A PROBABILISTIC APPROACH TO INTERMITTENCY

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Abstract. We present an original approach which allows to investigate the statistical properties of a non-uniform hyperbolic map of the interval. Based on a stochastic approximation of the deterministic map, this method gives essentially the optimal polynomial bound for the decay of correlations, the degree depending on the order of the tangency at the neutral fixed point.

CONTENT

§0 Introduction. Recently the study of the convergence to the equilibrium in hyperbolic systems has witnessed several new results ranging from new methods to treat systems with singularities [Li1], [Yo] or with partially hyperbolic behavior [BY], to methods for studying Anosov flows [Ch2], [Do], [Li2].

Thanks to such results we can now regard the study of the decay of correlations for uniformly hyperbolic systems as reasonably understood (albeit there is much room for improvements, especially as flows and dependence on smoothness of observable are concerned). On the contrary the available results on the convergence to the equilibrium in non-uniform hyperbolic systems are extremely unsatisfactory.

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The study of such systems stands as a challenge. In particular, it is evident the need to develop new strategies to investigate such problems.

This is the focus of the present paper where we study a one parameter family of intermittent maps. These applications are expanding, except at a neutral fixed point, where hyperbolicity is lost. The local behavior of the map at this point is responsible for various phenomenon. Let us denote by $1+\alpha$ the order of the tangency at the critical point. For $\alpha = 0$ we have a purely expanding map, which has a unique equilibrium state for the potential $\varphi = -\log DT$, with exponential decay of correlations. For $0 < \alpha < 1$, the map possess an absolutely continuous probability measure (SRB measure), which is still an equilibrium state (it is no more unique, since the Dirac mass at the origin is invariant and $\delta_0(\varphi) = 0$). For $\alpha \geq 1$, there are no absolutely continuous invariant probability measure, whereas one still has a σ-finite absolutely invariant measures [PS].

We focus here on the second region of the parameter, and propose to find the density of the invariant measure, and the rate of decay of the correlation functions. In this domain, one cannot expect a spectral gap for the Perron-Frobenius operator (see the end of section 4), therefore none of the usual strategies in this setting can be followed. Our approach is based on the following philosophy : sure, the map is not hyperbolic, but it is the case nearly everywhere; thus, if we perform a random perturbation of the map, the neutral fixed point should be lost in a cloud of hyperbolic points and the intermittent effect could be suppressed. This naive argument, rather surprisingly, works.

An interesting property of such a method is the following. For smooth expanding maps, the same idea can be carried out, but yields a sub-exponential rate of decay, while the decay is well known to be exponential. On the contrary, in the present case, the power law found appears to be near optimal, as remarked in section 4.

This is an indication that our crude approach performs better in the non-uniform case than in the uniform one. These considerations are at the base of our belief that this type of strategy could yield relevant results in more general situations.

The plan of the paper is as follows: In section one we present our model and discuss some related literature. Section two is devoted to the study of the invariant measure. The section may have an interest in itself since it gives a very direct approach to obtaining the invariant measure and its properties for such a map (for a comparison with other techniques see [CF,Th]). Section three introduces the key idea of the paper, that is the random perturbation and its instrumental properties. In section four we harvest the facts from the previous sections and obtain the announced result. In addition, we point out that our results suffice to establish the CLT for $C^{(1)}$ observable, provided that $\alpha < 1/2$ (Remark 4.2-(3)). The last section contains few considerations on how to treat the general problem of expanding maps with neutral fixed points. Since the focus of the present paper is on the method and not on the class of one-dimensional maps to which it can be applied, we content ourselves with few pointed, but sketchy, considerations.

§1 The model. Let us consider for $0 < \alpha < 1$ the map $T : [0,\, 1] \to [0,\, 1]$

$$T(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \forall x \in [0,\, 1/2) \\ 2x - 1 & \forall x \in [1/2,\, 1] \end{cases}$$

The importance of this kind of intermittent maps was addressed by Prellberg and Slawny in [PS], where the relationship with a statistical model introduced by Fisher [FF] and successively studied by Gallavotti [Ga] was emphasized. In the papers [GW], [W], the dynamical behavior of these maps was taken as a model for the intermittency of turbulent flows [PM]. In the paper [PS] several mathematical results were announced, concerning the ergodic and statistical properties of such maps, notably the presence of phase transitions for the topological pressure (see also [Lo]). The paper [W] deals with a piecewise linear version of the map and focus especially on the recurrence properties of the orbits. These latter properties have been put on a solid mathematical basis by Collet, Galves and Schmitt in [CGS] in the piecewise linear case, and by Campanino and Isola for the non-linear (σ-finite) case [CI1,CI2,CI3]. The problem of the decay of correlations was considered in the piecewise linear case and for the finite absolutely continuous (w.r.t. Lebesgue) measure in [LSV] and [Mo]. Both papers obtain an algebraic ($n^{-\gamma}$) upper bound for the decay of correlations, the first by using Markov approximations [some results were successively improved by Chernov in [Ch1]], the second by exhibiting the absence of spectral gap for the Perron-Frobenius operator using the induction procedure, already invoked in [PS] (it is interesting to remark that Mori's work follows the analysis of these maps carried out by Takahashi in a series of papers [Ta1,Ta2]).

The decay of correlations for the non-linear case is a more difficult problem. In our knowledge, the following methods have been proposed.

