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Almost sure invariance principle for random piecewise expanding maps

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Abstract

We prove a fiberwise almost sure invariance principle for random piecewise expanding transformations in one and higher dimensions using recent developments on martingale techniques.

Keywords: almost sure invariance principle, random dynamical systems, piecewise expanding maps

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1. Introduction

The objective of this note is to prove the almost sure invariance principle (ASIP) for a large class of random dynamical systems. The random dynamics is driven by an invertible, measure preserving transformation σ of $(\Omega, \mathcal{F}, \mathbb{P})$ called the base transformation. Trajectories in the phase space X are formed by concatenations $f_\omega^n := f_{\sigma^{n-1}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega$ of maps from a family of maps $f_\omega : X \to X$, $\omega \in \Omega$. For a systematic treatment of these systems we refer to [2]. For sufficiently regular bounded observables $\psi_\omega : X \to \mathbb{R}$, $\omega \in \Omega$, an almost sure invariance principle guarantees that the random variables $\psi_{\sigma^n\omega} \circ f_\omega^n$ can be matched with trajectories of a Brownian motion, with the error negligible compared to the length of the trajectory. In the present paper, we consider observables defined on some measure space (X, m) which is endowed with a notion of variation. In particular, we consider examples where the observables are functions of bounded variation or quasi-Hölder functions on a compact subset X of $\mathbb{R}^n$. Our setting is quite similar to that of [3], where the maps f_ω are called random Lasota–Yorke maps.

In a more general setting and under suitable assumptions, kifer proved in [13] central limit theorems (CLT) and laws of iterated logarithm; we will briefly compare Kifer's ssumptions with ours in remark 3 below. In [13, remark 2.7], Kifer claimed without proof (see [13, remark 4.1]) a random functional CLT, i.e. the weak invariance principle (WIP), and also a strong version of the WIP with almost sure convergence, namely the almost sure invariance principle (ASIP), referring to techniques of Philip and Stout [20].

Other early works on the ASIP for deterministic dynamical systems go back to Field, Melbourne and Török [8], Melbourne and Nicol [18, 19], and very recently to Korepanov [14, 15]. Another important contribution to this field, using a different approach with respect to the aforementioned papers, is Gouëzel's article [8]. All these papers also deal with the error term in the convergence of the process. The Gouëzel method was used in [1] to get the ASIP for the stationary random dynamical systems of *annealed* type, in contrast to the *quenched* systems which are the object of this paper.

In fact we present here a proof of the ASIP for our class of random transformations, following a method recently proposed by Cuny and Merlèvede [7]. This method is particularly powerful when applied to non-stationary dynamical systems; it was successfully used in [11] for a large class of *sequential* systems with some expanding features and for which only the CLT was previously known [6]. We stress that the ω-fibered random dynamical systems discussed above are also non-stationary since we use ω-dependent sample measures (see below) on the underlying probability space.

Although our method for establishing ASIP follows closely the strategy outlined in [11], the results in [11] deal with a different type of systems to the ones studied in the present paper. In [11] the authors consider sequential dynamical systems induced from a sequence of transformations $(T_k)_{k\in\mathbb{N}}$ which are then composed as:

$$\mathcal{T}_n := T_n \circ \ldots \circ T_1, \quad n \in \mathbb{N}. \tag{1}$$

In the present work the concatenations f_ω^n are driven by the ergodic, measure-preserving transformation σ on the base space $(\Omega, \mathcal{F}, \mathbb{P})$: we point out that no mixing hypotheses are imposed on σ. Our arguments exploit the fact that under the assumptions of our paper, the associated skew product transformation τ (see (3)) has a unique absolutely continuous invariant measure μ (see (6)), while in the context of sequential systems there is no natural notion of invariant distribution even after enlarging the space. In particular, the probability underlying our random processes will be given by the conditional measure μ_ω which exhibits the equivariance property, see section 2.1: this will allow us to prove the linear growth of the variance and finally to approach the $n^{1/4}$ rate for the ASIP error. These are considerable improvements over corresponding results for sequential systems, where one needs very strong assumptions to ensure the growth of the variance; see lemma 7.1 in [11].

The rate which we obtain by approximating our process with a sum of i.i.d. Gaussian variables (the content of the ASIP) is of order $n^{1/4}$ times a logarithmic correction, which is very close to the $n^{1/4}$ rate for various classes of dynamical systems, not necessarily uniformly expanding or uniformly hyperbolic. For instance the ASIP with rate $n^{1/4+\varepsilon}$ was proved in the scalar and vector cases respectively in [8] and [9]. A scalar ASIP with rate $n^{1/4}$ times logarithmic corrections is the result in [7], which inspired the present work. It would be interesting to see if it is possible to obtain sharper estimates in the random setting, either following the approach of [9], and therefore generalizing our results to vector ASIP, or the techniques of the recent work by Korepanov [15] which, in the context of nonuniformly expanding and nonuniformly hyperbolic transformations with exponential tails, improves the rates to n^{ε}, for any $\varepsilon > 0$. Recent results which claim "essentially optimal rates for slowly (polynomially) mixing deterministic dynamical systems, such as Pomeau–Manneville intermittent maps, with Hölder continuous observables" are given in [5], which contains also other references on previous results on the ASIP for different circumstances and techniques.

2. Preliminaries and statement of the main results

2.1. Preliminaries

We introduce in this section the fiber maps and the associated function spaces which we will use to form the random concatenations. We will call them *random expanding transformations*, or *random Lasota–Yorke maps*. We will refer to and use the general assumptions for these maps as proposed by Buzzi [3] in order to use his results on quenched decay of correlations. However, we will strengthen a few of those assumptions with the aim of obtaining limit theorems. Our additional conditions are similar to those called *Dec* and *Min* in the paper [6], where they were used to establish and recover a property akin to quasi-compactness for the composition of transfer operators.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\sigma : \Omega \to \Omega$ be an invertible $\mathbb{P}$-preserving transformation. We will assume that $\mathbb{P}$ is ergodic. Moreover, let $(X, \mathcal{B})$ be a measurable space endowed with a probability measure m and a notion of a variation $\text{var}\colon L^1(X, m) \to [0, \infty]$ which satisfies the following conditions:

(V1) $\text{var}(th) = |t| \, \text{var}(h)$;

(V2) $\text{var}(g + h) \leqslant \text{var}(g) + \text{var}(h)$;

(V3) $\|h\|_\infty \leqslant C_{\text{var}}(\|h\|_1 + \text{var}(h))$ for some constant $1 \leqslant C_{\text{var}} < \infty$;

(V4) for any $C > 0$, the set $\{h\colon X \to \mathbb{R} : \|h\|_1 + \text{var}(h) \leqslant C\}$ is $L^1(X,m)$-compact;

(V5) $\text{var}(1_X) < \infty$, where 1_X denotes the function equal to 1 on X;

(V6) $\{h\colon X \to \mathbb{R}_+ : \|h\|_1 = 1 \text{ and } \text{var}(h) < \infty\}$ is $L^1(X,m)$-dense in $\{h\colon X \to \mathbb{R}_+ : \|h\|_1 = 1\}$.

(V7) there exists $K_{\text{var}} < \infty$ such that

$$\text{var}(gh) + \|gh\|_1 \leqslant K_{\text{var}}(\text{var}(h) + \|h\|_1)(\text{var}(g) + \|g\|_1), \quad \text{for every } g, h \in BV, \tag{2}$$

where

$$BV = BV(X, m) = \{h \in L^1(X, m) : \text{var}(h) < \infty\};$$

(V8) for any $g \in L^1(X, m)$ such that $\operatorname{ess\,inf} g > 0$, we have $\text{var}(1/g) \leqslant \frac{\text{var}(g)}{(\operatorname{ess\,inf} \ g)^2}$.

We recall that BV is a Banach space with respect to the norm

$$\|h\|_{BV} = \text{var}(h) + \|h\|_1 .$$

On several occasions we will also consider the following norm

$$\|h\|_{\text{var}} = \text{var}(h) + \|h\|_\infty ,$$

on BV which (although different) is equivalent to $\|\cdot\|_{BV}$.

Let $f_\omega : X \to X$, $\omega \in \Omega$ be a collection of mappings on X. The associated skew product transformation $\tau : \Omega \times X \to \Omega \times X$ is defined by

$$\tau(\omega, x) = (\sigma\omega, f_\omega(x)), \tag{3}$$

where from now on we write $\sigma^k\omega$ instead of $\sigma^k(\omega)$ for each $\omega \in \Omega$ and $k \in \mathbb{Z}$. Each transformation f_ω induces the corresponding transfer operator $\mathcal{L}_\omega$ acting on $L^1(X,m)$ and defined by the following duality relation

$$\int_X (\mathcal{L}_\omega \phi)\psi \, \mathrm{d}m = \int_X \phi(\psi \circ f_\omega) \, \mathrm{d}m, \quad \phi \in L^1(X, m), \ \psi \in L^\infty(X, m). \tag{4}$$

For each $n \in \mathbb{N}$ and $\omega \in \Omega$, set

$$f_\omega^n = f_{\sigma^{n-1}\omega} \circ \cdots \circ f_\omega \quad \text{and} \quad \mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_\omega.$$

We say that the family of maps f_ω, $\omega \in \Omega$ (or the associated family of transfer operators $\mathcal{L}_\omega$, $\omega \in \Omega$) is *uniformly good* if:

(H1) the map $(\omega, x) \mapsto (\mathcal{L}_\omega H(\omega, \cdot))(x)$ is $\mathbb{P} \times m$-measurable, i.e. measurable on the space $(\Omega \times X, \mathcal{F} \times \mathcal{G})$ for every $\mathbb{P} \times m$-measurable function H such that $H(\omega, \cdot) \in L^1(X, m)$ for a.e. $\omega \in \Omega$;

(H2) there exists $C > 0$ such that

$$\|\mathcal{L}_\omega \phi\|_{BV} \leqslant C\|\phi\|_{BV}$$

for $\phi \in BV$ and $\mathbb{P}$ a.e. $\omega \in \Omega$.

(H3) For $\mathbb{P}$ a.e. $\omega \in \Omega$,

$$\sup_{n \geqslant 0, \|\phi\|_{BV} \leqslant 1} \|\phi \circ f_{\sigma^n\omega}\|_{BV} < \infty.$$

(H4) There exists $N \in \mathbb{N}$ such that for each $a > 0$ and any sufficiently large $n \in \mathbb{N}$, there exists $c > 0$ such that

$$\operatorname{ess\,inf} \mathcal{L}_\omega^{Nn} h \geqslant c/2 \, \|h\|_1, \quad \text{for every } h \in C_a \text{ and a.e. } \omega \in \Omega,$$

where $C_a := \{\phi \in BV : \phi \geqslant 0 \text{ and } \operatorname{var}(\phi) \leqslant a \int \phi \, \mathrm{d}m\}$.

(H5) There exist $K, \lambda > 0$ such that

$$\|\mathcal{L}_\omega^{(n)} \phi\|_{BV} \leqslant K\mathrm{e}^{-\lambda n}\|\phi\|_{BV},$$

for $n \geqslant 0$, $\mathbb{P}$ a.e. $\omega \in \Omega$ and $\phi \in BV$ such that $\int \phi \, \mathrm{d}m = 0$.

Remark 1. In sections 2.2 and 2.3 we provide explicit examples of random dynamical systems that satisfy (H1)–(H5). Using (H1), (H2) and (H5) we can prove the existence of a unique random absolutely continuous invariant measure for τ.

