Generalized Dynamical Variables and Measures for the Julia Sets.

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Summary. — We give rigorous estimates of the dimensions, entropies, characteristic exponents and scaling function of hyperbolic Julia sets, for any Gibbs measure, by the direct computations of the topological pressure.

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Introduction.

The thermodynamic formalism introduced by Ruelle *et al.* (1) has proved to be an essential scheme in order to understand and compute the dynamical and fractal properties of strange sets and in particular of the so-called mixing repellers (2). To this class the linear Cantor sets belong, for which all the relevant dynamical variables can be analytically computed, and the disconnected Julia

(1) D. RUELLE: *Thermodynamic Formalism* (Addison-Wesley, Reading, Mass., 1978); R. BOWEN: *Lecture Notes in Mathematics*, Vol. 470 (1975).
(2) D. RUELLE: *Ergod. Th. Dyn. Syst.*, 2, 99 (1982).

sets (3), for which an approximation scheme based on sequences of linear Cantors was developed and rigorous convergence proofs were given (4-6).

For the linear Cantor sets we first relate the generalized dimensions, entropies and Lyapunov indices to the free energy and show that, for the Gibbs measures, it can be expressed in terms of the pressure which is a purely topological quantity.

These relations are extended to the nonlinear Cantor sets, like the disconnected Julia sets, with a limit procedure that can be rigorously justified.

From a numerical point of view it is equivalent, and may be simpler, for the Julia sets not belonging to the real line, to compute the pressure using an expression which involves only the preimages of a given initial point. Moreover, the same method can be applied to connected Julia sets, provided that the hyperbolicity condition is satisfied (that is no critical point belongs to the Julia set). Even in this case we can relate to the pressure the generalized dimensions, entropies and Lyapunov indices for the one-parameter family of the Gibbs measures, among which the physically most interesting ones are the balanced measure with equal weights, the ordinary Gibbs measure and the Sinai-Bowen-Ruelle (SBR) measure.

1. – Gibbs measures and Cantor sets.

Consider an expanding map T, let J be its invariant set and μ an ergodic invariant measure on J, such that, for any measurable subset $A \subset J$, we have

$$\mu(T^{-1}A) = \mu(A).$$

An important subclass, to which the hyperbolic Julia sets belong, is given by the conformal mixing repellers. These are sets J, invariant with respect to maps T, which are uniformly expanding:

$$\|DT^n(x)\| > c\lambda^n, \quad c>0,\ \lambda>1,\ \forall x \in J,\ \forall n \in Z, \tag{1.1}$$

where $DT(x)$ denotes the tangent map, which is a scalar times an isometry, and enjoy the property

$$\text{closure}\,\{T^{-n}(x)\}_{n=0}^{\infty} = J, \qquad x \in J. \tag{1.2}$$

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(3) H. BROLIN: *Ark. für Math.*, **6**, 103 (1965).
(4) G. TURCHETTI and S. VAIENTI: to appear in *Phys. Lett. A* (1987).
(5) G. TURCHETTI and S. VAIENTI: *Generalized dimensions of strange sets and Cantorian approximation*, to appear on *Egypt. J. Phys.* (1988).
(6) S. VAIENTI: to appear in *J. Phys. A* (1988).

The hyperbolic Julia sets for polynomial maps are also defined as the closure of the repulsive fixed points, or the boundary of the basin of attraction of the point at infinity.

We introduce a one-parameter family of invariant ergodic measures $\mu_\sigma(x)$ on J, defined as follows: the pressure for the function $-\sigma \log\|DT(x)\|$ is given by

$$P(\sigma) = \limsup_{\mu \in \mathscr{M}(T,J)} \{h(\mu) - \sigma \int \log\|DT(x)\| \, d\mu(x)\}, \tag{1.3}$$

where $\mathscr{M}(T, J)$ is the space of all the invariant measures on J and h is the entropy. The measure for which the maximum of the functional in the r.h.s. of (1.3) is achieved will be denoted by $\mu_\sigma(x)$ and will be called Gibbs measure [1]:

$$P(\sigma) = h(\mu_\sigma) - \sigma \int \log\|DT(x)\| \, d\mu_\sigma(x). \tag{1.4}$$

For the rational Julia sets the integral term in (1.4) is the Lyapunov exponent Λ with respect to μ_σ and we can also write

$$P(\sigma) = h(\mu_\sigma) - \sigma\Lambda(\mu_\sigma). \tag{1.5}$$

A particular measure corresponding to $\sigma = 0$ is the balanced measure (we denote it with $\mu_B \equiv \mu_0$), for which

$$P(0) = h(\mu_B) = h_{top}, \tag{1.6}$$

where h_{top} is the topological entropy, that is the maximum of $h(\mu)$ with respect to all the measures in $\mathscr{M}(T, J)$.

