RETURN TIME STATISTICS FOR UNIMODAL MAPS

H. BRUIN AND S. VAIENTI

Abstract. We prove that a non-flat S-unimodal map satisfying a weak summability condition has exponential return time statistics on intervals around a.e. point. Moreover we prove that the convergence to the entropy in the Ornstein-Weiss formula enjoys normal fluctuations.

1. Introduction and statements of the results

The subject of this paper is the asymptotic distribution of return times in dynamical systems. We consider a measure preserving ergodic dynamical systems (X, f, μ). For $z \in X$, denote by U_z a measurable set containing z and by $\mu_{U_z} = \frac{\mu|U_z}{\mu(U_z)}$ the conditional measure on U_z. Let $\tau_{U_z}(x)$ be the first return of the point x into U_z: $\tau_{U_z}(x) = \inf(k > 0 \mid f^k(x) \in U_z, x \in U_z)$. We are interested in the distribution:

$$\mu_{U_z}(x \in U_z \mid \tau_{U_z}(x)\mu(U_z) > t), \tag{1}$$

as $\mu(U_z) \to 0$. We refer to [16, 8, 14] for the history and motivations of this question and for a presentation and discussion of the different techniques to solve it. Surprisingly, the limiting distribution of (1) shows a universal behavior for a wide class of dynamical systems with some degree of mixing, namely the distribution tends to e^{-t}. Under slightly stronger mixing conditions, one can also prove that the statistics of successive returns have a Poissonian distribution. [16, 14]. We want to stress that the convergence of the distribution (1) holds for μ-a.e. z and that the choice of the set U_z is also relevant. Proofs usually require U to be a cylinder set (w.r.t. some partition) or a ball; the use of balls is much more general and we will assume it in this paper. It is interesting that "return times to balls" recently allowed to reformulate several results of thermodynamic formalism in terms of return times statistics only, see [15, 2, 32, 13]. In [32] this approach was called "thermodynamics of return times".

In [8] we advocated the technique of inducing to explain the observed universality in the distribution of return times. Let us briefly recall the main result of that paper. Assume that X is a Riemannian manifold, consider an open set $\hat{X} \subset X$ and let $\hat{f} : \hat{X} \to \hat{X}$ the first return map. We will denote by $\hat{\mu}$ the conditional measure on $\hat{X}$, which can be proved to be $\hat{f}$-invariant and ergodic. Let $U_r(z)$ be the ball of radius r centered at $z \in X$ and let

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$\tau_{U_r(z)}(x)$ (resp. $\hat{\tau}_{U_r(z)}(x)$) be the first return time into $U_r(z)$ for f (resp. $\hat{f}$). As above we denote by μ_A (resp. $\hat{\mu}_A$) the conditional measure on A. We suppose that $(\hat{X}, \hat{\mu}, \hat{f})$ has return time statistics $\hat{g}(t)$, *i.e.*, for $\hat{\mu}$-a.e. $z \in \hat{X}$, there exists $\epsilon_z(r) > 0$ with $\epsilon_z(r) \to 0$ as $r \to 0$ such that:

$$\sup_{t \geq 0} |\hat{\mu}_{U_r(z)}(x \in U_r(z) \mid \hat{\tau}_{U_r(z)}(x) > \frac{t}{\hat{\mu}(U_r(z))}) - \hat{g}(t)| < \epsilon_z(r).$$

The key result of [8] is that the map $\hat{f}$ enjoys on $\hat{X}$ the same distribution as f:

Theorem 1. [8] *If the function $\hat{g}$ is continuous at 0, then there exists $g : \mathbb{R}^+ \to [0,1]$ such that $N := \{x \mid g(x) \neq \hat{g}(x)\}$ is countable and for μ-a.e. $z \in \hat{X}$ and $t \notin N$, there exists $\delta_{z,t}(r) > 0$ with $\delta_{z,t}(r) \to 0$ uniformly in t as $r \to 0$ such that:*

$$|\mu_{U_r(z)}(x \in U_r(z) \mid \tau_{U_r(z)}(x) > \frac{t}{\mu(U_r(z))}) - g(t)| < \delta_{z,t}(r).$$

This result is useful for dynamical systems that admit neighborhoods (around μ-a.e. point) where the induced (*i.e.* first return) map is hyperbolic. The distribution in the induced system can therefore computed by the usual techniques, quoted for example in [16], [14], [8], and then pushed back, by Theorem 1, to the original system. We gave in [8] a few applications; in particular we proved the exponential statistics, $g(t) = e^{-t}$ for C^3 interval maps of the interval such that the closure of the orbit of the critical points has zero measure.

The existence of a hyperbolic first return map around μ-a.e. point is, however, unlikely. In this paper we show how, for unimodal maps, this assumption can be discarded. Instead, we impose a summability condition which is used (among other things) to the guarantee the existence of an invariant probability measure μ which is absolutely continuous with respect to Lebesgue. This summability condition is considerably weaker than the Collet-Eckmann condition, which requires that the derivatives along the critical orbit grow exponentially fast. Hence Theorem 2 below extends the results in [30].

Let $f : I \to I$ be a C^3 S-unimodal map. The S stands for negative Schwarzian derivative, *i.e.*, $\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 < 0$ for all non-critical points. Let c be the critical point, and assume that it is non-flat. This means that there exists $\ell \in (1, \infty)$ such that $\lim_{x \to c} |f(x) - f(c)|/|x - c|^\ell$ exists and is positive. Denote $f^i(c)$ by c_i. In Section 2 we will prove the following theorem:

Theorem 2. *Let f be a non-flat C^3 S-unimodal map, satisfying*

$$\sum_{n \geq 1} |Df^{n-1}(c_1)|^{-\frac{1}{\ell}} < \infty. \tag{2}$$

Let μ be the invariant measure which is absolutely continuous with respect to Lebesgue. Then f has exponential return time statistics on intervals, i.e., *for μ-a.e. z*

$$\mu_U(\{y \mid \tau_U(y) > \frac{t}{\mu(U)}\}) \to e^{-t} \text{ as } \mu(U) \to 0,$$

where U are intervals around z.

The summability condition (2) is the same as was used by Nowicki & van Strien [28] to show the the existence of an absolutely continuous invariant probability measure μ. Clearly (2) is more inclusive than the Collet-Eckmann condition which were used in comparable results on return statistics, [3, 9]. In particular, Theorem 2 applies to systems where the decay of correlations is *not* exponentially fast, cf. [27].

As a second main result, we will prove the log-normal fluctuations for the entropy by the *Ornstein-Weiss* formula. Given a generating partition $\mathcal{C}$ and corresponding cylinder sets $C_n(x)$, this formula says that the first return of μ-a.e. x grows like e^{nh_μ}, where h_μ is the measure theoretic entropy of μ. The distribution of the process $\frac{\log \tau_{C_n(x)}(x) - nh_\mu}{\sqrt{n}}$, suitably normalized, will converge in law to the Gauss zero-one law.

