

Ergodic Properties of the Discontinuous Sawtooth Map

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Received May 15, 1991; final October 9, 1991

We prove the ergodicity for the discontinuous sawtooth map, adapting a technique previously used in billiard theory. The core of the proof is the construction of a Hopf chain passing through a countable dense set of discontinuity lines.

KEY WORDS: Ergodicity; stable and unstable manifolds; Sinai–Chernov theory; Hopf’s chain.

1. INTRODUCTION

The discontinuous sawtooth map (SM) is probably the easiest example of a low-dimensional discontinuous map for which the proof of the ergodicity is far from being trivial. This map is still under investigation both from mathematical and physical points of view since it is a relatively simple approximation for some physical problems^(1–4): plasma confinement in nuclear fusion, charged particles in magnetic fields, stochastic ionization, etc. In fact, it describes the motion, in the phase space, of a particle subject to a periodic one-dimensional potential or alternatively a rotor subject to a sequence of periodic impulses. The SM depends on a real positive constant K : when K is integer, the map reduces to a linear automorphism of the torus and the proof of the ergodicity is simple and standard⁽⁵⁾; also the statistical properties (for example, rate of decay of the correlations) can be worked out exactly,⁽⁶⁾ ultimately because a finite Markov partition can be easily constructed.

When $K \notin \mathbb{N}$, the situation changes considerably: the map becomes discontinuous and the torus is filled by a dense countable set of discontinuity lines for the powers of the map and its inverses. If this does not prevent the

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existence almost everywhere of local stable and unstable manifolds, nevertheless, since these manifolds can be arbitrarily short, it could become very difficult to apply the Hopf argument, which is the standard tool to prove ergodicity for such systems. The technique we use to overcome this difficulty is a direct application of the so-called Transversal Fundamental Theorem for billiards. It was firstly introduced by Sinai and Chernov⁽⁷⁾ to obtain local ergodicity of semidispersing billiards and was recently generalized by Kràmli *et al.*^(8,9) to prove the global ergodicity of systems of three and four elastic hard balls on the multidimensional torus. The same theorem has been utilized by Bunimovich⁽¹⁰⁾ to prove the ergodicity of two-dimensional hyperbolic billiards; we have also learned of a recent work by Liverani and Wojtkowski⁽¹¹⁾ to generalize it to a large class of symplectic maps with singularities.

The situations occurring there are much more difficult than our present case, especially because one must guarantee all the hyperbolic properties (for example, existence of the local stable and unstable manifolds, and absolute continuity) that are quite easy to prove for the SM.

However, the core of the proof of the ergodicity is essentially the same, although the limited amount of technical details makes the argument very transparent and, we hope, pedagogically clear.

We briefly recall that other techniques to prove the local or global ergodicity for smooth dynamical systems with singularities have been utilized, namely, in billiard theory by Sinai and Bunimovich^(16–18) and Gallavotti and Ornstein^(19,20); for mappings: by Collet and Levy,⁽²¹⁾ Young,⁽²²⁾ and Rychlik⁽²³⁾ for the Lozi map, by Wojtkowski⁽¹⁴⁾ for a class of piecewise linear continuous transformations of a torus, and by Burton and Easton⁽²⁴⁾ and Przytycki⁽²⁵⁾ for linked twist maps. An important contribution for the description of ergodic properties of smooth dynamical systems is the work of Katok and Strelcyn,⁽²⁶⁾ where the Pesin theory is generalized to maps with singularities in order to prove that the invariant sets with nonzero Lyapunov exponents have countably many ergodic components. This last fact is automatically true for the SM since it is almost hyperbolic for $K > 0$ in the sense defined in ref. 14. Recently a new technique to investigate the statistical properties of the discontinuous SM has been introduced,⁽¹²⁾ allowing us to compute, in particular, a rigorous finite-time estimate for the diffusion coefficient: such a technique requires a very precise description of the discontinuity lines, which we can skip here by working “in measure” and by using systematically the almost hyperbolicity of the map. We guess that those statistical properties would be completely understood if one could construct a (possibly infinite) Markov partition. We consider this the natural continuation of the present work.

The plan of the paper is the following: in Section 2 we characterize the

discontinuity lines and we construct almost everywhere the local stable and unstable manifolds.

In Section 3 we prepare the local coverings needed in the proof of the main result, Theorem 3.2, and we investigate the consequences of this theorem, namely: the construction of a Hopf chain in any open set whose closure is included in a finite number of polygonal regions covering (mod 0) the torus, and then the extension of the chain from a polygonal region to another one, thus realizing the global ergodicity. Finally, in Section 4 we prove the main theorem.

2. DISCONTINUITY LINES AND LOCAL STABLE AND UNSTABLE MANIFOLDS

The sawtooth map T is an area-preserving automorphism of the torus \mathbb{T}^2 with coordinates $(I, \theta) \text{ mod } 2\pi$, defined by

$$T(I, \theta) = (I + Kg(\theta), I + \theta + Kg(\theta))$$

where $K > 0$ and $g(\theta) = \theta$ for $\theta \in [-\pi, \pi)$. We denote by $\mu(dI d\theta) = dI d\theta/4\pi^2$ the Lebesgue measure on \mathbb{T}^2 preserved by T . The SM becomes discontinuous when $K \notin \mathbb{N}$, which is the case considered in the present paper. We put the coordinates $(\partial/\partial I, \partial/\partial \theta)$ on the tangent space of \mathbb{T}^2 and the metric naturally induced from the Euclidean structure of \mathbb{R}^2 . This metric defines a distance $\rho(\cdot, \cdot)$ and the corresponding length of a segment A will be denoted as $\text{length}(A)$. The differential of T , in the points $x \equiv (I, \theta)$ where it is defined, is given by the constant matrix

$$DT(x) = A_K = \begin{pmatrix} 1 & K \\ 1 & 1 + K \end{pmatrix}$$

which has eigenvalues $\lambda_{\pm} = 1 + [K \pm (4K + K^2)^{1/2}]/2$, $\lambda_- = \lambda_+^{-1} < 1$. We put $\lambda_- = \lambda$.

We denote by V_- and V_+ the corresponding eigenvectors.

In the following we will identify the tangent space of \mathbb{T}^2 , at each point, with its universal covering \mathbb{R}^2 . Let $V_-(I, \theta) = \{(I, \theta) + tV_-\}$, $t \in \mathbb{R}$, be the line at (I, θ) parallel to the eigenvector V_- .

