiani:

$$\mu(B) = C_r \sum_i \sum_{j=0}^{\tau_i - 1} \mu_I(T^{-j}(B) \cap Z_i), \tag{64}$$

where $B$ is any Borel set in [0, 1] and the first sum runs over the cylinders $Z_i$ with prescribed first return time $\tau_i$ and whose union gives $I$. The normalizing constant $C_r = \mu(I)$ satisfies $1 = C_r \sum_i \tau_i \mu_I(Z_i)$.

This immediately implies that by calling $\hat{\rho}$ the density of $\mu_I$ we have that $\rho(x) = C_r\hat{\rho}(x)$ for $m$-almost every $x \in I$ and therefore $\rho$ can be extended to a Lipschitz continuous function on $I$ as $\hat{\rho}$. A straightforward application of formula (64) gives [CHMV]:

$$\mu(a_{n-1}, a_n) = C_r \mu_I(Z_{n+1}), \tag{65}$$

$$\mu(a'_n, a'_{n-1}) = C_r \sum_{p=n+2}^{\infty} \mu_I(Z_p). \tag{66}$$

Let us now take a measurable $B \subset (a'_n, a'_{n_1})$; the formula above immediately implies that

$$\mu(B) = C_r \sum_{p=n+2}^{\infty} \mu_I(T^{-(p-n)} B \cap Z_p). \tag{67}$$

Passing to the densities we have

$$\int_B \rho\,(x)\,dx = C_r \sum_{p=n+2}^{\infty} \int_{T^{-(p-n)} B \cap Z_p} \hat{\rho}\,(x)\,dx \tag{68}$$

We now perform a change of variables by observing that the set $B$ is pushed backward $p - n - 2$ times by means of $T_1^{-1}$, then once by means of $T_2^{-1}$ and finally it splits into two parts according to the actions of $T_1^{-1}$ and $T_2^{-1}$. Therefore,

$$\sum_{p=n+2}^{\infty} \int_{T^{-(p-n)} B \cap Z_p} \hat{\rho}\,(x)\,dx = \sum_{p=n+2}^{\infty} \sum_{l=1,2} \int_B \frac{\hat{\rho}(T_l^{-1} T_2^{-1} T_1^{-(p-n-2)} y)}{|DT^{p-n}(T_l^{-1} T_2^{-1} T_1^{-(p-n-2)} y)|} dy. \tag{69}$$

Since $B$ is any measurable set in $(a'_n, a'_{n-1})$, we have for $m$-almost every $x \in (a'_n, a'_{n-1})$,

$$\rho(x) = C_r \sum_{p=n+2}^{\infty} \sum_{l=1,2} \frac{\hat{\rho}(T_l^{-1} T_2^{-1} T_1^{-(p-n-2)} x)}{|DT^{p-n}(T_l^{-1} T_2^{-1} T_1^{-(p-n-2)} x)|}$$
$$= C_r \sum_{m=2}^{\infty} \sum_{l=1,2} \frac{\hat{\rho}(T_l^{-1} T_2^{-1} T_1^{-(m-2)} x)}{|DT^m(T_l^{-1} T_2^{-1} T_1^{-(m-2)} x)|}. \tag{70}$$

This formula does not depend on the choice of the interval $(a'_n, a'_{n-1})$ and therefore it holds for $x \in (0, a'_0)$. For the cylinders $(a_{n-1}, a_n)$ we get similarly that, for $m$-almost any $x \in (a_0, 1)$,

$$\rho(x) = C_r \sum_{l=1,2} \frac{\hat{\rho}(T_l^{-1} x)}{|DT(T_l^{-1} x)|}. \tag{71}$$

Since $\hat{\rho}$ is Lipschitz continuous inside $I$ and the inverse branches of $T$ are $C^{1+\iota}$, we conclude that $\rho$ can be chosen as Lipschitz continuous over the disjoint open intervals $(0, a'_0) \cup (a_0, 1)$. It is now useful to observe that the right hand sides of (70) and (71) give exactly the expression of the Perron-Frobenius operator associated to the first return map and whenever $x$ is chosen into $I$. By the existence of the left (resp. right) limit of $\hat{\rho}$ in $a'_0$ (resp. $a_0$) we immediately obtain the continuity of $\rho$ in such points. We use now this result to prove the continuity of the density in $x_0$. We recall that such a density is the fixed point of the Perron-Frobenius operator, so that it verifies the following equation, for any $x \in [0, 1]$:

$$\rho(x) = \frac{\rho(T_1^{-1}(x))}{|DT\left(T_1^{-1}(x)\right)|} + \frac{\rho(T_2^{-1}(x))}{|DT\left(T_2^{-1}(x)\right)|}, \tag{72}$$

which gives, for $x = x_0$,

$$\rho(x_0) = \frac{\rho(a'_0)}{|DT\left(a'_0\right)|} + \frac{\rho(a_0)}{|DT\left(a_0\right)|}, \tag{73}$$

and this proves immediately the continuity in $x_0$.

We now observe that assumptions (52)–(55) together with the facts that $T_2(a_p) = a'_{p-1}$, $T_1(a'_p) = a'_{p-1}$ and $T_1 b_p = T_2 b_p = a_{p-1}$, allow to get easily the following asymptotic behaviors (for $p$ large) for the preimages of $x_0$ (again $c$ will denote a constant independent of $p$ and that could change from a formula to another):

$$a'_p \sim \frac{c}{(\alpha')^p};\ (1 - a_p) \sim \frac{c}{(\alpha')^p}, \tag{74}$$
$$(x_0 - b'_p) \sim \frac{c}{(\alpha')^{\frac{p}{B'}}};\ (b_p - x_0) \sim \frac{c}{(\alpha')^{\frac{p}{B}}}. \tag{75}$$

These formulas immediately imply that for $x = b_p$ (resp. $x = b'_p$) in a neighborhood of $x_0$ and for $p$ large the derivative behaves like $|DT(x)| \sim c\left(\alpha'\right)^{p(\frac{1}{B}-1)}$ (resp. $|DT(x)| \sim c(\alpha')^{p(\frac{1}{B'}-1)}$). Since $\hat{\rho}$ is bounded away from zero and infinity on $I$, by the preceding scalings on the growth of the derivative near $x_0$ we have that $\rho(x) \approx x^{\frac{1}{B^*}-1}$ for $x$ close to 0 and 1, which means that $\rho$ can be extended by continuity to zero on the right side of 0 and on the left side of 1. □

The preceding proposition suggests the following scaling for the density.

**Proposition 4.**

$$\rho(x) = c' x^a + o(x^a),\ x \to 0^+;\ a > 0, \tag{76}$$

$$\rho(x) = c''(1-x)^b + o((1-x)^b);\ x \to 1^-,\ b > 0, \tag{77}$$

*with*

$$a = b = \frac{1}{B^*} - 1, \tag{78}$$

*and the constant $c'$ and $c''$ verifying*

$$\left(\frac{1}{\alpha'}\right)^{\frac{1}{B^*}} + \left(\frac{1}{\alpha(\frac{c'}{c''})^{B^*}}\right)^{\frac{1}{B^*}} = 1. \tag{79}$$

*Proof.* We use again formula (72). By using for $T$ and its two inverse branches the asymptotic polynomial behaviors in 0 and 1 given in (52)-(55), we get at the lowest order in $x$ in the neighborhood of 0,

$$c' \left(\alpha'\right)^{-a-1} x^a + c''\alpha^{-b-1} x^b = c'' x^a. \tag{80}$$

Now, suppose $a < b$. Then $\left(\alpha'\right)^{-a-1} \approx 1$, which implies either $\alpha' = 1$ or $a = -1$ and both cases are excluded. On the contrary, if $a > b$, then $\alpha^{-b-1} x^b \sim 0$, implying $\alpha = 0$, which is still impossible. Hence, we necessarily have $a = b$. We now take the point $x$ in the neighborhood of 1. By making explicit $T_1^{-1}(x)$ and $T_2^{-1}(x)$ with respect to $x$ in the neighborhood of $x_0$ and substituting into the Perron-Frobenius equation we get, at the lowest order in $1-x$,

$$\frac{O(1)}{(1-x)^{\frac{B-1}{B}}} + \frac{O(1)}{(1-x)^{\frac{B'-1}{B'}}} = (1-x)^b, \tag{81}$$

from which we obtain

$$(1-x)^{-\frac{B^*-1}{B^*}} \approx (1-x)^b. \tag{82}$$

We finally conclude that $a = b = \frac{1}{B^*} - 1$. Substituting this common value into Eq. (80) we finally get the expression relating the constants $c'$ and $c''$. □

The latter relation is a good check for the validity of the shape of the density in 0 and 1.