In Section 3, we will show that this applies to unimodal maps and $\mathcal{C} = \{[c_2, c], [c, c_1]\}$, the partition into the two monotonicity intervals. To the absolutely continuous invariant measure μ we associate the variance σ_μ:

$$\sigma_\mu^2 = \sigma_\mu^2(\varphi) = \int \varphi^2 d\mu - (\int \varphi d\mu)^2 + 2\sum_{n=1}^{\infty}[\int \varphi(f^n)\varphi d\mu - (\int \varphi d\mu)^2], \quad (3)$$

where $\varphi = \log|f'| - \int \log|f'| d\mu$.

Theorem 3. *Let f be a non-flat C^3 S-unimodal interval map. Assume that $|Df^n(c_1)| \geq Cn^\tau$ for some $\tau > 4\ell - 3$ and $C > 0$. If the variance $\sigma_\mu^2 > 0$, then*

$$\mu(x \in I \mid \frac{\log \tau_{C_n(x)}(x) - nh_\mu}{\sigma_\mu \sqrt{n}} > u) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{x^2}{2}} dx. \quad (4)$$

The most difficult part in the proof will be the verification of the Central Limit Theorem for the potential φ, see Theorem 4. We think that this is interesting enough to state separately.

Theorem 4. *Under the assumptions of Theorem 3, φ satisfies the Central Limit Theorem.*

2. The exponential statistics

2.1. **The Hofbauer tower.** Let us come back to the unimodal map $f : I \to I$ defined in the introduction. If we denote $f^i(c)$ by c_i, we can rescale, without loss of generality, f such that $I = [c_2, c_1]$. Condition (2) implies that c is not attracted to a stable periodic orbit, neither is f is infinitely renormalizable.

We presently describe the canonical Markov extension (Hofbauer tower) of f. Let $D_2 = [c_2, c_1]$ and

$$D_{k+1} = \begin{cases} f(D_k) & \text{if } c \notin D_k, \\ [c_{k+1}, c_1] & \text{if } c \in D_k. \end{cases}$$

A short induction prove shows that $D_k = [c_k, c_{\beta(k)}]$ or $D_k = [c_{\beta(k)}, c_k]$, where $\beta(k) = k - \max\{i < k \mid D_i \ni c\}$. See [4] for a proof and additional information. The *Hofbauer tower* is the disjoint union $\hat{I} := \sqcup_{k \geq 2} D_k$. We can define the following map $\hat{g}$:

$$\hat{g}(x \in D_k) = f(x) \in \begin{cases} D_{k+1} & \text{if } c \notin [c_k, x], \\ D_{\beta(k)+1} & \text{if } c \in [c_k, x]. \end{cases}$$

By construction, $(\hat{I}, \hat{g})$ is a Markov map, in the sense that the image of each level D_k is precisely the union of levels D_i. Let $\pi : \hat{I} \to I$ be the natural projection. By construction (I, f) is a factor of $(\hat{I}, \hat{g})$: $\pi \circ \hat{g} = f \circ \pi$. Keller's result [17] states that the measure μ can be lifted to a measure $\hat{\mu}$ on $\hat{I}$ which is $\hat{g}$-invariant, absolutely continuous and satisfies $\mu = \hat{\mu} \circ \pi^{-1}$. In short, $\hat{\mu}$ is constructed as the vague limit of the sequence:

$$\begin{aligned} \hat{\mu}_1 | D_2 &= \text{normalized Lebesgue}, \quad \hat{\mu}_1 | D_k \equiv 0 \text{ for } k \geq 3, \\ \hat{\mu}_n &= \hat{\mu}_{n-1} \circ \hat{g}^{-1}. \end{aligned}$$

Several of our estimates are based of versions of the Koebe Lemma. We recall it as far as we use it. Originally this lemma is proved under the assumption that f has negative Schwarzian derivative. Recent work of Kozlovski shows that the C^3 assumption is sufficient, see [25, Section IV.3] and [21], but the focus of our results would not justify more complicated proofs avoiding the negative Schwarzian assumption.

Lemma 1. *Let $g : (a, b) \to \mathbb{R}$ be a monotone C^3 map with negative Schwarzian derivative. Then for any $\delta > 0$, there is a Koebe constant $K = K(\delta) = (\frac{1+\delta}{\delta})^2$ such that the following hold:*

- **(Koebe Principle)** *If $x \in (a, b)$ is such that $\frac{\min\{|g(b)-g(x)|, |g(x)-g(a)|\}}{|g(b)-g(a)|} \geq \delta$, then*
$$\frac{1}{K} \frac{|g(b) - g(a)|}{|b - a|} \leq |g'(x)| \leq K \frac{|g(b) - g(a)|}{|b - a|}.$$
- **(Macroscopic Koebe Principle)** *Under the same hypothesis,*
$$\frac{\min\{|b - x|, |x - a|\}}{|b - a|} \geq \frac{1}{K^2}.$$

Lemma 2. *Assume that f is conservative w.r.t. Lebesgue measure. There exists $\delta > 0$ such that for Lebesgue a.e. $x \in I$, there exist sequences of intervals $I_n \supset J_n \ni x$ satisfying*

- *both components of $I_n \setminus J_n$ have length $\geq \delta |I_n|$;*
- *$|I_n| \to 0$;*
- *$f^i(\partial J_n) \cap I_n^\circ = \emptyset$ for all $i \geq 1$. (Here I_n° denotes the interior.)*

The sets J_n (or rather certain lifts of them) will play the role of $\hat{X}$ from Theorem 1. The property $\text{orb}(\partial J_n) \cap J_n = \emptyset$ implies that any branch $f^s : H \subset J_n \to J_n$ of the first return map to J_n (*i.e.*, H is a maximal interval on which this first return map is continuous) satisfies $f^s(\partial H) \subset \partial J_n$. In particular, if $f^s|H$ is monotone, then $f^s(H) = J_n$. Most of these branches are extendible to an onto branch $f^s : H' \to I_n$, and then the Koebe Principle yields that the distortion of $f^s|H$ is uniformly bounded (by $K(\delta)$).

Proof. The proof depends on a result of Martens [24, Lemma 4.2] valid for conservative maps. It states that there exist symmetric intervals V, U around c, V compactly contained in U, such that both $f^i(\partial V)$ and $f^i(\partial U)$ are disjoint from the interior of U for all $i \geq 1$. Furthermore, for Lebesgue a.e. x there exists an integer sequence $k_n \to \infty$ and intervals $I_n \supset J_n \ni x$ such that f^{k_n} maps I_n monotonically onto U and J_n onto V. By the macroscopic Koebe Principle, there exists $\delta > 0$, depending only on V and U, such that both components of $I_n \setminus J_n$ have length $\geq \delta |I_n|$. Moreover, if $f^i(\partial J_n) \cap I_n^\circ \neq \emptyset$ for some $i > 0$, then $f^{k_n+i}(\partial J_n) = f^i(\partial V)$ intersects $f^{k_n}(I_n^\circ) = U^\circ$ contradicting the properties of V. Finally, due to the absence of wandering intervals (see [25, page 267]), $|I_n| \to 0$ as $n \to \infty$. Hence the I_n and J_n have all the asserted properties. □