In the paper [Yu], Yuri applied Markov approximations, generalizing the works of [LSV] and [Ch1]; the paper [Is] aims to extend to the non-linear case the approach of Mori, still inducing, and gives a description of the *zeta* function, and finally in [FS] the authors propose an interesting technique based on Hilbert metrics, yet the implementation of such an idea is still incomplete.

Let us go back to our particular model. In the next section, we will prove that there exists[1] a locally Lipschitz function $h \in C^{(0)}(]0,1]) \cap L^1([0,1])$ such that $Ph = h$.

§2 An invariant Cone.

If we define the cone $\mathcal{C}_0 = \{f \in C^{(0)}(]0,1]) \mid f \geq 0, f \text{ is decreasing}\}$ it is immediate to see that $\mathcal{C}_0$ is left invariant by the P-F operator. To see a bit more let us call X the identity, $X(x) = x$.

Lemma 2.1. *The cone* $\mathcal{C}_1 = \{f \in \mathcal{C}_0 \mid X^{\alpha+1} f \text{ is increasing}\}$*, is left invariant by the operator* P*.*

Proof. Let $f \in \mathcal{C}_1$, then

$$x^{\alpha+1} Pf(x) = \sum_{y \in T^{-1}x} \left(\frac{Ty}{y}\right)^{\alpha+1} \frac{y^{\alpha+1} f(y)}{D_y T}.$$

Setting $T^{-1}x = \{y_1, y_2\}$, $y_1 \leq y_2$, and $\xi = 2^\alpha y_1^\alpha$ we can write

$$x^{\alpha+1} Pf(x) = \frac{(1+\xi)^{\alpha+1}}{1+(\alpha+1)\xi}\, y_1^{\alpha+1} f(y_1) + \frac{1}{2}\left(\frac{2y_2-1}{y_2}\right)^{\alpha+1} y_2^{\alpha+1} f(y_2).$$

[1] Such a result can also be obtained by inducing [Th]. The method used here is more direct and provides additional informations on the properties of the invariant density, although such extra informations will not be essential in the following. See section five for a more detailed discussion.

Whence the result, since $X^{\alpha+1}f$, $x \mapsto y_1$, $x \mapsto \xi$, $x \mapsto y_2$ are increasing. □

Let us define

$$m(f) = \int_0^1 f(x)dx.$$

Obviously $m(Pf) = m(f)$.

The last interesting property is contained in the following.

Lemma 2.2. *The cone $\mathcal{C}_* = \{f \in \mathcal{C}_1 \cap L^1([0,1]) \mid f(x) \leq ax^{-\alpha}m(f)\}$ is invariant with respect to the operator P, provided a is chosen large enough.*

Proof. For each $f \in \mathcal{C}_*$ holds both

$$f(x) \leq ax^{-\alpha}m(f),$$

and

$$x^{\alpha+1}f(x) \leq f(1) \leq \int_0^1 f = m(f).$$

Let us suppose for simplicity that $m(f) = 1$. One has to find a constant a, independent of f, such that $Pf(x) \leq ax^{-\alpha}$ since $m(Pf) = m(f) = 1$.

$$\begin{aligned} Pf(x) =& \frac{f(y_1)}{D_{y_1}T} + \frac{f(y_2)}{D_{y_2}T} \\ \leq& \frac{ay_1^{-\alpha}}{D_{y_1}T} + \frac{y_2^{-\alpha-1}}{D_{y_2}T} \\ \leq& \left\{ \left(\frac{x}{y_1}\right)^\alpha \frac{1}{D_{y_1}T} + \frac{1}{a}\frac{x^\alpha}{y_2^{\alpha+1}D_{y_2}T} \right\} ax^{-\alpha}. \end{aligned}$$

The term in curly bracket is bounded by[2]

$$\frac{(1+\xi)^\alpha}{1+(\alpha+1)\xi} + \frac{2^\alpha}{a}\xi \leq \frac{1+\alpha\xi+\frac{2^\alpha}{a}\xi(1+(\alpha+1)\xi)}{1+(\alpha+1)\xi} \leq \frac{1+\left(\alpha+\frac{2^\alpha(\alpha+2)}{a}\right)\xi}{1+(\alpha+1)\xi} \leq 1.$$

Whenever $a \geq 2^\alpha(\alpha+2)$, from which the Lemma follows. □

Putting together all the previous estimates yields

Lemma 2.3. *There exists a locally Lipschitz function h such that $Ph = h$ and $h(x) \leq ax^{-\alpha}$.*

Proof. The operator P leaves invariant the set $K = \{f \in \mathcal{C}_* \mid m(f) = 1\}$. But $X^{\alpha+1}K$ consists of equibounded equicontinuous functions,[3] hence it is compact

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[2] Since $\frac{x^\alpha}{y_2^{\alpha+1}D_{y_2}T} \leq \frac{y_1^\alpha(1+\xi)^\alpha}{2^{-\alpha-1}2} \leq \xi(1+\xi)^\alpha \leq 2^\alpha\xi$.

[3] Let $f \in K$ and define $\phi(x) = x^{1+\alpha}f(x)$, then, for $x \geq y$, holds

$$\begin{aligned} 0 \leq \phi(x) - \phi(y) \leq (x^{1+\alpha} - y^{1+\alpha})f(x) &\leq a(1+\alpha)x^{-\alpha}\int_y^x \xi^\alpha d\xi \\ &\leq a(1+\alpha)|x-y|. \end{aligned}$$

in $C^{(0)}$. Accordingly, for each $f \in K$ the sequence $X^{\alpha+1}\frac{1}{n}\sum_{i=0}^{n-1} P^i f$ has accumulations points in $C^{(0)}$. Let $h_* \in K$ be such an accumulation point. Clearly, $h = X^{-1-\alpha}h_*$ is a fixed point of P, hence the result. The regularity of h is easily obtained by checking that $h \in \mathcal{C}_*$. □

Before introducing the random approximation to our dynamics let us remark a property of the functions in $\mathcal{C}_*$ that will be instrumental in the following.