Proposition 1. *Let f_ω, $\omega \in \Omega$ be a uniformly good family of maps on X. Then there exist a unique measurable and nonnegative function $h : \Omega \times X \to \mathbb{R}$ with the property that $h_\omega := h(\omega, \cdot) \in BV$, $\int h_\omega \, \mathrm{d}m = 1$, $\mathcal{L}_\omega(h_\omega) = h_{\sigma(\omega)}$ for a.e. $\omega \in \Omega$ and*

$$\operatorname{ess\,sup}_{\omega \in \Omega} \|h_\omega\|_{BV} < \infty. \tag{5}$$

Proof. Let

$$Y = \left\{ v\colon \Omega \times X \to \mathbb{R} : v \text{ measurable}, v_\omega := v(\omega, \cdot) \in BV \text{ and } \operatorname{ess\,sup}_{\omega\in\Omega} \|v_\omega\|_{BV} < \infty \right\}.$$

Then, Y is a Banach space with respect to the norm

$$\|v\|' := \operatorname{ess\,sup}_{\omega\in\Omega} \|v_\omega\|_{BV}.$$

Moreover, let Y_1 be the set of all $v \in Y$ such that $\int v_\omega \, dm = 1$ and $v_\omega \geqslant 0$ for a.e. $\omega \in \Omega$. It is easy to verify that Y_1 is a closed subset of Y and thus a complete metric space. We define a map $\mathcal{L}\colon Y_1 \to Y_1$ by

$$(\mathcal{L}(v))_\omega = \mathcal{L}_{\sigma^{-1}\omega} v_{\sigma^{-1}\omega}, \quad \text{for } \omega \in \Omega \text{ and } v \in Y_1.$$

The operator $\mathcal{L}_\omega$ was defined in (4); note that it follows from (H2) that

$$\|\mathcal{L}(v)\|' = \operatorname{ess\,sup}_{\omega\in\Omega} \|(\mathcal{L}(v))_\omega\|_{BV} \leqslant C \operatorname{ess\,sup}_{\omega\in\Omega} \|v_{\sigma^{-1}\omega}\|_{BV} = C\|v\|'.$$

Furthermore,

$$\int (\mathcal{L}(v))_\omega \, dm = \int \mathcal{L}_{\sigma^{-1}\omega} v_{\sigma^{-1}\omega} \, dm = \int v_{\sigma^{-1}\omega} \, dm = 1,$$

for a.e. $\omega \in \Omega$. Finally, since $v_\omega \geqslant 0$ for a.e. $\omega \in \Omega$ we have (using the positivity of operators $\mathcal{L}_\omega$) that $\mathcal{L}_{\sigma^{-1}\omega} v_{\sigma^{-1}\omega} \geqslant 0$ for a.e. $\omega \in \Omega$. Hence, we conclude that $\mathcal{L}$ is well-defined. Similarly,

$$\|\mathcal{L}(v) - \mathcal{L}(w)\|' \leqslant C\|v - w\|', \quad \text{for } v, w \in Y_1$$

which shows that $\mathcal{L}$ is continuous. Choose $n_0 \in \mathbb{N}$ such that $Ke^{-\lambda n_0} < 1$. Take arbitrary $v, w \in Y_1$ and note that by (H5),

$$\begin{aligned}\|\mathcal{L}^{n_0}(v) - \mathcal{L}^{n_0}(w)\|' &= \operatorname{ess\,sup}_{\omega\in\Omega} \left\| \mathcal{L}^{(n_0)}_{\sigma^{-n_0}\omega} (v_{\sigma^{-n_0}\omega} - w_{\sigma^{-n_0}\omega}) \right\|_{BV} \\ &\leqslant Ke^{-\lambda n_0} \operatorname{ess\,sup}_{\omega\in\Omega} \|v_{\sigma^{-n_0}\omega} - w_{\sigma^{-n_0}\omega}\|_{BV} = Ke^{-\lambda n_0} \|v - w\|'.\end{aligned}$$

Hence, $\mathcal{L}^{n_0}$ is a contraction on Y_1 and thus has a unique fixed point $\tilde{h} \in Y_1$. Set

$$h_\omega := \frac{1}{n_0}\tilde{h}_\omega + \frac{1}{n_0}\mathcal{L}_{\sigma^{-1}\omega}(\tilde{h}_{\sigma^{-1}\omega}) + \ldots + \frac{1}{n_0}\mathcal{L}^{(n_0-1)}_{\sigma^{-(n_0-1)}\omega}(\tilde{h}_{\sigma^{-(n_0-1)}\omega}) \quad \text{for } \omega \in \Omega,$$

and consider the map $h\colon \Omega \times X \to \mathbb{R}$ given by $h(\omega, \cdot) = h_\omega$ for $\omega \in \Omega$. Then, h is measurable, nonnegative, $\int h_\omega \, dm = 1$ and a simple computation yields $\mathcal{L}_\omega(h_\omega) = h_{\sigma\omega}$. Finally, by (H2) we have that

$$\operatorname{ess\,sup}_{\omega\in\Omega} \|h_\omega\|_{BV} \leqslant \frac{C^{n_0} - 1}{n_0(C - 1)} \operatorname{ess\,sup}_{\omega\in\Omega} \left\|\tilde{h}_\omega\right\|_{BV} < \infty.$$

Thus, we have established existence of h. The uniqueness is obvious since each h satisfying the assertion of the theorem is a fixed point of $\mathcal{L}$ and thus also of $\mathcal{L}^{n_0}$ which implies that it must be unique. □

At this stage, we point out that we will need (H5) for our later results; we use (H5) in the proof of proposition 1 only to give a simpler existence result for the random ACIM. With weaker control on the properties of f_ω, [3, 4] proved the above existence result; in particular, these results do not require (H5) and (H2) is allowed to hold with $C = C(\omega)$ such that $\log C \in L^1(\mathbb{P})$.

We define a probability measure μ on $\Omega \times X$ by

$$\mu(A \times B) = \int_{A\times B} h\, d(\mathbb{P} \times m), \quad \text{for } A \in \mathcal{F} \text{ and } B \in \mathcal{B}. \tag{6}$$

Then, it follows from proposition 1 that μ is invariant with respect to τ. Furthermore, μ is obviously absolutely continuous with respect to $\mathbb{P} \times m$. Finally, it follows from the uniqueness in proposition 1 that μ is the only measure with these properties.

Let us now consider for any $\omega \in \Omega$ the measures μ_ω on the measurable space $(X, \mathcal{B})$, defined by $d\mu_\omega = h_\omega dm$. We recall here two important properties of these measures, which are together equivalent to (6) and its invariance, see [2]. First, the so-called *equivariant property*: $f_\omega^* \mu_\omega = \mu_{\sigma\omega}$. Second, the disintegration of μ on the marginal $\mathbb{P}$; if A is any measurable set in $\mathcal{F} \times \mathcal{B}$, and $A_\omega = \{x; (\omega, x) \in A\}$, the *section* at ω, then $\mu(A) = \int \mu_\omega(A_\omega) d\mathbb{P}(\omega)$. The conditional (or sample) measure μ_ω will constitute the probability underlying our random processes.

We now describe a large class of examples of good families of maps f_ω, $\omega \in \Omega$. We first show that they satisfy properties (H1)–(H3); this will crucially depend on the choice of the function space. We then give additional requirements in order for those maps to satisfy a condition related to (H4), named *Min* when applied to sequential systems in [6], and condition (H5), called *Dec* in [6].

2.2. Example 1: random Lasota–Yorke maps

Take $X = [0, 1]$, a Borel σ-algebra $\mathcal{B}$ on $[0, 1]$ and the Lebesgue measure m on $[0, 1]$. Furthermore, let

$$\text{var}(g) = \inf_{h=g(\text{mod } m)} \sup_{0=s_0<s_1<\ldots<s_n=1} \sum_{k=1}^{n} |h(s_k) - h(s_{k-1})| .$$

It is well known that var satisfies properties (V1)–(V8) with $C_{\text{var}}, K_{\text{var}} = 1$. For a piecewise C^2 $f : [0, 1] \to [0, 1]$, set $\delta(f) = \text{ess inf}_{x\in[0,1]} |f'|$ and let $N(f)$ denote the number of intervals of monoticity of *f*. Consider now a measurable map $\omega \mapsto f_\omega$, $\omega \in \Omega$ of piecewise C^2 maps on $[0, 1]$ satisfying (H1) such that

$$N := \sup_{\omega\in\Omega} N(f_\omega) < \infty, \;\; \delta := \inf_{\omega\in\Omega} \delta(f_\omega) > 1, \;\; \text{and } D := \sup_{\omega\in\Omega} |f''_\omega|_\infty < \infty. \tag{7}$$

It is proved in [3] that the family f_ω, $\omega \in \Omega$ satisfies (H2) with

$$C = 4\left(\frac{N}{\delta} \vee 1\right)\left(\frac{D}{\delta^2} \vee 1\right)\left(\frac{1}{\delta} \vee 1\right), \tag{8}$$

where for any two real-valued functions g_1 and g_2, $g_1 \vee g_2 = \max\{g_1, g_2\}$, and (V8) has been used for the bound $\text{var}(1/f') \leqslant \frac{D}{\delta^2}$. We note that since $N < \infty$, condition (H3) holds.

We now discuss conditions that imply (H4). For each $\omega \in \Omega$, let b_ω be the number of branches of f_ω, so that there are essentially disjoint sub-intervals $J_{\omega,1}, \ldots, J_{\omega,b_\omega} \subset I$, with $\cup_{k=1}^{b_\omega} J_{\omega,k} = I$, so that $f_\omega|_{J_{\omega,k}}$ is C^2 for each $1 \leqslant k \leqslant b_\omega$. The minimal such partition $\mathcal{P}_\omega := \{J_{\omega,1}, \ldots, J_{\omega,b_\omega}\}$ is called the *regularity partition* for f_ω. We recall from classical results, e.g. [16], that whenever $\delta > 2$, and $\operatorname{ess\,inf}_{\omega\in\Omega} \min_{1\leqslant k\leqslant b_\omega} m(J_{\omega,k}) > 0$, there exist $\alpha \in (0,1)$ and $K > 0$ such that

$$\operatorname{var}(\mathcal{L}_\omega\phi) \leqslant \alpha\operatorname{var}(\phi) + K\|\phi\|_1, \quad \text{for} \phi \in BV \text{ and a.e. } \omega \in \Omega.$$

More generally, when $\delta < 2$, one can take an iterate $N \in \mathbb{N}$ so that $\delta^N > 2$. If the regularity partitions $\mathcal{P}^N_\omega := \{J^N_{1,\omega}, \ldots, J^N_{\omega,b^{(N)}_\omega}\}$ corresponding to the maps $f^{(N)}_\omega$ also satisfy $\operatorname{ess\,inf}_{\omega\in\Omega} \min_{1\leqslant k\leqslant b^{(N)}_\omega} m(J^N_{\omega,k}) > 0$, then there exist $\alpha^N \in (0,1)$ and $K^N > 0$ such that

$$\operatorname{var}(\mathcal{L}^N_\omega\phi) \leqslant \alpha^N\operatorname{var}(\phi) + K^N\|\phi\|_1, \quad \text{for } \phi \in BV \text{ and a.e. } \omega \in \Omega. \tag{9}$$

We will from now on assume that (9) holds for some $N \in \mathbb{N}$. Iterating, it is easy to show that

$$\operatorname{var}(\mathcal{L}^{RN}_\omega\phi) \leqslant (\alpha^N)^R\operatorname{var}(\phi) + C^N\|\phi\|_1, \quad \phi \in BV, \quad \omega \in \Omega, \quad R \in \mathbb{N}, \tag{10}$$

where $C^N = \frac{K^N}{1-\alpha^N}$. The proof of the following lemmas are deferred to appendices A.1 and A.2, respectively.

Lemma 1. *Suppose the following uniform covering condition holds:*

$$\text{For every subinterval } J \subset I, \exists k = k(J) \text{ s.t. for a.e. } \omega \in \Omega, f^{(k)}_\omega(J) = I. \tag{11}$$

Then, (H4) holds.