By the continuity of $\rho$, we could use the value of $a$ given above in terms of the map parameter $B^*$ to guess a functional expression for $\rho$. In agreement with the previous considerations, such an expression could be

$$\rho(x) = N(\gamma, \delta)\, e^{-\gamma x} x^\delta (1-x)^\delta, \tag{83}$$

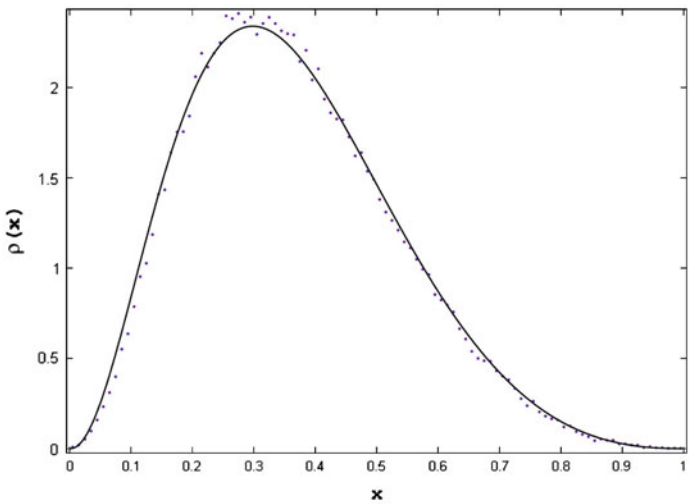



**Fig. 2.** Fit of the invariant density $\rho(x)$ for the map $T$ with the function given in (83)

where, if $I_\nu(z)$ is the modified Bessel function of the first kind,

$$N(\gamma,\delta)=\frac{\gamma^{\frac{1}{2}+\delta}e^{\frac{\gamma}{2}}}{\sqrt{\pi}\Gamma(1+\delta)I_{\frac{1}{2}+\delta}\left(\frac{\gamma}{2}\right)}, \tag{84}$$

with $\delta=a,\ c'=N(\gamma,\delta)$ and $c''=N(\gamma,\delta)e^{-\gamma}$.

Numerical computations performed on about $10^5$ values for Casimir maxima allowed us to estimate the parameters describing the local behavior of the map listed at the beginning of this section:

- $\alpha'\simeq 1.113,\ \alpha\simeq 0.4603$ ;
- $B'\simeq 0.3095,\ B\simeq 0.2856.$

Therefore, we get $B^*=B'$ and $\delta\simeq 2.2258$. The fit of the empirical stationary distribution function performed with such parameters comes out to be in good agreement with the functional expression for the invariant density (83) and the estimated value for $\gamma$ is $\gamma\simeq 4.26$.

An interesting question is to locate the maximum of the density $\rho$. Numerical investigations suggest that this maximum belongs to $\left[a'_0,x_0\right]$ (see Fig. 2) depending on the parameters which define the map $T$.

*3.1. Return times.* In Sect. II and in Sect. IV B of [PM], the periodic orbits of the system, due to its invariance under $R$, have been empirically classified by specifying that the initial condition belongs to the half space containing, say, the fixed point $c_1$ and the number of rotations they perform around the fixed point $c_2$ (cfr. [PM] Figs. 2 and 11b).

In particular, labeling as $\Sigma_+$ the portion of $\Sigma$ laying in the half space containing $c_1$, it can be shown by direct inspection that Fig. 11b in [PM], which represents the map of the set of maximum values of $C$ in itself associated to periodic trajectories starting from $\Sigma_+$ after $k$ rotations around $c_2$, is exactly the graph of the induced map of $T$ in the appropriate scale.

Therefore, the distribution of the number of times a trajectory of the system, starting from $\Sigma_+$, winds around $c_2$ before hitting again $\Sigma_+$ or equivalently, starting from $\Sigma_-$, winds around $c_1$ before hitting again $\Sigma_-$, is the same as that of the random variable $\tau_{(x_0,1)}(x)$, $x \in (x_0, 1)$, being the return time on $(x_0, 1)$ starting from $x$ under the dynamics induced by $T$. In terms of the already constructed invariant measure $\mu$, this probability is given by

$$\mu(\tau_{(x_0,1)}(x) \geq n\ ;\ x \in (x_0, 1)) = \sum_{l=n}^{\infty} \mu(\tau_{(x_0,1)}(x) = l\ ;\ x \in (x_0, 1)) \tag{85}$$

$$= \sum_{l=n}^{\infty} \mu(a_{n-2}, a_{n-1}).$$

But the sum on the r.h.s. can be computed using the corresponding expression evaluated in (6) and we finally get

$$\mu(\tau_{(x_0,1)}(x) \geq n; \quad x \in (x_0, 1)) \approx \left(\alpha'\right)^{-\frac{n}{B^*}}. \tag{86}$$

We remark that the distribution of $\tau_{(x_0,1)}(x)$, $x \in (x_0, 1)$, is the $\mu$ $a.s.$ limit of the empirical distribution of the points appearing in Fig. 2 of [PM].

We also take the occasion to remark that the average time between two crossings of $\Sigma$ corresponds to the gap between the filled bands of points appearing in Fig. 2 of [PM] which has been estimated to be about 0.66. Therefore, the period of the smallest periodic orbit of the Lorenz system is about $2 \cdot 0.66$ in complete agreement with what is predicted by the perturbation theory developed in [Lu] and the more rigorous estimate given in [GT].

*3.2. Statistical stability.* A slight change in the forcing term in the Lorenz equation will also change the shape of the associated map $T$ and therefore the invariant density associated to it, which will exists provided the perturbed map still satisfies (52–55). At the end of the section we will give two examples of such a perturbation of the forcing contribution to the Lorenz field, the first preserving the original symmetry of the Lorenz system, and the second breaking it. As already remarked in the Introduction, this last type of perturbation has been empirically shown in [CMP] to model the impact of anthropogenic forcing to climate dynamics of the northern hemisphere as well as the effect of the sea surface temperature on the Indian summer monsoon rainfall variability [KDC].

Let us denote by $T_\epsilon$ the perturbed map. We show in this section that under suitable assumptions the density $\rho_\epsilon$ of the perturbed measure will converge to the density $\rho$ of the unperturbed one in the $L^1_m$ norm. This kind of property is know as *statistical stability*. A former paper by Alves and Viana [AV], see also the successive paper by Alves [Al], addressed the question of the statistical stability for a wide class of non-uniformly expanding maps. Their result is based on two assumptions: (i) the perturbed map belongs to an open neighborhood of the unperturbed one in the $C^k$ topology with $k \geq 2$ and (ii) the two maps are compared throughout their first return maps defined on the *same* subset where the first return maps are uniformly expanding, with bounded distortion and long branches. Moreover, the structural parameters of the perturbed map (especially those bounding the derivative and the distortion) could be chosen uniformly in a $C^k$ neighborhood of the unperturbed map. The main result of those papers is that when the perturbed map converges to the unperturbed one in the $C^k$ topology then the density of the absolutely continuous invariant perturbed measure converges to the density of the unperturbed measure in the $L^1_m$ norm. Here we prove the same result but allowing the perturbed map to be close to the unperturbed one in the $C^0$ topology only. We will make use of induction but, in order to preserve the Markov structure of the first return map, we will compare the perturbed and the unperturbed first return maps on *different* induction subsets. The difficulty will therefore arise in the comparison of the Perron-Frobenius operators, which will now be defined on different functional spaces. The proof we give is inspired by the recent work [BV], but it contains the important improvement of changing the domains of inductions. Contrary to [AV] we are not able to establish the continuity of the map $T_\epsilon \mapsto \rho_\epsilon$ and this is surely due to the fact that we only require the maps $C^0$ close. On the other hand, discarding regularity allows us to cover a much wider class of examples; we believe in fact that our techniques could be used to prove the statistical stability for general classes of maps with some sort of criticalities and singularities.

*Assumptions on the perturbed map.*

*Assumption A* $T_\epsilon$ is a Markov map of the unit interval which is one-to-one and onto on the intervals $[0, x_{\epsilon,0})$ and $(x_{\epsilon,0}, 1]$, convex on both sides and of class $C^{1+\iota_\epsilon}$ on the open interval $(0, x_{\epsilon,0}) \cup (x_{\epsilon,0}, 1)$.

*Assumption B* Let $\|\cdot\|_0$ denote the $C^0$-norm on the unit interval, then

$$\lim_{\epsilon\to 0} \|T_\epsilon - T\|_0 = 0. \tag{87}$$

Moreover, $\forall x \in [0,1]$, $x \neq x_0$, we can find $\epsilon(x)$ such that, $\forall \epsilon < \epsilon(x)$, $DT_\epsilon$ exists and is finite and we have

$$\lim_{\epsilon\to 0} DT_\epsilon(x) = DT(x). \tag{88}$$

Furthermore,

$$\lim_{x\to x_0^+} \lim_{\epsilon\to 0} \frac{DT_\epsilon(x)}{DT(x)} = \lim_{x\to x_0^-} \lim_{\epsilon\to 0} \frac{DT_\epsilon(x)}{DT(x)} = 1. \tag{89}$$

*Assumption C* Let us denote by $C_{h,\epsilon}$ and $\iota_\epsilon$ respectively the Hölder constant and the Hölder exponent for the derivative of $T_\epsilon$ on the open interval $(0, x_{\epsilon,0}) \cap (x_{\epsilon,0}, 1)$; namely: $|DT_\epsilon(x) - DT_\epsilon(y)| \leq C_{h,\epsilon}|x-y|^{\iota_\epsilon}$ for any $x, y$ either in $(0, x_{\epsilon,0})$ or in $(x_{\epsilon,0}, 1)$. We assume $C_{h,\epsilon}$ and $\iota_\epsilon$ to converge to the corresponding quantities for $T$ in the limit $\epsilon \to 0$.

*Assumption D* Let us set $d_{(\epsilon,1,0)} := \inf_{(b_{\epsilon,1}, a_{\epsilon,0})} |DT_\epsilon(x)|$. We assume $d_{(\epsilon,1,0)} > 1$ and that there exists a constant $d_c$ and $\epsilon_c = \epsilon(d_c)$ such that, $\forall \epsilon < \epsilon_c$, $|d_{(1,0)} - d_{(\epsilon,1,0)}| < d_c$.