Given $I_n \supset J_n$, let $\hat{J}_n = \sqcup\{D_k \cap J_n \mid D_k \supset I_n\} \subset \hat{I}$. Note that in general, $\pi^{-1}(J_n)$ strictly contains $\hat{J}_n$. Only if $x \notin \overline{\text{orb}(c)}$, then the two sets are equal for n sufficiently large. Define $\hat{F}_n : \hat{J}_n \to \hat{J}_n$ to be the first return map. It is known (see [17, Theorem 1(3)]) that $(\hat{I}, \hat{g})$ is ergodic and conservative. Therefore $\hat{F}_n$ is defined $\hat{\mu}$-a.e. on $\hat{J}_n$. By the Markov property of the Hofbauer tower and the construction of J_n it follows that $\hat{F}_n$ is an *extendible Markov map* in the sense that if $y \in \hat{J}_n$ and $\hat{F}_n|\{y\} = \hat{g}^i$, then there exists an interval $V \ni y$ such that $\hat{g}^i$ maps V continuously and monotonically onto an interval W such that $\pi(W) = I_n$. These ideas were discussed in detail in [5]. It follows by the Koebe Principle that all branches of all iterates of $\hat{F}_n$ have the same distortion bounds. Moreover, there exists k_0 such that $\inf_{k \geq k_0} \inf_{y \in \hat{J}_n} |D\hat{F}_n^k(y)| > 1$.

Lemma 3. *Let U be any subinterval of J_n and $\hat{U}_n := \pi^{-1}(U) \cap \hat{J}_n$. Then the system $(\hat{J}_n, \hat{\mu}_{\hat{J}_n}, \hat{F}_n)$ has exponential return time statistics on $\hat{U}_n$*

Proof. The map $\hat{F}_n$ corresponds to a jump transformation F_n on J_n, defined by $F_n(y) = \pi \circ \hat{F}_n \circ \pi^{-1}(y)$. The above described properties of $\hat{F}_n$ show that F_n is well defined. In fact $F_n(y) = f^{\tilde{\tau}(y)}(y)$, where $\tilde{\tau}(y)$ is the first *n-decent* return to J_n, defined as

$$\tilde{\tau}(y) = \tilde{\tau}_U(y) = \min\{i > 0, \begin{array}{l} f^i(y) \in U \text{ and there exists an interval} K \ni y \\ \text{such that } f^i : K \to I_n \text{ is monotone onto.} \end{array} \}$$

As was shown in [5], $\tilde{\tau}(y) = \hat{\tau}(\hat{y})$ for any $\hat{y} \in \hat{J}_n$ and $\hat{\tau}$ the first return time to $\hat{J}_n$. The map F_n is a Rychlik map as defined in [8]. The results of that paper yield the exponential return statistics. □

2.2. **Proof of Theorem 2.** Let us first collect some facts of the invariant density $h(y) = \frac{d\mu(y)}{dy}$ of μ. Define $\delta_k = |Df^{k-1}(c_1)|^{-1/\ell}$. It was shown by Nowicki [26] that for some constant $C > 0$,

$$h(y) \leq C \sum_{k \geq 1} \delta_k |y - c_k|^{-(1-\frac{1}{\ell})}. \tag{5}$$

In particular

$$\sum_k \delta_k |c_k - x|^{\frac{1}{\ell}-1} < \infty \quad \text{for } \mu\text{-a.e. } x. \tag{6}$$

If, in addition, f satisfies the Collet-Eckmann condition (*i.e.*, $\delta_k \to 0$ exponentially), then more precise estimates are known [33, 19]. Keller & Nowicki showed that $h(y) = \psi(y) \sum_{k \geq 1} \delta_k |y - c_k|^{-(1-\frac{1}{\ell})}$, where ψ has bounded variation. For our purpose we only need the upper bound in (5). It immediately follows that the density h is finite at c. Indeed, the Chain Rule and non-flatness give

$$|Df^k(c_1)| = |Df^{k-1}(c_1)|\,|Df(c_k)| = \mathcal{O}(\ell)|Df^{k-1}(c_1)|\,|c - c_k|^{\ell-1}.$$

Therefore

$$\begin{aligned} h(c) &\leq C\sum_k |Df^{k-1}(c_1)|^{-\frac{1}{\ell}} |c - c_k|^{1-\frac{1}{\ell}} \\ &\leq \mathcal{O}(C\ell) \sum_k |Df^k(c_1)|^{-\frac{1}{\ell}} < \infty. \end{aligned} \tag{7}$$

We want to consider "intermediate" first return maps to J_n, where J_n are neighbourhoods of x as introduced in Lemma 2, and eventually we take limits $n \to \infty$, *i.e.*, $|J_n| \to 0$. Assume $x \notin \text{orb}(c)$. Since endpoints of level D_k belong to orb(c), every level D_k that contains x also contains I_n for n large. Therefore more returns to a small neighborhood U, $x \in U \subset I_n$, will be n-decent if n gets larger (so the intervals U we can consider depends on n). We will need a precise on the proportion of decent branches for the return map to U.

Lemma 4. *For any x satisfying Lemma 2 and formula (6)*

$$\lim_{n\to\infty} \sup_{x \in U \subset J_n} \mu_U(\{y \mid \tau_U(y) \text{ is not } n\text{-decent}\}) = 0.$$

Proof. As in Lemma 3, set $\hat{U}_n := \pi^{-1}(U) \cap \hat{J}_n$. Set $\hat{h}(y) = \frac{d\hat{\mu}(y)}{dy}$ and let $\hat{h}_k = \hat{h}|D_k$ be the densities on levels of $\hat{I}$. Clearly $h = \sum_k \hat{h}_k$. Moreover, $\hat{h}$ is a fixed point of the Perron-Frobenius operator on $\hat{I}$ and the points $c \in D_k$ are the only critical points of $\hat{g}$.

Due to the Markov property of $(\hat{I}, \hat{g})$, $\hat{h}_k$ has only two singularities, namely at the endpoints c_k and $c_{\beta(k)}$. More precisely, letting $a_k = \hat{h}_k(c)$ when $c \in D_k$ and $a_k = 0$ otherwise, we have for $y \in D_k = [c_k, c_{\beta(k)}]$:

$$\hat{h}_k(y) \leq C(\delta_k |y - c_k|^{-(1-\frac{1}{\ell})} + a_{k-\beta(k)} \delta_{\beta(k)} |y - c_{\beta(k)}|^{-(1-\frac{1}{\ell})}).$$

The first term is obvious because $\hat{h}_k \leq h$; the second term arises because the endpoint $c_{\beta(k)}$ is the image of $c \in D_{k-\beta(k)}$ under $\hat{g}^{k-\beta(k)}$.