Lemma 2. *There exist $K, \lambda > 0$ such that (H5) holds.*

2.3. Example 2: random piecewise expanding maps

In higher dimensions, the properties (V1)–(V8) can be checked for the so-called quasi-Hölder space $\mathcal{B}_\beta$, which in particular is injected in L^∞ (condition (V3)) and has the algebra property (V7). Originally developed by Keller [12] for one-dimensional dynamics, we refer the reader to the Saussol paper [21] for a detailed presentation of that space in higher dimensions, as well as for the proof of its main properties. In particular, using the same notations as in [21], we use the following notion of variation:

$$\operatorname{var}_\beta(\phi) = \sup_{0<\varepsilon\leqslant\varepsilon_0} \varepsilon^{-\beta} \int_{\mathbb{R}^n} \operatorname{osc}(\phi, B_\varepsilon(x))\,\mathrm{d}x,$$

where $\varepsilon_0 > 0$ is sufficiently small,

$$\operatorname{osc}(\phi, B_\varepsilon(x)) = \operatorname{ess\,sup}_{B_\varepsilon(x)}\phi - \operatorname{ess\,inf}_{B_\varepsilon(x)}\phi,$$

and we define the norm (we use the notation introduced in section 2.1), $\|\phi\|_{BV} := \operatorname{var}_\beta(\phi) + \|\phi\|_1$, which makes the set $\{\phi \in L^1(m) | \operatorname{var}_\beta(\phi) < \infty\}$, a Banach space $\mathcal{B}_\beta$. In [21] it is proved that this notion of variation satisfies (V1)–(V3) and (V5)–(V7) and noted that (V4) is proven in [12] (theorem 1.13). We prove here that (V8) holds too. Observe that

$$\begin{aligned}\operatorname{osc}(1/\phi, B_\varepsilon(x)) &= \operatorname{ess\,sup}_{B_\varepsilon(x)}(1/\phi) - \operatorname{ess\,inf}_{B_\varepsilon(x)}(1/\phi)\\ &= 1/\operatorname{ess\,inf}_{B_\varepsilon(x)}(\phi) - 1/\operatorname{ess\,sup}_{B_\varepsilon(x)}(\phi)\\ &= \frac{\operatorname{ess\,sup}_{B_\varepsilon(x)}(\phi) - \operatorname{ess\,inf}_{B_\varepsilon(x)}(\phi)}{(\operatorname{ess\,sup}_{B_\varepsilon(x)}(\phi))(\operatorname{ess\,inf}_{B_\varepsilon(x)}(\phi))}\\ &\leqslant \frac{\operatorname{osc}(\phi, B_\varepsilon(x))}{(\operatorname{ess\,inf}_{B_\varepsilon(x)}(\phi))^2} \leqslant \frac{\operatorname{osc}(\phi, B_\varepsilon(x))}{(\operatorname{ess\,inf}\ \phi)^2},\end{aligned}$$

which readily implies that $\operatorname{var}_\beta(1/\phi) \leqslant \frac{\operatorname{var}_\beta(\phi)}{(\operatorname{ess\,inf}\ \phi)^2}$.

We now describe the family of maps which we will endow later with a uniformly good structure; this class has been considered in [21, section 2] and [10, definition 2.9]. Let M be a compact subset of $\mathbb{R}^N$ which is the closure of its non-empty interior. We take a map $f: M \to M$ and let $\mathcal{A} = \{A_i\}_{i=1}^m$ be a finite family of disjoint open sets such that the Lebesgue measure of $M \setminus \bigcup_i A_i$ is zero, and there exist open sets $\tilde{A}_i \supset \overline{A_i}$ and $C^{1+\gamma}$ maps $f_i : \tilde{A}_i \to \mathbb{R}^N$, for some real number $0 < \gamma \leqslant 1$ and some sufficiently small real number $\varepsilon_1 > 0$ such that

1. $f_i(\tilde{A}_i) \supset B_{\varepsilon_1}(f(A_i))$ for each i, where $B_\varepsilon(V)$ denotes a neighborhood of size ε of the set V. The maps f_i are the local extensions of f to the $\tilde{A}_i$;
2. there exists a constant C_1 so that for each i and $x, y \in f(A_i)$ with $\operatorname{dist}(x, y) \leqslant \varepsilon_1$,
$$|\det Df_i^{-1}(x) - \det Df_i^{-1}(y)| \leqslant C_1 |\det Df_i^{-1}(x)| \operatorname{dist}(x, y)^\gamma;$$
3. there exists $s = s(f) < 1$ such that $\forall x, y \in f(\tilde{A}_i)$ with $\operatorname{dist}(x, y) \leqslant \varepsilon_1$, we have
$$\operatorname{dist}(f_i^{-1}x, f_i^{-1}y) \leqslant s \operatorname{dist}(x, y);$$
4. each ∂A_i is a codimension-one embedded compact piecewise C^1 submanifold and
$$s^\gamma + \frac{4s}{1-s} Z(f) \frac{\Gamma_{N-1}}{\Gamma_N} < 1 \tag{12}$$
where $Z(f) = \sup_x \sum_i \#\{\text{smooth pieces intersecting } \partial A_i \text{ containing } x\}$ and Γ_N is the volume of the unit ball in $\mathbb{R}^N$[5].

We now consider a family of maps $\{f_\omega\}_{\omega\in\Omega}$ satisfying the above conditions for a.e. $\omega \in \Omega$, and with uniform constants $\varepsilon_1, C_1, \gamma, s$, and $Z := \operatorname{ess\,sup}_{\omega\in\Omega} Z(f_\omega)$. The hypotheses (H1)–(H3) follow as in the one-dimensional case, provided the function $\omega \mapsto f_\omega$ is measurable. In order to satisfy hypotheses (H4) and (H5), as in the one-dimensional case we impose further conditions on the above class of maps. A Lasota–Yorke inequality is guaranteed for each map f_ω by [21, lemma 4.1] or [10, proposition 3.1], and this ensures the following uniform Lasota–Yorke inequality holds, for some $0 < \alpha < 1, K > 0$,

$$\operatorname{var}_\beta(\mathcal{L}_\omega\phi) \leqslant \alpha \operatorname{var}_\beta(\phi) + K\|\phi\|_1, \quad \text{for } \phi \in \mathcal{B}_\beta \text{ and a.e. } \omega \in \Omega. \tag{13}$$

As in the one-dimensional case, we can obtain an N-fold Lasota–Yorke inequality (of the type (9) using var_β in place of var) provided that the regularity partitions of a.e. N-fold composition of maps have positive Lebesgue measure. The proof of the following lemma is deferred until appendix B.

[5] Condition (12) can be considerably weakened, see [21], but its right statement requires additional definitions; for smooth boundaries assumption (12) is perfectly adapted to our purposes.

Lemma 3. *Suppose the uniform covering condition holds for the above class of maps:*

for any open set $J \in M$, there exists $k = k(J)$ such that for a.e. $\omega \in \Omega, f_\omega^k(J) = M$.

Then conditions (H4) and (H5) are satisfied.

2.4. Further properties of the random ACIM

Let μ_ω be, as above, the measure on X given by $\mathrm{d}\mu_\omega = h_\omega \mathrm{d}m$ for $\omega \in \Omega$. We have the following important consequence of (H5), which establishes the appropriate decay of correlations result that will be used later on.

Lemma 4. *There exists* $K > 0$ *and* $\rho \in (0, 1)$ *such that*

$$\left| \int \mathcal{L}_\omega^{(n)}(\phi h_\omega)\psi \, \mathrm{d}m - \int \phi \, \mathrm{d}\mu_\omega \cdot \int \psi \, \mathrm{d}\mu_{\sigma^n\omega} \right| \leqslant K\rho^n \, \|\psi\|_\infty \cdot \|\phi\|_{\mathrm{var}} , \tag{14}$$

for $n \geqslant 0$, $\psi \in L^\infty(X, m)$ *and* $\phi \in BV(X, m)$.

Proof. We consider two cases. Assume first that $\int \phi \, \mathrm{d}\mu_\omega = \int \phi h_\omega \, \mathrm{d}m = 0$. Then, it follows from (H5) that

$$\begin{aligned} &\left| \int \mathcal{L}_\omega^{(n)}(\phi h_\omega)\psi \, \mathrm{d}m - \int \phi \, \mathrm{d}\mu_\omega \cdot \int \psi \, \mathrm{d}\mu_{\sigma^n\omega} \right| = \left| \int \mathcal{L}_\omega^{(n)}(\phi h_\omega)\psi \, \mathrm{d}m \right| \\ &\qquad \leqslant \|\psi\|_\infty \cdot \left\| \mathcal{L}_\omega^{(n)}(\phi h_\omega) \right\|_1 \leqslant \|\psi\|_\infty \cdot \left\| \mathcal{L}_\omega^{(n)}(\phi h_\omega) \right\|_{BV} \leqslant K\mathrm{e}^{-\lambda n} \, \|\phi h_\omega\|_{BV} \cdot \|\psi\|_\infty , \end{aligned}$$

and thus (14) follows from (2) and (5). Now we consider the case when $\int \phi \, \mathrm{d}\mu_\omega \neq 0$. We have

$$\begin{aligned} &\left| \int \mathcal{L}_\omega^{(n)}(\phi h_\omega)\psi \, \mathrm{d}m - \int \phi \, \mathrm{d}\mu_\omega \cdot \int \psi \, \mathrm{d}\mu_{\sigma^n\omega} \right| \\ &= \left| \int \mathcal{L}_\omega^{(n)}(\phi h_\omega)\psi \, \mathrm{d}m - \int \phi h_\omega \, \mathrm{d}m \cdot \int \psi h_{\sigma^n\omega} \, \mathrm{d}m \right| \\ &\leqslant \|\psi\|_\infty \cdot \int \left| \left(\mathcal{L}_\omega^{(n)}(\phi h_\omega) - \left(\int \phi h_\omega \, \mathrm{d}m \right) h_{\sigma^n\omega} \right) \right| \, \mathrm{d}m \\ &= \|\psi\|_\infty \cdot \left| \int \phi h_\omega \, \mathrm{d}m \right| \cdot \int |\mathcal{L}_\omega^n(\Phi - h_\omega)| \, \mathrm{d}m \\ &\leqslant \|\psi\|_\infty \cdot \left| \int \phi h_\omega \, \mathrm{d}m \right| \cdot \|\mathcal{L}_\omega^n(\Phi - h_\omega)\|_{BV} , \end{aligned}$$

where

$$\phi h_\omega = \left(\int \phi h_\omega \, \mathrm{d}m \right) \Phi.$$

Note that $\int (\Phi - h_\omega) \, \mathrm{d}m = 0$ and thus using (H5),

$$\|\psi\|_\infty \cdot \left|\int \phi h_\omega \,\mathrm{d}m\right| \cdot \left\|\mathcal{L}_\omega^{(n)}(\Phi - h_\omega)\right\|_{BV} \leqslant K\mathrm{e}^{-\lambda n}\|\psi\|_\infty \cdot \left|\int \phi h_\omega \,\mathrm{d}m\right| \cdot \|\Phi - h_\omega\|_{BV}$$
$$\leqslant K\mathrm{e}^{-\lambda n}\|\psi\|_\infty \cdot \left\|\left(\phi - \int \phi h_\omega \,\mathrm{d}m\right)h_\omega\right\|_{BV}.$$

Hence, it follows from (2) and (5) that

$$\left|\int \mathcal{L}_\omega^{(n)}(\phi h_\omega)\psi \,\mathrm{d}m - \int \phi \,\mathrm{d}\mu_\omega \cdot \int \psi \,\mathrm{d}\mu_{\sigma^n\omega}\right| \leqslant K'\mathrm{e}^{-\lambda n}\|\psi\|_\infty \cdot \|\phi\|_{BV}$$

for some $K' > 0$ and thus (14) follows readily by recalling that $\|\cdot\|_{BV} \leqslant \|\cdot\|_{\mathrm{var}}$. □

Remark 2. We would like to emphasize that (14) is a special case of a more general decay of correlation result obtained in [3] which does not require (H5) and yields (14) but with $K = K(\omega)$.

Finally, we prove that condition (H4) implies that we have a uniform lower bound for h_ω.