For a polynomial map of degree s the balanced measure enjoys the following property on any set A on which T is injective (with unique inverse) [3,7]:

$$\mu_B(TA) = s\mu_B(A). \tag{1.7}$$

When σ is equal to the Hausdorff dimension D_H, where the pressure vanishes, according to the well-known Bowen-Ruelle formula [2]

$$P(D_H) = h(\mu_{D_H}) - D_H\Lambda(\mu_{D_H}) = 0, \tag{1.8}$$

we call the corresponding measure μ_{D_H} the uniform Gibbs measure, since it is equivalent to the D_H-Hausdorff measure of J [8].

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(7) M. JU. LJUBICH: *Ergod. Th. Dyn. Syst.*, 3, 351 (1983).

(8) K. J. FALCONER: *The Geometry of Fractal Sets* (Cambridge University Press, Cambridge, 1985).

Finally, when $\sigma = d$, where $d = 1, 2$ is the dimension of the space where the set J is embedded, the measure μ_d becomes the SBR measure [9] and one has $P(d) = -\alpha_{\mathrm{TH}}$, the theoretical escape rate [6,9].

For the linear Cantor sets the Gibbs measures can be explicitly computed. Let us consider a linear Cantor C on the real line, $C \subset [0, 1]$, with diam $C = 1$ and a map $L(x)$ which is piecewise linear on $L^{-1}([0, 1])$ such that

$$L^{-1}([0, 1]) = \sum_{k=1}^{s} I_k, \qquad I_k \cap I_j = \emptyset \text{ for } k \neq j. \tag{1.9}$$

We denote with $L_k^{-1}(x)$ the inverse of $L(x)$ on I_k, where it is injective. The diameter of I_k, that is its length, is equal to the slope of the linear function $L_k^{-1}(x)$, and is called a scale λ_k of the Cantor C:

$$\operatorname{diam}(I_k) = \operatorname{diam}(L_k^{-1}([0, 1])) = \lambda_k. \tag{1.10}$$

Since $\lambda_k < 1$, one has $L^{-n}([0, 1]) \subset L^{-m}([0, 1])$ for $m < n$, and the Cantor is defined by

$$C = \lim_{n \to \infty} L^{-n}([0, 1]). \tag{1.11}$$

The set $L^{-n}([0, 1])$ consists of s^n disjoint preimages of $[0, 1]$ denoted with

$$I_{k_1 \dots k_n} = L_{k_n}^{-1} \dots L_{k_1}^{-1}([0, 1]). \tag{1.12}$$

From these intervals we immediately obtain an order n partition $\mathcal{A}^{(n)}$ of C according to

$$\mathcal{A}^{(n)} = \bigcup_{k_1 \dots k_n} A_{k_1 \dots k_n}, \qquad A_{k_1 \dots k_n} = I_{k_1 \dots k_n} \cap C. \tag{1.13}$$

It is easy to check that

$$\operatorname{diam}(I_{k_1 \dots k_n}) = \operatorname{diam}(A_{k_1 \dots k_n}) = \lambda_{k_1} \dots \lambda_{k_n}. \tag{1.14}$$

We introduce a sequence of measures $\mu_{(n)}(x)$, uniform on the intervals $I_{k_1 \dots k_n}$, defined by

$$\mu_{(1)}(I_k) = p_k, \qquad \sum_{k=1}^{s} p_k = 1 \tag{1.15}$$

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(9) T. Bohr and D. Rand: *Physica (Utrecht) D*, **25**, 387 (1987).

and

$$\mu_{(n)}(I_{k_1\ldots k_n}) = p_{k_1} \cdot\ldots\cdot p_{k_n}. \tag{1.16}$$

For a detailed analysis of these measures we refer to [10].

This sequence of measures $\mu_{(n)}(x)$ can also be represented numerically since, for any finite n, their density $\rho_{(n)}(x)$ is given by

$$\rho_{(n)}(x) = \begin{cases} 0, & x \notin I_{k_1\ldots k_n}. \\ \dfrac{p_{k_1}\ldots p_{k_n}}{\lambda_{k_1}\ldots\lambda_{k_n}}, & x \in I_{k_1\ldots k_n}. \end{cases} \tag{1.17}$$

Then we observe that, letting $\mu(x)$ be an invariant measure on J, and requiring

$$\mu(A_{k_1\ldots k_n}) = \mu_{(n)}(I_{k_1\ldots k_n}), \tag{1.18}$$

the measure μ will be completely defined and will be an invariant ergodic measure of J.