Now let x satisfy Lemma 2 (so that the sets $I_n \supset J_n \ni x$ are well-defined), and also (6). Take an interval $U = (u_0, u_1)$ such that $x \in U \subset J_n$. Let $\hat{\Phi} : \pi^{-1}(U) \to \pi^{-1}(U)$ be the first return map. By $\hat{\Phi}$-invariance of $\hat{\mu}_{\pi^{-1}(U)}$

we find

$$\begin{aligned}\mu(\{y \in U \mid \tau_U \text{ is not } n\text{-decent}\}) &= \hat{\mu}(\hat{\Phi}^{-1}(\pi^{-1}(U) \setminus \hat{U}_n)) \\ &= \hat{\mu}(\pi^{-1}(U) \setminus \hat{U}_n).\end{aligned}$$

We know (see [18, 6]) that h is bounded away from 0, say $h_0 = \inf_{\text{supp}(\mu)} h > 0$. Then we can integrate

$$\begin{aligned}\frac{\hat{\mu}(\pi^{-1}(U) \cap D_k)}{\mu(U)} &\leq \frac{\hat{\mu}(\pi^{-1}(U) \cap D_k)}{h_0|u_1 - u_0|} \\ &\leq \frac{C}{h_0|u_1 - u_0|} \cdot \\ &\quad \int_{u_0}^{u_1} (\delta_k|c_k - y|^{\frac{1}{\ell}-1} + \delta_{\beta(k)} a_{k-\beta(k)} |c_{\beta(k)} - y|^{\frac{1}{\ell}-1}) dy \\ &\leq \frac{C\ell}{h_0}(\delta_k|c_k - x|^{\frac{1}{\ell}-1} + \delta_{\beta(k)} a_{k-\beta(k)} |c_{\beta(k)} - x|^{\frac{1}{\ell}-1}).\end{aligned}$$

As $x \neq c_k$ for any $k \geq 0$, there exists $N = N(n)$ such that all returns to levels D_k with $k \leq N$ are n-decent, and $N(n) \to \infty$ for $n \to \infty$. Therefore

$$\frac{\mu(\{y \in U \mid \tau_U \text{ is not } n\text{-decent}\})}{\mu(U)} \leq$$
$$\frac{C\ell}{h_0} \sum_{k \geq N} \delta_k|c_k - x|^{\frac{1}{\ell}-1} + \delta_{\beta(k)} a_{k-\beta(k)} |c_{\beta(k)} - x|^{\frac{1}{\ell}-1}.$$

As $\sum_k a_k = h(c) < \infty$ and because of (6), both $\delta_k|c_k - x|^{\frac{1}{\ell}-1}$ and a_k are summable. Therefore

$$\lim_{n \to \infty} \frac{\mu(\{y \in U \mid \tau_U \text{ is not } n\text{-decent}\})}{\mu(U)} = 0,$$

uniformly in U. This proves the lemma. □

Now we are ready to prove the main theorem.

Proof of Theorem 2. Let $\alpha_n = \sup_{x \in U \subset J_n} \frac{\hat{\mu}(\hat{U}_n)}{\mu(U)}$. As we have seen in Lemma 4, $\lim_n \alpha_n = 1$. Because $f \circ \pi = \pi \circ \hat{g}$ we have

$$\mu_U(\{y \mid \tau_U(y) > \frac{t}{\mu(U)}\}) = \hat{\mu}_{\pi^{-1}(U)}(\{\hat{y} \mid \hat{\tau}_{\pi^{-1}(U)}(\hat{y}) > \frac{t}{\mu(U)}\}).$$

The right hand side can be rewritten as a sum of three terms:

$$\begin{aligned}\text{r.h.s.} &\leq \hat{\mu}_{\pi^{-1}(U)}(\pi^{-1}(U) \setminus \hat{U}_n) \\ &\quad + \hat{\mu}_{\pi^{-1}(U)}(\{\hat{y} \in \hat{U}_n \mid \hat{\tau}_{\hat{U}_n}(\hat{y}) > \frac{t}{\mu(U)}\}) \\ &\quad + \hat{\mu}_{\pi^{-1}(U)}(\{\hat{y} \in \hat{U}_n \mid \hat{\tau}_{\hat{U}_n}(\hat{y}) > \tau_{\pi^{-1}(U)}(\hat{y})\}) \\ &= I + II + III.\end{aligned}$$

We have the estimates

$$I = \frac{\hat{\mu}(\pi^{-1}(U) \setminus \hat{U}_n)}{\hat{\mu}(\pi^{-1}(U))} = \frac{\hat{\mu}(\pi^{-1}(U) \setminus \hat{U}_n)}{\mu(U)} \leq 1 - \alpha_n \to 0.$$

Next

$$II = \alpha_n \hat\mu_{\hat U_n}(\{\hat y \mid \hat\tau_{\hat U_n}(\hat y) > \frac{t}{\mu(U)}\}) = \alpha_n \hat\mu_{\hat U_n}(\{\hat y \mid \hat\tau_{\hat U_n}(\hat y) > \frac{\tilde t}{\hat\mu(\hat U_n)}\})$$

for $\tilde t = t\alpha_n$. The main result of [8] says that the return statistics of an induced transformation coincides with the return statistics of the original system. In this case, it means that the system $(\hat I, \hat\mu, \hat g)$ has the same return statistics on $\hat U_n$ as the induced system $(\hat J_n, \hat\mu_{\hat J_n}, \hat F_n)$. By Lemma 3, these statistics have the exponential distribution. Hence II tends to $\alpha_n e^{-\tilde t}$ as $\mu(U) \to 0$, and then to e^{-t} as $n \to \infty$. The third term

$$\begin{aligned} III &= \hat\mu_{\pi^{-1}(U)}(\hat\Phi^{-1}(\pi^{-1}(U) \setminus \hat U_n) \cap \hat U_n) \\ &\le \hat\mu_{\pi^{-1}(U)}(\pi^{-1}(U) \setminus \hat U_n) = I \to 0, \end{aligned}$$

as $n \to \infty$. This gives the required upper bound for $\mu_U(\{y \mid \tau_U(y) > \frac{t}{\mu(U)}\})$. Now for the lower bound

$$\begin{aligned} \text{r.h.s.} &\ge \hat\mu_{\pi^{-1}(U)}(\{\hat y \in \hat U_n \mid \hat\tau_{\hat U_n}(\hat y) > \frac{t}{\mu(U)}\}) \\ &\quad -\hat\mu_{\pi^{-1}(U)}(\{\hat y \in \hat U_n \mid \tau_{\hat U_n}(\hat y) > \tau_{\pi^{-1}(U)}(\hat y)\}) \\ &= II - III. \end{aligned}$$

The above arguments show that this also tends to e^{-t} as $\mu(U) \to 0$ and $n \to \infty$. This finishes the proof. □

Remark 1. *It has been shown in* [16] *that the e^{-t} statistics for local return times imply the analogous statistics for* entrance times. *Therefore for our unimodal maps we get the additional interesting result:*

$$|\mu(x \in X \mid \tau_{U_r(z)}(x) > \frac{t}{\mu(U_r(z))}) - e^{-t}| \to 0$$

when $r \to 0$ and for μ-almost all $z \in X$.