Lemma 5. *We have*

$$\operatorname{ess\,inf} h_\omega \geqslant c/2, \quad \text{for a.e. } \omega \in \Omega. \tag{15}$$

Proof. We first note that it follows from proposition 1 that there exists $a > 0$ such that $h_\omega \in C_a$ for a.e. $\omega \in \Omega$. Hence, it follows from (H4) applied to $h = h_\omega$ that for n large,

$$\operatorname{ess\,inf} h_{\sigma^{Nn}\omega} \geqslant c/2 \quad \text{for a.e. } \omega \in \Omega,$$

which implies the desired conclusion. □

Remark 3. We now briefly compare our setting, results and assumptions with those in [13]. In the latter, the space X is replaced by a foliation $\{\Xi_\omega\}_{\omega\in\Omega}$. On the fibered subset $\Xi := \{(\omega, \xi) : \omega \in \Omega, \xi \in \Xi_\omega\}$ one can consider the skew map $\tau(\omega, \xi) = (\sigma\omega, f_\omega\xi)$ with the associated *fiber maps* $f_\omega : \Xi_\omega \to \Xi_{\sigma\omega}$. In our situation the Ξ_ω's for all ω coincide with the set X and all $f_\omega : X \to X$ are endomorphisms of X with some regularity property. Since this situation covers the applications we have in mind (random composition of maps), we do not treat the case of ω-dependent fibers, but in principle, all the arguments in the present paper also extend to this more general setting. Kifer used a martingale approximation, where the martingale approximation error in [13] is given in terms of an infinite series (see the error g_ω in equation (4.18) in [13]), which appears difficult to estimate under general assumptions. Instead, our martingale approximation error term is explicitly given in terms of a finite sum (see (24)), and, as mentioned above, it can be bound without difficulty in our setting. Furthermore, Kifer invoked a rate of mixing, but to deal with it he assumed strong conditions (ϕ-mixing and α-mixing), which are difficult to check on concrete examples of dynamical systems. We use instead quenched decay of correlations on a space of regular observables, for example, bounded variation or quasi-Hölder and L^∞ functions (exponential decay was shown by Buzzi [3]), with an addition: the constant that scales the norm of the observable in the decay rate is independent of the noise ω; we can then satisfy the hypotheses of Sprindzuk's result, see below.

2.5. Statement of the main result

We are now ready to state our main result. We will consider an observable $\psi\colon \Omega \times X \to \mathbb{R}$. Let $\psi_\omega = \psi(\omega, \cdot)$, $\omega \in \Omega$ and assume that

$$\sup_{\omega\in\Omega} \|\psi_\omega\|_{BV} < \infty. \tag{16}$$

We also introduce the centered observable

$$\tilde{\psi}_\omega = \psi_\omega - \int \psi_\omega \, \mathrm{d}\mu_\omega, \quad \omega \in \Omega$$

and consider the associated Birkhoff sum $\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k$, and the variance

$$\tau_n^2 = \mathbb{E}_\omega \left(\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k \right)^2. \tag{17}$$

The almost sure invariance principle is a matching of the trajectories of the dynamical system with a Brownian motion in such a way that the error is negligible in comparison with the Birkhoff sum. Other limit theorems such as the central limit theorem, the functional central limit theorem and the law of the iterated logarithm will be consequences (see [20]) of our proof of the ASIP and therefore they will hold for random Lasota–Yorke maps.

Theorem 1. *Let us consider the family of uniformly good random Lasota–Yorke maps. Then there exists $\Sigma^2 \geqslant 0$ such that $\lim_{n\to\infty} \frac{1}{n}\tau_n^2 = \Sigma^2$. Moreover one of the following cases hold:*

(i) $\Sigma = 0$, *and this is equivalent to the existence of $\phi \in L^2(\Omega \times X)$ such that (co-boundary condition)*

$$\tilde{\psi} = \phi - \phi \circ \tau. \tag{18}$$

(ii) $\Sigma^2 > 0$ *and in this case for $\mathbb{P}$ a.e. $\omega \in \Omega$, $\forall \varepsilon > \frac{5}{4}$, by enlarging the probability space $(X, \mathcal{B}, \mu_\omega)$ if necessary, it is possible to find a sequence $(Z_k)_k$ of independent centered (i.e. of zero mean) Gaussian random variables such that*

$$\sup_{1\leqslant k\leqslant n} \left| \sum_{i=1}^{k} (\tilde{\psi}_{\sigma^i\omega} \circ f_\omega^i) - \sum_{i=1}^{k} Z_i \right| = O(n^{1/4} \log^{\varepsilon}(n)), \quad \text{m a.s.} \tag{19}$$

Remark 4. We notice that the statement (ii) of the theorem 1 also holds μ_ω-a.s. and for $\mathbb{P}$ a.e. ω, since the measures μ_ω and m are equivalent by lemma 5.

3. Reverse martingale difference construction

In this section we construct the reverse martingale difference and establish various useful estimates that will play an important role in the rest of the paper. Indeed, the proof of our main result (theorem 1) will be obtained as a consequence of the recent result by Cuny and Merlevède [7] applied to our reverse martingale difference.

For $\omega \in \Omega$ and $k \in \mathbb{N}$, let

$$\mathcal{T}_\omega^k := (f_\omega^k)^{-1}(\mathcal{B}).$$

Furthermore, for a measurable map $\phi\colon X\to\mathbb{R}$ and a σ-algebra $\mathcal{H}$ on X, $\mathbb{E}_\omega(\phi|\mathcal{H})$ will denote the conditional expectation of ϕ with respect to $\mathcal{H}$ and the measure μ_ω. We begin with the following technical lemma.

Lemma 6. *We have*

$$\mathbb{E}_\omega(\phi\circ f_\omega^l|\mathcal{T}_\omega^n)=\left(\frac{\mathcal{L}_{\sigma^l\omega}^{(n-l)}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}}\right)\circ f_\omega^n, \tag{20}$$

for each $\omega\in\Omega$ *and* $0\leqslant l\leqslant n$.

Proof. We note that the right-hand side of (20) is measurable with respect to $\mathcal{T}_\omega^n$. Take now an arbitrary $A\in\mathcal{T}_\omega^n$ and write it in the form $A=(f_\omega^n)^{-1}(B)$ for some $B\in\mathcal{B}$. We have

$$\begin{aligned}\int_A\phi\circ f_\omega^l\,\mathrm{d}\mu_\omega&=\int_X(\phi\circ f_\omega^l)\mathbf{1}_A\,\mathrm{d}\mu_\omega\\&=\int_X(\phi\circ f_\omega^l)\cdot(\mathbf{1}_B\circ f_\omega^n)\,\mathrm{d}\mu_\omega=\int_X\phi(\mathbf{1}_B\circ f_{\sigma^l\omega}^{n-l})\,\mathrm{d}\mu_{\sigma^l\omega}\\&=\int_X h_{\sigma^l\omega}\phi(\mathbf{1}_B\circ f_{\sigma^l\omega}^{n-l})\,\mathrm{d}m=\int_X\mathcal{L}_{\sigma^l\omega}^{(n-l)}(h_{\sigma^l\omega}\phi)\mathbf{1}_B\,\mathrm{d}m\\&=\int_X\frac{\mathcal{L}_{\sigma^l\omega}^{(n-l)}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}}\mathbf{1}_B\,\mathrm{d}\mu_{\sigma^n\omega}=\int_X\left[\left(\frac{\mathcal{L}_{\sigma^l\omega}^{(n-l)}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}}\right)\circ f_\omega^n\right](\mathbf{1}_B\circ f_\omega^n)\,d\mu_\omega\\&=\int_X\left[\left(\frac{\mathcal{L}_{\sigma^l\omega}^{(n-l)}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}}\right)\circ f_\omega^n\right]\mathbf{1}_A\,d\mu_\omega=\int_A\left(\frac{\mathcal{L}_{\sigma^l\omega}^{(n-l)}(h_{\sigma^l\omega}\phi)}{h_{\sigma^n\omega}}\right)\circ f_\omega^n\,d\mu_\omega,\end{aligned}$$

which proves (20). □

We now return to the observable ψ_ω introduced in (16) and its centered companion $\tilde\psi_\omega=\psi_\omega-\int\psi_\omega\,\mathrm{d}\mu_\omega,\quad\omega\in\Omega$.

Set

$$M_n=\tilde\psi_{\sigma^n\omega}+G_n-G_{n+1}\circ f_{\sigma^n\omega},\quad n\geqslant 0, \tag{21}$$

where $G_0=0$ and

$$G_{k+1}=\frac{\mathcal{L}_{\sigma^k\omega}(\tilde\psi_{\sigma^k\omega}h_{\sigma^k\omega}+G_kh_{\sigma^k\omega})}{h_{\sigma^{k+1}\omega}},\quad k\geqslant 0. \tag{22}$$

We emphasize that M_n and G_n depend on ω. However, in order to avoid complicating the notation, we will not make this dependence explicit. In preparation for the next proposition we need the following elementary result.

Lemma 7. *We have*

$$\mathcal{L}_\omega((\psi\circ f_\omega)\phi)=\psi\mathcal{L}_\omega\phi,\quad\text{for }\phi\in L^1(X,m)\text{ and }\psi\in L^\infty(X,m).$$

Proof. For an arbitrary $z\in L^\infty(X,m)$, we have (using (4)) that

$$\int_X(\psi\mathcal{L}_\omega\phi)z\,\mathrm{d}m=\int_X z\psi\mathcal{L}_\omega\phi\,\mathrm{d}m=\int_X(z\circ f_\omega)(\psi\circ f_\omega)\phi\,\mathrm{d}m=\int_X\mathcal{L}_\omega((\psi\circ f_\omega)\phi)z\,\mathrm{d}m,$$

which readily implies the desired conclusion. □

Now we prove that the sequence $(M_n \circ f_\omega^n)_n$ is a reversed martingale (or the reversed martingale difference) with respect to the sequence of σ-algebras $(\mathcal{T}_\omega^n)_n$.

Proposition 2. *We have*

$$\mathbb{E}_\omega(M_n \circ f_\omega^n | \mathcal{T}_\omega^{n+1}) = 0.$$

Proof. It follows from lemma 6 that

$$\mathbb{E}_\omega(M_n \circ f_\omega^n | \mathcal{T}_\omega^{n+1}) = \left(\frac{\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega} M_n)}{h_{\sigma^{n+1}\omega}} \right) \circ f_\omega^{n+1}. \tag{23}$$

Moreover, by (21) we have

$$\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega} M_n) = \mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega}\tilde{\psi}_{\sigma^n\omega} + h_{\sigma^n\omega} G_n - h_{\sigma^n\omega}(G_{n+1} \circ f_{\sigma^n\omega})).$$

By lemma 7,

$$\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega}(G_{n+1} \circ f_{\sigma^n\omega})) = G_{n+1}\mathcal{L}_{\sigma^n\omega} h_{\sigma^n\omega} = G_{n+1} h_{\sigma^{n+1}\omega},$$

and thus it follows from (22) that

$$\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega} M_n) = 0.$$

This conclusion of the lemma now follows readily from (23). □

We now establish several auxiliary results that will be used in the following section. These results estimate various norms of functions related to M_n and G_n, defined in (21) and (22), respectively.