Lemma 2.4. *For each function $f \in \mathcal{C}_*$*

$$\inf_{x\in[0,1]} f(x) = f(1) \geq \min\left\{ a, \left[\frac{\alpha(1+\alpha)}{a^\alpha}\right]^{\frac{1}{1-\alpha}} \right\} \int_0^1 f.$$

Proof. It clearly suffices to consider the case $\int_0^1 f = 1$. We have already seen that

$$f(x) \leq a x^{-\alpha},$$
$$f(x) \leq x^{-1-\alpha} f(1).$$

We introduce the point $x_* = a^{-1}f(1)$. On its left the first inequality is stricter and the opposite holds on its right. If $x_* > 1$, then $f(1) > a$, otherwise

$$1 = \int_0^1 f = \int_0^{x_*} f + \int_{x_*}^1 f \leq \int_0^{x_*} a\xi^{-\alpha} + \int_{x_*}^1 \xi^{-1-\alpha} f(1) \leq \frac{a^\alpha}{\alpha(1-\alpha)} f(1)^{1-\alpha},$$

from which the lemma follows. □

§3 A Random Perturbation.

For simplicity let us identify $[0,1]$ with the circle S^1, on S^1 the map is not smooth but it is continuous (this is not essential but it will make our life a bit easier). Let us define the "ball" $B_\varepsilon(x) = \{y \in S^1 \mid |x-y| \leq \varepsilon\}$ and the averaging operator[4]

$$\mathbb{A}_\varepsilon f(x) = \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} f(y)dy.$$

It is now possible to define the perturbed operator

$$\mathbb{P}_\varepsilon = P^{n_\varepsilon}\mathbb{A}_\varepsilon,$$

where $n_\varepsilon \in \mathbb{N}$ will be specified later.

The following Lemma shows that the perturbed operator is not too different from the original one, provided we consider observables in $\mathcal{C}_*$.

[4] Let us remark that this particular choice of $\mathbb{A}_\varepsilon$ has nothing special, any other "reasonable" choice would do as well.

Lemma 3.1. *For each $f \in \mathcal{C}_*$*

$$\|P^{n_\varepsilon} f - \mathbb{P}_\varepsilon f\|_1 \leq c_1 \|f\|_1 \varepsilon^{1-\alpha}.$$

Where $c_1 = \frac{10a}{\alpha(1-\alpha)}$.

Proof. We assume that $f \in \mathcal{C}_*$ and $\int f = 1$. First, observe that

$$\|P^{n_\varepsilon} f - \mathbb{P}_\varepsilon f\|_1 \leq \|f - \mathbb{A}_\varepsilon f\|_1.$$

Next, we recall the estimates

$$\begin{aligned} f(x) \leq& a x^{-\alpha} \\ \frac{f(y)}{f(x)} \leq& \left(\frac{x}{y}\right)^{1+\alpha} \qquad \forall x \geq y. \end{aligned}$$

This allows us to bound the L^1 norm of the difference between the function f and its average.

$$\begin{aligned} \|f - \mathbb{A}_\varepsilon f\|_1 \leq& \frac{1}{2\varepsilon}\int_\varepsilon^{1-\varepsilon} dx \int_{B_\varepsilon(x)} dy\ |f(y) - f(x)| + \int_{B_\varepsilon(0)} dx \left| f(x) - \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} dy\ f(y)\right| \\ \leq& \frac{1}{2\varepsilon}\int_\varepsilon^{1-\varepsilon} dx \left\{ \int_{x-\varepsilon}^{x} dy[f(y) - f(x)] + \int_x^{x+\varepsilon} dy[f(x) - f(y)]\right\} \\ &+ \int_{B_\varepsilon(0)} f(x)dx + \int_{B_{2\varepsilon}(0)} f(y)dy \\ \leq& \frac{1}{2\varepsilon}\int_\varepsilon^{1-\varepsilon} dx \int_{x-\varepsilon}^{x} dy f(y)\left[1 - \frac{f(y+\varepsilon)}{f(y)}\right] + 4a\int_0^{2\varepsilon} x^{-\alpha} dx \\ \leq& \frac{a}{2(1-\alpha)\varepsilon}\int_\varepsilon^{1-\varepsilon} dx \left(2x^{1-\alpha} - (x+\varepsilon)^{1-\alpha} - (x-\varepsilon)^{1-\alpha}\right) \\ &+ \frac{a}{2\alpha}\int_\varepsilon^{1-\varepsilon} dx \left(x^{-\alpha} - (x+\varepsilon)^{-\alpha}\right) + \frac{4a}{1-\alpha}(2\varepsilon)^{1-\alpha} \\ \leq& \frac{a}{2(1-\alpha)(2-\alpha)\varepsilon}\left\{(2\varepsilon)^{2-\alpha} - 2\varepsilon^{2-\alpha} - (1-2\varepsilon)^{2-\alpha} - 1 + 2(1-\varepsilon)^{2-\alpha}\right\} \\ &+ \frac{a}{2\alpha(1-\alpha)}\{2^{1-\alpha}\varepsilon^{1-\alpha} - 1 - \varepsilon^{1-\alpha} + (1-\varepsilon)^{1-\alpha}\} + \frac{8a}{1-\alpha}\varepsilon^{1-\alpha} \\ \leq& \frac{10}{\alpha(1-\alpha)}\varepsilon^{1-\alpha}. \end{aligned}$$