We set $W_-(I, \theta) = \{V_-(I, \theta) - \text{mod } \mathbb{T}^2\}$, the curve emanating from $(I, \theta) \in \mathbb{R}^2$ on the square $[-\pi, \pi) \times [-\pi, \pi)$ identified with \mathbb{T}^2 with the natural projection. If $\tilde{W}_-(I, \theta)$ denotes the largest connected subset of $W_-(I, \theta)$ containing (I, θ) and where T and hence DT are continuous, then $T\tilde{W}_-(I, \theta) \subset \tilde{W}_-(T(I, \theta))$. Moreover, if $A_{12} \subset \tilde{W}_-(I_1, \theta_1)$ is the segment joining (I_1, θ_1) to $(I_2, \theta_2) \in \tilde{W}_-(I_1, \theta_1)$, we have $\text{length}(TA_{12}) = \lambda \text{length}(A_{12})$.

In a similar way, we define $V_+(I, \theta) = \{(I, \theta) + tV_+\}$, $t \in \mathbb{R}$, the line at $(I, \theta) \in \mathbb{R}^2$ parallel to the eigenvector V_+ , and the sets $W_+(I, \theta) = \{V_+(I, \theta) - \text{mod } \mathbb{T}^2\}$ and $\tilde{W}_+(I, \theta)$ are correspondingly defined. In particular, if $A_{12} \subset \tilde{W}_+(I_1, \theta_1)$ is the segment joining (I_1, θ_1) to $(I_2, \theta_2) \in \tilde{W}_+(I_1, \theta_1)$, we have $\text{length}(T^{-1}A_{12}) = \lambda \text{length}(A_{12})$.

It follows from the definition that the discontinuity lines D_n for the mapping T^n and the discontinuity lines D_{-n} for T^{-n} are respectively given by

$$D_n = \bigcup_{s=0}^{n-1} T^{-s}\gamma \quad \text{and} \quad D_{-n} = \bigcup_{s=0}^{n-1} T^{(s+1)}\gamma$$

where $T^{-1}A$ denotes the preimage of A and γ is the set $\theta = \pm\pi$, $-\pi \leq I < \pi$. We put $\gamma_n = T^{-n}\gamma$ and $\bar{\gamma}_n = T^n\gamma$. We are not really interested in the detailed structure of these sets. Apart from $T\gamma = \bar{\gamma}_1$, which is a closed curve on the torus intersecting γ transversally, each set of the type γ_n (for $\bar{\gamma}_n$ the argument is similar) is the (disjoint) union of segments parallel to each other and whose endpoints lie on the same segment belonging to γ_p , $p < n$, or on two different segments belonging to γ_p and $\gamma_{p'}$ with $p' \leq p < n$. We will refer to these segments as the connected components of γ_n (or $\bar{\gamma}_n$).

Then if V_1 and V_2 are two linear subspaces of \mathbb{R}^2 , we call $\Omega(V_1, V_2)$ the smallest positive angle between them.

Lemma 2.1. If $\Omega(\gamma_n, V_-)$ denotes the angle between any connected component of γ_n and any local stable manifold (i.e., the direction V_-), then $\Omega(\gamma_n, V_-)$ converges to zero for $n \rightarrow +\infty$; similarly, $\Omega(\bar{\gamma}_n, V_+) \rightarrow 0$ and the convergence is monotone. Finally, γ_n and γ_m always intersect transversally for $n \neq m$, and the same holds for $\bar{\gamma}_n$ and $\bar{\gamma}_m$.

Proof. The lemma simply follows upon observing that $\Omega(\gamma, V^\pm) \neq 0$ and if we call V_γ a vector in the direction γ , then $A_K V_\gamma$ is in the direction $T\gamma$ (the transformation $T|_{\theta=\pi}: \gamma \rightarrow \mathbb{T}^2$ is in fact a continuous linear function of $I \in \mathbb{T}^1$). Then, if we set $V_\gamma = C_+ V_+ + C_- V_-$, with $C_+, C_- \in \mathbb{R}$, we have

$$\Omega(\gamma_n, V_-) = \Omega(A_K^{-n}V_\gamma, V_-) = \Omega(C_+ \lambda^n V_+ + C_- \lambda^{-n} V_-, V_-) \xrightarrow{n \rightarrow \infty} 0$$

Moreover,

$$\begin{aligned} \Omega(\gamma_n, \gamma_m) &= \Omega(A_K^{-n}V_\gamma, A_K^{-m}V_\gamma) = \Omega(A_K^{-(n-m)}\tilde{V}_\gamma, \tilde{V}_\gamma) \\ &\text{where } n > m \text{ and } \tilde{V}_\gamma = A_K^{-m}V_\gamma \end{aligned}$$

But the r.h.s. of this equality is different from zero since V_γ is not an eigenvector of A_K . Similar considerations apply to $\bar{\gamma}_n$.

We now prove that the convergence of $\Omega(\gamma_n, V_-)$ and $\Omega(\bar{\gamma}_n, V_+)$ to zero is monotone. Let in fact $Z_n = \tan \alpha_n$ be the tangent of the angle α_n between $\bar{\gamma}_n$ and the θ axis. Then we apply A_K to $\bar{\gamma}_n: Z_{n+1} = 1 - 1/(Z_n + 1 + K)$, $Z_0 = +\infty$. Since $Z_2 < Z_1$, it is easy to show by induction that $Z_{n+1} < Z_n, \forall n > 0$.

In the same way, applying A_K^{-1} to γ_n , we get, setting again $Z_n = \tan \alpha_n$, where α_n is the angle between γ_n and the θ axis,

$$Z_{n+1} = \frac{(1 + K) Z_n - K}{1 - Z_n}, \quad Z_0 = -(1 + K)$$

Since $Z_2 > Z_1$, by induction it follows that $Z_{n+1} > Z_n, \forall n > 0$.

Now we want to show, using a classical argument,⁽¹⁶⁾ that for almost all the points in \mathbb{T}^2 , the segment $\tilde{W}_-(I, \theta)$ [resp. $\tilde{W}_+(I, \theta)$] contains a local stable (resp. unstable) manifold of finite length. We specialize the proof for the stable manifold; the argument for the unstable manifold is similar.

We start by defining the sets, for $n > 0$,

$$\mathcal{A}_n(\sigma) = \left\{ x \in \mathbb{T}^2; \rho(T^n x, \gamma) < \frac{\sigma c}{(n + 1)^2} \right\}$$

where we put for simplicity $x \equiv (I, \theta)$; $\sigma > 0$ and c is a fixed constant.