Lemma 8. *We have that*

$$\sup_{n\geqslant 0} \|G_n\|_{BV} < \infty.$$

Proof. By iterating (22), we obtain

$$G_n = \frac{1}{h_{\sigma^n\omega}} \sum_{j=0}^{n-1} \mathcal{L}_{\sigma^j\omega}^{(n-j)}(\tilde{\psi}_{\sigma^j\omega} h_{\sigma^j\omega}), \quad n \in \mathbb{N}. \tag{24}$$

We note that

$$\int \tilde{\psi}_{\sigma^j\omega} h_{\sigma^j\omega}\, \mathrm{d}m = \int \tilde{\psi}_{\sigma^j\omega}\, \mathrm{d}\mu_{\sigma^j\omega} = 0, \tag{25}$$

and thus it follows from (H5) that

$$\left\| \sum_{j=0}^{n-1} \mathcal{L}_{\sigma^j\omega}^{(n-j)}(\tilde{\psi}_{\sigma^j\omega} h_{\sigma^j\omega}) \right\|_{BV} \leqslant K \sum_{j=0}^{n-1} \mathrm{e}^{-\lambda(n-j)} \left\| \tilde{\psi}_{\sigma^j\omega} h_{\sigma^j\omega} \right\|_{BV},$$

for each $n \in \mathbb{N}$ which together with (V8), (2), (5), (15) and (16) implies the conclusion of the lemma. □

Lemma 9. *We have that*

$$\sup_{n\geqslant 0} \|M_n^2\|_{BV} < \infty.$$

Proof. In view of (16), (21) and lemma 8, it is sufficient to show that

$$\sup_{n\geqslant 0} \|G_{n+1} \circ f_{\sigma^n\omega}\|_{BV} < \infty.$$

However, this follows directly from (H3) and lemma 8. □

Lemma 10. *We have that*

$$\sup_{n\geqslant 0} \left\|\mathbb{E}_\omega(M_n^2 \circ f_\omega^n | \mathcal{T}_\omega^{n+1})\right\|_\infty < \infty.$$

Proof. It follows from lemma 6 that

$$\mathbb{E}_\omega(M_n^2 \circ f_\omega^n | \mathcal{T}_\omega^{n+1}) = \left(\frac{\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega} M_n^2)}{h_{\sigma^{n+1}\omega}}\right) \circ f_\omega^{n+1},$$

and thus, recalling (15),

$$\sup_{n\geqslant 0} \left\|\mathbb{E}_\omega(M_n^2 \circ f_\omega^n | \mathcal{T}_\omega^{n+1})\right\|_\infty \leqslant \frac{2}{c} \left\|\mathcal{L}_{\sigma^n\omega}(h_{\sigma^n\omega} M_n^2)\right\|_\infty .$$

Taking into account (2), (H2), (5), lemma 9 and the fact that $\|\cdot\|_\infty \leqslant C_{\text{var}} \|\cdot\|_{BV}$ (see (V3)) we obtain the conclusion of the lemma. □

4. Sprindzuk's theorem and consequences

The main tool in establishing the almost sure invariance principle is the recent result by Cuny and Merlevède (quoted in our theorem 3 in section 5). However, in order to verify the assumptions of that theorem we will first need to apply the following classical result due to Sprindzuk [22].

Theorem 2. *Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $(f_k)_k$ be a sequence of nonnegative and measurable functions on Ω. Moreover, let $(g_k)_k$ and $(h_k)_k$ be bounded sequences of real numbers such that $0 \leqslant g_k \leqslant h_k$. Assume that there exists $C > 0$ such that*

$$\int \left(\sum_{m<k\leqslant n} (f_k(x) - g_k)\right)^2 \mathrm{d}\mu(x) \leqslant C \sum_{m<k\leqslant n} h_k \tag{26}$$

for $m, n \in \mathbb{N}$ such that $m < n$. Then, for every $\varepsilon > 0$

$$\sum_{1\leqslant k\leqslant n} f_k(x) = \sum_{1\leqslant k\leqslant n} g_k + O(\Theta^{1/2}(n) \log^{3/2+\varepsilon} \Theta(n)),$$

for μ *a.e.* $x \in \Omega$, *where* $\Theta(n) = \sum_{1 \leqslant k \leqslant n} h_k$.

Lemma 11. *For each* $\varepsilon > 0$,

$$\sum_{k=0}^{n-1} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) = \sum_{k=0}^{n-1} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) + O(\Theta^{1/2}(n) \log^{3/2+\varepsilon} \Theta(n)),$$

for μ *a.e.* $\omega \in \Omega$, *where*

$$\Theta(n) = \sum_{k=0}^{n-1} (\mathbb{E}_\omega(M_k^2 \circ f_\omega^k) + \|M_k^2\|_{\mathrm{var}}). \tag{27}$$

Proof. Fix $\omega \in \Omega$. We want to apply theorem 2 to

$$f_k = \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) \quad \text{and} \quad g_k = \mathbb{E}_\omega(M_k^2 \circ f_\omega^k).$$

We have that

$$\begin{aligned}
&\int \Bigg[\sum_{m<k\leqslant n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) - \sum_{m<k\leqslant n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \Bigg]^2 \mathrm{d}\mu_\omega \\
&= \int \Bigg(\sum_{m<k\leqslant n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) \Bigg)^2 \mathrm{d}\mu_\omega - \Bigg(\sum_{m<k\leqslant n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \Bigg)^2 \\
&= \sum_{m<k\leqslant n} \int \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1})^2 \, \mathrm{d}\mu_\omega \\
&+ 2 \sum_{m<i<j\leqslant n} \int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j | \mathcal{T}_\omega^{j+1}) \, \mathrm{d}\mu_\omega \\
&- \sum_{m<k\leqslant n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k)^2 - 2 \sum_{m<i<j\leqslant n} \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j) \\
&\leqslant \sum_{m<k\leqslant n} \int \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1})^2 \, \mathrm{d}\mu_\omega \\
&+ 2 \sum_{m<i<j\leqslant n} \int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j | \mathcal{T}_\omega^{j+1}) \, \mathrm{d}\mu_\omega \\
&- 2 \sum_{m<i<j\leqslant n} \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j).
\end{aligned} \tag{28}$$

On the other hand, it follows from lemma 6 that for $i < j$ we have

$$
\begin{aligned}
&\int \mathbb{E}_\omega(M_i^2\circ f_\omega^i|\mathcal{T}_\omega^{i+1})\cdot\mathbb{E}_\omega(M_j^2\circ f_\omega^j|\mathcal{T}_\omega^{j+1})\,\mathrm{d}\mu_\omega\\
&=\int\left[\left(\frac{\mathcal{L}_{\sigma^i\omega}(h_{\sigma^i\omega}M_i^2)}{h_{\sigma^{i+1}\omega}}\right)\circ f_\omega^{i+1}\right]\cdot\left[\left(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega}M_j^2)}{h_{\sigma^{j+1}\omega}}\right)\circ f_\omega^{j+1}\right]\mathrm{d}\mu_\omega\\
&=\int\left(\frac{\mathcal{L}_{\sigma^i\omega}(h_{\sigma^i\omega}M_i^2)}{h_{\sigma^{i+1}\omega}}\right)\cdot\left[\left(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega}M_j^2)}{h_{\sigma^{j+1}\omega}}\right)\circ f_{\sigma^{i+1}\omega}^{j-i}\right]\mathrm{d}\mu_{\sigma^{i+1}\omega}\\
&=\int\mathcal{L}_{\sigma^i\omega}(h_{\sigma^i\omega}M_i^2)\cdot\left[\left(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega}M_j^2)}{h_{\sigma^{j+1}\omega}}\right)\circ f_{\sigma^{i+1}\omega}^{j-i}\right]\mathrm{d}m\\
&=\int\mathcal{L}_{\sigma^i\omega}^{(j-i+1)}(h_{\sigma^i\omega}M_i^2)\cdot\left(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega}M_j^2)}{h_{\sigma^{j+1}\omega}}\right)\mathrm{d}m.
\end{aligned}
$$

Moreover

$$
\mathbb{E}_\omega(M_i^2\circ f_\omega^i)=\int(M_i^2\circ f_\omega^i)\,\mathrm{d}\mu_\omega=\int M_i^2\,\mathrm{d}\mu_{\sigma^i\omega}
$$

and

$$
\begin{aligned}
\mathbb{E}_\omega(M_j^2\circ f_\omega^j)&=\int(M_j^2\circ f_\omega^j)\,\mathrm{d}\mu_\omega=\int M_j^2\,\mathrm{d}\mu_{\sigma^j\omega}=\int M_j^2h_{\sigma^j\omega}\,\mathrm{d}m\\
&=\int\mathcal{L}_{\sigma^j\omega}(M_j^2h_{\sigma^j\omega})\,\mathrm{d}m=\int\frac{\mathcal{L}_{\sigma^j\omega}(M_j^2h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}}\,\mathrm{d}\mu_{\sigma^{j+1}\omega}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int \mathbb{E}_\omega(M_i^2\circ f_\omega^i|\mathcal{T}_\omega^{i+1})\cdot\mathbb{E}_\omega(M_j^2\circ f_\omega^j|\mathcal{T}_\omega^{j+1})\,\mathrm{d}\mu_\omega-\mathbb{E}_\omega(M_i^2\circ f_\omega^i)\cdot\mathbb{E}_\omega(M_j^2\circ f_\omega^j)\\
&=\int\mathcal{L}_{\sigma^i\omega}^{(j-i+1)}(h_{\sigma^i\omega}M_i^2)\cdot\left(\frac{\mathcal{L}_{\sigma^j\omega}(h_{\sigma^j\omega}M_j^2)}{h_{\sigma^{j+1}\omega}}\right)\mathrm{d}m-\int M_i^2\,\mathrm{d}\mu_{\sigma^i\omega}\cdot\int\frac{\mathcal{L}_{\sigma^j\omega}(M_j^2h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}}\,\mathrm{d}\mu_{\sigma^{j+1}\omega}.
\end{aligned}
$$

Therefore, it follows from lemma 4 that

$$
\begin{aligned}
&\int \mathbb{E}_\omega(M_i^2\circ f_\omega^i|\mathcal{T}_\omega^{i+1})\cdot\mathbb{E}_\omega(M_j^2\circ f_\omega^j|\mathcal{T}_\omega^{j+1})\,\mathrm{d}\mu_\omega-\mathbb{E}_\omega(M_i^2\circ f_\omega^i)\cdot\mathbb{E}_\omega(M_j^2\circ f_\omega^j)\\
&\leqslant K\rho^{j-i+1}\left\|\frac{\mathcal{L}_{\sigma^j\omega}(M_j^2h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}}\right\|_\infty\cdot\left\|M_i^2\right\|_{\mathrm{var}}.
\end{aligned}
$$

Furthermore,

$$
\int\mathbb{E}_\omega(M_k^2\circ f_\omega^k|\mathcal{T}_\omega^{k+1})^2\,\mathrm{d}\mu_\omega\leqslant\left\|\mathbb{E}_\omega(M_k^2\circ f_\omega^k|\mathcal{T}_\omega^{k+1})\right\|_\infty\cdot\mathbb{E}_\omega(M_k^2\circ f_\omega^k).
$$

Thus, the last two inequalities combined with (28) imply that

$$\int \left[\sum_{m<k\leqslant n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) - \sum_{m<k\leqslant n} \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) \right]^2 \mathrm{d}\mu_\omega$$
$$\leqslant \sum_{m<k\leqslant n} \int \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1})^2 \,\mathrm{d}\mu_\omega$$
$$+ 2 \sum_{m<i<j\leqslant n} \int \mathbb{E}_\omega(M_i^2 \circ f_\omega^i | \mathcal{T}_\omega^{i+1}) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j | \mathcal{T}_\omega^{j+1}) \,\mathrm{d}\mu_\omega$$
$$- 2 \sum_{m<i<j\leqslant n} \mathbb{E}_\omega(M_i^2 \circ f_\omega^i) \cdot \mathbb{E}_\omega(M_j^2 \circ f_\omega^j).$$
$$\leqslant \sum_{m<k\leqslant n} \left\| \mathbb{E}_\omega(M_k^2 \circ f_\omega^k | \mathcal{T}_\omega^{k+1}) \right\|_\infty \cdot \mathbb{E}_\omega(M_k^2 \circ f_\omega^k)$$
$$+ 2K \sum_{m<i<j\leqslant n} \rho^{j-i+1} \left\| \frac{\mathcal{L}_{\sigma^j\omega}(M_j^2 h_{\sigma^j\omega})}{h_{\sigma^{j+1}\omega}} \right\|_\infty \cdot \left\| M_i^2 \right\|_{\mathrm{var}},$$

which combined with (H2), (5), (15) and lemmas 9 and 10 implies that (26) holds with

$$h_k = \mathbb{E}_\omega(M_k^2 \circ f_\omega^k) + \left\| M_k^2 \right\|_{\mathrm{var}}.$$

The conclusion of the lemma now follows directly from theorem 2. □

5. Proof of the main theorem

The goal of this section is to establish the almost sure invariance principle by proving theorem 1. It is based on the following theorem due to Cuny and Merlevède.