*Remark on the notation.* To simplify the notations we will set: $W'_n = (a'_n, a'_{n-1})$; $W_n = (a_n, a_{n+1})$ and we will denote by $W'_{\epsilon,n} = (a'_{\epsilon,n}, a'_{\epsilon,n-1})$ and $W_{\epsilon,n} = (a_{\epsilon,n}, a_{\epsilon,n+1})$ the corresponding perturbed intervals, where $a'_{\epsilon,n}$ and $a_{\epsilon,n}$, $n > 1$, are the preimages of the maximum point $x_{\epsilon,0}$. We also set

$$Z_{\epsilon,1} = Z^1_{\epsilon,1} \cup Z^2_{\epsilon,1}\ ;\ Z^1_{\epsilon,1} := (a'_{\epsilon,0}, b'_{\epsilon,1}),\ Z^2_{\epsilon,1} := (b_{\epsilon,1}, a_{\epsilon,0}) \tag{90}$$

and

$$Z_{\epsilon,n} = Z^1_{\epsilon,n} \cup Z^2_{\epsilon,n} \; ; \; Z^1_{\epsilon,n} := (b'_{\epsilon,n-1}, b'_{\epsilon,n}), \; Z^2_{\epsilon,n} := (b_{\epsilon,n}, b_{\epsilon,n-1}), \tag{91}$$

where $T_\epsilon b'_{\epsilon,n} = T_\epsilon b_{\epsilon,n} = a_{\epsilon,n-1}$. The same notation will be used for the corresponding unperturbed intervals. We denote by $I_\epsilon := (a'_{\epsilon,0}, a_{\epsilon,0}) \setminus \{x_{\epsilon,0}\}$ the interval where we will induce with the first perturbed return map. From now on, we will denote by $F$ the first return map of $T$ over $I$, by $F_\epsilon$ the first return map of $T_\epsilon$ on $I_\epsilon$ and by $P$ and $P_\epsilon$ the Perron-Frobenius operators associated respectively with $F$ and $F_\epsilon$. If $t \in T^{-n}z$, where $t = T^{-1}_{i_n} \circ T^{-1}_{i_{n-1}} \cdots \circ T^{-1}_{i_1} z$ with $i_k = 1$ or 2, we will call the sequence $i_1, \ldots, i_n$ the *signature* of $t$ relative to $z$.

*Remark 5.* The preceding assumptions imply that the order of tangency of $T_\epsilon$ in 0, $x_0$ and 1 tends, in the limit $\epsilon \to 0$, to that of $T$. This is the first requirement to get again Lemma 1 for the perturbed map. The other requirement is expressed by Assumption D which guarantees the condition (i) in Lemma 1. Notice that this condition cannot be deduced by Assumptions (A)–(C). On the other hand, the assumptions (ii) and (iii) of Lemma 1 are still valid for the perturbed map since we have only to control a finite number of relations among the corresponding derivatives. For instance, by using Assumptions B and C, we have

$$|DT(a'_l) - DT_\epsilon(a'_{\epsilon,l})| \le |DT_\epsilon(a'_l) - DT_\epsilon(a'_{\epsilon,l})| + |DT_\epsilon(a'_l) - DT(a'_l)|. \tag{92}$$

The first term on the right hand side of this inequality is controlled by the Hölder continuity of the derivative of $T_\epsilon$ while the second one is controlled by the local convergence to $DT$ of $DT_\epsilon$ in the limit $\epsilon \to 0$. However, we need some more information for the first return maps, which are summarized in the following lemma.

**Lemma 6.** *(i) For any $n \ge 0$, let $t_n$ and $t_{\epsilon,n}$ be two preimages of order $n$ of $x_0$ and $x_{\epsilon,0}$ respectively with the same signature with respect to these two points. Then, $\lim_{\epsilon\to 0} t_{\epsilon,n} = t_n$.*

*(ii) For any $n > 0$ we have:*

$$\lim_{\epsilon\to 0} \left\| T^n_\epsilon - T^n \right\|_0 = 0. \tag{93}$$

*(iii) For any $x \neq \cup_{k=0}^\infty T^{-k} x_0$, $n > 0$ there exists $\epsilon(x,n)$ such that, for any $\epsilon < \epsilon(x,n)$, the derivative $DT^n_\epsilon(x)$ exists and is finite and moreover*

$$\lim_{\epsilon\to 0} DT^n_\epsilon(x) = DT^n(x). \tag{94}$$

*(iv) For any $n \ge 1$, let $[u_n, v_n]$, $[u_{\epsilon,n}, v_{\epsilon,n}] \subset [0,1]$ be such that $u_{\epsilon,n} \to u_n$, $v_{\epsilon,n} \to v_n$ in the limit $\epsilon \to 0$ and $T^n_\epsilon \upharpoonright_{[u_{\epsilon,n}, v_{\epsilon,n}]}$, $T^n \upharpoonright_{[u_n,v_n]}$ are injective on the respective images. Then, setting for any $y \in T^n_\epsilon([u_{\epsilon,n}, v_{\epsilon,n}]) \cap T^n([u_n, v_n])$,*

$$T^{-(n)}_\epsilon := (T^n_\epsilon \upharpoonright_{[u_{\epsilon,n}, v_{\epsilon,n}]})^{-1}, \; T^{-(n)} := (T^n \upharpoonright_{[u_n,v_n]})^{-1}, \tag{95}$$

*$T^{-(n)}_\epsilon(y) \to T^{-(n)}(y)$ in the limit $\epsilon \to 0$.*

*Proof.* (i) We prove it for $n = 0$, for $n \ge 1$ the proof will follow by induction. Suppose $x_{\epsilon,0}$ does not converge to $x_0$, then passing to subsequences, by compactness, there exists a subsequence $\epsilon_n$ and a point $\tilde{x} \neq x_0$ such that $x_{\epsilon_n,0} \to \tilde{x}$ for $n \to \infty$. In such a point $T(\tilde{x}) < 1$ since $T$ has only one maximum located at $x_0$. Now, $|T_{\epsilon_n}(x_{\epsilon_n,0}) - T(\tilde{x})| = |1 - T(\tilde{x})| > 0$. We now fix $\sigma > 0$ and choose $n$ large enough, depending on $\sigma$, in such a way that for uniform convergence we get

$$\begin{aligned}\left|T_{\epsilon_n}(x_{\epsilon_n,0})-T(\tilde{x})\right| &= \left|T_{\epsilon_n}(x_{\epsilon_n,0})-T_{\epsilon_n}(\tilde{x})+T_{\epsilon_n}(\tilde{x})+T(x_{\epsilon_n,0})\right.\\ &\quad \left.-T(x_{\epsilon_n,0})+T(\tilde{x})\right|\\ &\le 2\left\|T_{\epsilon_n}-T\right\|_0+|T(x_{\epsilon_n,0})-T_{\epsilon_n}(\tilde{x})|\\ &\le 2\sigma+|T(x_{\epsilon_n,0})-T_{\epsilon_n}(\tilde{x})|. \end{aligned} \tag{96}$$

In the limit $n\to\infty$ the second term on the right-hand side of the previous inequality goes to zero by Assumption B and by the continuity of $T$. We finally send $\sigma$ to zero getting a contradiction with the above strictly positive lower bound.

(ii) The proof is standard and by induction and it uses the uniform continuity of $T^n$ on the closed unit interval.

(iii) We use induction again. Suppose the limit holds for $n$. Then we write

$$\begin{aligned}|DT_\epsilon^{n+1}(x)-DT^{n+1}(x)| &= \left|DT_\epsilon(T_\epsilon^n(x))DT_\epsilon^n(x)-DT(T^n(x))DT^n(x)\right.\\ &\quad \left.+DT_\epsilon(T^n(x))DT_\epsilon^n(x)-DT_\epsilon(T^n(x))DT_\epsilon^n(x)\right|. \end{aligned} \tag{97}$$

Now, we know that: (a) $x\ne\cup_{k=0}^{n-1}T^{-k}x_0$ by assumption, and also (b) $x\ne\cup_{k=0}^{n-1}T_\epsilon^{-k}x_{\epsilon,0}$, since by the induction assumption the derivative $DT_\epsilon^n(x)$ is well defined at $x\ne\cup_{k=0}^{\infty}T^{-k}x_0$. We need to take $\epsilon$ even smaller, than a certain $\epsilon(x,n)$, to guarantee that $|DT_\epsilon^{n+1}(x)|$ is well defined too. This is easily achieved since the preimages of $x_0$ and $x_{\epsilon,0}$ converge to each other according to signature and by choosing $\epsilon$ small enough depending on $x$ and $n$ we could just get (a) and (b) at the same time and for any fixed $n$. We can now bound the previous expression by:

$$\begin{aligned}&|DT_\epsilon^n(x)||DT_\epsilon(T_\epsilon^n(x))-DT_\epsilon(T^n(x))|\\ &+|DT_\epsilon(T^n(x))DT_\epsilon^n(x)-DT(T^n(x))DT^n(x)|. \end{aligned} \tag{98}$$

The second term converges to zero by the induction assumption. The first term can be bounded making use of the Hölder continuity assumption on the derivative, namely

$$|DT_\epsilon(T_\epsilon^n(x))-DT_\epsilon(T^n(x))|\le C_{h,\epsilon}|T_\epsilon^n(x))-T^n(x)|^{\iota_\epsilon}, \tag{99}$$

and of Assumption C assuring $C_{h,\epsilon}$ and $\iota_\epsilon$ to converge to the corresponding quantities given for $T$.