Lemma 2.2. If $x \notin \mathcal{A}(\sigma) = \bigcup_{n \geq 0} \mathcal{A}_n(\sigma)$, then x has a local stable manifold of length at least σ .

Proof. We put $l(T^n x) = \tilde{W}_-(T^n x)$; clearly, $\text{length}(\tilde{W}_-(T^n x)) \geq \sigma c / (n + 1)^2$; then we denote by $\hat{l}(x_{n-m})$ the connected component of $T^{-m}l(T^n x)$ containing $T^{n-m}x, 0 \leq m \leq n$. Note that $\hat{l}(x_{n-m}) \subset \tilde{W}_-(T^{n-m}x)$. Then let m^* be the last time that $\hat{l}(x_{n-m})$ intersects γ ; since $x \notin \mathcal{A}(\sigma)$, it follows that the length of $\hat{l}(x_{n-m^*})$ is bigger than $\sigma c (n - m^* + 1)^{-2}$. If we now set $l_n^-(x) = T^{-(n-m^*)}\hat{l}(x_{n-m^*})$, we have

$$\text{length}(l_n^-(x)) \geq \sigma c \lambda^{-n+m^*} (n - m^* + 1)^{-2} \geq \sigma$$

provided we choose $c^{-1} = \inf_{x > 0} \lambda^{-x} (x + 1)^{-2}$ [the minimum being attained for $x = -2/(\log \lambda) - 1$]. The local stable manifold at $x, W_-^{\text{loc}}(x)$, is clearly given by $W_-^{\text{loc}}(x) = \bigcap_{n \geq 0} l_n^-(x)$.

From now on we also set $W_+^{\text{loc}}(x)$, the local unstable manifold at x . The only question left is: how big is $\mathcal{A}(\sigma)$?

Lemma 2.3:

$$\mu(\mathcal{A}(\sigma)) \leq c' \sigma \quad \text{for some } c' > 0$$

Proof. Since T is area-preserving, we easily get

$$\mu(\mathcal{A}(\sigma)) \leq \sum_{n=0}^{\infty} \mu(\mathcal{A}_n(\sigma)) \leq \sum_{n=0}^{\infty} 2Lc\sigma(n+1)^{-2} \leq c'\sigma$$

where L denotes the length of the discontinuity line γ .

Consequently, the points that have a local stable manifold of positive size are all the points $x \notin \bigcap_{k=1}^{\infty} \mathcal{A}(1/k)$. But

$$\mu\left(\bigcap_{k=1}^{\infty} \mathcal{A}\left(\frac{1}{k}\right)\right) \leq \lim_{k \rightarrow \infty} \mu\left(\mathcal{A}\left(\frac{1}{k}\right)\right) = \lim_{k \rightarrow \infty} \frac{c'}{k} = 0$$

3. CONSTRUCTION OF THE HOPF CHAIN

In this section we begin the proof of the ergodicity: first of all we show how to construct sufficiently many open neighborhoods each of which belongs to an ergodic component of T ; then we extend the ergodicity to the whole torus. The standard tool in this kind of proof is the classical Hopf argument⁽¹³⁾: the central step is to join almost all the couples of points by a chain of local stable and unstable manifolds, provided the (Lebesgue) measure μ is absolutely continuous along the partitions given by the local stable and unstable manifolds, which is true in our case since the local manifolds are segments and are defined almost everywhere.

Now if $f(x)$ is a continuous function on \mathbb{T}^2 and if we define the “time averages” in the future and in the past, respectively, as

$$f^+(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x); \quad f^-(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{-i} x)$$

then by the Birkhoff–Khinchin ergodic theorem, $f^+(x)$ and $f^-(x)$ exist and are equal to each other for x belonging to a subset of \mathbb{T}^2 of full measure. Moreover, $f^+(x)$ [resp. $f^-(x)$] is constant, when it exists, on the local stable (resp. unstable) manifold at x , by uniform continuity. This fact and the existence of the chain between almost all the couples of points ensure that in these points $f^+(x)$ and $f^-(x)$ assume the same value. Clearly the existence of local manifolds almost everywhere does not guarantee the Hopf chain, since, for example, given two points x and y , the chains leaving x and y could accumulate in a point without intersecting: this could happen if the local manifolds are of arbitrarily small length, which is our present case.

The first step is then the construction of open domains where the time averages $f^+(x)$ and $f^-(x)$ assume the same value on a set of full measure.

We put $\psi = \Omega(V_-, V_+)$; then we fix a positive number ε also satisfying the following conditions:

- (i) $\varepsilon < \frac{\sin \psi}{27}$ and $\varepsilon < \psi$
- (ii) $\frac{\sin \psi \sin \varepsilon}{\sin(\psi - \varepsilon)} \leq 2\varepsilon$

This last condition can be satisfied since the function, in ε , on the left-hand side is convex increasing with derivative in zero equal to one. The prescriptions (i) and (ii) will be justified in the following.

By Lemma 2.3 there will be a number $n_\varepsilon > 0$ such that $\Omega(\gamma_n, V_-) < \varepsilon$ and $\Omega(\bar{\gamma}_n, V_+) < \varepsilon$ for $n > n_\varepsilon$. Then we choose any connected open set U whose closure does not intersect any γ_n and $\bar{\gamma}_n$ for $n < n_\varepsilon$; then U will be contained in the interior of a closed polygonal region whose sides are just connected pieces of γ_n and $\bar{\gamma}_n$, for $n < n_\varepsilon$. Next, for each $\delta \in \mathbb{R}^+$ sufficiently small, we construct a covering $\mathcal{G}(U, \delta)$ of U with the following properties:

(a) The elements $G_i \in \mathcal{G}(U, \delta)$ are rectangles with each pair of opposite sides parallel to V_+ and V_- , respectively, and of equal length δ . Moreover, $G_i \cap \gamma_n = G_i \cap \bar{\gamma}_n = \emptyset$ for each i and $n < n_\varepsilon$. These rectangles are clearly the image of true rectangles in \mathbb{R}^2 under the natural projection.

(b) The elements G_i must satisfy some intersection properties, which can be formulated in a very general setting.⁽⁷⁻¹⁰⁾ Instead, we prefer to construct explicitly the covering $\mathcal{G}(U, \delta)$ and those properties will be a direct consequence of our construction. To do this, we cover U with a grid of rectangles of size $\alpha\delta$, $0 < \alpha < 1$, satisfying (a) and then we enlarge each rectangle to size δ , provided (a) is still satisfied. Then we put $\alpha = 1 - 3\varepsilon/\sin \psi$; this and the bounds (i) on ε imply that each rectangle intersects eight other rectangles of the covering and for two intersecting rectangle G_i and G_j which have in common the two stable (resp. unstable) sides, i.e., the sides in the direction V_- (resp. V_+), we have $\mu(G_i \cap G_j) = 3\varepsilon\delta^2$ (see Fig. 1). As we will see in a moment, these properties largely suffice for our purposes.