Remark 2. *A lot of work has recently been devoted to the estimation of the error of the limiting distribution e^{-t}, see for example* [16], [14], [31], [11], [1], [30]. *This error is related to the rate of mixing of the systems and sometimes to the degree of hyperbolicity* [16]. *In particular for some Gibbsian sources, the error can be obtained by looking at the asymptotic behavior of the smallest return of the set U_z to itself,* [31]. *An interesting improvement of our techniques would be to push back the error term from the induced system (where it can be very often computed) to the original one: this will probably require a more constructive way of inducing.*

Remark 3. *As already pointed out in* [8], *the techniques of inducing can be easily adapted to the statistics of successive return times. This means with the obvious change of notations to study the asymptotic distribution of the quantity:*

$$\mu_{U_r(z)}(x \in U_r(z) \mid \tau_{U_r}^{k-1}\mu(U_r) \le t < \tau_{U_r}^{k}\mu(U_r))$$

as $r \to 0$, where τ_A^k is k-th return time to A. Under additional mixing conditions (α-mixing) one can prove that the limiting distribution is the Poisson law: $\frac{t^k}{k!}e^{-t}$ for μ-a.e. z.

3. Fluctuations of the entropy

One of the most interesting consequences of the exponential statistics for return times is the behavior of the fluctuations in the convergence to the measure theoretic entropy as given by Ornstein-Weiss formula, see [10] and [20] for the first works in this direction. Let $C_n(x)$ be the unique element of $\mathcal{C}_n = \bigvee_{i=1}^{n} T^{-(i-1)}\mathcal{C}$, which contains the point $x \in X$, where $\mathcal{C}$ is a finite generating partition of our ergodic system (X, f, μ). Assume that the sum $H_\mu(\mathcal{C}) = -\sum_{C\in\mathcal{C}} \mu(C)\log\mu(C)$ is finite. Ornstein-Weiss Theorem [29] asserts that:

$$\lim_{n\to\infty} \frac{1}{n} \log \tau_{C_n(x)}(x) = h_\mu \text{ for } \mu\text{-a.e. } x \in X,$$

where h_μ is the measure theoretical entropy of μ. We will define in a moment a convenient partition for our class of unimodal maps. Assume that the variance $\sigma_\mu^2(\varphi)$ as defined in (3) is positive. We are interested in showing the following convergence in distribution when $n \to \infty$:

$$\mu(x \in X \mid \frac{\log \tau_{C_n(x)}(x) - nh_\mu}{\sigma_\mu\sqrt{n}} > u) \Rightarrow \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{x^2}{2}} dx. \tag{8}$$

Theorem 5 uses a sufficient condition due to Saussol [31] to prove the convergence (4). Saussol's result is however much more general since it states the equivalence of the fluctuations to the entropy according to the Ornstein-Weiss and Shannon-McMillan formulas provided the e^{-t} law holds for local returns time uniformly in t.

Theorem 5. [31] *Define the error to the asymptotic distribution of the return times into cylinders as:*

$$E_\mu(C_n(x)) = \sup_{t\geq 0} |\mu_{C_n(x)}(x \in C_n(x) \mid \tau_{C_n(x)}\mu(C_n(x)) > t) - e^{-t}|.$$

Suppose that:
(i) $E_\mu(C_n(x)) \to 0$ for μ-a.e. x as $\mu(C_n(x)) \to 0$;
(ii) the fluctuations in the Shannon-McMillan Theorem are log-normal, i.e.

$$\mu\{x \in X \mid \frac{-\log\mu(C_n(x)) - nh_\mu}{\sigma_\mu\sqrt{n}} > u\} \Rightarrow \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{x^2}{2}} dx,$$

where $0 < \sigma_\mu < \infty$. Then the limit (4) follows.

3.1. **Proof of Theorem 3.** We verify the assumptions of the preceding theorem. Assumption (i) is just the content of the exponential statistics established in Section 2. It remains to prove assumption (ii). This will be achieved in two steps, whose details will be given below, namely:

- The invariant measure μ enjoys a *weak-Gibbs* property. This means that for μ-a.e. x we have:

  $$\mu(C_n(x)) \in [c_1(n)|Df^n(x)|^{-1}, c_2(n)|Df^n(x)|^{-1}],$$

  where $c_1(n)$ and $c_2(n)$ decrease (resp. increase) subexponentially in n. To be precise, we need

  $$e^{-n^\alpha} \leq c_1(n) \leq c_2(n) \leq e^{n^\alpha},$$

for some $0 < \alpha < 1/2$. (We refer to [10] and [15] for the details on Gibbs sources and weak-Gibbs measures.) This allows us to replace the measure of a cylinder of length n with the sum $S_n := \sum_{l=0}^{n-1} \log |Df(f^l)|$. Then the Cesaro mean converges to the positive Lyapunov exponent of the measure μ which is also equal to the measure theoretic entropy h_μ [22]: $h_\mu = \int \log |f'| d\mu$. More precisely, we have for the upper bound:

$$\begin{aligned} & \mu(x \in X \mid \frac{-\log \mu(C_n(x)) - nh_\mu}{\sigma_\mu \sqrt{n}} > u) \\ \leq \; & \mu(x \in X \mid \frac{\log |Df^n(x)| - nh_\mu}{\sigma_\mu \sqrt{n}} > u + \frac{\log c_1(n)}{\sigma_\mu \sqrt{n}}) \\ \leq \; & \mu(x \in X \mid \frac{\sum_{i=0}^{n-1} \log |Df(f^i(x))| - nh_\mu}{\sigma_\mu \sqrt{n}} > u - \frac{n^\alpha}{\sigma_\mu \sqrt{n}}) \\ \leq \; & \mu(x \mid \frac{\sum_{i=0}^{n-1} \log |Df(f^i(x))| - nh_\mu}{\sigma_\mu \sqrt{n}} > u - \delta) \\ = \; & \mu(x \mid \frac{\sum_{i=0}^{n-1} \varphi(f^i(x))}{\sigma_\mu \sqrt{n}} > u - \delta), \end{aligned}$$

where δ is any positive number bigger than $\frac{n^\alpha}{\sigma_\mu \sqrt{n}}$. Since $\alpha < \frac{1}{2}$, we can take $\delta \to 0$ as $n \to \infty$. Similar estimates hold for the lower bound.

- Let us suppose now that the Central Limit Theorem holds for the potential $\varphi = \log |f'| - \int \log |f'| d\mu$ with variance $\sigma_\mu > 0$. In this case we can continue the lower bound of the above item as:

$$\liminf_{n \to \infty} \mu(x \mid \mu(C_n(x)) < e^{-nh_\mu - \sigma_\mu u \sqrt{n}}) \geq \frac{1}{\sqrt{2\pi}} \int_{u-\delta}^{\infty} e^{-\frac{x^2}{2}} dx,$$

which gives the desired result for the lower bound when $\delta \to 0$.

The proofs of the weak-Gibbs property of μ and the Central Limit Theorem are given in the next subsections. □

The weak-Gibbs property and the Central Limit Theorem for the potential φ are interesting in their own right, as they allow application of thermodynamic formalism to our unimodal maps and they establish finer fluctuations behaviors. We think for example of he large deviations property for the random process: $\log \mu(C_n(x))$ and $\log \tau_{C_n(x)}(x)$. See [15] for related results in the case of $(\varphi - f)$-mixing systems).