This proves Lemma 3.1. $\square$

Next,

$$\begin{aligned}\mathbb{P}_\varepsilon f(x) &= \sum_{y\in T^{-n_\varepsilon}x} \frac{1}{2\varepsilon D_y T^{n_\varepsilon}} \int_0^1 dz \chi_{B_\varepsilon(y)}(z) f(z)\\ &= \sum_{y\in T^{-n_\varepsilon}x} \frac{1}{2\varepsilon D_y T^{n_\varepsilon}} \int_0^1 dz \chi_{B_\varepsilon(z)}(y) f(z)\\ &= \frac{1}{2\varepsilon}\int_0^1 dz P^{n_\varepsilon} \chi_{B_\varepsilon(z)}(x) f(z)\\ &:= \int_0^1 \mathcal{K}_\varepsilon(x, z) f(z) dz.\end{aligned}$$

Our task is to find a lower bound for the kernel $\mathcal{K}_\varepsilon(x, z)$. For this purpose, let us define T_1 to be the map T restricted to the interval $[0, 1/2]$ and $a_n = T_1^{-n}1$. We have the following asymptotic bound for the sequence a_n.

Lemma 3.2. *For all integer $n > 0$ the following holds*

$$a_n \le 2^{\frac{1}{\alpha^2}+\frac{1}{\alpha}} n^{-\frac{1}{\alpha}}.$$

Proof. The Lemma is proven by induction. First it is clearly satisfied for $n = 1$. Next, let us suppose that $a_n < cn^{-\frac{1}{\alpha}}$, and let us prove that $a_{n+1} < c(n+1)^{-\frac{1}{\alpha}}$. If it is false, then

$$a_n = a_{n+1}(1 + 2^\alpha a_{n+1}^\alpha) \ge c(n+1)^{-\frac{1}{\alpha}}(1 + 2^\alpha c^\alpha (n+1)^{-1}).$$

By the assumption on a_n we obtain

$$n^{-\frac{1}{\alpha}} \ge (n+1)^{-\frac{1}{\alpha}}(1 + \frac{2^\alpha c^\alpha}{n+1})$$

or equivalently

$$(1 + \frac{1}{n})^{\frac{1}{\alpha}} \ge 1 + \frac{2^\alpha c^\alpha}{n+1}.$$

By convexity it follows

$$(2^{\frac{1}{\alpha}} - 1)\frac{1}{n} \ge \frac{2^\alpha c^\alpha}{n+1},$$

that is

$$c^\alpha \le 2^{-\alpha}(2^{\frac{1}{\alpha}} - 1)\frac{n+1}{n} \le 2^{-\alpha+1}(2^{\frac{1}{\alpha}} - 1),$$

which is contradictory if we choose $c = 2^{\frac{1}{\alpha^2}+\frac{1}{\alpha}}$. □

We define $\Delta_k = [a_k, a_{k-1}]$ for each $k > 0$. We are now able to prove

Proposition 3.3. *There exists $\gamma > 0$ such that for each $\varepsilon > 0$, x, $z \in S^1$*

$$\mathcal{K}_\varepsilon(x, z) \geq \gamma,$$

provided we choose $n_\varepsilon = [2^{2+\frac{1}{\alpha}}\varepsilon^{-\alpha}] + 1$.[5]

Proof. First of all, we choose $k_0 = 3$. Next, notice that for each interval J and integer m

$$(P^m\chi_J)(x) \geq \chi_{T^mJ}(x) \inf_{y\in J}(D_yT^m)^{-1}.$$

Let $\delta_0 = a_{k_0} - a_{k_0+1}$. By Lemma 2.4 it is obvious that there exists n_0 and c_0 such that for all intervals I of size larger than δ_0 holds

$$P^n\chi_I \geq c_0$$

provided $n \geq n_0$. Thus the task is to control the $\inf_{y\in J}(D_yT^m)^{-1}$, where m is the time needed for the interval J to become an interval of size δ_0. Let $I_0 = [0, a_{k_0}]$. Let J be an interval, three possibilities can occur :

(1) $J \cap I_0 = \emptyset$.
(2) $J \cap I_0 \neq \emptyset$ and J contains, at most, one a_k for $k > k_0$.
(3) J contains more than one a_k for $k > k_0$.

We can associate to each J a sequence $n_1, k_1, \ldots, k_{p-1}, n_p$ of integers (n_1 may be null), retracing the trajectory of J in the following way : For a time n_1 (1) holds, then the image of J enters the intermittent region I_0 and (2) holds with $k = k_1 + k_0$, so after k_1 iterations it exits from I_0. Then the image of J stays in the hyperbolic region for n_2 iterations, and so on... Finally we end when the size of the interval becomes larger than δ_0 or if case (3) happens. Let us see what happens in this regimes.