Given an element G_i of the covering, we call E_i^- a neighborhood of its stable faces, namely $E_i^- = E_i^-(1) \cup E_i^-(2)$, where one side of $E_i^-(j)$, $j = 1, 2$, is a stable side of G_i and the other is along the unstable side and of length $3\varepsilon\delta/\sin \psi$. In the same way we define a neighborhood E_i^+ of the unstable faces and the length along the stable sides of each component of E_i^+ is still $3\varepsilon\delta/\sin \psi$. Then $\mu(E_i^\pm(j)) = 3\delta^2\varepsilon$, $j = 1, 2$ (see Fig. 2).

Definition 3.1. We call stable (resp. unstable)-proper an element $G_i \in \mathcal{G}(U, \delta)$ if the measure of the points in $E_i^-(j)$ [resp. $E_i^+(j)$], $j = 1, 2$,

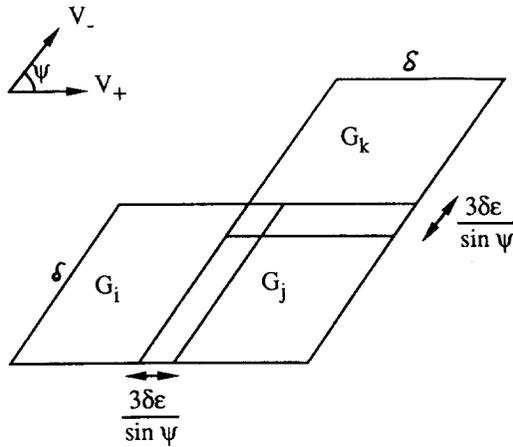


Fig. 1. Construction of the covering.

for which $W_-^{loc}(x)$ [resp. $W_+^{loc}(x)$] intersects both the unstable (resp. stable) sides of G_i is positive. If this is not the case, the rectangle is said to be stable (resp. unstable)-improper.

Finally, we call $\mathcal{G}_I^-(U, \delta)$ the union of the stable-improper rectangles and $\mathcal{G}_I^+(U, \delta)$ the union of the unstable-improper ones, and $\mathcal{G}_I(U, \delta) = \mathcal{G}_I^-(U, \delta) \cup \mathcal{G}_I^+(U, \delta)$.

Our main result (the Transversal Fundamental Theorem for the sawtooth map) can now be stated as follows.

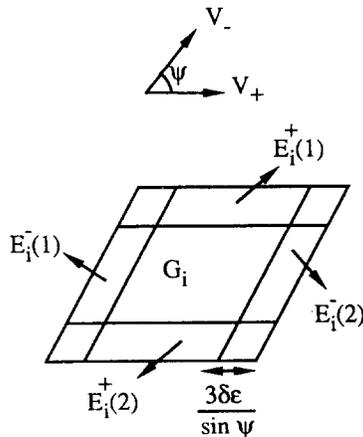


Fig. 2. Stable and unstable faces of a rectangle.

Theorem 3.2:

$$\lim_{\delta \rightarrow 0^+} \frac{\mu(\mathcal{G}_\delta(U, \delta))}{\delta} = 0$$

The proof is given in Section 4; now we want to derive and discuss the main consequences about ergodicity.

Remark 3.3. Theorem 3.2 and the absolute continuity of μ imply ergodicity,

It can be shown that a partition of the torus into segments of local stable and unstable manifolds is measurable, with conditional measures equivalent to arc length.⁽¹⁵⁾ Here we will simply show that an application of the Fubini theorem on the rectangular domains of the covering will suffice for our purposes.

We note that the preceding prescriptions on the covering $\mathcal{G}(U, \delta)$ imply that if G_i and G_j are two stable proper rectangles (for the unstable ones, the argument is similar) intersecting in, say, $G_{ij} = E_i^-(1) = E_j^-(2)$, then there is a set of positive measure of points $x \in G_{ij}$ for which

$$\text{length}(W_-^{\text{loc}}(x) \cap G_i) = \text{length}(W_-^{\text{loc}}(x) \cap G_j) = \delta$$

If we consider any interesting couple $E_i^-(l_1), E_j^+(l_2)$ belonging or not to the same proper (both stable and unstable) rectangle, denoting with V_i^- and V_j^+ the subsets, respectively, of $E_i^-(l_1)$ and $E_j^+(l_2)$ of positive measure prescribed by the definition of properness, we want to prove that $\mu(V_i^- \cap V_j^+) > 0$ (see Fig. 3).

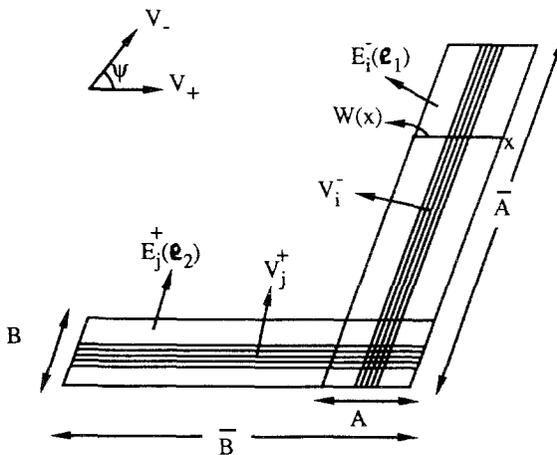


Fig. 3. Construction for the absolute continuity of the measure.