(iv) Let us set $y_n:=T^{-(n)}(y)\in[u_n,v_n]$ and $y_{\epsilon,n}:=T_\epsilon^{-(n)}(y)\in[u_{\epsilon,n},v_{\epsilon,n}]$. Suppose $y_{\epsilon,n}$ does not converge to $y_n$. Then, by passing again to subsequences and by compactness, we can find $\tilde{y}\ne y_n$ such that $\lim_{k\to\infty}y_{\epsilon_k,n}=\tilde{y}$. But $y=T_{\epsilon_k}^n(y_{\epsilon_k,n})=T_{\epsilon_k}^n(y_{\epsilon_k,n})-T_{\epsilon_k}^n(\tilde{y})+T_{\epsilon_k}^n(\tilde{y})$. For $k$ going to infinity the last term tends to a value different from $y$ since $T$ is injective over $[u_n,v_n]$, while the first difference goes to zero by (ii) above. □

It is clear that with the previous assumptions the map $T_\epsilon$ will admit a unique absolutely continuous invariant measure with density $\rho_\epsilon$. This density will be related to the invariant density $\hat{\rho}_\epsilon$ of the first return map $F_\epsilon$ on $I_\epsilon$ by the formula (70), with normalizing constant $C_{\epsilon,r}$. Our next result will be to prove the statistical stability of the unperturbed density, namely

**Proposition 7.**

$$\lim_{\epsilon \to 0^+} \|\rho - \rho_\epsilon\|_{L^1_m} = 0. \tag{100}$$

*Proof.* The proof is divided into two parts. The second part, which concerns the comparison of the invariant densities outside the regions of induction, will follow closely the proof of an analogous result given in [BV], but in our case the proof will be easier since the quantities we are going to consider have an exponential tail contrarily to the corresponding ones analysed in [BV] where the presence of a neutral fixed point forced those quantities to decay polynomially fast. The first part concerns the comparison of the invariant densities inside the regions of induction and this part is new.

*First part.* Let us suppose without restriction that the induction sets $I = (a_0', a_0)\backslash\{x_0\}$, $I_\epsilon = (a_{\epsilon,0}', a_{\epsilon,0})\backslash\{x_{\epsilon,0}\}$ verify $a_{\epsilon,0}' < a_0'$, $a_{\epsilon,0} < a_0$. In the following, to ease the notation we will simply write $dx$ instead of $dm(x)$ for the (normalized) Lebesgue measure on $[0, 1]$ and, for any interval $J \subset [0, 1]$, we will set $|J| := m(J)$. We begin by bounding

$$\int_{I \cap I_\epsilon} |\hat{\rho}(x) - \hat{\rho}_\epsilon(x)|dx. \tag{101}$$

In footnote 3 we defined the Banach spaces $B(I)$ and $B(I_\epsilon)$, which are invariant respectively under the action of the Perron-Frobenius operators $P$ and $P_\epsilon$. The densities $\hat{\rho}$ and $\hat{\rho}_\epsilon$ belong respectively to these spaces and they are Lipschitz continuous on the open intervals $(a_0', x_0) \cup (x_0, a_0)$ and $(a_{\epsilon,0}', x_{\epsilon,0}) \cup (x_{\epsilon,0}, a_{\epsilon,0})$. In fact we have to consider the action of the Perron-Frobenius operators on a larger functional space, namely that of functions of bounded variation. It is a standard result that the Perron-Frobenius operator associated to Gibbs-Markov maps with bounded distortion leaves invariant this space and moreover it satisfies a Lasota-Yorke inequality for the complete norm given by the sum of the total variation and the $L^1_m$ norm, see for instance [B] for an account of these results. We denote by $BV(I)$ and $BV(I_\epsilon)$ the Banach spaces of functions of bounded variations defined respectively on the induction sets $I$ and $I_\epsilon$, and by $\|\cdot\|_{BV(I)}$, $\|\cdot\|_{BV(I_\epsilon)}$ the respective norms. We remark that the Lebesgue measure associated to these norms should be understood as normalized to the sets $I$ and $I_\epsilon$. Since the Perron-Frobenius operators $P$ and $P_\epsilon$ are quasi-compact on respectively $BV(I)$ and $BV(I_\epsilon)$, we know that, in the limit $n \to \infty$,

$$\left\|P^n \mathbf{1}_I - \hat{\rho}\right\|_{BV(I)} \to 0, \tag{102}$$

$$\left\|P_\epsilon^n \mathbf{1}_{I_\epsilon} - \hat{\rho}_\epsilon\right\|_{BV(I_\epsilon)} \to 0. \tag{103}$$

It will be important in what follows that the convergence of the two previous limits are uniform with respect to $\epsilon$ in the $L^\infty_m$, and therefore in the $L^1_m$, norms. This is guaranteed by the results in [LSV], in particular Lemmas 4.8 and 4.11. As a matter of fact, our first return Gibbs-Markov maps fit the assumptions of the *covering systems* with countably many branches investigated in [LSV]. In particular, it can be proven that there exist two constants $C$ and $\Lambda$ such that $\left\|P^n \mathbf{1}_I - \hat{\rho}\right\|_\infty \le C\Lambda^n$, where the constant $C$ and the rate $\Lambda$ have an explicit and $C^\infty$ dependence on some parameters characterizing the map and its expanding properties.[4] Therefore, given $\eta > 0$ we can choose $n$ large enough, depending on $\eta$, and such that

$$\int_{I\cap I_\epsilon} |\hat{\rho} - \hat{\rho}_\epsilon| dx = \int_{I\cap I_\epsilon} \left|\hat{\rho} - P^n \mathbf{1}_I + P^n \mathbf{1}_I + P_\epsilon^n \mathbf{1}_{I_\epsilon} - P_\epsilon^n \mathbf{1}_{I_\epsilon} - \hat{\rho}_\epsilon\right| dx$$
$$\leq 2\eta + \int_{I\cap I_\epsilon} \left|P^n \mathbf{1}_I - P_\epsilon^n \mathbf{1}_{I_\epsilon}\right| dx. \tag{104}$$

Let us introduce, for $n \geq 2$,

$$\hat{\rho}_n := P^{n-1}\mathbf{1}_I \ ; \ \hat{\rho}_{\epsilon,n} := P_\epsilon^{n-1}\mathbf{1}_{I_\epsilon}, \tag{105}$$

and finally $\tilde{\rho}_n := \hat{\rho}_n$ on $I \cap I_\epsilon$ and $\tilde{\rho}_n := a_n$, on $I_\epsilon \backslash (I \cap I_\epsilon)$, where $a_n = \lim_{x\to a_0'^+} \hat{\rho}_n(x)$. Notice that this right limit exists since $\hat{\rho}_n$ is Lipschitz continuous on $(a_0', x_0)$ and moreover $\tilde{\rho}_n \in BV(I_\epsilon)$ as proven in Sect. 2. We remark that the need of considering $BV\,(I_\epsilon)$ follows by the fact that $\tilde{\rho}_n$ could be discontinuous in $x_{\epsilon,0}$. Let us rewrite the second term in (104) as

$$\int_{I\cap I_\epsilon} \left|P^n \mathbf{1}_I - P_\epsilon^n \mathbf{1}_{I_\epsilon}\right| dx = \int_{I\cap I_\epsilon} |P\hat{\rho}_n - P_\epsilon \hat{\rho}_{\epsilon,n}| dx$$
$$\leq \int_{I\cap I_\epsilon} \left|P\hat{\rho}_n - P_\epsilon \tilde{\rho}_n\right| dx + \int_{I\cap I_\epsilon} |P_\epsilon\tilde{\rho}_n - P_\epsilon \hat{\rho}_{\epsilon,n}| dx. \tag{106}$$

We now consider the term $\int_{I\cap I_\epsilon} |P_\epsilon\tilde{\rho}_n - P_\epsilon \hat{\rho}_{\epsilon,n}| dx$; by the positivity and the contraction in $L^1_m$ of the Perron-Frobenius operator, we have

$$\int_{I\cap I_\epsilon} |P_\epsilon\tilde{\rho}_n - P_\epsilon \hat{\rho}_{\epsilon,n}| dx \leq \int_{I_\epsilon} |P_\epsilon\tilde{\rho}_n - P_\epsilon \hat{\rho}_{\epsilon,n}| dx \leq \int_{I_\epsilon} |\tilde{\rho}_n - \hat{\rho}_{\epsilon,n}| dx$$
$$\leq \int_{I_\epsilon\cap I} |\hat{\rho}_n - \hat{\rho}_{\epsilon,n}| dx + \int_{I_\epsilon\backslash(I_\epsilon\cap I)} |a_n - \hat{\rho}_{\epsilon,n}| dx$$
$$= \int_{I_\epsilon\cap I} |P\hat{\rho}_{n-1} - P_\epsilon \hat{\rho}_{\epsilon,n-1}| dx + \int_{I_\epsilon\backslash(I_\epsilon\cap I)} |a_n - \hat{\rho}_{\epsilon,n}| dx$$
$$\leq \int_{I_\epsilon\cap I} |P\hat{\rho}_{n-1} - P_\epsilon \hat{\rho}_{\epsilon,n-1}| dx + m(I_\epsilon\backslash(I_\epsilon \cap I))(||\hat{\rho}_n||_\infty + ||\hat{\rho}_{\epsilon,n}||_\infty), \tag{107}$$

where the $L^\infty_m$-norm should be understood in terms of the normalized Lebesgue measures respectively on $I$ and $I_\epsilon$. But each of these norms is

[4] These constants can be explicitly computed using the Hilbert metric approach. In particular $C = (1 + a)D_H e^{D_H \Lambda^{-2N_0}} \Lambda^{-2N_0}$, and $\Lambda = \left(\tanh \frac{D_H}{4}\right)^{\frac{1}{N_0}}$. The integer $N_0$ insures that the hyperbolic diameter of the iterate $P^{N_0}$ of a certain cone of bounded variation functions is finite and bounded by $D_H$. In particular, $a$, $\Delta$ and $D_H$ are smooth functions of the quantities $\nu$ and $D$ entering the Lasota-Yorke inequality (see the next footnote).