(1) Let $D = \sup_{y\in[a_{k_0+1},1]} \frac{D_y^2T}{D_yT^2}$ and $r = (D_{a_{k_0+1}}T)^{-1} < 1$. For $n \leq n_1$, the usual distortion estimates yield, for each $y \in J$, with $c_2 = D/(1-r)$,

$$\frac{D_yT^n|J|}{|T^nJ|} \leq \text{Exp}[c_2|T^nJ|].$$

(2) Let $J_1 = T^{n_1}J$. Let us see what happens in the intermittent region. Suppose that $J_1 \subset]a_{k+1}, a_{k-1}[$, i.e. (2) occurs with $k = k_0 + k_1$. In this case a direct computation for $j \leq k_1$ and $y_1, y_2 \in J_1$ implies

$$\begin{aligned}\frac{D_{y_1}T^j}{D_{y_2}T^j} \leq & \text{Exp}\left[\sum_{i=1}^{k_1} \frac{\sup_{\xi\in[a_{k-i+2},a_{k-i}]} D_\xi^2T}{\inf_{\xi\in[a_{k-i+2},a_{k-i}]} D_\xi T}|T^iy_2 - T^iy_1|\right] \\ \leq & \text{Exp}\left[\sum_{i=1}^{k_1} \alpha(1+\alpha)a_{k-i+2}^{\alpha-1}(D_{a_{k+2-i}}T^{k_1-i})^{-1}|T^{k_1}J_1|\right] \\ \leq & \text{Exp}\left[\alpha(\alpha+1)\sum_{q=k_0+2}^{k_0+k_1+1} a_q^{\alpha-1}(D_{a_q}T^{q-(k_0+2)})^{-1}|T^{k_1}J_1|\right].\end{aligned}$$

[5] Here the square braket stands for the integer part.

Since the first branch of the map is convex, we have

$$D_{a_q}T^{q-(k_0+2)} \geq r\frac{a_{k_0+1}-a_{k_0+2}}{a_{q-1}-a_q} = ra_{k_0+2}^{1+\alpha}a_q^{-(1+\alpha)}.$$

Moreover, Lemma 3.2 gives $a_q^{2\alpha} \leq 2^{2+2/\alpha}q^{-2}$. This yields, setting $c_3 = \dfrac{\alpha(1+\alpha)2^{2+2/\alpha}}{ra_{k_0+2}^{1+\alpha}}$,

$$\frac{D_{y_1}T^j}{D_{y_2}T^j} \leq \mathrm{Exp}\left[c_3|T^{k_1}J_1|\right].$$

(3) Finally, let us see what happens if (3) holds for some iterate $K = T^jJ$. If more than one third of the size of K is in $[a_1, 1]$, then we consider $K \cap [a_1, 1]$ and case (1) will hold for ever, loosing just factor 1/3. Else we cut K into pieces $\Delta_{k_-}, \cdots, \Delta_{k_+}$ such that the union of them is of size bigger than $|K|/3$. For these Δ_k, the preceding computation yields

$$P^{k-k_0}\chi_{\Delta_k} \geq \chi_{\Delta_{k_0}}\frac{\mathrm{Exp}[-c_3|\Delta_{k_0}|]}{|\Delta_{k_0}|}|\Delta_k|.$$

Therefore, with $l = n_0 + k_+ - k_0$,

$$P^l\chi_K \geq \sum_{k=k_-}^{k_+} P^{l+k_0-k}P^{k-k_0}\chi_{\Delta_k} \geq \sum_{k=k_-}^{k_+} c_0\frac{\mathrm{Exp}[-c_3\delta_0]}{\delta_0}|\Delta_k| \geq c_0\frac{\mathrm{Exp}[-c_3\delta_0]}{\delta_0}\frac{|K|}{3}.$$

Since we control what happens on each region, it is possible to estimate the total distortion after $m = n_1 + k_1 + \cdots + n_p + l$ iterations, where $l = n_0$ if case (3) never happens ($l = n_0 + k_+ - k_0$ if case (3) occurs).

$$\begin{aligned}
P^m\chi_J \geq& P^lP^{n_p}P^{k_{p-1}}\cdots P^{n_2}P^{k_1}P^{n_1}\chi_J\\
\geq& |J|\frac{c_0}{3\delta_0}\mathrm{Exp}\left[-c_3\delta_0 - c_2|T^{n_p+\cdots+k_1+n_1}J| - \cdots - c_3|T^{k_1+n_1}J| - c_2|T^{n_1}J|)\right]\\
\geq& |J|\frac{c_0}{3\delta_0}\mathrm{Exp}\left[-(c_2+c_3)\delta_0(1 + r^{n_p} + r^{n_p+n_{p-1}} + \cdots + r^{n_p+n_{p-1}+\cdots+n_2})\right]\\
\geq& |J|\frac{c_0}{3\delta_0}\mathrm{Exp}\left[\frac{-(c_2+c_3)\delta_0 r}{1-r}\right] := \gamma|J|.
\end{aligned}$$

To conclude, we need to fix n_ε. We choose the supremum over all possible values of $m = n_1 + k_1 + \cdots + n_p + l$, associated to intervals J of size 2ε. It is immediate to see that the worst case scenario is when case (3) happens at the beginning, and $J =]-2\varepsilon/3, 4\varepsilon/3[$. In this case, m is such that $a_{k_0+m} \leq 2\varepsilon/3$. Clearly, $n_\varepsilon = [2^{2+\frac{1}{\alpha}}\varepsilon^{-\alpha}] + 1$ is large enough and the Lemma is proven. □

§4 Decay of Correlations.

Proposition 3.3 allows immediately to conclude that $\mathbb{P}_\varepsilon$ has an invariant density h_ε to which it converges exponentially fast[6] in L^1. In the following we will call μ the invariant measure $d\mu = hdx$. Using all the above facts, we are able to prove our main result.