Theorem 3 ([7]). *Let $(X_n)_n$ be a sequence of square integrable random variables adapted to a non-increasing filtration $(\mathcal{G}_n)_n$. Assume that $\mathbb{E}(X_n|\mathcal{G}_{n+1}) = 0$ a.s.,*

$$v_n^2 := \sum_{k=1}^{n} \mathbb{E}(X_k^2) \to \infty \quad \text{when } n \to \infty \tag{29}$$

and that $\sup_n \mathbb{E}(X_n^2) < \infty$. Moreover, let $(a_n)_n$ be a non-decreasing sequence of positive numbers such that the sequence $(a_n/v_n^2)_n$ is non-increasing, (a_n/v_n) is non-decreasing and such that:

1.

$$\sum_{k=1}^{n} (\mathbb{E}(X_k^2|\mathcal{G}_{k+1}) - \mathbb{E}(X_k^2)) = o(a_n) \quad a.s.; \tag{30}$$

2.

$$\sum_{n\geqslant 1} a_n^{-v} \mathbb{E}(|X_n|^{2v}) < \infty \quad \text{for some } 1 \leqslant v \leqslant 2. \tag{31}$$

Then, enlarging our probability space if necessary, it is possible to find a sequence $(Z_k)_k$ of independent centered (i.e. of zero mean) Gaussian variables with $\mathbb{E}(X_k^2) = \mathbb{E}(Z_k^2)$ such that

$$\sup_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_i-\sum_{i=1}^{k}Z_i\right|=o((a_n(\left|\log(v_n^2/a_n)\right|+\log\log a_n))^{1/2}),\quad \text{a.s.}$$

Notations. In what follows, we write $a_n \sim b_n$ if there exists $c \in \mathbb{R}\setminus\{0\}$ such that $\lim_{n\to\infty} a_n/b_n = c$.

Proof of theorem 1. Part (i) will be proved in proposition 3 below; we now show part (ii). Let us first suppose that by using theorem 3, we could obtain the almost sure invariance principle for the sequence $(X_k)_k = (M_k \circ f_\omega^k)_k$. Combining this with lemma 8, the almost sure invariance principle for the sequence $(\tilde{\psi}_{\theta^k\omega} \circ f_\omega^k)_k$, stated in theorem 1, follows since (21) implies that

$$\sum_{k=0}^{n-1}X_k=\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega}\circ f_\omega^k - G_n\circ f_\omega^n.$$

We are now left with the proof of the ASIP for for

$$X_n = M_n\circ f_\omega^n \quad \text{and} \quad \mathcal{G}_n=\mathcal{T}_\omega^n,$$

and we will apply theorem 3 directly to these quantities. We note that it follows from lemma 11 that

$$\sum_{k=1}^{n}(\mathbb{E}(X_k^2|\mathcal{G}_{k+1})-\mathbb{E}(X_k^2))=O(b_n),$$

with

$$b_n=\Theta^{1/2}(n)\log^{3/2+\varepsilon}\Theta(n), \tag{32}$$

for any positive ϵ and where $\Theta(n)$ is given by (27). On the other hand, it follows from lemma 9 that $\Theta(n)\leqslant Dn$ for some $D>0$ and every $n\in\mathbb{N}$. The last part of this section will be devoted to prove that in our case

$$v_n^2=\sum_{k=0}^{n-1}\mathbb{E}_\omega(M_k^2\circ f_\omega^k)\sim n\Sigma^2,$$

where Σ^2, whose existence is ensured by lemma 12, is assumed strictly positive in part (ii) of this theorem. We now put

$$a_n=v_n\log^{\varepsilon'}(v_n^2),\ \varepsilon'\geqslant\frac{3}{2}+\varepsilon.$$

In this way and noticing that v_n is increasing, the monotonicity assumption on a_n/v_n is immediately satisfied. To deal with the other condition $\frac{a_n}{v_n^2}=\frac{\log^{\varepsilon'}(v_n^2)}{v_n}$, we observe that the function $x\to\frac{\log^{\varepsilon'}(x^2)}{x}, x>0$ has negative derivative for x large enough depending on the value of ε'. This implies in our situation that the monotonicity of $\frac{a_n}{v_n^2}$ is ensured for n large enough. To deal with the (finitely many) lower values of n we use remark 2.4 in [7], which asserts that the condition on $\frac{a_n}{v_n^2}$ can be replaced with the following one: there exists a constant $\tilde{C}$ such that for any $n \geqslant 1$, $\sup_{k\geqslant 1}(\frac{a_k}{v_k^2}) \leqslant \tilde{C}\frac{a_n}{v_n^2}$: the easy details are left to the reader. We have now to show that (31) holds with $v = 2$.

Since $\sup_n \|M_n\|_\infty < \infty$, we have that $\sup_n \|X_n\|_\infty < \infty$ and thus

$$\sum_{n\geqslant 1} a_n^{-2}\mathbb{E}_\omega(|X_n|^4) \leqslant C\sum_{n\geqslant 2} a_n^{-2} \sim C\sum_{n\geqslant 2}\frac{1}{n\log^{2\varepsilon'} n} < \infty,$$

since $2\varepsilon' > 1$. We finally notice that with our choice of a_n and by renaming ϵ as $\frac{1+\varepsilon}{2}$, we get the error term claimed in (19). □

As we said above, in the last part of this section we will prove the linear growth of the variance.

Lemma 12. *There exists* $\Sigma^2 \geqslant 0$ *such that*

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_\omega\left(\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega}\circ f_\omega^k\right)^2 = \Sigma^2, \quad \text{for a.e. } \omega \in \Omega. \tag{33}$$

Proof. Note that

$$\begin{aligned}\mathbb{E}_\omega\left(\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega}\circ f_\omega^k\right)^2 &= \sum_{k=0}^{n-1}\mathbb{E}_\omega(\tilde{\psi}^2_{\sigma^k\omega}\circ f_\omega^k) + 2\sum_{0\leqslant i<j\leqslant n-1}\mathbb{E}_\omega((\tilde{\psi}_{\sigma^i\omega}\circ f_\omega^i)(\tilde{\psi}_{\sigma^j\omega}\circ f_\omega^j))\\ &= \sum_{k=0}^{n-1}\mathbb{E}_\omega(\tilde{\psi}^2_{\sigma^k\omega}\circ f_\omega^k) + 2\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f^{j-i}_{\sigma^i\omega})).\end{aligned}$$

Set $g(\omega) = \mathbb{E}_\omega(\tilde{\psi}^2_\omega)$, $\omega \in \Omega$. By applying the Birkhoff's ergodic theorem for g over the ergodic measure-preserving system $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$, we find that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}_\omega(\tilde{\psi}^2_{\sigma^k\omega}\circ f_\omega^k) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}g(\sigma^k\omega) = \int_\Omega g(\omega)\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_\Omega\int_X\tilde{\psi}(\omega,x)^2\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega) = \int_{\Omega\times X}\tilde{\psi}(\omega,x)^2\,\mathrm{d}\mu(\omega,x),\end{aligned}$$

for a.e. $\omega \in \Omega$. Furthermore, set

$$\Psi(\omega) = \sum_{n=1}^{\infty}\int_X\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,\mathrm{d}\mu_\omega(x) = \sum_{n=1}^{\infty}\int_X\mathcal{L}_\omega^n(\tilde{\psi}_\omega h_\omega)\tilde{\psi}_{\sigma^n\omega}\,\mathrm{d}m.$$

By (14) and (16), we have

$$|\Psi(\omega)| \leqslant \sum_{n=1}^{\infty}\left|\int_X\mathcal{L}_\omega^{(n)}(\tilde{\psi}_\omega h_\omega)\tilde{\psi}_{\sigma^n\omega}\,\mathrm{d}m\right| \leqslant \tilde{K}\sum_{n=1}^{\infty}\rho^n = \frac{\tilde{K}\rho}{1-\rho},$$

for some $\tilde{K} > 0$ and a.e. $\omega \in \Omega$. In particular, $\Psi \in L^1(\Omega)$ and thus it follows again from Birkhoff's ergodic theorem that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\Psi(\sigma^i\omega)=\int_\Omega \Psi(\omega)\,\mathrm{d}\mathbb{P}(\omega)=\sum_{n=1}^{\infty}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,\mathrm{d}\mu(\omega,x), \tag{34}$$

for a.e. $\omega\in\Omega$. In order to complete the proof of the lemma, we are going to show that

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f_{\sigma^i\omega}^{j-i}))-\sum_{i=0}^{n-1}\Psi(\sigma^i\omega)\right)=0, \tag{35}$$

for a.e. $\omega\in\Omega$. Using (14), we have that for a.e. $\omega\in\Omega$,

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f_{\sigma^i\omega}^{j-i}))-\sum_{i=0}^{n-1}\Psi(\sigma^i\omega)\right|\\
&=\left|\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f_{\sigma^i\omega}^{j-i}))-\sum_{i=0}^{n-1}\sum_{k=1}^{\infty}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^{k+i}\omega}\circ f_{\sigma^i\omega}^{k}))\right|\\
&\leqslant\sum_{i=0}^{n-1}\sum_{k=n-i}^{\infty}\left|\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^{k+i}\omega}\circ f_{\sigma^i\omega}^{k}))\right|=\sum_{i=0}^{n-1}\sum_{k=n-i}^{\infty}\left|\int_X\mathcal{L}_{\sigma^i\omega}^{(k)}(\tilde{\psi}_{\sigma^i\omega}h_{\sigma^i\omega})\tilde{\psi}_{\sigma^{k+i}\omega}\,\mathrm{d}m\right|\\
&\leqslant\tilde{K}\sum_{i=0}^{n-1}\sum_{k=n-i}^{\infty}\rho^k=\tilde{K}\frac{\rho}{(1-\rho)^2},
\end{aligned}$$

which readily implies (35). It follows from (34) and (35) that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\mathbb{E}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^i\omega}(\tilde{\psi}_{\sigma^j\omega}\circ f_{\sigma^i\omega}^{j-i}))=\sum_{n=1}^{\infty}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,\mathrm{d}\mu(\omega,x)$$

for a.e. $\omega\in\Omega$ and therefore (33) holds with

$$\Sigma^2=\int_{\Omega\times X}\tilde{\psi}(\omega,x)^2\,\mathrm{d}\mu(\omega,x)+2\sum_{n=1}^{\infty}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^n(\omega,x))\,\mathrm{d}\mu(\omega,x). \tag{36}$$

Finally, we note that it follows readily from (33) that $\Sigma^2\geqslant 0$ and the proof of the lemma is completed. □

We now present necessary and sufficient conditions under which $\Sigma^2=0$. We note that a similar result is stated in [13, 2.10] with $\tilde{\psi}\circ\tau$ instead of $\tilde{\psi}$ in (18).