First of all note that $V_i^- \cap V_j^+ \neq \emptyset$ and moreover $V_i^- \cap V_j^+$ can be written as the Cartesian product $\tilde{A} \times \tilde{B}$, where $\tilde{A} = A \cap V_i^-$ and $\tilde{B} = B \cap V_j^+$, with A an unstable side of $E_i^-(l_1)$ and B a stable side of $E_j^+(l_2)$. If μ_A and μ_B denote, respectively, the Lebesgue measures restricted to A and B , we put $\lambda_A = \mu_A$ and $\lambda_B = \sin \psi \mu_B$ and we call them conditional (with respect to μ) along segments in the directions V_{\pm} . Since $V_i^- = \tilde{A} \times \tilde{A}$ and $V_j^+ = \tilde{B} \times \tilde{B}$, where \tilde{A} and \tilde{B} are, respectively, the stable side of $E_i^-(l_1)$ and the unstable side of $E_j^+(l_2)$, we have by the Fubini theorem that $\mu(V_i^-) = \delta \sin \psi \lambda_A(\tilde{A}) > 0$ and $\mu(V_j^+) = \delta \lambda_B(\tilde{B}) > 0$ and this implies

$$\mu(V_i^- \cap V_j^+) = \mu(\tilde{A} \times \tilde{B}) = \lambda_A(\tilde{A}) \lambda_B(\tilde{B}) > 0$$

We call the BK set (from Birkhoff–Khinchin) the set of full measure where the time averages in the future and in the past exist and are equal to each other. The result, proved previously, reads: for λ_A -almost the points $x \in \tilde{A}$, λ_B -almost the points in $W_i^-(x) \cap V_j^+$ belong to the BK set and this shows how to construct a chain of local stable and unstable manifolds whose intersection points belong to the BK set. But this is clearly restricted to the set V_i^{\pm} of positive measure in the proper rectangles. To extend it, we note that an argument like that used before shows that for λ_A -almost the points $x \in \tilde{A}$, we have that $\lambda_{W(x)}$ -almost the points in $W(x) \cap V_i^-$ belong to the BK set: $\lambda_{\tilde{A}}$ is the conditional measure on the stable side \tilde{A} of $E_i^-(l_1)$ and $\lambda_{W(x)}$ the conditional measure on the segment $W(x)$ passing through x in the direction V_+ and touching the opposite sides of $E_i^-(l_1)$; note that $\lambda_{W(x)}(W(x) \cap V_i^-) = \lambda_A(\tilde{A})$ (see Fig. 3).

A similar construction holds for the set of type $E_i^+(\cdot)$. Now we exclude from U the set U_0 composed by the union of all the segments $W_{\pm}(x) \cap U$, $x \in U$ [see Section 2 for the definition of the line $W_{\pm}(\cdot)$], each of which includes subsets of the complement of the BK set of positive conditional measure. The set U_0 is of zero μ -measure again by the Fubini theorem. Then, for any other point of U/U_0 not belonging to the sets $\{V_i^{\pm}\}$ but with a stable or unstable local manifold intersecting both the opposite sides of a set of type $E_i^{\pm}(\cdot)$, the local manifold will meet the corresponding V_i^{\pm} in a set of positive conditional measure, from which one can start the chain. We can effectively connect any two points in U/U_0 with the chain, making use of the following argument. We will prove in a moment that, if $\mathcal{G}_p^c(U, \delta)$ denotes the largest connected union of proper (both stable and unstable) rectangles, then

$$\mu(U/\mathcal{G}_p^c(U, \delta)) \xrightarrow{\delta \rightarrow 0^+} 0$$

We claim that, given two points in U/U_0 , their local manifolds will meet the set $\mathcal{G}_p^c(U, \delta)$ [in particular, intersecting completely a set of type

$E_i^\pm(\cdot)$], for δ sufficiently small. If this is not the case, two situations could occur:

(a) For any δ , the shortest of the local manifolds of two points of length, say, A is covered by improper rectangles. But the measure of these rectangles is at least $\delta^2 \sin \psi [(A + \eta)/(\delta - \eta)]$, where $\eta = 4\epsilon\delta/\sin \psi$ arises from the mutual intersection of the rectangles, and so, dividing by δ and sending δ to zero, we get a term of order 1, which contradicts Theorem 3.2.

(b) For any δ the same shortest local manifold is covered by improper and proper rectangles not belonging to $\mathcal{G}_p^c(U, \delta)$. These proper rectangles belong to a connected component which, however, cannot cover all of U : then each of these connected components will be “encircled” by improper rectangles, thus leading to a situation of the type described in (a) in the limit $\delta \rightarrow 0$.

What is left to prove is then:

Proposition 3.4:

$$\mu(U/\mathcal{G}_p^c(U, \delta)) \xrightarrow{\delta \rightarrow 0^+} 0$$

Proof. If the statement is not true, there is a sequence $\{\delta_\eta\}$ decreasing to zero for which

$$\mu(\mathcal{G}_p^c(U, \delta_\eta)) \xrightarrow{\eta \rightarrow \infty} g < \mu(U)$$

We treat now the case $g > 0$; the case $g = 0$ is essentially similar and we refer to Appendix A for the details. Let us consider the rectangles of $\mathcal{G}_p^c(U, \delta_\eta)$ whose sides are not completely included in another element of $\mathcal{G}_p^c(U, \delta_\eta)$ and among these sides choose those, $\{l_\alpha\}$, not belonging to the complement of U . These sides will intersect an improper rectangle of $\mathcal{G}(U, \delta_\eta)$ and we put $L(\delta_\eta) = \sum_\alpha \text{length}(l_\alpha)$. The measure of the improper rectangles intersecting the $\{l_\alpha\}$ is surely bounded from below by $\frac{1}{4}L(\delta_\eta) \delta_\eta \sin \psi d(\epsilon)$, where the factor 1/4 arises from the fact that we could have four transversal l_α giving only one improper rectangle, and $d(\epsilon)$ is a positive factor given by the measure of the intersections of the improper rectangles [for example, $d(\epsilon)$ can be chosen as $1 - 27\epsilon/\sin \psi$ with $\epsilon < (\sin \psi)/27$; see condition (i) on the choice of ϵ]. Now there exists a subsequence δ_{η_i} for which $L(\delta_{\eta_i}) \xrightarrow{i \rightarrow \infty} \zeta > 0$, because otherwise we could find another subsequence δ_{η_j} for which $L(\delta_{\eta_j}) \xrightarrow{j \rightarrow \infty} 0$ and consequently

$$\mu(\mathcal{G}_p^c(U, \delta_{\eta_j})) \xrightarrow{j \rightarrow \infty} \mu(U)$$

In fact, the limit $L(\delta_{\eta_j}) \rightarrow 0$ implies that the complement of $\mathcal{G}_p^c(U, \delta_{\eta_j})$ in U has measure tending to zero in the limit $j \rightarrow \infty$, by the isoperimetric theorem. Then, for the subsequence δ_{η_i} , the measure of the improper rectangles considered before gives

$$\lim_{i \rightarrow \infty} \frac{L(\delta_{\eta_i}) \delta_{\eta_i} \sin \psi d(\varepsilon)}{4\delta_{\eta_i}} > 0$$

which contradicts Theorem 3.2.

Now we are ready to prove the global ergodicity.