bounded by the Banach norm and in particular for $\hat{\rho}_n$ we have, by the Lasota-Yorke inequality,

$$\left\|\hat{\rho}_n\right\|_\infty \leq \left\|\hat{\rho}_n\right\|_{BV(I)} \leq \left\|P^{n-1}\mathbf{1}_I\right\|_{BV(I)} \leq \nu^{n-1}\left\|\mathbf{1}_I\right\|_{BV(I)} + D. \tag{108}$$

This last quantity, for all $n$ large enough, is less than a constant $C_2$ and the same argument also applies to $\left\|\hat{\rho}_{\epsilon,n}\right\|_\infty$. Moreover, setting $C_2$ and $C_{\epsilon,2}$ the constants bounding (108) in the unperturbed and perturbed case, for $\epsilon$ sufficiently small, we have that the difference $|C_2 - C_{\epsilon,2}|$ is bounded by a constant independent of $\epsilon$.[5] By setting

$$G_l := \int_{I_\epsilon \cap I} |P\hat{\rho}_l - P_\epsilon \tilde{\rho}_l| dx \tag{109}$$

with $l = 1, \ldots, n$ and $\hat{\rho}_1 := \mathbf{1}_I,\ \hat{\rho}_\epsilon, 1 := \mathbf{1}_{I\epsilon}$ we have

$$\int_{I\cap I_\epsilon} |P\hat{\rho}_n - P_\epsilon \hat{\rho}_{\epsilon,n}| dx = \sum_{l=1}^{n} G_l + (n-1)C_2 m(I_\epsilon \backslash (I_\epsilon \cap I)), \tag{110}$$

where $m(I_\epsilon \backslash (I_\epsilon \cap I)) = O(\epsilon)$.
In order to compute the term $G_l$ we have to use the explicit structure of the Perron-Frobenius operator. In particular we have

$$\begin{aligned}\int_{I_\epsilon\cap I} |P\hat{\rho}_n - P_\epsilon\tilde{\rho}_n| dx &= \int_{I_\epsilon\cap I} \left| \sum_{i\geq 1} \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF(F_i^{-1}x)|} - \sum_{i\geq 1} \frac{\tilde{\rho}_n(F_{\epsilon,i}^{-1}x)}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|} \right| dx \\ &\leq \int_{I_\epsilon\cap I} \sum_{i\geq 1} \left| \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF(F_i^{-1}x)|} - \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|} + \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|} - \frac{\tilde{\rho}_n(F_{\epsilon,i}^{-1}x)}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|} \right| dx.\end{aligned} \tag{111}$$

Actually what we want to do is to compare the preimages of the perturbed and of the unperturbed first return maps whose direct images are defined on cylinders with the *same* return times. This can always be done and in particular we will consider points $x$ whose perturbed and unperturbed preimages are both defined. In this regard, it will be enough to erase from

[5] The constants $\nu < 1$ and $D$ are in fact explicitly determined in terms of the map; we defer to [B] for the details. To compare with what is stated in [B], we need to show that there exists a power $n_0$ of the first return map $F$ having the absolute value of its derivative uniformly larger than 2. In the case of interest, this follows easily from the proof of Lemma 1 by combining the Markov structure of $F$ with the lower bound for the absolute value of its derivative which is uniformly larger than 1 and which is an explicit function of the parameters describing the local behavior of the map $T$, in particular $\alpha', \alpha$ and $d_{(1,0)}$. The quantities $\nu$ and $D$ are then functions of the lower bound of $\left|DF^{n_0}\right|$ and of the constant, which we denote by $D'$, appearing in Adler's condition. This last condition is equivalent to prove that $T$ has bounded distortion and we defer to [CHMV] where the constant bounding the distortion is explicitly determined as a function of the parameters defining the map. In the present case, a simple inspection of the proof in [CHMV] shows that such a constant is a multiple of $d_{(0,1)}$. Hence, a contribution to $D'$ comes from $d_{(0,1)}$, while the other one [CHMV] comes from the divergent behavior of the second derivative close to the fixed point. However, in our case, the Hölder continuity assumption on the first derivative of the map and the exponential decay of the length of $Z_i$ makes this second contribution simply bounded by 1.

$I_\epsilon \cap I$ the open interval with endpoints $x_{\epsilon,0}, x_0$ whose measure goes to zero in the limit $\epsilon \to 0$. We will prove that the sum in (111) is bounded uniformly in $\epsilon$ in order to exchange the sum with the limit $\epsilon \to 0$. We recall that the perturbed and unperturbed induced first return maps are Gibbs-Markov and have bounded distortion and that $\|\hat{\rho}_n\|_\infty < C_2$. Therefore, on each interval $Z_i^j$ (resp. $Z_{\epsilon,i}^j$) $i \geq 1$, $j = 1, 2$, where $F$ (resp. $F_{\epsilon,i}$) is injective we have:

- For any $i \geq 1$; $j = 1, 2$ and $\forall x, y \in F(Z_i^j)$, we have $\frac{|DF(F_i^{-1}x)|}{|DF(F_i^{-1}y)|} \leq D_1$, and $\forall x, y \in F(Z_{\epsilon,i}^j)$ we have $\frac{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|}{|DF_\epsilon(F_{\epsilon,i}^{-1}y)|} \leq D_2$.
- For $\epsilon$ small enough, by the argument developed in footnote 4, the difference $|D_1 - D_2|$ is bounded by a constant independent of $\epsilon$.
- There exists $y \in Z_i^j$ (resp. $Z_{\epsilon,i}^j$) such that $|DF(y)| = \frac{|F(Z_i^j)|}{|Z_i^j|}$ (resp. $|DF_\epsilon(y)| = \frac{|F_\epsilon(Z_{\epsilon,i}^j)|}{|Z_{\epsilon,i}^j|}$).

This immediately implies that the first term in (111) is bounded by

$$\int_{I_\epsilon \cap I} \sum_{i\geq 1} \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF(F_i^{-1}x)|} dx \leq C_2 D_1 \sum_{i\geq 1} \frac{|Z_i|}{|F(Z_i)|}. \tag{112}$$

Similar bounds hold also for the other three terms in (111). We recall that the images of the $Z_i$ have length $(x_0 - a_0')$ and the sum over the $|Z_i|$'s gives the length of $I$. We can therefore take the limit $\epsilon \to 0$ in (111). Let us consider the first two terms in (111),

$$\sum_{i\geq 1} \left| \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF(F_i^{-1}x)|} - \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|} \right|$$
$$= \sum_{i\geq 1} \left| \frac{\hat{\rho}_n(F_i^{-1}x)}{|DF(F_i^{-1}x)|} \right| \left| 1 - \frac{|DF(F_i^{-1}x)|}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|} \right|. \tag{113}$$

We can bound this quantity making use of Lemma (6) part (iii) and part (iv) first and then by observing that the point $F_i^{-1}x$ does not coincide with $x_0$. Let us set $w := F_i^{-1}x$; $w_\epsilon := F_{\epsilon,i}^{-1}x$ and $F = T^i$. Then,

$$\left| \frac{DF(F_i^{-1}x)}{DF_\epsilon(F_{\epsilon,i}^{-1}x)} \right| = \prod_{m=0}^{i-1} \left| \frac{DT(T^m w) DT_\epsilon(T^m w)}{DT_\epsilon(T_\epsilon^m w_\epsilon) DT_\epsilon(T^m w)} \right|. \tag{114}$$

We notice that $|T^m w - T^m w_\epsilon|$ goes to zero by the continuity of $T^m$. The intervals with endpoints $T^i w$ and $T_\epsilon^i w_\epsilon$ do not contain $x_{\epsilon,0}$ and their length tends to zero when $\epsilon$ vanishes. Therefore,

$$\prod_{m=0}^{i-1} \left| \frac{DT(T^m w)}{DT_\epsilon(T^m w)} \right| \exp \left[ \sum_{m=0}^{i-1} \frac{1}{|DT_\epsilon(y)|} C_{h,\epsilon} (\|T^m - T_\epsilon^m\|_0^{\iota_\epsilon} + |T^m w - T^m w_\epsilon|) \right], \tag{115}$$

where $y$ is a point between $T^i w$ and $T^i_\epsilon w_\epsilon$. Hence, by Assumption B, this term tends to 1 in the limit $\epsilon \to 0$.[6]
Moreover, the other couple of terms in (111),

$$\sum_{i\geq 1}\left|\frac{\hat{\rho}_n(F_i^{-1}x)}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|}-\frac{\tilde{\rho}_n(F_{\epsilon,i}^{-1}x)}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|}\right| \leq \sum_{i\geq 1}\frac{1}{|DF_\epsilon(F_{\epsilon,i}^{-1}x)|}|\hat{\rho}_n(F_i^{-1}x)-\tilde{\rho}_n(F_{\epsilon,i}^{-1}x)|. \quad (116)$$

We remark that the function $\tilde{\rho}_n$ is a continuous extension of $\hat{\rho}$ to $I_\epsilon\backslash(I\cap I_\epsilon)$, and therefore we can rewrite

$$|\hat{\rho}_n(F_i^{-1}x)-\tilde{\rho}_n(F_{\epsilon,i}^{-1}x)| = |\tilde{\rho}_n(F_i^{-1}x)-\tilde{\rho}_n(F_{\epsilon,i}^{-1}x)|, \quad (117)$$

where $\tilde{\rho}_n$ is now defined on $I \cup I_\epsilon$. This function is continuous on $I \cup I_\epsilon\backslash\{x_0\}$ and, by part (iv) of Lemma (6), $\lim_{\epsilon\to 0^+}|\tilde{\rho}_n(F_i^{-1}x) - \tilde{\rho}_n(F_{\epsilon,i}^{-1}x)| = 0$.
To resume: for $n$ larger than a certain $n(\eta)$,

$$(104) \leq 2\eta + \sum_{l=1}^{n} G_l + (n-1)O(\epsilon)\ ; \quad (118)$$

each $G_l$ is bounded uniformly w.r.t. $\epsilon$ and tends to zero for $\epsilon$ tending to zero. Therefore we can pass to the limit $\epsilon \to 0$ and then $\eta \to 0$.