If we choose

$$p_k = \frac{\lambda_k^\sigma}{\sum\limits_{k=1}^{s} \lambda_k^\sigma}, \tag{1.19}$$

then we obtain the Gibbs measures on C. The balanced measure is given for $\sigma = 0$, that is

$$p_k = \frac{1}{s}, \tag{1.20}$$

while the uniform Gibbs measure is given by [8]

$$p_k = \lambda_k^{D_H}, \qquad \sum_{k=1}^{s} \lambda_k^{D_H} = 1. \tag{1.21}$$

The same analysis applies to any linear Cantor set in $\boldsymbol{R}^n$ and obviously to Cantor sets in C. We consider a map $L(\boldsymbol{x})$ defined on an open set $\Omega \supset C$, $\operatorname{diam}\Omega = \operatorname{diam} C$, such that $L^{-1}(\Omega)$ is the union of s disjoint sets $I_1, \ldots, I_s$ and the inverse of $L(\boldsymbol{x})$ restricted to I_k is linear. Such inverses, denoted by $L_k^{-1}(\boldsymbol{x})$, are given by

$$L_k^{-1}(\boldsymbol{x}) = \boldsymbol{a}_k + \lambda_k R_k \boldsymbol{x}, \tag{1.22}$$

[10] M. F. Barnsley and S. Denko: *Proc. R. Soc. London, Sect. A*, **399**, 243 (1985).

where R_k are rotation matrices and $|\lambda_k|<1$. With such a condition one has $L^{-n}(\Omega)\subset L^{-m}(\Omega)$ for $m<n$ and C is defined by $C=\lim_{n\to\infty} L^{-n}(\Omega)$.

Given any disconnected J and a nonlinear map $T(x)$, we choose a disk $\Omega\supset J$ with the same diameter as J and such that it does not contain any critical point of T. Then we consider s linear maps $L_k^{(1)^{-1}}$ which transform the disk Ω into s disks Ω_k which contain $T_k^{-1}(\Omega)$ and such that $\operatorname{diam}\Omega_k=\operatorname{diam}T_k^{-1}(\Omega)$.

At order n there are s^n such maps $L_{k_1\ldots k_n}^{(n)^{-1}}$, for $k_1, \ldots, k_n=1,\ldots,s$, which transform Ω into s^n disks containing $T_{k_n}^{-1}\ldots T_{k_1}^{-1}(\Omega)$ and having the same diameter.

As a consequence one associates to (J,T) a sequence $(C_n, L^{(n)})$ of linear Cantors and the approximation theorems for the pressure and the relevant dynamical variables have been proved for the balanced measures ([4,6]). Indeed one can introduce on C_n a Gibbs measure $\mu_\sigma^{(n)}$ and prove that in the limit $n\to\infty$ they go into the Gibbs measure μ_σ on J (see sect. 3 and Appendix A); one can also prove, using only arguments based on fractal geometry, that the Hausdorff distance of C_n from J goes to zero and that the Hausdorff dimension of C_n, $D_{\mathrm{H}}^{(n)}$, becomes the Hausdorff dimension D_{H} of J as n goes to ∞ (see appendix B).

The geometric interpretation of this construction is simple for the 1-dimensional Cantor sets: in fact, in this case, the linear maps are simply defined by

$$L_{k_1\ldots k_n}^{(n)^{-1}}([0,\,1])=T_{k_1\ldots k_n}^{-n}([0,\,1])\,. \tag{1.23}$$

2. – Free energy, pressure and generalized variables.

Consider an hyperbolic totally disconnected set J (Cantor set) and an open set $\Omega\supset J$ which does not intersect any critical point of the map $T(x)$ with respect to which J is invariant. Let $\mathcal{A}^{(1)}$ be a partition of J:

$$\mathcal{A}^{(1)}=(T^{-1}\Omega)\cap J=\bigcup_{k=1}^{s}A_k\,,\qquad A_k\cap A_j=\emptyset \text{ if } k\neq j\,. \tag{2.1}$$

Denoting with T_k^{-1} the inverse of $T(x)$ on the set B_k, where $T^{-1}(\Omega)=\bigcup_k B_k$ and $A_k=B_k\cap J$, the refinement $\mathcal{A}^{(n)}$ of the partition at order n is defined by

$$\mathcal{A}^{(n)}=(T^{-n}\Omega)\cap J=\bigcup_{k_1\ldots k_n=1}^{s}A_{k_1\ldots k_n}\,, \tag{2.2}$$

where

$$A_{k_1\ldots k_n}=B_{k_1\ldots k_n}\cap J\,,\qquad B_{k_1\ldots k_n}=T_{k_n}^{-1}\ldots T_{k_1}^{-1}(\Omega) \tag{2.3}$$

are all disjoint sets.