Proposition 3.5. The transformation T is ergodic on the torus \mathbb{T}^2 with respect to the Lebesgue measure.

Proof. We know that the closure of U is included in the interior of a polygonal region, say R , bounded by connected segments of γ_n and $\bar{\gamma}_n$, for $n < n_\varepsilon$. Among them take a connected subset $\tilde{\gamma}_m \subset \gamma_m$, $m < n_\varepsilon$, such that we can find an open connected set O including $\tilde{\gamma}_m$ but not intersecting any $\bar{\gamma}_n$ for $n < n_\varepsilon$. On the set O we can perform the covering in rectangles of size δ described above and find that the largest connected union of proper “unstable” rectangles will cover $O \pmod{0}$ in the limit $\delta \rightarrow 0$. Then there will be sets of points of positive Lebesgue measure whose local unstable manifolds connect open domains in R and in the complement of R , thus continuing the Hopf chain outside U .

Proposition 3.6. The transformation T is Bernoulli.

Proof. Since Proposition 3.5 can be easily extended to all the powers of T and our map is a piecewise linear almost hyperbolic transformation, we can apply the results in ref. 14 to conclude that it is also Bernoulli.

4. PROOF OF THEOREM 3.2

We perform the proof only with respect to the stable direction, the other case being completely equivalent.

The strategy of the proof is the following: we start by computing the measure of the set of rectangles, denoted by $\mathcal{G}_{<N}(\delta)$, crossed by two or more transversals γ_n , $n_\varepsilon < n \leq N$. We include these rectangles in the stable-improper set $\mathcal{G}_I^-(U, \delta)$ and we get $\mu(\mathcal{G}_{<N}(\delta)) \leq C_1(N) \delta^2$, where $C_1(N)$ depends only on N (and ε , which is fixed). An important remark is that δ can be chosen depending on N so small that no rectangle contains parallel components of γ_n for $n \leq N$.

This suggests that we define N as a function of δ . Let us in fact call $\nu(N)$ the minimum distance between parallel components of γ_n , $n_\varepsilon < n \leq N$;

clearly $v(N)$ is a positive function going to 0 for $N \rightarrow \infty$. At the same time the function $C_1(N)$ introduced above (see later for its precise definition) is a finite function of N growing to ∞ for $N \rightarrow \infty$. This makes it possible to find, for δ smaller than a $\bar{\delta}$ sufficiently small, a function $N(\delta) > n_\varepsilon$ growing to infinity for $\delta \rightarrow 0$ and satisfying, for $\delta < \bar{\delta}$, $v(N(\delta)) > \text{diameter}(G_i)$ and, for example, $C_1(N(\delta)) < \delta^{-\beta}$, $\beta < 1$, implying $C_1(N(\delta)) \delta \rightarrow 0$ when $\delta \rightarrow 0$. (*Proof*: see Appendix B.)

We will use these facts in a moment; from now on we fix $\delta < \bar{\delta}$, and for simplicity of notation we put $N \equiv N(\delta)$. Note, however, that we could keep N constant; in fact, the relation (**) below shows that we can send $N \rightarrow \infty$ at the end (since $\lambda < 1$).

Then, given a single $\gamma_n, n_\varepsilon < n \leq N$, intersecting a rectangle, we compute the measure of the set of points whose local stable manifolds are prevented by γ_n from crossing entirely the rectangle.

The measure of this set is at most $2\varepsilon\delta^2$ and if this set belongs, in the worst case, to a set of type $E_i^-(\cdot)$, we see that we still have a region of measure $\varepsilon\delta^2$ that needs to be filled with too short local stable manifolds in order to declare the rectangle stable-improper.

Then the set of points with too short stable manifold in a rectangle crossed by one $\gamma_n, n_\varepsilon < n \leq N$, belongs to $D = \bigcup_{p=1}^\infty D_{N+p}$, where D_{N+p} is defined as follows: given $y \in \mathbb{T}^2$, call $\alpha_{N+p}(y)$ the point of the set $W_-(y) \cap \gamma_{N+p}$ closest to y , and $l_{y, \alpha_{N+p}(y)}$ the segment of endpoints y and $\alpha_{N+p}(y)$ in $W_-(y)$. Then

$$D_{N+p} = \{y \in \mathbb{T}^2; \text{length}(l_{y, \alpha_{N+p}(y)}) \leq \delta, \\ \text{and } l_{y, \alpha_{N+p}(y)} \cap \gamma_j = \emptyset, 0 \leq j \leq N+p-1\}$$

We will show that $\mu(D) \leq 4\pi\delta\lambda^N/(1-\lambda)$; moreover, we include a rectangle G_i in $\mathcal{G}_I^-(U, \delta)$ if

$$\mu(G_i \cap D) \geq \delta^2\varepsilon \geq \frac{\varepsilon}{\sin \psi} \mu(G_i) \tag{*}$$

The last information allows us to compute the measure of the union of all the rectangles, denoted $\mathcal{G}_{>N}(\delta)$, satisfying (*), namely $\mu(\mathcal{G}_{>N}(\delta)) \leq C_2\delta\lambda^N$, where C_2 is a constant depending only on ε and λ .

In conclusion, the measure of all the stable-improper rectangles will be bounded by

$$\mu(\mathcal{G}_I^-(U, \delta)) \leq \mu(\mathcal{G}_{<N}(\delta)) + \mu(\mathcal{G}_{>N}(\delta)) \leq C_1(N) \delta^2 + C_2\delta\lambda^N \tag{**}$$

Dividing by δ , sending δ to zero, and remembering the properties of $N = N(\delta)$, we get

$$\lim_{\delta \rightarrow 0} \frac{\mu(\mathcal{G}_I^-(U, \delta))}{\delta} = 0$$

Now we illustrate the simple geometric considerations to justify the preceding arguments.