*Second part.* According to the assumptions made at the beginning of the first part and without loss of generality, we will assume that all the $W_{\epsilon,n}$'s lie to the left of the corresponding $W_n$. Therefore we have

$$\begin{aligned}\int_{[0,1]}|\rho-\rho_\epsilon|dx &= \int_{I\cap I_\epsilon}|\rho-\rho_\epsilon|dx + \int_{I\cap W_{\epsilon,1}}|\rho-\rho_\epsilon|dx \\ &+ \int_{I_\epsilon\cap W_1'}|\rho-\rho_\epsilon|dx \\ &+ \sum_{l=1}^{\infty}\left\{\int_{W_l\cap W_{\epsilon,l}}|\rho-\rho_\epsilon|dx + \int_{W_l\backslash(W_l\cap W_{\epsilon,l})}|\rho-\rho_\epsilon|dx\right\} \\ &+ \sum_{l=1}^{\infty}\left\{\int_{W_l'\cap W_{\epsilon,l}'}|\rho-\rho_\epsilon|dx + \int_{W_l'\backslash(W_l'\cap W_{\epsilon,l}')}|\rho-\rho_\epsilon|dx\right\}. \quad (119)\end{aligned}$$

The densities are given in terms of the corresponding densities of the induced subsets and of the multiplicative constants $C_r$ and $C_{\epsilon,r}$. Hence,

6 Actually $y$ depends on $\epsilon$, $y = y_\epsilon$. Setting $y^* := \lim_{\epsilon\to 0} y_\epsilon$, we get

$$|DT_\epsilon(y_\epsilon) - DT(y^*)| \leq |DT_\epsilon(y_\epsilon) - DT_\epsilon(y^*)| + |DT_\epsilon(y^*) - DT(y^*)|.$$

The first term on the r.h.s. can be bounded making use of the Hölder continuity assumption on $DT_\epsilon$, the second making use of Assumption B.

we should first compare the latter. Since they are surely smaller than 1, we have

$$|C_r - C_{\epsilon,r}| \leq \sum_{i=1}^{\infty} i \left| \int_{Z_i^1} \hat{\rho} \frac{dx}{m(I)} - \int_{Z_{\epsilon,i}^1} \hat{\rho}_\epsilon \frac{dx}{m(I_\epsilon)} \right|. \tag{120}$$

The same bound holds also choosing $Z_i^2$ ($Z_{\epsilon,i}^2$) instead of $Z_i^1$ ($Z_{\epsilon,i}^1$). The sum converges uniformly as a function of $\epsilon$ since the $L_m^\infty$ norms of $\hat{\rho}$ and $\hat{\rho}_\epsilon$ are bounded by $C_2$ and the lengths of the $Z_i^1$ and $Z_{\epsilon,i}^1$ decay exponentially fast. We now show that passing to the limit $\epsilon \to 0$ inside the sum this vanishes. In this regard we rewrite the previous bound as

$$\begin{aligned} &\sum_{i=1}^{\infty} i \left| \int_{Z_i^1 \cap Z_{\epsilon,i}^1} \hat{\rho} \frac{dx}{m(I)} + \int_{Z_i^1 \setminus (Z_i^1 \cap Z_{\epsilon,i}^1)} \hat{\rho} \frac{dx}{m(I)} \right. \\ &\quad \left. - \int_{Z_{\epsilon,i}^1 \cap Z_i^1} \hat{\rho}_\epsilon \frac{dx}{m(I_\epsilon)} - \int_{Z_{\epsilon,i}^1 \setminus (Z_i^1 \cap Z_{\epsilon,i}^1)} \hat{\rho}_\epsilon \frac{dx}{m(I_\epsilon)} \right| \\ &\leq \sum_{i=1}^{\infty} i \left[ 2C_2 m(Z_i^1 \Delta Z_{\epsilon,i}^1) + C_2 \left| \frac{1}{m(I)} - \frac{1}{m(I_\epsilon)} \right| \right. \\ &\quad \left. + \int_{Z_i^1 \cap Z_{\epsilon,i}^1} |\hat{\rho} - \hat{\rho}_\epsilon| \frac{dx}{m(I_\epsilon)} \right]. \end{aligned} \tag{121}$$

Each term in the last sum vanishes in the limit $\varepsilon \to 0$; in particular the third term tends to zero by what is stated in the first part of the proof. Moreover, by Lemma 1 and by the fact that the derivatives of the maps $T$ and $T_\epsilon$ are strictly expanding in the neighborhood of $x_0$, for $x \in (0, a_0')$, we get

$$\sum_{m=2}^{\infty} \sum_{l=1,2} \frac{1}{|DT^m(T_l^{-1} T_2^{-1} T_1^{-(m-2)} x)|} \leq C_3 \frac{1}{(\alpha\alpha' \log \alpha')} := C_4. \tag{122}$$

Furthermore, for $x \in (0, a_0)$,

$$\sum_{l=1,2} \frac{1}{|DT(T_l^{-1} x)|} \leq \left( \min_{(b_2,b_1) \cup (b_1',b_2')} |DT| \right)^{-1} := C_5. \tag{123}$$

Analogous bounds hold also for the perturbed map, so we can choose the constants $C_4, C_5$ independent of $\epsilon$. Let us call $\rho_s$ (resp. $\rho_r$), the representations of the invariant density on $(0, x_0)$ (resp. $(x_0, 1)$) without the normalizing factor $C_r$. By the previous bounds on the derivatives of $T$ and the boundness of the densities on the induced spaces, it follows immediately that there exists a constant $C_6$ such that the $L_m^\infty$ norms of $\rho_s$ and $\rho_r$ are bounded by $C_6$. The same argument also holds for $\rho_{\epsilon,s}$ and $\rho_{\epsilon,r}$ and, since $C_6$ can be chosen independent of $\epsilon$, $\|\rho_{\epsilon,s}\|_\infty, \|\rho_{\epsilon,r}\|_\infty \leq C_6$.

We can now proceed to bound each term in (119). For the first one we get

$$\int_{I\cap I_\epsilon}|\rho-\rho_\epsilon|dx \le |C_r-C_{\epsilon,r}|\int_{I\cap I_\epsilon}\hat\rho dx + C_{\epsilon,r}\int_{I\cap I_\epsilon}|\hat\rho-\hat\rho_\epsilon|dx, \tag{124}$$

which can be bounded uniformly in $\epsilon$ by arguing as in the previous computations. For the second term (the third one can be bounded in the same way) we have

$$\int_{I\cap W_{\epsilon,1}}|\rho-\rho_\epsilon|dx \le |C_r-C_{\epsilon,r}|\int_{I\cap W_{\epsilon,1}}\hat\rho dx + C_{\epsilon,r}\int_{I\cap W_{\epsilon,1}}|\hat\rho-\rho_{\epsilon,s}|dx. \tag{125}$$

The right-hand side is uniformly bounded in $\epsilon$, in particular

$$C_{\epsilon,r}\int_{I\cap W_{\epsilon,1}}|\hat\rho-\rho_{\epsilon,s}|dx \le (C_2+C_6)m(I\cap W_{\epsilon,1}), \tag{126}$$

where vanishes $m(I\cap W_{\epsilon,1})$ in the limit $\epsilon\to 0$.

We now consider the last sum in (119). Similar arguments allow to bound the remaining sum which is even easier to handle. We first have

$$\sum_{l=1}^{\infty}\int_{W'_l\backslash(W'_l\cap W'_{\epsilon,l})}|\rho-\rho_\epsilon|dx \le \sum_{l=1}^{\infty}\left[|C_r-C_{\epsilon,r}|\int_{W'_l\backslash(W'_l\cap W'_{\epsilon,l})}\rho_s dx + C_{\epsilon,r}\int_{W'_l\backslash(W'_l\cap W'_{\epsilon,l})}|\rho_s-\rho_{\epsilon,s}|dx\right]. \tag{127}$$

The sum is uniformly convergent as a function of $\epsilon$ since $W'_l\backslash(W'_l\cap W'_{\epsilon,l})\subset W'_l$ and the length of such an interval decays exponentially fast with rate independent of $\epsilon$. Finally, previous considerations imply that each term into the sum goes to zero in the limit $\epsilon\to 0$.