3.2. **The weak-Gibbs property.** Let $\mathcal{C} = \{[c_2, c], [c, c_1]\}$ be the partition into the two monotonicity intervals of f. In the absence of attracting periodic points, this partition is generating. Let as before $\mathcal{C}_n = \mathcal{C} \vee f^{-1}(\mathcal{C}) \vee \cdots \vee f^{-(n-1)}(\mathcal{C})$, and let $C_n(x)$ be the n-cylinder containing x.

Lemma 5. *Suppose f satisfies condition (2), and let μ be its absolutely continuous invariant measure. Then for any $\varepsilon > 0$ and μ-a.e. x there exists n_0 such that*

$$\frac{b}{n^{3(\ell+1)}} \leq \frac{b}{n^{2(\ell+1)}} |f^n(C_n(x))| \leq \mu(C_n(x)) |Df^n(x)| \leq n^{2(1+\varepsilon)}$$

for all $n \geq n_0$ and $b = \frac{1}{2} \inf_{supp(\mu)} \frac{d\mu(y)}{dy} > 0$.

Proof. The upper bound: We have the invariant measure μ and Lebesgue measure m. The arguments use the Borel-Cantelli Lemma applied to m. Let W_n be the collection of n-cylinders C such that $\mu(C) > n^{1+\varepsilon} m(C)$. Write $A_n = \cup_{W_n} C$. Since μ is a probability measure, $1 \geq \sum_{C \in W_n} \mu(C) \geq n^{1+\varepsilon} \sum_{C \in W_n} m(C)$, hence $m(A_n) \leq n^{-(1+\varepsilon)}$. The Borel-Cantelli Lemma implies that m-a.e. x belongs to A_n only for finitely many n. Next, for each n-cylinder C, let $U(C) = \{x \in C \mid |Df^n(x)| > n^{1+\varepsilon}|C|^{-1}\}$. Write $B_n = \cup_{C \in \mathcal{C}_n} U(C)$. Then $1 \geq \int_{U(C)} |Df^n(x)|\, dx \geq \frac{n^{1+\varepsilon}}{|C|} m(U(C))$, so that $m(B_n) = \sum_{C \in \mathcal{C}_n} m(U(C)) \leq n^{-(1+\varepsilon)} \sum_{C \in \mathcal{C}_n} |C| = n^{-(1+\varepsilon)}$. Again the Borel-Cantelli Lemma gives that m-a.e. x belongs to B_n for only finitely many n. Therefore, for m-a.e. x, and because $\mu \ll m$ also for μ-a.e. x, there exists $n_0 = n_0(x)$ such that $x \notin A_n \cup B_n$ for all $n \geq n_0$. It follows that

$$\mu(C_n(x)) \cdot |Df^n(x)| \leq n^{1+\varepsilon} \cdot |C_n(x)| \cdot \frac{n^{1+\varepsilon}}{|C_n(x)|} \leq n^{2(1+\varepsilon)}.$$

The lower bound(s): Here we will apply the Borel-Cantelli Lemma to the invariant measure μ. Let $\hat{V}_n$ be the set of $\hat{x} \in \hat{I}$ such that $d(\hat{x}, \partial D_i) \leq n^{-(\ell+1)}|D_i|$ for some $i \in \mathbb{N}$. Recall that the densities h_k on the levels D_k satisfy $h_k(y) \leq \delta_k |y - c_k|^{\frac{1}{\ell}-1} + a_{k-\beta(k)} \delta_{\beta(k)} |y - c_\beta(k)|^{\frac{1}{\ell}-1}$. Therefore

$$\begin{aligned}
\hat{\mu}(\hat{V}_n) &\leq \sum_k \int_{c_k}^{c_k + n^{-(\ell+1)}} + \int_{c_{\beta(k)} - n^{-(\ell+1)}}^{c_{\beta(k)}} \\
&\qquad \delta_k |y - c_k|^{\frac{1}{\ell}-1} + a_{k-\beta(k)} \delta_{\beta(k)} |y - c_\beta(k)|^{\frac{1}{\ell}-1} dy \\
&\leq \sum_k \frac{2}{\ell} (\delta_k + a_{k-\beta(k)} \delta_{\beta(k)})\, n^{-\frac{\ell+1}{\ell}} \\
&\leq Cn^{-\frac{\ell+1}{\ell}},
\end{aligned}$$

for some constant $C < \infty$. By the Borel-Cantelli Lemma, for μ-a.e. x, $f^n(\hat{x}) \in \hat{V}_n$ (for $\hat{x} = \pi^{-1}(x) \cap D_2$) for only finitely many n. Take such x. Since $f^n(C_n(x)) = D_k$ if and only if $\hat{g}(\hat{x}) \in D_k$, we conclude that for n sufficiently large, there is Koebe space of relative length $n^{-(\ell+1)}$ around $f^n(x)$. This gives

$$|Df^n(x)| \geq \left(\frac{n^{-(\ell+1)}}{1 + n^{-(\ell+1)}} \right)^2 \frac{|f^n(C_n(x))}{|C_n(x)|} \geq \frac{1}{2n^{2(l+1)}} \frac{|f^n(C_n(x))|}{|C_n(x)|}.$$

It follows

$$\mu(C_n(x)) \cdot |Df^n(x)| \geq \frac{b}{n^{2(\ell+1)}} |f^n(C_n(x))|.$$

Finally we give a lower bound for $|C_n(x)|$. Take $\hat{U}_n = \sqcup\{D_k \mid |D_k| < n^{-(\ell+1)}\}$. Then

$$\hat{\mu}(\hat{U}_n) \leq \sum_{D_k \subset \hat{U}_n} \int_{D_k} h_k(y) dy \leq \frac{n^{-\frac{\ell+1}{\ell}}}{\ell} \sum_k (\delta_k + a_{k-\beta(k)} \delta_{\beta(k)}) \leq Cn^{-\frac{\ell+1}{\ell}}.$$

Using Borel-Cantelli again, we find that for μ-a.e. x, $f^n(C_n(x)) \leq n^{-(\ell+1)}$ with at most finitely many exceptions. This concludes the proof. □

3.3. **The Central Limit Theorem.** We begin with a characterization of the potential $\varphi = \log|f'| - \int \log|f'| d\mu$, which will be useful in order to apply a theorem by Gordin (see below).

Lemma 6. *The potential* $\varphi = \log|f'| - \int \log|f'| d\mu$ *belongs to* $L^2(\mu)$.