Let us suppose a rectangle G_i intersects two connected components of γ_n , $n_\varepsilon < n \leq N$; we call γ_m and γ_p the straight lines starting from the point 0 at angle ϕ_{mp} and intersecting G_i . The problem is now to find a circle of center 0 including G_i . Referring to the situation sketched in Fig. 4, where the distance between 0 and DD' is larger than the distance between 0 and PP' , since the convergence of the γ_n is monotone by Lemma 2.1 and supposing that the angle (measured counterclockwise) between the two oriented vectors $P\vec{A}$ and γ_n in Fig. 4 is smaller than ε , there are two cases to consider, namely

$$(i) \begin{cases} \gamma_m \cap DD' \neq \emptyset \\ \gamma_p \cap DD' \neq \emptyset \end{cases} \quad \text{and} \quad (ii) \begin{cases} \gamma_m \cap DD' = \emptyset \\ \gamma_p \cap DD' \neq \emptyset \end{cases}$$

The largest segment, of length S , intercepted by γ_m and γ_p on the straight line DD' occurs when $\Omega(PA, \gamma_m) = 0$ and $\Omega(V_-, \gamma_p)$ approaches 0. A simple geometrical construction gives

$$S = \delta \left[1 + \frac{\sin \varepsilon}{\sin(\psi - \varepsilon)} \right] = \delta S_1(\varepsilon, \psi)$$

The two cases considered above correspond to two different types of triangles, those having a vertex at 0, one side along γ_p , another side joining 0 to D [case (i)] or 0 to E (not to D , just to simplify the computation) [case (ii)], and finally the third side along DD' . A straightforward investigation shows that the largest of the two sides intersecting at 0 (which

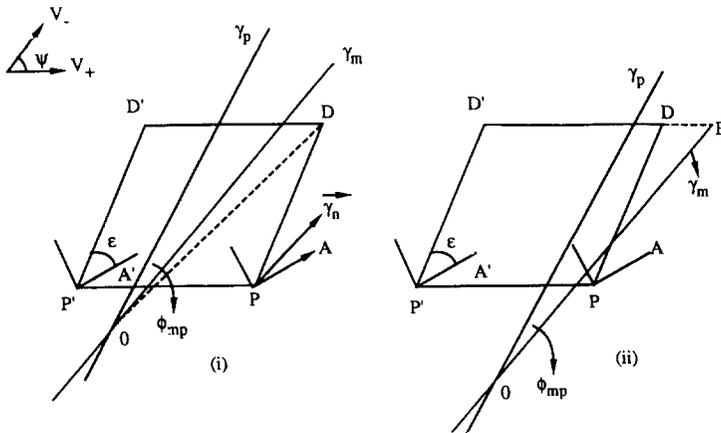


Fig. 4. Possible intersections of a rectangle with two discontinuity lines.

will be the radius of the circle we are looking for) is, in both cases, bounded by $\delta S_1(\varepsilon, \psi)[\cot \phi_{mp} + 1]$. This allow us to compute the measure of the set of rectangles in U which are intersected by more than one (non-parallel) $\gamma_n, n_e < n \leq N$; we have

$$\mu(\mathcal{G}_{<N}(\delta)) \leq \delta^2 C_1(N)$$

where

$$C_1(N) = \pi S_1^2(\varepsilon, \psi) \sup \{ (\cot \phi_{mp} + 1)^2; n_e < m, p \leq N \}$$

$$\text{and } \gamma_p \cap U \neq \emptyset, \gamma_m \cap U \neq \emptyset \}$$

$$\cdot \# \{ \text{intersection points in } U \text{ among the } \gamma_n, n_e < n \leq N \}$$

Note that the cardinality of the intersection points is finite since the γ_n have finite length and meet transversally.

The next step is to estimate the area intercepted by one singularity line $\gamma_n, n_e < n \leq N$, in the rectangle G_i . Since $\Omega(V_-, \gamma_n) < \varepsilon$, it is an easy exercise to show that this area is always bounded by $(\delta^2 \sin \psi \sin \varepsilon) / \sin(\psi - \varepsilon)$ for the two possible cases sketched in Fig. 5. By suitable choice of ε , that area is smaller than $2\varepsilon\delta^2$ and this justifies the prescription (ii) on the choice of ε .

Finally, we have to estimate the measure of the set $D = \bigcup_{p=1}^{\infty} D_{N+p}$. Clearly

$$D_{N+p} \subset B_{N+p} = \{ y; \rho(T^{N+p}y, \gamma) \leq \lambda^{N+p}\delta \}$$

and, by the invariance of the measure,

$$\mu(B_{N+p}) \leq 4\pi\lambda^{N+p}\delta, \quad \text{so that } \mu(D) \leq \frac{4\pi\delta\lambda^N}{1-\lambda}$$

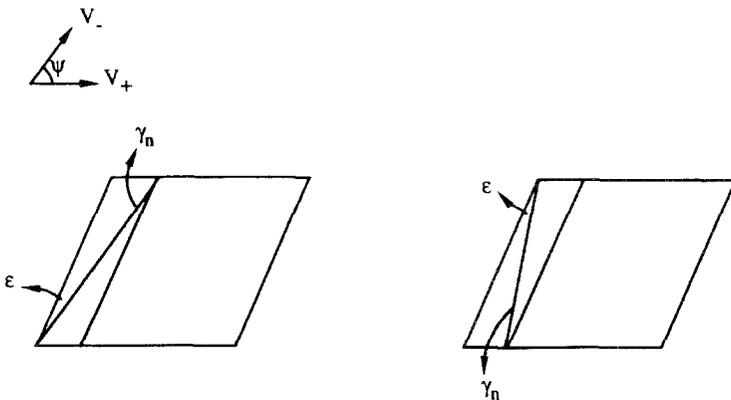


Fig. 5. Possible intersections of a rectangle with one discontinuity line.

As we have already said, we consider G_i stable-improper if $\mu(G_i \cap D) \geq (\varepsilon/\sin \psi) \mu(G_i)$: we call it G'_i . Then the measure of all the stable improper rectangles of this type satisfies, using the property (b) of the covering,

$$\begin{aligned} \mu(\mathcal{G}_{>N}(\delta)) &\leq \frac{\sin \psi}{\varepsilon} \sum_i \mu(G'_i \cap D) \\ &\leq \frac{8 \sin \psi}{\varepsilon} \mu(\mathcal{G}_{>N}(\delta) \cap D) \leq \frac{32\pi \sin \psi \delta \lambda^N}{\varepsilon(1-\lambda)} \end{aligned}$$

which concludes the proof.