Proposition 3. *We have that* $\Sigma^2=0$ *if and only if there exists* $\phi\in L^2(\Omega\times X)$ *such that*

$$\tilde{\psi}=\phi-\phi\circ\tau. \tag{37}$$

Proof. We first observe that

$$\begin{aligned}
&\int_{\Omega\times X}\left(\sum_{k=0}^{n-1}\tilde{\psi}(\tau^k(\omega,x))\right)^2 \mathrm{d}\mu(\omega,x)\\
&\quad=\sum_{k=0}^{n-1}\int_{\Omega\times X}\tilde{\psi}^2(\tau^k(\omega,x))\,\mathrm{d}\mu(\omega,x)+2\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}\int_{\Omega\times X}\tilde{\psi}(\tau^k(\omega,x)\tilde{\psi}(\tau^j(\omega,x))\,\mathrm{d}\mu(\omega,x)\\
&\quad=n\int_{\Omega\times X}\tilde{\psi}^2(\omega,x)\,\mathrm{d}\mu(\omega,x)+2\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^{k-j}(\omega,x))\,\mathrm{d}\mu(\omega,x)\\
&\quad=n\int_{\Omega\times X}\tilde{\psi}^2(\omega,x)\,\mathrm{d}\mu(\omega,x)+2\sum_{k=1}^{n-1}(n-k)\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^k(\omega,x))\,\mathrm{d}\mu(\omega,x),
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\Omega\times X}\left(\sum_{k=0}^{n-1}\tilde{\psi}(\tau^k(\omega,x))\right)^2 \mathrm{d}\mu(\omega,x)=n\left(\int_{\Omega\times X}\tilde{\psi}^2(\omega,x)\,\mathrm{d}\mu(\omega,x)+2\sum_{k=1}^{n-1}\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^k(\omega,x))\,\mathrm{d}\mu(\omega,x)\right)\\
&-2\sum_{k=1}^{n-1}k\int_{\Omega\times X}\tilde{\psi}(\omega,x)\tilde{\psi}(\tau^k(\omega,x))\,\mathrm{d}\mu(\omega,x).
\end{aligned}$$

Assume now that $\Sigma^2=0$. Then, it follows from the above equality and (36) that

$$\int_{\Omega\times X}\left(\sum_{k=0}^{n-1}\tilde{\psi}\circ\tau^k\right)^2\mathrm{d}\mu=-2n\sum_{k=n}^{\infty}\int_{\Omega\times X}\tilde{\psi}(\tilde{\psi}\circ\tau^k)\,\mathrm{d}\mu-2\sum_{k=1}^{n-1}k\int_{\Omega\times X}\tilde{\psi}(\tilde{\psi}\circ\tau^k)\,\mathrm{d}\mu. \tag{38}$$

On the other hand, by (14) we have that $\int_{\Omega\times X}\tilde{\psi}(\tilde{\psi}\circ\tau^k)\,\mathrm{d}\mu\to 0$ exponentially fast when $k\to\infty$ and hence, it follows from (38) that the sequence $(X_n)_n$ defined by

$$X_n(\omega,x)=\sum_{k=0}^{n-1}\tilde{\psi}(\tau^k(\omega,x)),\quad \omega\in\Omega,\ x\in X$$

is bounded in $L^2(\Omega\times X)$. Thus, it has a subsequence $(X_{n_k})_k$ which converges weakly to some $\phi\in L^2(\Omega\times X)$. We claim that ϕ satisfies (37). Indeed, take an arbitrary $g=\mathbf{1}_{A\times B}$, where $A\in\mathcal{F}$ and $B\in\mathcal{B}$ and observe that $g\in L^2(\Omega\times X)$ and

$$\begin{aligned}
\int_{\Omega\times X}g(\tilde{\psi}-\phi+\phi\circ\tau)&=\lim_{k\to\infty}\int_{\Omega\times X}g(\tilde{\psi}-X_{n_k}+X_{n_k}\circ\tau)\,\mathrm{d}\mu\\
&=\lim_{k\to\infty}\int_{\Omega\times X}g(\tilde{\psi}\circ\tau^{n_k})\,\mathrm{d}\mu=0,
\end{aligned}$$

where in the last equality we used (14) again. Therefore, $\tilde{\psi}-\phi+\phi\circ\tau=0$ which readily implies (37).

Suppose now that there exists $\phi\in L^2(\Omega\times X)$ satisfying (37). Then,

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\tilde{\psi}\circ\tau^k=\frac{1}{\sqrt{n}}(\phi-\phi\circ\tau^n),$$

and thus

$$\left\|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\tilde{\psi}\circ\tau^k\right\|_{L^2(\Omega\times X)}\leqslant\frac{2}{\sqrt{n}}\|\phi\|_{L^2(\Omega\times X)}\to 0,$$

when $n\to\infty$. Therefore, it follows by integrating (33) over Ω that

$$\Sigma^2=\lim_{n\to\infty}\left\|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\tilde{\psi}\circ\tau^k\right\|^2_{L^2(\Omega\times X)}=0.$$

This concludes the proof of the proposition. □

In the rest of the paper we assume that $\Sigma^2>0$. We also need the following lemmas.

Lemma 13. *We have that*

$$\mathbb{E}_\omega(X_iX_j)=0,\quad \text{for } i<j.$$

Proof. By lemma 2, we conclude that $\mathbb{E}_\omega(M_i\circ f^i_\omega|\mathcal{T}^{i+1}_\omega)=0$. Moreover, we note that $M_j\circ f^j_\omega$ is measurable with respect to $\mathcal{T}^{i+1}_\omega$ and thus

$$\mathbb{E}_\omega((M_j\circ f^j_\omega)(M_i\circ f^i_\omega)|\mathcal{T}^{i+1}_\omega)=(M_j\circ f^j_\omega)\mathbb{E}_\omega(M_i\circ f^i_\omega|\mathcal{T}^{i+1}_\omega)=0,$$

which immediately implies desired conclusion. □

We now recall that v_n^2 is given by (29).

Lemma 14. *We have that $v_n^2\to\infty$ as $n\to\infty$.*

Proof. We already established from (21) that $\sum_{k=0}^{n-1}X_k=\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega}\circ f^k_\omega-G_n\circ f^n_\omega$; therefore

$$\left(\sum_{k=0}^{n-1}X_k\right)^2=\left(\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega}\circ f^k_\omega\right)^2-2(G_n\circ f^n_\omega)\left(\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega}\circ f^k_\omega\right)+(G_n^2\circ f^n_\omega). \tag{39}$$

By lemma 12 and the assumption $\Sigma^2>0$,

$$\tau_n^2:=\mathbb{E}_\omega\left(\sum_{k=0}^{n-1}\tilde{\psi}_{\sigma^k\omega}\circ f^k_\omega\right)^2\to\infty. \tag{40}$$

On the other hand, it follows from (16), (39) and lemma 8 that

$$\mathbb{E}_\omega\left(\sum_{k=0}^{n-1}X_k\right)^2\sim\tau_n^2. \tag{41}$$

By lemma 13 and (41), we have that

$$v_n^2 = \sum_{k=0}^{n-1} \mathbb{E}_\omega(X_k^2) = \mathbb{E}_\omega\left(\sum_{k=0}^{n-1} X_k\right)^2 \sim \tau_n^2, \tag{42}$$

which together with (40) implies the desired conclusion of lemma 14. □

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Appendix A. Verification of Hypotheses (H4) and (H5) for random Lasota–Yorke maps

A.1. Verification of Hypothesis (H4)

Lemma A.1. *For sufficiently large $a > 0$, we have that $\mathcal{L}_\omega^{RN} C_a \subset C_{a/2}$, for any sufficiently large $R \in \mathbb{N}$ and a.e. $\omega \in \Omega$.*

Proof. Choose $\phi \in C_a$. Then, it follows from (10) that

$$\text{var}(\mathcal{L}_\omega^{RN}\phi) \leqslant (a(\alpha^N)^R + C^N)\, \|\phi\|_1 \leqslant a/2\, \|f\|_1\,,$$

whenever R is such that $(\alpha^N)^R < 1/2$ and $a \geqslant \frac{C^N}{1/2-(\alpha^N)^R}$. □

Proof of lemma 1. Let us assume without loss of generality, that $\int \phi\, \mathrm{d}m = 1$. Following [17] we claim that for every $\phi \in C_a$ there exists an interval $J = [\frac{j-1}{n}, \frac{j}{n}] \subset I$ with $n = \lceil 2a \rceil, 1 \leqslant j < n$, such that $\text{ess inf}_{x\in J}\phi \geqslant \frac{1}{2}\int \phi\, \mathrm{d}m$.

Note that $\int_J \phi\, \mathrm{d}m \leqslant |J|\text{ess sup }(\phi|_J) \leqslant |J|(\text{ess inf }(\phi|_J) + \text{var}(\phi_{\text{int}(J)}))$. In particular, if the claim did not hold, we would have

$$1 = \int_I \phi \, \mathrm{d}m = \sum_{j=1}^{n} \int_{[\frac{j-1}{n}, \frac{j}{n}]} \phi \, \mathrm{d}m < \frac{1}{2} + \frac{1}{n} \mathrm{var}(\phi) \leqslant 1,$$

which is a contradiction. Hence, the claim holds.

Now assume (11) holds. Let $\phi \in C_a$ (with $\int \phi \, \mathrm{d}m = 1$) and let n, J be as in the claim above. Let $k = \max_{1 \leqslant j < n} k([\frac{j-1}{n}, \frac{j}{n}])$, as guaranteed by (11). Then, for a.e. $\omega \in \Omega$, $f_\omega^{(k)}(J) = I$. From the definition of $\mathcal{L}$ we get

$$\text{ess inf } \mathcal{L}_\omega^{(k)} \phi \geqslant \text{ess inf } |f_\omega^{(k)}|^{-1} \text{ess inf } (\phi|_J) \geqslant \frac{1}{2} \text{ess inf } |f_\omega^{(k)}|^{-1} =: \alpha_0^*,$$

where α_0^* is independent of ω (recall that $\text{ess sup}_{x \in I, \omega \in \Omega} |f'_\omega(x)| < \infty$).

To finish the proof, let N be as in (10), and R be sufficiently large so that $NR > k$ and the conclusion of lemma A.1 holds. Let $c = 2\alpha_0^* \cdot \text{ess inf}_{x \in I, \omega \in \Omega} |f_\omega^{(NR-k)}|^{-1}$. Then, for every $\phi \in C_a$, $\text{ess inf } \mathcal{L}_\omega^{(NR)} \phi \geqslant c/2$. In addition, by lemma A.1, $\mathcal{L}_\omega^{(NR)} \phi \in C_a$ and $\int \mathcal{L}_\omega^{(NR)} \phi \, \mathrm{d}m = 1$. Hence by induction, we conclude that for every $n \geqslant R$, $\phi \in C_a$ and $\mathbb{P}$ a.e. $\omega \in \Omega$,

$$\text{ess inf } \mathcal{L}_\omega^{Nn} \phi \geqslant c/2 \, \|\phi\|_1 .$$

□

A.2. Verification of Hypothesis (H5)

We will now establish several auxiliary results that will show that (H4) and (10) imply (H5). We begin by recalling the notion of a Hilbert metric on C_a. For $\phi, \psi \in BV$ we write $\phi \preceq \psi$ if $\psi - \phi \in C_a \cup \{0\}$. Furthermore, for $\phi, \psi \in BV$ we define

$$\Xi(\phi, \psi) := \sup\{\lambda \in \mathbb{R}^+ : \lambda\phi \preceq \psi\} \quad \text{and} \quad \Upsilon(\phi, \psi) := \inf\{\mu \in \mathbb{R}^+ : \psi \preceq \mu\phi\},$$

where we take $\Xi(\phi, \psi) = 0$ and $\Upsilon(\phi, \psi) = \infty$ if the corresponding sets are empty. Finally, set

$$\Theta_a(\phi, \psi) = \log \frac{\Upsilon(\phi, \psi)}{\Xi(\phi, \psi)}, \quad \text{for } \phi, \psi \in C_a.$$

We recall (see [3, 17]) that for $\psi \in C_{\nu a}$ for $\nu \in (0, 1)$ such that $\|\psi\|_1 = 1$, we have

$$\Theta_a(g, 1) \leqslant \log \frac{(1+\nu)(1+V)\text{ess sup } \psi}{(1-\nu)(1-V)\text{ess inf } \psi}, \quad \text{where } V = \frac{\text{var}(1_X)}{a}. \tag{A.1}$$

Lemma A.2. *Assume that $\varphi, \psi \in C_a$, $\int \phi = \int \psi = 1$. Then,*

$$\|\varphi - \psi\|_{BV} \leqslant 2(1+a)\Theta_a(\varphi, \psi). \tag{A.2}$$

Proof. Let $r, s \geqslant 0$ such that $r \leqslant 1 \leqslant s$ and $r\varphi \preceq \psi \preceq s\varphi$. Note that if such r or s do not exist, we have that $\Theta_a(\varphi, \psi) = \infty$ and there is nothing to prove. Then, we have

$$\|\psi - \varphi\|_1 \leqslant \int |\psi - r\varphi| + \int (1-r)\varphi = 2(1-r).$$

Furthermore,

$$\mathrm{var}(\psi - \varphi) \leqslant \mathrm{var}(\psi - r\varphi) + (1 - r)\mathrm{var}(\varphi) \leqslant a(1 - r) + (1 - r)a = 2a(1 - r).$$

The above two estimates imply that

$$\|\psi - \varphi\|_{BV} \leqslant 2(1 - r)(1 + a).$$

Since $1 - r \leqslant -\log r \leqslant \log s/r$, we conclude the required inequality from the definition of Θ_a. ☐