Proof of Lemma 6. First observe that there exists K such that $\mu((c-\varepsilon, c+\varepsilon)) < K\varepsilon$ for all ε. Indeed, by (5), and analogous to the computations in Lemma 4,

$$\begin{aligned}
\mu((c-\varepsilon, c+\varepsilon)) &\leq C \sum_k \int_{c-\varepsilon}^{c+\varepsilon} |y - c_k|^{\frac{1}{\ell}-1} |Df^{k-1}(c_1)|^{-\frac{1}{\ell}} dy \\
&\leq 2C\,\varepsilon \sum_k |Df^{k-1}(f(c))|^{-1/\ell} \int_0^{\epsilon} |y|^{1-1/\ell} dy \\
&\leq K\varepsilon^{1/\ell},
\end{aligned}$$

similar to (7). As $\log|f'| = \mathcal{O}(\ell) \log|x - c|$,

$$\begin{aligned}
\int (\log|f'|)^2 d\mu &\leq \sum_n K 2^{(1-n)/\ell} |\mathcal{O}(\ell) \log 2^{-n}|^2 \\
&\leq 2K \log 2\mathcal{O}(\ell^2) \sum_n 2^{-n/\ell} n^2 < \infty,
\end{aligned}$$

proving the lemma. □

Let us now prove Theorem 4, which states that the Central Limit Theorem holds for φ. The condition that the variance is positive is non-trivial, but can, as usual, be reduced to the condition that φ is not cohomologous to 0. Notably, $\sigma^2_\mu(\varphi) = 0$ for the full quadratic map $f : x \mapsto 4x(1-x)$. The reason is that f is differentiably conjugate to the full tent-map $T : x \mapsto 1 - 2|x|$. The logarithm of the derivative of the conjugacy $\psi := \log h'$ satisfies $\varphi = \psi - \psi \circ f = 0$, which in turn implies that $\sigma^2_\mu(\varphi) = 0$. In this case, the lengths of cylinder sets are too regular, so the fluctuation of return time statistics is less than expected. The precise form of the fluctuation in this case follows from the results in [10]. In general, if $\varphi = \psi - \psi \circ f$ has a solution, Gordin's [12] results imply that $\psi = \sum_j P^j(\varphi) \in L^1(\mu)$, where P is the Perron Frobenius operator. If in addition $\exp(\psi) \in L^1(\mu)$, then the integral of $\exp(\psi)$ conjugates f to a tent map. We conjecture that $\sigma^2_\mu(\varphi) = 0$ is very unlikely in the class of maps under consideration.

Proof of Theorem 3. In [7], it was shown that Hölder observables satisfy the Central Limit Theorem. In our case, φ is not Hölder, and not even bounded. However $\varphi \in L^2(\mu)$, see Lemma 6, and the singularity of φ is localized and of logarithmic order. In the proofs of [35] and [7] combined, it turns out that when φ is lifted to an appropriate tower, it satisfies the required Hölder-like property. Let us discuss the construction more precisely.

In [7] a jump transformation $F : \cup_i \omega_i \subset \Omega_0 \to \Omega_0$ is constructed, where Ω_0 is an interval, and $\{\omega_i\}_i$ in interval partition of Ω_0 such that

- The Lebesgue measure $m(\Omega_0 \setminus \cup_i \omega_i) = 0$,

- For each i, $F|\omega_i = f^{R_i}|\omega_i$ for some $R_i \geq 1$, and $F : \omega_i \to \Omega_0$ is a diffeomorphism with uniformly (in i) bounded distortion.
- The *induce times* R_i are summable:

$$\sum_i R_i \, m(\omega_i) < \infty. \tag{9}$$

Assuming that $|Df^n(c_1)| \geq Cn^\tau$ for some $\tau > 4\ell - 3$, we have that the tail $\sum_{R_i > n} m(\omega_i) \leq \mathcal{O}(n^{-\alpha})$ for any $\alpha \in (3, \frac{\tau-1}{\ell-1} - 1)$. Under Hölder conditions, the Central Limit Theorem is claimed.

In [7], Ω_0 is a neighborhood of the critical point, but the same construction would be valid if Ω_0 is a neighborhood bounded away from c. In addition, the construction of F, (see the definition of the binding period $p(x)$ in [7]) implies that there is a constant $\kappa > 0$ such that

$$\inf\{|c - y| \mid y \in f^j(\omega_i)\} > \kappa |f^j(\omega_i)| \tag{10}$$

for all i and $0 \leq j < R_i$.

The paper [7] continues to invoke [35] for the following tower construction:

$$\Delta = \sqcup_i \sqcup_{j=0}^{R_j - 1} \omega_{i,j},$$

where $\omega_{i,j} = \omega_i$ for each j. Equip Δ with Lebesgue measure $\tilde{m}$ as reference measure: $\tilde{m}|\omega_{i,j} = m|\omega_i$. By (9), $\tilde{m}$ is a finite measure. Define a map $\tilde{f} : \Delta \to \Delta$ by

$$\tilde{f}(x \in \omega_{i,j}) = \begin{cases} x \in \omega_{i,j+1} & \text{if } j+1 < R_i, \\ f^{R_i}(x) \in \Omega_0 & \text{if } j+1 = R_i, \end{cases}$$

where $\Omega_0 = \cup_i \omega_{i,0}$ mod m. The system $(\Delta, \tilde{f})$ is a countable Markov system, and using (9), a standard argument produces an $\tilde{f}$-invariant probability measure $\tilde{\mu}$, such that, for each $\omega_{i,j}$, $\frac{d\tilde{\mu}}{d\tilde{m}}$ is bounded and bounded away from 0. Define the projection $\pi : \Delta \to I$ by $\pi(x \in \omega_{i,j}) = f^j(x)$. Lift φ to the tower, *i.e.*, take $\tilde{\varphi}|\omega_{i,j} = \varphi \circ f^j$. The f-invariant measure μ on I satisfies $\mu = \tilde{\mu} \circ \pi^{-1}$.

We need a theorem of Gordin [12], cited in [34], which translates in our notation to

Theorem 6 (Gordin). *Suppose $\tilde{\varphi} \in L^2(\tilde{\mu})$, and*

$$\sum_{n \geq 0} \sqrt{\int |\mathbb{E}(\tilde{\varphi} | \tilde{f}^{-n}\mathcal{B})|^2 \, d\tilde{\mu}} < \infty,$$

where $\mathcal{B}$ is the algebra of Lebesgue measurable sets on Δ, and $\mathbb{E}(\tilde{\varphi}|\tilde{f}^{-n}\mathcal{B})$ is the conditional expectation with respect to $\tilde{\mu}$. Then $\tilde{\varphi}$ satisfies the Central Limit Theorem.

Let $\tilde{h} = \frac{d\tilde{\mu}}{d\tilde{m}}$ be the density of $\tilde{\mu}$. If $U(\tilde{\varphi}) := \tilde{\varphi} \circ \tilde{f}$ and U^* its dual operator (w.r.t. $\tilde{\mu}$), then we can compute (cf. [34]) $U^{*n}(\tilde{\varphi}) = P^n(\tilde{\varphi}\tilde{h})/\tilde{h}$, where P

denotes the Perron-Frobenius operator. Furthermore,

$$\mathbb{E}(\tilde\varphi|\tilde f^{-n}\mathcal{B})(x) = U^n U^{n*}(\tilde\varphi)(x) = \frac{1}{\tilde h\circ \tilde f^n(x)} \sum_{\tilde f^n(y)=\tilde f^n(x)} \left|\frac{(\tilde\varphi\tilde h)(y)}{D\tilde f^n(y)}\right|.$$