The order n partition function is defined by [11]

(2.4) $$Z_n(\beta, \alpha; \mu) = \sum_{k_1 \dots k_n = 1}^{s} \frac{\mu(A_{k_1 \dots k_n})^\beta}{[\operatorname{diam}(A_{k_1 \dots k_n})]^\alpha}$$

and the free energy [11] is given by the thermodynamic limit

(2.5) $$F(\beta, \alpha; \mu) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta, \alpha; \mu).$$

The pressure is then given by [1]

(2.6) $$P(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{k_1 \dots k_n} [\operatorname{diam} A_{k_1 \dots k_n}]^\alpha$$

and is related to the free energy according to

(2.7) $$P(\alpha) = F(0, -\alpha; \mu).$$

The generalized dimensions D_q are given by [12,13]

(2.8) $$D_q(\mu) = \frac{\tau_q(\mu)}{q-1},$$

τ_q being the point where the partition function $Z_n(q, \tau; \mu)$ is of order one when $n \to \infty$. Indeed, for any real $\tau \neq \tau_q$, the limit of the partition function $Z_n(q, \tau; \mu)$ is 0 or ∞. It is easy to check that τ_q is also the unique solution of the implicit equation

(2.9) $$F(q, \tau; \mu) = 0.$$

The generalized Renyi entropies are defined by [14]

(2.10) $$h_q(\mu) = \frac{1}{1-q} \lim_{n \to \infty} \frac{1}{n} \log \sum_{k_1 \dots k_n = 1}^{s} [\mu(A_{k_1 \dots k_n})]^q$$

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(11) P. COLLET, J. L. LEBOWITZ and A. PORZIO: *J. Stat. Phys.*, 47, 609 (1987); E. VUL, K. KHANIN and Y. SINAI: *Russ. Math. Survey*, 39, 1 (1984).
(12) D. BESSIS, G. PALADIN, G. TURCHETTI and S. VAIENTI: to appear in *J. Stat. Phys.* (1988).
(13) T. C. HALSEY, M. H. JENSEN, L. P. KADANOFF, I. PROCACCIA and B. J. SHRAIMAN: *Phys. Rev. A*, 33, 1141 (1986); H. G. HENTSCHEL and I. PROCACCIA: *Physica (Utrecht) D*, 8, 435 (1983); P. GRASSBERGER: *Phys. Lett. A*, 107, 101 (1983).
(14) J. P. ECKMANN and D. RUELLE: *Rev. Mod. Phys.*, 57, 617 (1985); L. K. KADANOFF and C. TANG: *Proc. Nat. Acad. Sci. USA*, 81, 1276 (1984).

and consequently are related to the free energy by

$$h_q(\mu) = \frac{1}{1-q} F(q, 0; \mu). \tag{2.11}$$

Finally the generalized Lyapunov indices are given by [12,15]

$$\mathscr{L}_q(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \int_J \|DT^n(x)\|^q \, \mathrm{d}\mu(x). \tag{2.12}$$

It can be proved that (a sketch of the proof for the Gibbs measures will be given in the next section)

$$\mathscr{L}_q(\mu) = F(1, q; \mu). \tag{2.13}$$

For the Gibbs measures μ_σ defined in the previous section, it can be shown (see next section for the proof) that

$$F(\beta, \alpha; \mu_\sigma) = P(\beta\sigma - \alpha) - \beta P(\sigma). \tag{2.14}$$

As a consequence, for these measures, the generalized dimensions, entropies and Lyapunov indices can be expressed in terms of the pressure only, that is a function of purely topological nature.

Indeed, from (2.8), (2.9) and (2.14), we have that the generalized dimensions D_q are given by

$$P(q\sigma - (q-1)\, D_q(\mu_\sigma)) - qP(\sigma) = 0. \tag{2.15}$$

The Renyi entropies become

$$h_q(\mu_\sigma) = \frac{1}{1-q} [P(q\sigma) - qP(\sigma)] \tag{2.16}$$

and the generalized Lyapunov indices

$$\mathscr{L}_q(\mu_\sigma) = P(\sigma - q) - P(\sigma). \tag{2.17}$$

For the balanced measure μ_B with $\sigma = 0$, we recover the formulae quoted in [12], namely

$$\begin{cases} P(-(q-1)\, D_q(\mu_B)) = qP(0), \\ h_q(\mu_B) = P(0) \equiv h_{top}, \\ \mathscr{L}_q(\mu_B) = P(-q) - P(0), \end{cases} \tag{2.18}$$

[15] R. BENZI, G. PALADIN, G. PARISI and A. VULPIANI: *J. Phys. A*, **17**, 3521 (1984).

while for the uniform Gibbs measure $\mu_{D_{\rm H}}$ we have

$$(2.19)\qquad \begin{cases} D_q(\mu_{D_{\rm H}}) = D_{\rm H}\,, \\ h_q(\mu_{D_{\rm H}}) = \dfrac{P(qD_{\rm H})}{1-q}\,, \\ \mathscr{L}_q(\mu_{D_{\rm H}}) = P(D_{\rm H}-q)\,, \end{cases}$$

where we have used (1.8) and observed that

$$P(qD_{\rm H} - (q-1)\,D_q(\mu_{D_{\rm H}})) = qP(D_{\rm H}) = 0$$

which implies the first of (2.18). For the SBR measure we have only to choose $\sigma = d$, where $d = 1$ for Julia sets on the real line and $d = 2$ for Julia sets extending on the complex plane.