Finally we have

$$\sum_{l=1}^{\infty}\int_{W'_l\cap W'_{\epsilon,l}}|\rho-\rho_\epsilon|dx \le \sum_{l=1}^{\infty}\left[|C_r-C_{\epsilon,r}|\int_{W'_l\cap W'_{\epsilon,l}}\rho_s dx + C_{\epsilon,r}\int_{W'_l\cap W'_{\epsilon,l}}|\rho_s-\rho_{\epsilon,s}|dx\right]. \tag{128}$$

The preceding considerations also apply to the first sum in this formula proving this to vanish in the limit $\epsilon\to 0$. For the second sum we make use of the representations of $\rho_s$ and $\rho_{\epsilon,s}$ in terms of the density on the induced space. Thus we have

$$\begin{aligned}
&\int_{W'_l\cap W'_{\epsilon,l}} |\rho_s - \rho_{\epsilon,s}|dx \\
&\leq \int_{W'_l\cap W'_{\epsilon,l}} \sum_{p=l+2}^{\infty}\sum_{k=1,2}\left|\frac{\hat\rho(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x)}{|DT^{p-l}(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x)|} - \frac{\hat\rho_\epsilon(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x)}{|DT_\epsilon^{p-l}(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x)|}\right|dx \\
&\leq \int_{W'_l\cap W'_{\epsilon,l}} \sum_{p=l+2}^{\infty}\sum_{k=1,2}\left|\frac{\hat\rho(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x)}{|DT^{p-l}(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x)|} - \frac{\hat\rho_\epsilon(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x)}{|DT^{p-l}(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x)|}\right|dx \\
&+ \int_{W'_l\cap W'_{\epsilon,l}} \sum_{p=l+2}^{\infty}\sum_{k=1,2}\left|\frac{\hat\rho_\epsilon(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x)}{|DT^{p-l}(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x)|} - \frac{\hat\rho_\epsilon(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x)}{|DT_\epsilon^{p-l}(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x)|}\right|dx \\
&= Q_{1,l} + Q_{2,l}.
\end{aligned} \tag{129}$$

We further decompose $Q_{1,l}$ as

$$\begin{aligned}
Q_{1,l} = \int_{W'_l\cap W'_{\epsilon,l}} \sum_{p=l+2}^{\infty}\sum_{k=1,2} &\frac{1}{|DT^{p-l}(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x)|} \\
&\times \Big|\hat\rho(T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x) - \hat\rho(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x) \\
&+\hat\rho(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x) - \hat\rho_\epsilon(T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}x)\Big|.
\end{aligned} \tag{130}$$

Changing variables, setting $y_k := T_k^{-1}T_2^{-1}T_1^{-(p-l-2)}x$ and $y_{\epsilon,k} := y_\epsilon(y_k) = T_{\epsilon,k}^{-1}T_{\epsilon,2}^{-1}T_{\epsilon,1}^{-(p-l-2)}(T^p y_k)$, since $y_k$ and $y_{\epsilon,k}$ belong to $Z_p^k \cup Z_{\epsilon,p}^k$, we get

$$Q_{1,l} = \sum_{k=1,2}\sum_{p=l+2}^{\infty}\int_{Z_p^k} |\hat\rho(y_k) - \hat\rho(y_{\epsilon,k}) + \hat\rho(y_{\epsilon,k}) - \hat\rho_\epsilon(y_{\epsilon,k})|dy_k. \tag{131}$$

But,

$$\sum_{l=1}^{\infty} Q_{1,l} \leq 4C_2\sum_{l=1}^{\infty}\sum_{p=l+2}^{\infty} m(Z_p), \tag{132}$$

which is clearly convergent because the measure of $Z_p$ is exponentially decreasing. Moreover, by what has been shown in the first part of the proof,

$$\lim_{\epsilon\to 0}\int_{Z_p^k} |\hat\rho(y_{\epsilon,k}) - \hat\rho_\epsilon(y_{\epsilon,k})|dy_k = 0. \tag{133}$$

On the other hand, we first take $\epsilon$ small enough to get $y_{\epsilon,k}$ on the same side of $x_0$ as $y_k$ and then we use the Lipschitz continuity property of $\hat\rho$

to conclude, by observing that $y_{\epsilon,k}$ tends to $y_k$ when $\epsilon$ tends to zero, that also

$$\lim_{\epsilon \to 0} \int_{Z_p^k} |\hat{\rho}(y_k) - \hat{\rho}(y_{\epsilon,k})| dy_k = 0. \tag{134}$$

We now consider $Q_{2,l}$ and show that it is uniformly bounded in $\epsilon$. As a matter of fact,

$$Q_{2,l} = \int_{W_l' \cap W_{\epsilon,l}'} \sum_{p=l+2}^{\infty} \sum_{k=1,2} \left| \frac{\hat{\rho}_\epsilon (T_{\epsilon,k}^{-1} T_{\epsilon,2}^{-1} T_{\epsilon,1}^{-(p-l-2)} x)}{\left| DT^{p-l}(T_k^{-1} T_2^{-1} T_1^{-(p-l-2)} x) \right|} \right| \times \left| 1 - \frac{\left| DT^{p-l}(T_k^{-1} T_2^{-1} T_1^{-(p-l-2)} x) \right|}{\left| DT_\epsilon^{p-l}(T_{\epsilon,k}^{-1} T_{\epsilon,2}^{-1} T_{\epsilon,1}^{-(p-l-2)} x) \right|} \right| dx. \tag{135}$$

We bound it by the sum of its two parts: the density will have bounded infinity norm; the sums in $p$ over the inverses of the derivatives are bounded by a constant since the derivatives decay exponentially fast and the sums over $l$ will be controlled by the measure of $W_l'$. Finally the same arguments that led to bound (114) apply also to the second factor in the previous expression proving it tends to zero in the limit $\epsilon \to 0$.

This concludes the proof. □

We end up our analysis considering two examples of the perturbed Lorenz system giving rise to perturbed versions of the map $T$ of the kind discussed in this section.

*Example 8.* Let us consider a perturbation of the Lorenz field (Fig. 3) obtained by adding the constant forcing field $(0, 0, -\epsilon\beta(\rho + \sigma))$, $\epsilon > 0$. The perturbation is easily seen

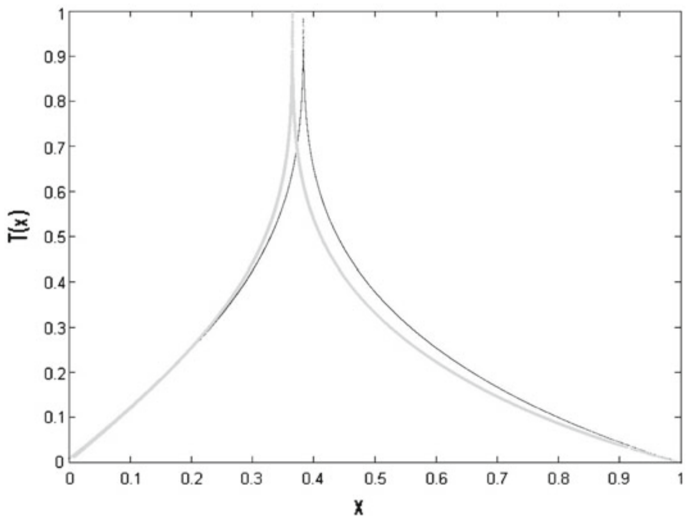



**Fig. 3.** $T_\epsilon$ (thick line), $T$ (thin line)

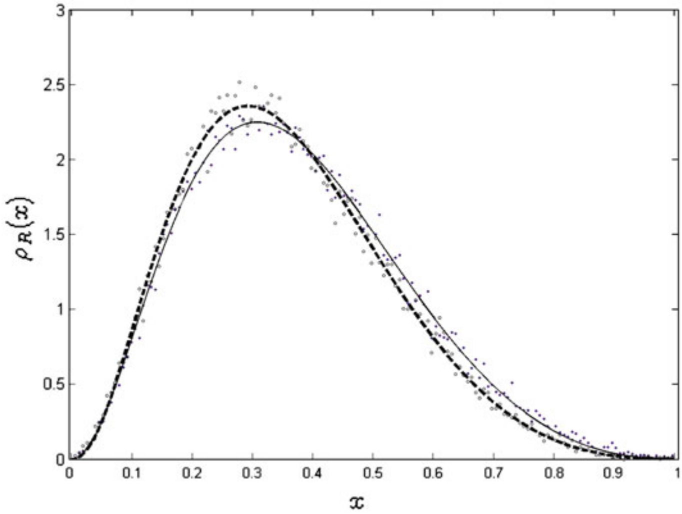



**Fig. 4.** Fit of the invariant density for the evolution under $T$ (solid line) and under $T_\epsilon$, $\epsilon = 0.5$, (dashed line)

to preserve the symmetry under the involution $R$ of the unperturbed field. Arguing as in the first section, for $\epsilon$ sufficiently small, the perturbed system will keep the same features of the unperturbed one, hence map $T_\epsilon$ is easily seen to satisfy (52–55) as well as Assumptions A–D. Here it follows the plot of $T_\epsilon$, for $\epsilon = 0.5$, and the plot of the fit of the invariant density $\rho_R$ for the evolution under the maps $T$ and $T_\epsilon$, corresponding respectively to the choice of the Poincaré surfaces $\Sigma_+$, $\Sigma_+^\epsilon$, the last one being constructed as in the unperturbed case (Fig. 4).