Theorem 4.1. *For all $g \in L^\infty$, $f \in \mathcal{C}^{(1)}([0,1])$ such that $\int f d\mu = 0$ the following holds*

$$\left| \int g \circ T^n f d\mu \right| \leq c_4 C\left(\|f\|_{C^{(1)}}\right) \|g\|_\infty n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}.$$

Where $C : I\!R \to I\!R$ is affine.

Proof. Let $f \in \mathcal{C}_* + I\!R$, $\int_0^1 f = 0$ and $g \in L^\infty$, $\|g\|_\infty = 1$. For each $n \in \mathbb{N}$ let us write $n = kn_\varepsilon + m$, $k \in \mathbb{N}$ and $m < n_\varepsilon$. Remembering Lemma 3.1, we have

$$\begin{aligned}
\left| \int_0^1 g P^n f \right| \leq & \|P^n f - \mathbb{P}_\varepsilon^k P^m f\|_1 + \|\mathbb{P}_\varepsilon^k P^m f\|_1 \\
\leq & \sum_{i=0}^{k-1} \|P^{(i+1)n_\varepsilon} P^m f - \mathbb{P}_\varepsilon P^{in_\varepsilon} P^m f\|_1 + \mathrm{Exp}\left[-\gamma k\right] \|f\|_1 \\
\leq & 2c_1 \|f\|_1 \frac{n}{n_\varepsilon} \varepsilon^{1-\alpha} + e^\gamma \mathrm{Exp}\left[-\gamma \frac{n}{n_\varepsilon}\right] \|f\|_1 \\
\leq & c_4 \|f\|_1 n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}.
\end{aligned}$$

by choosing $\varepsilon = n^{-\frac{1}{\alpha}} (\log n^{(1-\gamma)^{-1}(\frac{1}{\alpha}-1)2^{2+\frac{1}{\alpha}}})^{\frac{1}{\alpha}}$.

This is not yet the decay of correlation with respect to the absolutely continuous invariant measure $d\mu = hdx$ of our dynamical system. To get such a result, we need to notice that if $f \in C^{(1)}$, then we can choose $\lambda, \nu, \delta \in I\!R$ such that $f_{\lambda,\nu,\delta}(x) := (f(x) + \lambda x + \nu)h(x) + \delta \in \mathcal{C}_*$, and $(\lambda x + \nu)h(x) + \delta \in \mathcal{C}_*$, the dependence of the parameters with respect to the $C^{(1)}$ norm of f being affine. Finally, the decay of correlations with respect to μ, for each $f \in C^{(1)}$, $\int f d\mu = 0$ and $g \in L^\infty$ can be estimated as follows

$$\left| \int g \circ T^n f d\mu \right| \leq c_4 C\left(\|f\|_{C^{(1)}}\right) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}.$$

Where $C : I\!R \to I\!R$ is affine. □

[6] Let us briefly recall the argument: set $\Omega = [0,1]$ and consider $f \in L^1(\Omega)$ with $\int_\Omega f = 0$. Remember that $\mathbb{P}_\varepsilon 1 = \mathbb{P}_\varepsilon^* 1 = 1$ and let $\Omega_\varepsilon^+ = \{x \in \Omega \mid \mathbb{P}_\varepsilon f \geq 0\}$; $\Omega_+ = \{x \in \Omega \mid f \geq 0\}$, then

$$\begin{aligned}
\|\mathbb{P}_\varepsilon f\|_1 &= 2\int_{\Omega_\varepsilon^+} dx \int_\Omega dy \mathcal{K}_\varepsilon(x,y) f(y) = 2\int_\Omega dy f(y) \int_{\Omega_\varepsilon^+} dx [\mathcal{K}_\varepsilon(x,y) - \gamma] \\
&\leq 2\int_{\Omega_+} dy f(y) \int_\Omega dx [\mathcal{K}_\varepsilon(x,y) - \gamma] = (1-\gamma) 2 \int_{\Omega_+} dy f(y) \\
&= (1-\gamma)\|f\|_1.
\end{aligned}$$

Accordingly, for each $f \in L^1(\Omega)$ and defining $\Pi f = \int_\Omega f$, holds

$$\|(\mathbb{P}_\varepsilon - \Pi)^n f\|_1 = \|\mathbb{P}_\varepsilon^n (1\!\!1 - \Pi) f\|_1 \leq (1-\gamma)^n \|f\|_1.$$

Remark 4.2.

(1) *The proof of Theorem 4.1 allows to estimate the difference between* h_ε *and* h*; namely, to prove the estimate:*[7] $\|h_\varepsilon - h\|_1 \leq const.\varepsilon^{1-\alpha}(\log \varepsilon^{-1})^{\frac{1}{\alpha}}$.

(2) *Theorem 4.1 allows to get an estimate for the rate of decay for Hölder continuous observables. More precisely, given a* β*-Hölder continuous function* f*, we can bound the correlations by* $n^{-\beta(\frac{1}{\alpha}-1)}$*, up to some logarithmic correction.*[8]

(3) *We have obtained a polynomial decay, with a bound comparable to the one found in* [Mo] *for the piecewise linear case and the one stated in* [Is] *for the general case. Moreover, compared with numerical simulations* [LSV]*, our bound appears to be extremely close to optimal: the expected one is the same apart from the logarithmic correction.*

(4) *As a side consequence of our work we have that* $\sum_{n=0}^{\infty} P^n f$ *converges in* L^1 *provided* $\alpha < \frac{1}{2}$*. According to* [Li3] *this estimate suffices to prove the Central Limit Theorem. That is, given* $f \in \mathcal{C}^{(1)}([0,1])$ *such that* $\int_0^1 f d\mu = 0$

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n} f \circ T^i$$

converges, in distribution, to a Gaussian variable with variance $\int_0^1 f^2 d\mu + 2\sum_{i=1}^{\infty} \int_0^1 f f \circ T^i d\mu$.