The first couple of relations (2.18) and (2.19) were already given in [9,11,16], while the third one, concerning the Lyapunov exponents, was given in [12]. The extension to any Gibbs measure was proposed in [17].

Here a unified derivation from the free energy is presented and an explicit and simple construction of the measures, using the linear Cantorian approximation, is given.

We recall, finally, that we can easily evaluate the scaling function of any measure μ_σ, once we know the pressure. Let $\overline{E}(\alpha, \mu_\sigma)$ be the subset of J given by all the points x such that (denoting with $B(x, l)$ the sphere of radius l centred at x)

$$(2.20)\qquad \limsup_{l\to 0} \frac{\log \mu_\sigma(B(x, l))}{\log l} = \alpha$$

and $\underline{E}(\alpha, \mu_\sigma)$ the set obtained by replacing lim sup with lim inf in (2.20). Then the scaling function $f(\alpha; \mu_\sigma)$ is the Hausdorff dimension of the sets $\underline{E}(\alpha; \mu_\sigma)$, $\overline{E}(\alpha, \mu_\sigma)$ and is related to $\tau_q(\mu_\sigma)$ in (2.8) by the equations [11,13]

$$(2.21)\qquad \begin{cases} \alpha_q(\mu_\sigma) = \dfrac{\mathrm{d}}{\mathrm{d}q}\,\tau_q(\mu_\sigma)\,, \\ q\alpha_q(\mu_\sigma) - f(\alpha_q(\mu_\sigma); \mu_\sigma) = \tau_q(\mu_\sigma)\,. \end{cases}$$

To conclude this section we recall that, using the well-known relation

$$(2.22)\qquad \alpha_{TH} = -P(d)\,,$$

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[16] M. J. Feigenbaum: *J. Stat. Phys.*, **46**, 919 (1987).
[17] S. Vaienti: to appear in *J. Phys. A* (1988).

one recovers, for the SBR measure, other relations already quoted in the literature [6,9].

We can also observe that, differentiating (2.15) with respect to q and evaluating the derivative at $q=1$, one has

$$D_1(\mu_\sigma)=\sigma-\frac{P(\sigma)}{P'(\sigma)}=\frac{h(\mu_\sigma)}{\Lambda(\mu_\sigma)}. \tag{2.23}$$

Indeed $P'(\sigma)$ is equal to the Lyapunov exponent $\Lambda(\mu_\sigma)=-\,\mathrm{d}\mathscr{L}_q/\mathrm{d}q\;(\mu_\sigma)|_{q=0}$ as can be seen from (2.17) and $\sigma\Lambda(\mu_\sigma)+P(\sigma)=h(\mu_\sigma)$ is the Kolmogorov entropy according to (1.5) or to (2.16) if the limit $q\to 1$ is taken.

In [18] it was shown that the r.h.s. of (2.23) is exactly the Hausdorff dimension of the measure μ_σ defined as in [14]:

$$D_{\mathrm{H}}(\mu_\sigma)=\lim_{\substack{A\subset J\\ \mu_\sigma(A)=1}}\{\text{Hausdorff dimension of } A\}. \tag{2.24}$$

For connected Julia sets one has $D_{\mathrm{H}}(\mu_0)=1$ for the balanced measure, since [18] $h(\mu_0)=\Lambda(\mu_0)$. In the literature $D_{\mathrm{H}}(\mu_0)$ is called «information dimension».

3. – The linear Cantor sets.

Let us consider a linear Cantor set $C\subset[0,1]$ with s scales $\lambda_1,\ldots,\lambda_s$ and let $L(x)$ be the piecewise linear map on $L^{-1}([0,1])$. Letting $p_1,\ldots,p_s$ be the weights defining a measure μ according to (1.15), (1.16) and (1.18), it is easy to compute the free energy defined by (2.5) and the result reads

$$F(\beta,\alpha;\mu)=\log\left(\sum_{k=1}^{s}p_k^\beta\lambda_k^{-\alpha}\right). \tag{3.1}$$

The pressure is given by

$$P(\alpha)=\log\left(\sum_{k=1}^{s}\lambda_k^\alpha\right). \tag{3.2}$$

The Renyi entropies read

$$h_q(\mu)=\frac{1}{1-q}\log\left(\sum_{k=1}^{s}p_k^q\right). \tag{3.3}$$

and the Kolmogorov entropy is the limit for $q\to 1$

$$h(\mu)\equiv h_1(\mu)=-\sum_{k=1}^{s}p_k\log p_k. \tag{3.4}$$

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[18] A. MANNING: *Ann. Math.*, **119**, 425 (1984).