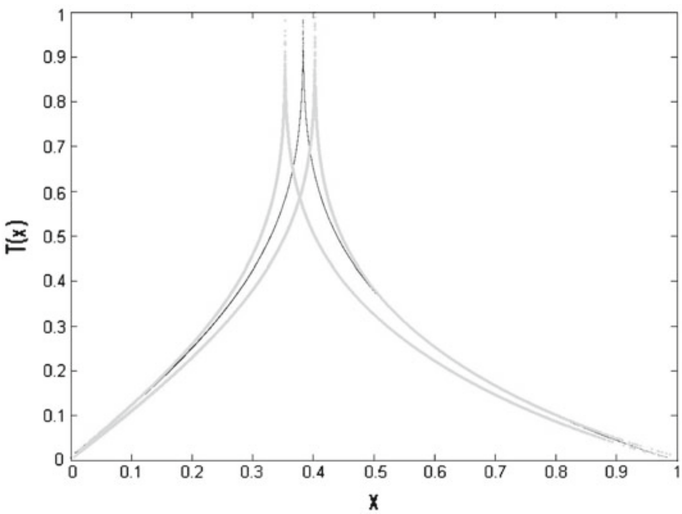



**Fig. 5.** $T$ (thin line), $T_\epsilon^+$ (thick line to the left of $T$), $T_\epsilon^-$ (thick line to the right of $T$)

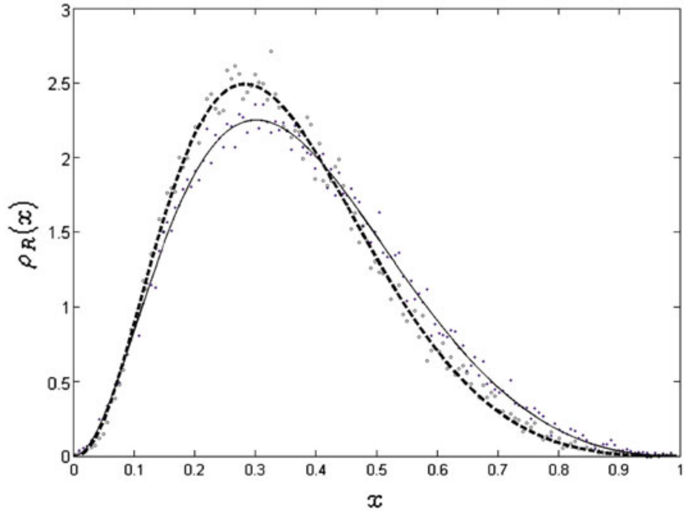



**Fig. 6.** Fit of the invariant densities for the evolution under $T$ (solid line) and under $T_\epsilon^+$ (dashed line)

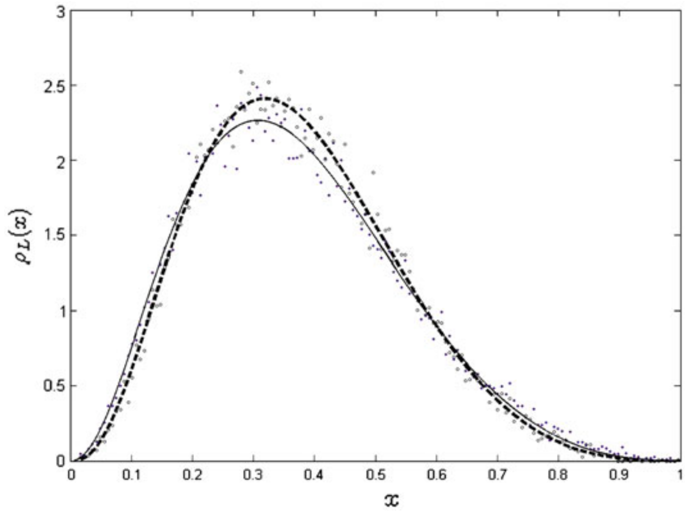



**Fig. 7.** Fit of the invariant densities for the evolution under $T$ (solid line) and under $T_\epsilon^-$ (dashed line)

*Example 9.* We now consider the following perturbation of the Lorenz field (Fig. 5) realized by adding the field $(\epsilon \cos\theta, \epsilon \sin\theta, 0)$ where $\epsilon > 0$ and $\theta \in [0, 2\pi)$. The perturbed system is not $R$-invariant anymore, anyway, for $\epsilon$ sufficiently small, the system will still have a saddle fixed point $c_0^\epsilon$ and two unstable fixed points $c_1^\epsilon, c_2^\epsilon$. Hence, for any $\theta \in [0, 2\pi)$, we have two different $T_\epsilon$, namely $T_\epsilon^+, T_\epsilon^-$, both satisfying (52–55) as well as Assumptions A–D corresponding respectively to the choice of the Poincaré surfaces $\Sigma_+^\epsilon, \Sigma_-^\epsilon$, which can be constructed as in the unperturbed case. To obtain meaningful plots of the deviation from $T$ of the perturbed maps as well as of the deviation of the associated invariant densities from the unperturbed one (Figs. 6 and 7), $\epsilon$ has been set equal to 2.5 and $\theta$ to 70° as in [CMP].

*Acknowledgement.* S. Vaienti wishes to thank G. Cristadoro, W. Bahsoun, R. Aimino and I. Melbourne for useful discussions and comments.

## References


[A] Arnold, V.I.: *Mathematical Methods in Classical Mechanics.* Second edition, Berlin-Heidelberg-NewYork: Springer, 1989

[Al] Alves, J.F.: Strong statistical stability of non-uniformly expanding maps. Nonlinearity, **17**(4), 1193–1215 (2004)

[ABS] Afraimovich, V.S., Bykov, V.V., Shil'nikov, L.P.: On the appearence and structure of the Lorenz attractor. Dokl. Acad. Sci. USSR, **234**, 336–339 (1977)

[AD] Aaronson, J., Denker, M.: Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. Stoch. Dyn. **1**, 193–237 (2001)

[AV] Alves, J.F., Viana, M.: Statistical stability for robust classes of maps with non-uniform expansion. Erg. Th. Dyn. Syst. **22**(1), 1–32 (2002)

[B] Broise, A.: Transformations dilatantes de l'intervalle et théorèmes limites. Asterisque **238**, 5–110 (1996)

[BH] Bhansali, J.R., Holland, M.: Frequency analysis of chaotic intermittency maps with slowly decaying correlations. Stat. Sinica **17**, 15–41 (2007)

[BV] Bahsoun, W., Vaienti, S.: Metastability of certain intermittent maps. Nonlinearity **25**(1), 107–124 (2012)

[CHMV] Cristadoro, G-P., Haydn, N., Marie, Ph., Vaienti, S.: Statistical properties of intermittent maps with unbounded derivative. Nonlinearity **23**, 1071–1095 (2010)

[CMP] Corti, S., Molteni, F., Palmer, T.N.: Signature of recent climate change in frequencies of natural atmospheric circulation regimes. Nature **398**, 799–802 (1999)

[FJKTV] Foias, C., Jolly, M.S., Kukavica, I., Titi, E.S.: The Lorenz equations as a metaphore for the Navier-Stokes equations. Disc. Con. Dyn. Sys. **7**(2), 403–429 (2001)

[G] Guckenheimer, J.: *A strange, strange attractor*. In: *The Hopf Bifurcation and its Applications*, J. E. Marsden, M. McCracken, eds., New York: Springer-Verlag, 1976

[GT] Galias, Z., Tucker, W.: *Short periodic orbits for the Lorenz system*. In: Procedings of the IEEE International Conference on Signals and Electronic Systems 2008 (ICSES 08), Piscataway, NJ: IEEE, 2008, pp. 285–288

[GW] Guckenheimer, J., Williams, R.F.: Structural stability of Lorenz attractors. Publ. Math. IHES **50**, 307–320 (1979)

[K] Keller, G.: Generalized bounded variation and applications to piecewise monotonic transformations. Z. Wahrsch. Verw. Gebiete **69**(3), 461–478 (1985)

[KDC] Kumar Mittal, A., Dwivedi, S., Chandra Pandey, A.: Bifurcation analysis of a paradigmatic model of monsoon prediction. Nonlinear Proc. in Geophys. **12**, 707–715 (2005)

[KDO] Diaz-Ordaz, K.: Decay of correlations for non-Hölder observables for one-dimensional expanding Lorent-like maps. Disc. Con. Dyn. Sys. **15**, 159–176 (2006)

[L] Lorenz, E.N.: Deterministic Nonperiodic Flow. J. Atmos. Sci. **20**, 130–141 (1963)

[LM] Labarca, R., Moreira, C.G.: Essential Dynamics for Lorenz maps on the real line and the Lexicographical World. Ann. I. H. Poincaré **23**, 683–694 (2006)

[Lu] Lucarini, V.: Evidence of Dispersion Relations for the Nonlinear Response of the Lorenz 63 System. J. Stat. Phys. **134**, 381–400 (2009)

[LSV] Liverani, C., Saussol, B., Vaienti, S.: Conformal measures and decay of correlations for covering weighted systems. Erg. Th. Dyn. Syst. **18**, 1399–1420 (1998)

[LSY] Young, L.-S.: Recurrence times and rates of mixing. Israel J. Math. **110**, 153–188 (1999)

[MR] Marsden, J.E., Ratiu, T.S.: *Indroduction to Mechanics and Symmetry*. Second edition, Berlin-Heidelberg-NewYork: Springer, 1998

[OHL] Diaz-Ordaz, K., Holland, M.P., Luzzatto, S.: Statistical properties of one-dimensional maps with critical points and singularities. Stoch. and Dyn. **6**, 423–458 (2006)

[PM] Pelino, V., Maimone, F.: Energetics, skeletal dynamics, and long term predictions on Kolmogorov-Lorenz systems. Phys. Rev. E **76**, 046214 (2007)

[PP1] Pasini, A., Pelino, V.: A unified view of Kolmogorov and Lorenz systems. Phys. Lett. A **275**, 435–445 (2000)

[PP2] Pelino, V., Pasini, A.: Dissipation in Lie–Poisson systems and the Lorenz-84 model. Phys. Lett. A **291**, 389–396 (2001)

[Se] Selten, F.M.: An Efficient Description of the Dynamics of Barotropic Flow. J. of Atm. Sci. **52**(7), 915–936 (1995)

[Sp] Sparrow, C.: *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. Berlin-Heidelberg-NewYork: Springer, 1982

[T] Tucker, W.: A rigorous ODE solver and Smale's 14th problem. Found. Comp. Math. **2:1**, 53–117 (2002)

[V] Viana, M.: What's new on Lorenz strange attractors? Math. Intell. **22**(3), 6–19 (2000)

[Z] Zeitlin, V.: Self-Consistent Finite-Mode Approximations for the Hydrodynamics of an Incompressible Fluid on Nonrotating and Rotating Spheres. Phys. Rev. Lett. **93**, 264501 (2004)

Communicated by G. Gallavotti