§5 General Considerations.

The reader may be under the impression that the proposed approach is specific to the special maps studied here. In particular, section two seems quite model dependent. In fact, while the estimates done there would apply certainly to similar maps, it is true that some more work is needed to present them in a completely general fashion[9].

Nevertheless, section two is not completely necessary. Its aim was to make the paper self contained and to emphasize the existence of a very direct method of

[7] It suffices to write

$$\begin{aligned} \|h - h_\varepsilon\|_1 &\leq \|P^n 1 - h\|_1 + \|\mathbb{P}_\varepsilon^{\frac{n}{n\varepsilon}} 1 - h_\varepsilon\|_1 + \|\mathbb{P}_\varepsilon^{\frac{n}{n\varepsilon}} 1 - P^n 1\|_1 \\ &\leq \text{const.}\{n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}} + (1-\gamma)^{n\varepsilon^\alpha} + n\varepsilon\} \end{aligned}$$

and to choose $n \sim \varepsilon^{-\alpha}\log\varepsilon^{-1}$.

[8] The result is easily obtained by approximation. If f is a β-Hölder function, then we can approximate it in L^1 by functions $f_\varepsilon \in C^{(1)}$ such that $\|f - f_\varepsilon\|_1 \leq \|f\|_\beta \varepsilon^\beta$ and $\|f'\|_\infty \leq \|f\|_\beta \varepsilon^{\beta-1}$ (where $\|f\|_\beta$ is the β-Hölder norm). Accordingly,

$$\left|\int f \circ T^n g\right| \leq \|g\|_\infty \|f - f_\varepsilon\|_1 + (c_1\|f'_\varepsilon\|_\infty + c_2\|f\|_\infty) n^{-\frac{1}{\alpha}+1},$$

and the result follows by choosing ε judiciosly.

[9] Yet, the results extend immediately to any map of the interval which is $\mathcal{C}^{(1)}$–conjugate to our model. More precisely, suppose that $\widetilde{T} : [0,1] \to [0,1]$ and $\Phi \in \text{Diff}^{(1)}([0,1])$ satisfies $\widetilde{T} \circ \Phi = \Phi \circ T$. Then, for the absolutely continuous $\widetilde{T}$–invariant measure $\widetilde{\mu}$ defined by $\widetilde{\mu}(f) = \mu(f \circ \Phi)$, it is straightforward to see that the power law decay is the same for T and $\widetilde{T}$ for $\mathcal{C}^{(1)}$ observable.

studying the invariant measure. If one is willing to make same assumptions on the invariant measure (which can eventually be proven separately) sections two can be greatly simplified.

Here we discuss briefly what is really essential in order to apply the present method.

Let us consider a map $T : [0, 1] \to [0, 1]$ expanding but for the fixed point 0. where $D_0T = 1$. Assume D^2T continuous but in the point 0 and

$$\begin{aligned} |D_x^2 T| &\leq C x^{\alpha-1} \\ |D_x T| &= 1 + c x^{\alpha} + o(x^{\alpha}). \end{aligned} \tag{5.1}$$

Assume further that there exists an invariant probability measure absolutely continuous with respect to Lebesgue (see the discussion at the end of this section). Calling h its density we have $Ph = h$. Then, in analogy with Lemma 2.1 holds[10]

Lemma 5.1. *There exist $a > \alpha$ and $b > 0$ such that the cone*

$$\mathcal{C} = \left\{ f \in C^{(1)}(]0, 1]) \;\middle|\; 0 \leq f(x) \leq 2h(x) \int_0^1 f;\ |f'(x)| \leq \frac{a + bx}{x} f(x) \right\},$$

is invariant with respect to the Perron-Frobenius operator.

Proof. Obviously, the first condition is invariant by P. If f belongs to the cone

$$\begin{aligned} |(Pf)'(x)| &= \left| \sum_{y \in T^{-1}x} \frac{D_y^2 T}{|D_y T|^3} f(y) + \frac{1}{|D_y T|^2} f'(y) \right| \\ &\leq \sum_{y \in T^{-1}x} \left(\frac{2C y^{\alpha-1}}{|D_y T|^2} + \frac{a + by}{y |D_y T|} \right) \frac{f(y)}{|D_y T|} \\ &\leq \frac{a + bx}{x} Pf(x) \sup_{y \in [0,1]} \left[\frac{2C}{|D_y T|^2} \frac{y^{\alpha-1} T(y)}{a + bT(y)} + \frac{T(y)}{y |D_y T|} \frac{a + by}{a + bT(y)} \right] \end{aligned}$$

Let $\Omega(y)$ being the term in brackets. We have

$$T(y) = \int_0^y D_t T dt = \int_0^y (1 + ct^{\alpha} + o(t^{\alpha})) dt = y + \frac{c}{\alpha + 1} y^{\alpha+1} + o(y^{\alpha+1}).$$