The generalized Lyapunov exponents read

$$\mathscr{L}_q(\mu) = \log\left(\sum_{k=1}^{s} p_k \lambda_k^{-q}\right) \tag{3.5}$$

and consequently the ordinary Lyapunov exponent is given by

$$\Lambda(\mu) \equiv \lim_{q\to 0} \frac{\mathscr{L}_q(\mu)}{q} = -\sum_{k=1}^{s} p_k \log \lambda_k . \tag{3.6}$$

From (3.6) and (3.4) we obtain

$$h(\mu) - \sigma\Lambda(\mu) = \sum_{k=1}^{s} p_k \log \frac{\lambda_k^\sigma}{p_k}, \tag{3.7}$$

and it follows that, for

$$p_k = \frac{\lambda_k^\sigma}{\sum_{k=1}^{s} \lambda_k^\sigma}, \tag{3.8}$$

the r.h.s. of (3.7) becomes exactly $P(\sigma)$ so that (3.8) is the Gibbs measure μ_σ for the linear Cantor. Replacing (3.8) into (3.1) we immediately obtain that for the measure μ_σ the relation between the free energy and the pressure is given by (2.14):

$$F(\beta, \alpha; \mu_\sigma) = P(\beta\sigma - \alpha) - \beta P(\sigma).$$

4. – Computation of the pressure.

Disconnected sets. The basic formula we use to compute the pressure of a disconnected hyperbolic set of the complex plane (Julia set) is given by (2.6). Introducing the scales $\lambda_{k_1 \ldots k_n}^{(n)}$ of the associated linear Cantors C_n according to

$$\lambda_{k_1 \ldots k_n}^{(n)} = \frac{\operatorname{diam}(A_{k_1 \ldots k_n})}{\operatorname{diam}(\Omega \cap J)}, \tag{4.1}$$

the pressure can be written as [4,6]

$$P(\alpha) = \lim_{n\to\infty} \frac{1}{n} P_n(\alpha), \tag{4.2}$$

where

$$P_n(\alpha) = \log\left[\sum_{k_1 \ldots k_n} (\lambda_{k_1 \ldots k_n}^{(n)})^\alpha\right] \tag{4.3}$$

can be interpreted as the pressure for the linear Cantor C_n associated to the linear maps $L^{(n)}(x)$ defined at the end of sect. 2. The thermodynamic limit (4.2) is the central step of the linear Cantorian approximation which was developed for the conformed disconnected repellers. In the same way we can write the free energy for the Gibbs measures μ_σ as

$$F(\beta,\,\alpha;\,\mu_\sigma)=\lim_{n\to\infty}\frac{1}{n}\,F_n(\beta,\,\alpha;\,\mu_\sigma^{(n)})\,, \tag{4.4}$$

where F_n is the free energy of a linear Cantor set with scales $\lambda^{(n)}_{k_1\dots k_n}$, given by (4.1), and weights

$$p_{k_1\dots k_n}=\mu_\sigma^{(n)}(B_{k_1\dots k_n})=\frac{[\lambda^{(n)}_{k_1\dots k_n}]^\sigma}{\sum\limits_{j_1\dots j_n}[\lambda^{(n)}_{j_1\dots j_n}]^\sigma}\,, \tag{4.5}$$

namely, according to (3.1),

$$F_n(\beta,\,\alpha;\,\mu_\sigma^{(n)})=\log\left(\sum_{k_1\dots k_n}\frac{[\lambda^{(n)}_{k_1\dots k_n}]^{\beta\sigma-\alpha}}{\left(\sum\limits_{j_1\dots j_n}[\lambda^{(n)}_{j_1\dots j_n}]^\sigma\right)^\beta}\right)=P_n(\beta\sigma-\alpha)-\beta P_n(\sigma)\,. \tag{4.6}$$

It is, therefore, sufficient to use the convergence theorem on the pressure to prove that (2.14) holds.

Connected sets. If the Julia set is connected, we have to replace the partition $\mathscr{A}^{(1)}$ with any Markov partition of J (1) and the limit (4.2) is still true. However this limit is difficult to compute numerically since the Markov partitions are hard to construct. Nevertheless we have another useful method to compute the pressure, whose motivation is in the Walter's theory (19) of the Ruelle-Perron-Frobenius operator (17).

Let x be any nonexcluded point of the complex plane (for polynomial maps there are at most two such points, one of which is the point at infinity); then

$$P(\alpha)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{y\in T^{-n}(x)}\frac{1}{|DT^n(y)|^\alpha}\,. \tag{4.7}$$

The existence of limit (4.2) for Markov partitions, the techniques of theorem (4.6) in (6) and all the relations between the pressure and the dynamical variables written in sect. 2 apply also to the connected hyperbolic Julia sets.

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(19) P. WALTERS: *Trans. Am. Math. Soc.*, **236**, 121 (1978).

5. – Numerical results.

We have computed the pressure for several Julia sets (connected and disconnected) using (4.2) and (4.7), with a maximum iteration order $N = 14$. A good stability is observed, not only for the pressure, but also for all the computed dynamical variables, that is the generalized dimensions, entropies and the scaling functions.