APPENDIX A

We prove Proposition 3.4 when there exists a nonincreasing sequence $\{\delta_\eta\}$ for which $\mu(\mathcal{G}_p(U, \delta_\eta)) \rightarrow_{\eta \rightarrow \infty} 0$. In this case we have to change a little the argument in the proof of Proposition 3.4. If $\mathcal{G}_p(U, \delta_\eta)$ denotes the union of the proper, both stable and unstable, rectangles, we have two cases to consider:

- (i) $\limsup_{\eta \rightarrow \infty} \mu(\mathcal{G}_p(U, \delta_\eta)) < \mu(U)$
- (ii) $\limsup_{\eta \rightarrow \infty} \mu(\mathcal{G}_p(U, \delta_\eta)) = \mu(U)$

In the first case the total measure of the improper rectangles is asymptotically positive, so that, dividing by δ_η and taking the limit for $\eta \rightarrow +\infty$, we get a contradiction with Theorem 3.2. The case (ii) requires a close investigation.² Let δ_{η_k} be a subsequence for which we have the existence of the limit in (ii); then we estimate the measure of the set of improper rectangles:

$$\mu(\mathcal{G}_i(U, \delta_{\eta_k})) \geq \delta_{\eta_k} \frac{\sin \psi}{4} \frac{d(\varepsilon)}{\sum_{i=1}^{N(\delta_{\eta_k})} L_i(\delta_{\eta_k})}$$

where $N(\delta_{\eta_k})$ is the number of connected components of $\mathcal{G}_p(U, \delta_{\eta_k})$, including also simple rectangles, and $L_i(\delta_{\eta_k})$ is defined in the following way. Let $\mathcal{G}_{p,i}(U, \delta_{\eta_k})$ be the i th connected component of $\mathcal{G}_p(U, \delta_{\eta_k})$; then consider those rectangles of $\mathcal{G}_{p,i}(U, \delta_{\eta_k})$ whose sides are not completely included in another element of $\mathcal{G}_{p,i}(U, \delta_{\eta_k})$ and among these sides choose those,

² We note at this point that it is sufficient to prove the result for an open subdomain $U' \subset U$ made of rectangles belonging to a grid of fixed size $\bar{\delta}$ with the boundary sides taken off; in this way the boundary of U' is a piecewise linear curve and the geometric considerations are very easy.

$\{l_{i,\delta_{\eta_k},\alpha}\}_\alpha$, not belonging to the complement of U . We now assume that none of the boundary sides of the $\mathcal{G}_{p,i}(U, \delta_{\eta_k})$, $i = 1, \dots, N(\delta_{\eta_k})$, belongs to the complement of U : this restriction can be easily removed by observing that in the limit $k \rightarrow +\infty$ the sum of the length of the $l_{i,\delta_{\eta_k},\alpha}$ is larger than or equal to the sums of the length of the sides belonging to the complement of U , so that the r.h.s. in the bound (o) below must be multiplied by 1/2. Then we put $L_i(\delta_{\eta_k}) = \sum_\alpha \text{length}(l_{i,\delta_{\eta_k},\alpha})$. But $L_i(\delta_{\eta_k})$ is larger than the perimeter enclosing $\mathcal{G}_{p,i}(U, \delta_{\eta_k})$, denoted $P_i(\delta_{\eta_k})$. Now let us suppose we glue together at the boundary, in some way, all the $\mathcal{G}_{p,i}(U, \delta_{\eta_k})$: doing this, we get a polygon whose perimeter is less than

$$\sum_{i=1}^{N(\delta_{\eta_k})} P_i(\delta_{\eta_k})$$

but with the same area of the (disconnected) union of the $\mathcal{G}_{p,i}(U, \delta_{\eta_k})$. Then by the isoperimetric theorem we get

$$\mu(\mathcal{G}_I(U, \delta_{\eta_k})) \geq \delta_{\eta_k} \frac{\sin \psi d(\varepsilon)}{2} [\pi \mu(\mathcal{G}_p(U, \delta_{\eta_k}))]^{1/2} \quad (o)$$

which again implies that $\mu(\mathcal{G}_I(U, \delta_{\eta_k}))/\delta_{\eta_k}$ is different from zero in the limit $k \rightarrow \infty$.

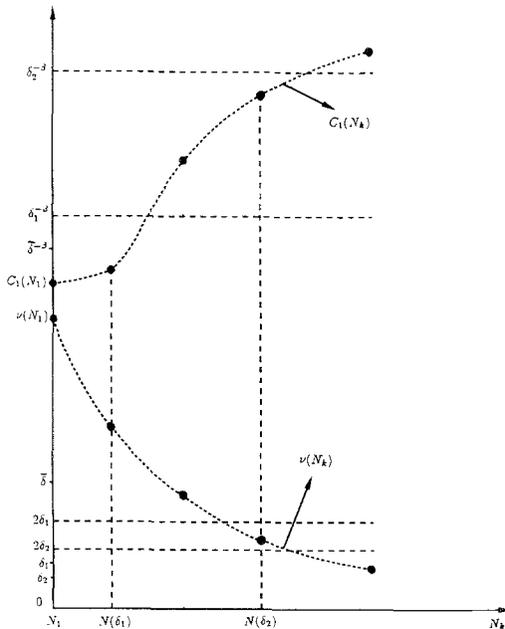


Fig. 6. Graph of $C_1(N_k)$ and $\nu(N_k)$ versus N_k and construction of the function $N(\delta)$.

APPENDIX B

We know that $\lim_{N \rightarrow \infty} C_1(N) = +\infty$ and $\lim_{N \rightarrow \infty} \nu(N) = 0$. Then it is always possible to choose a subsequence $N_k, k \geq 1$, with the following properties: (i) $C_1(N_k)$ and $\nu(N_k)$ go to the respective limits monotonically; (ii) $C_1(N_1) \geq \nu(N_1)$; (iii) $N_1 > n_e$. Now we fix $\bar{\delta}$ and β in such a way that $\bar{\delta} < \nu(N_1)/2$ (we use the fact that the diameter of a rectangle G_i of size δ is simply bounded by 2δ), and $\bar{\delta}^{-\beta} > C_1(N_1)$; for example, we can take $\bar{\delta} < \inf\{1/C_1(N_1); \nu(N_1)/2\}$ and then $[\log C_1(N_1)/\log \bar{\delta}^{-1}] < \beta < 1$. Then for any $\delta < \bar{\delta}$ we can always find an $N_e, e \geq 1$, such that $C_1(N_e) < \delta^{-\beta}$ and $\nu(N_e) > 2\delta$ (see Fig. 6). Choose finally, for the given δ , $N(\delta) = N_e$ in such a way that $N(\delta)$ grows to ∞ for $\delta \rightarrow 0^+$: this is surely possible by the already stated asymptotic properties of the sequences $C_1(N_k)$ and $\nu(N_k)$.

ACKNOWLEDGMENTS

This work would have been impossible without the constant help, concrete collaboration, and encouragement of my friend Carlangelo Liverani. He taught me the Sinai–Chernov theory that is the starting point of this work. I also to thank N. Simányi for very useful discussions and the Department of Mathematics of the University of Southern California (Los Angeles), where this work was initiated. This work was supported by Contract CEE n. SC1*0281.

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