Therefore

$$\begin{aligned}\int_\Delta |\mathbb{E}(\tilde\varphi|\tilde f^{-n}\mathcal{B})|^2\, d\tilde\mu &= \int_\Delta |U^nU^{*n}(\tilde\varphi)|^2\, d\tilde\mu = \int_\Delta |U^{*n}(\tilde\varphi)|^2\, d\tilde\mu \\ &= \int_\Delta |U^nU^{*n}(\tilde\varphi)\cdot\tilde\varphi|\, d\tilde\mu. \qquad (11)\end{aligned}$$

Let $\Delta_n := \pi^{-1}(B(c;L^{-n}))$ be the lift of the L^{-n}-ball around c, where $L = \sup|f'|$. If $f^j(\omega_i)\cap B(c;L^{-n}) \neq \emptyset$, then by (10), $|f^j(\omega_i)| \leq 1/(\kappa L^n)$, and $|f^{j+n}(\omega_i)| \leq L^{n-1}|f^{j+1}(\omega_i)| \leq (1/\kappa L)|f'(\xi)| = \mathcal{O}(L^{-n(\ell-1)}) \ll |\Omega_0|$ for some $\xi\in f^j(\omega_i)$. Therefore $R_i > j+n$. It follows that if $y\in\omega_{i,j}$, then $\tilde f^{-n}\circ\tilde f^n(y) = \{y\}$, and also $\frac{\tilde h(y)}{|D\tilde f^n(y)|\tilde h\circ \tilde f^n(y)} = 1$. Hence for such y, the integrand $|U^nU^{*n}(\tilde\varphi)\cdot\tilde\varphi|(y) = |\tilde\varphi(y)|^2$.

Next let $\Delta'_n := \pi^{-1}(B(c;n^{-5})\setminus B(c;L^{-n}))$. Then $|\tilde\varphi| \leq \mathcal{O}(n(\ell-1)\log L)$ on Δ'_n, whereas $|\tilde\varphi| \leq \mathcal{O}(5(\ell-1))\log n$ on $\pi^{-1}(I\setminus B(c;n^{-5}))$. If $y\in\Delta'_n$, then $\tilde f^{-n}\circ\tilde f^n(y)\cap\Delta_n = \emptyset$. Therefore

$$\begin{aligned}\int_{\Delta'_n\setminus\Delta_n} |U^nU^{*n}(\tilde\varphi)\cdot\tilde\varphi(x)|d\tilde\mu &= \int_{\Delta'_n\setminus\Delta_n} \frac{\tilde\varphi(x)}{\tilde h\circ\tilde f^n(x)} \sum_{\tilde f^n(y)=\tilde f^n(x)} \left|\frac{\tilde\varphi(y)}{D\tilde f(y)}\right| d\tilde\mu \\ &\leq \int_{\Delta'_n\setminus\Delta_n} \mathcal{O}(n^2) d\tilde\mu.\end{aligned}$$

Combining these estimates, we can continue (11) as

$$\begin{aligned}&\leq \int_{\Delta_n} |\tilde\varphi(y)|^2\, d\tilde\mu + \int_{\Delta'_n\setminus\Delta_n} \mathcal{O}(n^2)\, d\tilde\mu + \mathcal{O}(\log n)\int_\Delta |U^nU^{*n}(\tilde\varphi)|\, d\tilde\mu \\ &\leq \mathcal{O}(L^{-n}n^2) + \mathcal{O}(n^{-3}) + \mathcal{O}(\log n)\int_\Delta |U^{*n}(\tilde\varphi)|\, d\tilde\mu \\ &= \mathcal{O}(L^{-n}n^2) + \mathcal{O}(n^{-3}) + \mathcal{O}(\log n)\int_\Delta |P^n(\tilde\varphi\tilde h)|\, d\tilde m.\end{aligned}$$

The next argument is to prove that $\tilde\varphi\tilde h$ has sufficient Hölder properties. We chose Ω_0 bounded away from c, so φ is C^2 on Ω_0. Given $x,y\in\omega_{i,j}$, the *separation time* $s(x,y)$, as defined in [35, Section 1.1], counts the minimal number of returns to Ω_0 before x and y belong to different partition elements ω_k. Because F is expanding and $f^{R_i-j}: f^j(\omega_i)\to\Omega_0$ has bounded distortion, it is not hard to check that $|x-y|/|\omega_{i,j}|$ is exponentially small in $s(x,y)$, see also [7, Lemma 4.5].

Write $\tilde\varphi\tilde h = C\cdot(\frac{d\lambda}{d\tilde m} - \frac{d\lambda'}{d\tilde m})$ as the scaled difference of two probability densities, for example $\frac{d\lambda}{d\tilde m} = \frac{1}{C}(1+\max\{\tilde\varphi\tilde h, 0\})$ and $\frac{d\lambda'}{d\tilde m} = \frac{1}{C}(1-\min\{\tilde\varphi\tilde h,0\})$. Here $\frac{d\lambda'}{d\tilde m}$ has the logarithmic singularity, and $C>0$ is a normalizing constant. On each interval $\omega_{i,j}$, $\tilde h(x)$ is Hölder continuous, bounded and bounded away from 0. Therefore, there exists $\beta \in (0,1)$ such that for all $x, y \in f^j(\omega_i)$,

$$\begin{aligned} \left| \frac{\frac{d\lambda'}{d\tilde{m}}(x)}{\frac{d\lambda'}{d\tilde{m}}(y)} - 1 \right| &\leq \mathcal{O}(1) \left| \frac{\log|x-c|}{\log|y-c|} - 1 \right| \\ &\leq \mathcal{O}(1) \frac{|x-y|}{|y-c|} \\ &\leq \mathcal{O}(\frac{1}{\kappa}) \frac{|x-y|}{|f^j(\omega_i)|} \leq \text{Const } \beta^{s(x,y)}. \end{aligned}$$

The estimates for $\frac{d\lambda}{d\tilde{m}}$ are similar, and give the same upper bound. Hence both $\frac{d\lambda}{d\tilde{m}}$ and $\frac{d\lambda'}{d\tilde{m}}$ belong to $C_\beta^+(\Delta)$ as in [35]. Following the [35] argument, we obtain for the correlations

$$\int_\Delta |P^n(\tilde{\varphi}\tilde{h})| \, d\tilde{m} = |\tilde{f}_*^n \lambda - \tilde{f}_*^n \lambda'|(\Delta) \leq \mathcal{O}(n^{-(\alpha-1)})$$

for the $\alpha > 3$ from above. Therefore $\sqrt{\int_\Delta |\mathbb{E}(\tilde{\varphi}|\tilde{f}^{-n}\mathcal{B})|^2 \, d\tilde{\mu}}$ is indeed summable, and by Gordin's Theorem, $\tilde{\varphi}$ satisfies the Central Limit Theorem. The same is true for the original observable φ. □

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Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, the Netherlands

E-mail address: bruin@math.rug.nl

URL: http://www.math.rug.nl/~bruin

PHYMAT, Université de Toulon, Centre de Physique Théorique and FRUMAM, Fédération de Recherche des Unités de Mathématiques de Marseille, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France

E-mail address: vaienti@cpt.univ-mrs.fr

URL: http://www.cpt.univ-mrs.fr/