Lemma A.3. *For any $a \geqslant 2\mathrm{var}(1_X)$, we have that $\mathcal{L}_\omega^{RN}$ is a contraction on C_a, for any sufficiently large $R \in \mathbb{N}$ and a.e. $\omega \in \Omega$.*

Proof. We follow closely [3, lemma 2.5.]. Let R be given by lemma A.1. We will assume that $n = R$ also satisfies (H4) (with respect to some c). Take now $\phi, \psi \in C_a$. It is sufficient to consider the case when $\|\phi\|_1 = \|\psi\|_1 = 1$. Then,

$$\operatorname{ess\,sup} \mathcal{L}_\omega^{RN}\phi \leqslant \left\|\mathcal{L}_\omega^{RN}\phi\right\|_1 + C_{\mathrm{var}}\mathrm{var}(\mathcal{L}_\omega^{RN}\phi) \leqslant (1 + C_{\mathrm{var}}a/2)\,\|\phi\|_1 = 1 + C_{\mathrm{var}}a/2.$$

By (H4),

$$\operatorname{ess\,inf} \mathcal{L}_\omega^{RN}\phi \geqslant c/2.$$

Using (A.1), we obtain that

$$\Theta_a(\mathcal{L}_\omega^{RN}\phi, 1) \leqslant \log \frac{3/2(1 + \mathrm{var}(1_X)/a)(1 + C_{\mathrm{var}}a/2)}{c(1 - \mathrm{var}(1_X)/a)/4}.$$

Since $a \geqslant 2\mathrm{var}(1_X)$, we have

$$\Theta_a(\mathcal{L}_\omega^{RN}\phi, 1) \leqslant \log \frac{9(1 + C_{\mathrm{var}}a/2)}{c/2}.$$

Using triangle inequality,

$$\Theta_a(\mathcal{L}_\omega^{RN}\phi, \mathcal{L}_\omega^{RN}\psi) \leqslant 2\log \frac{9(1 + C_{\mathrm{var}}a/2)}{c/2} =: \Delta < \infty.$$

This implies that $\mathcal{L}_\omega^{RN}$ is a contraction with coefficient $\kappa \in (0, 1)$ satisfying $\kappa \leqslant \tanh(\Delta/4)$. ☐

Let BV^0 denote the space of all functions in BV of zero mean.

Lemma A.4. *For each $\phi \in BV^0$, there exist $K = K(\phi), \lambda = \lambda(\phi) > 0$ such that*

$$\|\mathcal{L}_\omega^n\phi\|_{BV} \leqslant K\mathrm{e}^{-\lambda n}\,\|\phi\|_{BV} \quad \text{for a.e. } \omega \in \Omega \text{ and } n \in \mathbb{N}.$$

Proof. Without any loss of generality, we can assume that $\|\phi\|_1 = 2$ and thus $\|\phi^+\|_1 = \|\phi^-\|_1 = 1$. Obviously, there exists $a \geqslant 2\mathrm{var}(1_X)$ such that $\phi^+, \phi^- \in C_a$. Assume that R is given by previous lemma and set $M = RN$. Write $n = kM + r$ for $k \in \mathbb{N}_0$ and $0 \leqslant r < M$. It follows from (H2), (A.2) and lemma A.3 that

$$
\begin{aligned}
\|\mathcal{L}_\omega^n \phi\|_{BV} &= \left\|\mathcal{L}_\omega^{kM+r}\phi\right\|_{BV} \\
&= \left\|\mathcal{L}_{\sigma^{kM}\omega}^r \mathcal{L}_\omega^{kM} f\right\|_{BV} \\
&\leqslant 2C^r(1+a)\Theta_a(\mathcal{L}_\omega^{kM}\phi^+, \mathcal{L}_\omega^{kM}\phi^-) \\
&\leqslant 2C^{M-1}(1+a)\Theta_a(\mathcal{L}_\omega^{kM}\phi^+, \mathcal{L}_\omega^{kM}\phi^-) \\
&\leqslant K\kappa^k \\
&\leqslant \frac{K}{2}\kappa^k \|\phi\|_1 \\
&\leqslant \frac{K}{2}(\kappa^{1/M})^n \cdot \kappa^{-r/M} \|\phi\|_{BV},
\end{aligned}
$$

for some $K > 0$ which readily implies the desired conclusion. □

Finally, we obtain (5) by removing ϕ-dependence of K and λ in lemma A.3.

Proof of lemma 2. Let l^1 denote the space of all sequences $\Phi = (\phi_n)_{n\geqslant 1} \subset BV^0$ such that

$$
\|\Phi\|_1 = \sum_{n\geqslant 1} \|\phi_n\|_{BV} < \infty.
$$

Then, $(l^1, \|\cdot\|_1)$ is a Banach space. For each $\omega \in \Omega$ and $n \in \mathbb{N}$, we define a linear operator $T(\omega, n)\colon BV^0 \to l^1$ by

$$
T(\omega, n)\phi = (\mathcal{L}_\omega(\phi), \mathcal{L}_\omega^2(\phi), \ldots, \mathcal{L}_\omega^n(\phi), 0, 0, \ldots), \quad \phi \in BV^0.
$$

We note that $T(\omega, n)$ is bounded operator. Indeed, it follows from (H2) that

$$
\|T(\omega, n)\phi\|_1 = \sum_{k=1}^n \left\|\mathcal{L}_\omega^k(\phi)\right\|_{BV} \leqslant \sum_{k=1}^n C^k \|\phi\|_{BV} \leqslant nC^n \|\phi\|_{BV}.
$$

Hence, $T(\omega, n)$ is bounded. Furthermore, note that it follows from previous lemma that

$$
\|T(\omega, n)\phi\|_1 = \sum_{k=1}^n \left\|\mathcal{L}_\omega^k(\phi)\right\|_{BV} \leqslant \frac{K(\phi)}{1-\mathrm{e}^{-\lambda(\phi)}} \|\phi\|_{BV} = C(\phi)\|\phi\|_{BV},
$$

where

$$
C(\phi) := \frac{K(\phi)}{1-\mathrm{e}^{-\lambda(\phi)}}.
$$

Hence, for each $\phi \in BV^0$, we have

$$
\sup\{\|T(\omega, n)\phi\|_1 : \omega \in \Omega,\ n \in \mathbb{N}\} < \infty.
$$

It follows from the uniform boundedness principle that there exists $L > 0$ independent on ω and n such that

$$
\|T(\omega, n)\| \leqslant L, \quad \omega \in \Omega,\ n \in \mathbb{N}.
$$

Hence,

$$\sum_{k=1}^{n} \left\| \mathcal{L}_\omega^k(\phi) \right\|_{BV} \leqslant L \left\|\phi\right\|_{BV}, \quad \omega \in \Omega,\ \phi \in BV^0,\ n \in \mathbb{N}.$$

In particular,

$$\left\|\mathcal{L}_\omega^n(\phi)\right\|_{BV} \leqslant L \left\|\phi\right\|_{BV}, \quad \omega \in \Omega,\ \phi \in BV^0,\ n \in \mathbb{N}. \tag{A.3}$$

Using (A.3), for $\omega \in \Omega$ and $1 \leqslant k \leqslant n$,

$$\left\|\mathcal{L}_\omega^n(\phi)\right\|_{BV} = \left\|\mathcal{L}_{\sigma^k\omega}^{n-k}\mathcal{L}_\omega^k(\phi)\right\|_{BV} \leqslant L \left\|\mathcal{L}_\omega^k(\phi)\right\|_{BV}.$$

Summing over k,

$$n \left\|\mathcal{L}_\omega^n(\phi)\right\|_{BV} \leqslant L \sum_{k=1}^{n} \left\|\mathcal{L}_\omega^k(\phi)\right\|_{BV} \leqslant L^2 \left\|\phi\right\|_{BV},$$

and thus

$$\left\|\mathcal{L}_\omega^n(\phi)\right\|_{BV} \leqslant \frac{L^2}{n} \left\|\phi\right\|_{BV}.$$

We conclude that there exists $N_0 \in \mathbb{N}$ independent on ω such that

$$\left\|\mathcal{L}_\omega^{N_0}(\phi)\right\| \leqslant \frac{1}{e} \left\|\phi\right\|, \quad \omega \in \Omega,\ \phi \in BV^0.$$

Take now any $n \in \mathbb{N}$ and write it as $n = kN_0 + r$, $k \in \mathbb{N}_0$ and $0 \leqslant r < N_0$. Using (H2) and the inequality above,

$$\left\|\mathcal{L}_\omega^n \phi\right\|_{BV} \leqslant \frac{C^{N_0}}{e^k} \left\|\phi\right\|_{BV}, \quad \omega \in \Omega,\ \phi \in BV^0,$$

which readily implies (H5). □

Appendix B. Verification of Hypothesis (H4) for multidimensional random piecewise expanding maps

Define the cone $\mathcal{C}_a := \{\phi \in \mathcal{B}_\beta; \phi \geqslant 0; \mathrm{var}_\beta(\phi) \leqslant a\, \mathbb{E}_m(\phi)\}$. The following lemma is the multidimensional version of the first part of the proof of lemma 1.

Lemma B.1. *For each element $\phi \in \mathcal{C}_a$, there is an open set J on which ϕ is bounded from below by $\mathbb{E}_m(\phi)/2$.*

Proof of lemma 19. Take a function $\phi \in \mathcal{C}_a$ with $\int_M \phi \mathrm{d}m = 1$. Let us consider a sufficiently fine, finite partition Q of M into cubes (intersected with M). We recall that ε_0 is the constant entering the seminorm var_β. Consider $\varepsilon' < \min\{\varepsilon_0, \frac{1}{2a}\}$ and assume all elements in Q have diameter less than ε'. Then, for every $x \in B_k$ we have that $B_k \subset B_{\varepsilon'}(x)$. In particular, $\mathrm{osc}(\phi, B_k) \leqslant \mathrm{osc}(\phi, B_{\varepsilon'}(x))$ for every $x \in B_k$. Then,

$$1 = \int_M \phi \mathrm{d}m = \sum_k \int_{B_k} \phi \mathrm{d}m \leqslant \sum_k \int_{B_k} \sup_{B_k} \phi \mathrm{d}m \leqslant \sum_k \int_{B_k} (\inf_{B_k} \phi + \operatorname{osc}(\phi, B_k))\mathrm{d}m$$
$$\leqslant \sum_k \int_{B_k} \left(\inf_{B_k} \phi + \operatorname{osc}(\phi, B_{\varepsilon'}(x))\right) \mathrm{d}m(x) \leqslant \left(\sum_k \int_{B_k} (\inf_{B_k} \phi)\mathrm{d}m\right) + \int_M \operatorname{osc}(\phi, B_{\varepsilon'}(x))\mathrm{d}m(x). \tag{B.1}$$

Notice that $\int_M \operatorname{osc}(\phi, B_{\varepsilon'}(x))\mathrm{d}m(x) \leqslant \varepsilon' \mathrm{var}_\beta(\phi) \leqslant \varepsilon' \ a \int_M \phi \mathrm{d}m < \frac{1}{2}$. Hence, there exists a cube B_j where the essential infimum of *f* is at least 1/2. Indeed, if this were not the case, the first term on the rhs of (B.1) would be bounded above by 1/2, and (B.1) would not be satisfied. □

Proof of lemma 3. The proof follows closely the strategy from appendix A. We first deal with hypothesis (H4). Lemma A.1 follows similarly in multidimensional setting. Lemma B.1 proves the first part of lemma 1 in our multidimensional setting, and the rest of the proof of lemma 1 proceeds verbatim. Now we turn to Hypthesis (H5). In the multidimensional setting, we replace lemma A.2 with lemma 3.8 [10]. The proofs of lemmas A.3–A.4 remain valid in the multidimensional setting using var_β instead of var. The proof of lemma 2 then proceeds verbatim. We thus obtain (H5) in our multidimensional setting. □

We note that results similar to lemmas A.3 and A.4 in the multidimensional setting are contained in lemma 3.6 and theorem 2.17 [10], respectively.

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