For each case three different Gibbs measures μ_σ were considered, namely the balanced measure $\sigma = 0$, the uniform Gibbs measure $\sigma = D_{\rm H}$ and the SBR measure $\sigma = 1$ for the Julia set on the real line, $\sigma = 2$ for the Julia sets on the plane. Comparison with rigorous bounds given in ([20]) were satisfactory.

The numerical values were checked with the theoretical bounds ([20])

$$\log s - \alpha \log \nu_{\max} \leqslant P(\alpha) \leqslant \log s - \alpha \log \nu_{\min}\,, \tag{5.1}$$

where

$$\nu_{\min} = \min_{x \in J} |DT(x)|\,, \qquad \nu_{\max} = \max_{x \in J} |DT(x)| \tag{5.2}$$

and s is the degree of the mapping. When $\nu_{\min} < 1$, we look for the smallest iteration T^m of T such that $|DT^m(x)| = \rho_m > 1$ for $x \in J$ and replace (5.2) with $\nu_{\min} = \rho_m^{1/m}$.

We have considered three Julia sets of the quadratic map

$$T(z) = z^2 - p \tag{5.3}$$

with

i) $p = 3$: the Julia set is a totally disconnected subset of the real line,

ii) $p = 0.15$: the Julia set is homeomorphic to the unit circle,

iii) $p = -2$: the Julia set is totally disconnected and extends onto the complex plane.

For the case i) the pressure was already computed, with a very good accuracy, using (4.2) and (4.3), in ([4]) when the linear Cantorian approximation was first proposed. When $|p|$ is small as in case ii) one can compute the pressure using a perturbation expansion proposed by Ruelle ([2]):

$$P(\alpha) = \log 2 + \frac{|p|^2}{4} - \alpha \log 2 + O(|p|^3) \tag{5.4}$$

and the generalized dimensions read

$$D_q(\mu_\sigma) = 1 + \frac{|p|^2}{4 \log 2} + O(|p|^3)\,, \tag{5.5}$$

([20]) S. Vaienti: *Nuovo Cimento*, **99**, 77 (1987).

while the Renyi entropies are

$$h_q(\mu_\sigma) = \log 2 + \frac{|p|^2}{4}\,. \tag{5.6}$$

For the quadratic maps the bounds (5.1) are easily computed for $0<p<3/4$; setting $u=1/2+\sqrt{1/4+p}$ and $u'=\sqrt{u-p}$, for $0<\alpha<\log 2/\log 2u$ one has $\nu_{\max}=2u$ and $\nu_{\min}=2u'$.

In tables I-III we quote, for some values of α, the pressure $P_n(\alpha)$ computed by (4.7) compared with a linear extrapolation $n\to\infty$ obtained using only the last two terms of the sequence P_n, while in table IV we show the extrapolated values obtained by the Thiele algorithm applied to the whole sequence up to $n=16$.

We point out that the two methods agree very well but the first is much more regular than the second; so we always use for $P(\alpha)$ the linearly extrapolated results.

In fig. 1 we plot, for $-5<\alpha<5$, the pressure $P(\alpha)$ for the Julia sets i), ii), iii) and, for the second case, the bounds (5.1) just described are plotted as dashed lines in fig. 3, while fig. 2 shows the difference $\Delta P(\alpha)$ between the pressure and the Ruelle approximation (5.4).

For the maps i), ii), iii), we show in fig. 4-6 the generalized dimensions $D_q(\mu_\sigma)$ for $-5<q<5$ and $\sigma=0$, $D_{\rm H}$, d; in fig. 7-9 we give the plots of the Renyi entropies for the same maps and the same measures. Finally in fig. 10-12 we quote the scaling functions $f(\alpha;\mu_\sigma)$ following the same scheme.

We point out that some dimensions for complex Julia sets for $p=-0.32-$

TABLE I. – *We compare the pressure $P(\alpha)$ computed with 2^n preimages (left columns) with the linear extrapolations for $n=\infty$ obtained from the $(n-1)$-th and n-th terms (right columns), where $1\leqslant n\leqslant 14$. The map is $z'=z^2+2$ and the quoted values of α are -3, -1, 1 and 3.*