Which shows that $\Omega(y) \leq 1 + \left(\frac{2C}{a} + c\left(\frac{1}{\alpha+1} - 1\right) \right) y^{\alpha} + o(y^{\alpha})$. Therefore there exists $\delta > 0$ such that $\Omega(y) \leq 1$ on $[0, \delta]$ provided a is big enough. Next, outside this neighborhood, we have $y > \delta$ and $|D_y T| > \gamma > 1$, so

$$\Omega(y) \leq \frac{1}{a} \frac{2C\delta^{\alpha-1}}{\gamma^2} + \frac{a}{b} \frac{1}{\delta\gamma} + \frac{1}{\gamma} < 1$$

[10] Such a result should extends easily to maps expanding but for some fixed points $\{p_j\}$ where $D_{p_j} T = 1$. We can define $\theta(x) = \min_j |x - p_j|$ and assume D^2T continuous but in the points $\{p_j\}$ and

$$\begin{aligned} |D_x^2 T| &\leq C\theta(x)^{\alpha-1} \\ |D_x T| &= 1 + c_j \theta(x)^{\alpha} + o(\theta(x)^{\alpha}). \end{aligned}$$

Then the same cone with $\frac{a + b\theta(x)}{\theta(x)}$ instead of $\frac{a+bx}{x}$ should be invariant.

provided a and b/a are big enough. □

In addition, we assume that for some $0 < \beta < 1$, $0 < \gamma_0 < 1$ and $c_* > 0$,

$$\gamma_0 \leq h(x) \leq c_* x^{-\beta}. \tag{5.2}$$

It follows that a sharper cone is invariant. Note that in the process of proving the next lemma we will establish the analogous of Lemma 2.4.

Lemma 5.2. *There exists* $\delta > 0$ *and* $0 < \gamma < \gamma_0$ *such that*

$$\mathcal{C}_* = \{f \in \mathcal{C} \mid f(x) \geq \gamma \int_0^1 f \text{ for } x \leq \delta\},$$

is invariant with respect to the Perron-Frobenius operator.

Proof. The proof starts by establishing the analogous of Lemma 2.4

Sub-Lemma 5.3. *If* $f \in \mathcal{C}$, *then*

$$\min_{[\delta,1]} f \geq \frac{\delta^a}{2e^b} \int_0^1 f,$$

provided δ *is chosen small enough.*

Proof. We have the bounds

$$f(x) \leq 2c_* x^{-\beta} \int_0^1 f$$

and

$$|f'(x)| \leq (ax^{-1} + b) f(x)$$

coming from the cone and the assumed estimate on the invariant measure. For $x \geq y > \delta$ the second bound yields

$$\left(\frac{y}{x}\right)^a e^{-b(x-y)} \leq \frac{f(x)}{f(y)} \leq \left(\frac{x}{y}\right)^a e^{b(x-y)}.$$

Accordingly, normalizing $\int f = 1$,

$$1 = \int f = \int_0^\delta f + \int_\delta^1 f \leq \frac{2c_*}{1-\beta}\delta^{1-\beta} + \delta^{-a} e^b \min_{x \in [\delta,\, 1]} f.$$

Next, by choosing δ sufficiently small, we have $\frac{2c_*}{1-\beta}\delta^{1-\beta} \leq \frac{1}{2}$ from which the result follows. □

Let $\mu = \|DT\|_\infty$. We choose δ so small that for all $y \leq \delta$ hold $|D_y T|^{-1} \geq 1 - 2c\delta^\alpha$ and Sub-Lemma 5.3 together with $1 - 2c\delta^\alpha + \mu^{-1} > 1$. Let $x \leq \delta$, then $T^{-1}x$ will consist of, at least, two points $y_1 \leq \delta$ and $y_2 \geq \delta$. By choosing γ small, holds

$$\begin{aligned} Pf(x) \geq & |D_{y_1} T|^{-1} f(y_1) + \mu^{-1} f(y_2) \geq \left[(1 - 2c\delta^\alpha)\gamma + \mu^{-1} \min\left\{\gamma, \frac{\delta^a}{2e^b}\right\}\right] \int_0^1 Pf \\ & \geq \gamma \int_0^1 Pf. \end{aligned}$$

□

This is enough to show that $\inf f \geq \gamma \int f$ whenever $f \in \mathcal{C}_*$, which implies (since the constant function 1 belongs to the cone $\mathcal{C}_*$)

$$\inf_{n\geq 0} \inf_{[0,1]} P^n 1 \geq \gamma > 0. \tag{5.3}$$

We have so generalized the two ingredients of section two used in the paper: the existence of an invariant cone for P that allows to obtain the estimates used to prove Lemma 3.1 and the analogous of Lemma 2.4, necessary in proving Proposition 3.3.

The reader can then easily generalize section three, since the distortion estimates depend only on the behavior of the neutral fixed point which is ensured by our assumption on D^2T. Section four follows exactly in the same way yielding a polynomial decay depending on α and β.

In conclusion, the following more general result holds:

Proposition 5.4. *Given a map T satisfying (5.1), if it has an absolutely continuous invariant probability measure with density satisfying (5.2), then for all $g \in L^\infty$, $f \in C^{(1)}$, $\int f = 0$, $0 < \gamma < \frac{1}{\alpha} - \frac{\beta}{\alpha}$, holds*

$$\left| \int f g \circ T^u \right| \leq c_4 C(\|f\|_{C^{(1)}}) \|g\|_\infty n^{-\gamma}$$

for some constants c_4 and $C(\|f\|_{C^{(1)}})$.

Here we do not address how to obtain the needed estimate on the invariant density, although several approaches (besides the ones in the style of what we have done in section two) are possible (see, for example, [Th] where a result of the type (5.2) is obtained by inducing).

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