n	-3		-1		1		3	
1	3.73537		1.70722		−0.32093		−2.34908	
2	3.91959	4.10380	1.76863	1.83003	−0.38233	−0.44374	−2.53329	−2.71751
3	3.99980	4.16023	1.79355	1.84341	−0.40543	−0.45162	−2.59715	−2.72487
4	4.04424	4.17756	1.80662	1.84582	−0.41653	−0.44984	−2.62413	−2.70506
5	4.07171	4.18161	1.81456	1.84632	−0.42309	−0.44932	−2.63933	−2.70014
6	4.09022	4.18275	1.81987	1.84641	−0.42745	−0.44928	−2.64953	−2.70050
7	4.10348	4.18303	1.82366	1.84643	−0.43057	−0.44929	−2.65687	−2.70093
8	4.11343	4.18311	1.82651	1.84644	−0.43291	−0.44929	−2.66238	−2.70100
9	4.12118	4.18312	1.82873	1.84644	−0.43473	−0.44929	−2.66667	−2.70098
10	4.12737	4.18313	1.83050	1.84644	−0.43619	−0.44929	−2.67010	−2.70097
11	4.13244	4.18313	1.83195	1.84644	−0.43738	−0.44929	−2.67291	−2.70097
12	4.13666	4.18313	1.83315	1.84644	−0.43837	−0.44929	−2.67525	−2.70097
13	4.14024	4.18313	1.83418	1.84644	−0.43921	−0.44929	−2.67723	−2.70097
14	4.14330	4.18313	1.83505	1.84644	−0.43993	−0.44929	−2.67892	−2.70097

TABLE II. – *The same as table* I *for the map* $z' = z^2 - 0.15$.

n	-3		-1		1		3	
1	0.69315		0.69315		0.69315		0.69315	
2	1.22970	1.76625	0.86061	1.02807	0.53761	0.38208	0.26071	−0.17173
3	1.60583	2.35810	0.98019	1.21936	0.42357	0.19549	−0.07066	−0.73339
4	1.85815	2.61511	1.06215	1.30801	0.34347	0.10314	−0.30835	−1.02145
5	2.03235	2.72916	1.11991	1.35096	0.28576	0.05491	−0.48213	−1.17726
6	2.15700	2.78023	1.16188	1.37173	0.24319	0.03035	−0.61148	−1.25820
7	2.24933	2.80329	1.19330	1.38183	0.21099	0.01778	−0.70996	−1.30087
8	2.31988	2.81377	1.21748	1.38675	0.18603	0.01137	−0.78660	−1.32308
9	2.37529	2.81856	1.23656	1.38915	0.16626	0.00811	−0.84750	−1.33470
10	2.41984	2.82075	1.25193	1.39032	0.15028	0.00645	−0.89683	−1.34075
11	2.45637	2.82176	1.26457	1.39090	0.13713	0.00560	−0.93747	−1.34390
12	2.48686	2.82222	1.27512	1.39118	0.12613	0.00517	−0.97148	−1.34555
13	2.51268	2.82244	1.28406	1.39132	0.11681	0.00495	−1.00032	−1.34640
14	2.53481	2.82253	1.29172	1.39138	0.10881	0.00484	−1.02507	−1.34685

TABLE III. – *The same as table* I *for the map* $z' = z^2 - 3$.

n	-3		-1		1		3	
1	4.46969		1.95200		−0.56570		−3.08340	
2	4.50722	4.54474	1.91880	1.88560	−0.47694	−0.38817	−2.67998	−2.27657
3	4.53579	4.59294	1.90369	1.87349	−0.45383	−0.40761	−2.64194	−2.56586
4	4.55420	4.60943	1.89546	1.87075	−0.44210	−0.40691	−2.60199	−2.48212
5	4.56631	4.61476	1.89039	1.87011	−0.43504	−0.40680	−2.58316	−2.50784
6	4.57467	4.61646	1.88698	1.86996	−0.43034	−0.40683	−2.56932	−2.50011
7	4.58072	4.61699	1.88455	1.86992	−0.42698	−0.40682	−2.55976	−2.50245
8	4.58527	4.61716	1.88272	1.86992	−0.42446	−0.40682	−2.55251	−2.50174
9	4.58882	4.61722	1.88130	1.86991	−0.42250	−0.40682	−2.54689	−2.50195
10	4.59166	4.61723	1.88016	1.86991	−0.42093	−0.40682	−2.54239	−2.50189
11	4.59399	4.61724	1.87923	1.86991	−0.41965	−0.40682	−2.53871	−2.50191
12	4.59593	4.61724	1.87845	1.86991	−0.41858	−0.40682	−2.53565	−2.50190
13	4.59757	4.61724	1.87779	1.86991	−0.41767	−0.40682	−2.53305	−2.50191
14	4.59897	4.61724	1.87723	1.86991	−0.41690	−0.40682	−2.53083	−2.50191

$-0.043i$ have already been computed in ([21]) and the scaling function was evaluated for a Julia set close to the unit circle, namely $p = \pm 0.15$, in ([22]).

It can be observed that, while the dimensions D_q for the uniform Gibbs measure are obviously constant, for the other two measures the variation is not negligible and can reach 10% in the interval $-5 < q < 5$. The entropies are

([21]) D. SAUPE: *Physica (Utrecht) D*, **28**, 358 (1987).
([22]) M. H. JENSEN, L. KADANOFF and I. PROCACCIA: *Phys. Rev. A*, **36**, 1409 (1987).

TABLE IV. – *For the same values of* α *as in table* I, II, III *and for the maps* $z' = z^2 - p$ *with* $p = -2$, $p = 0.15$, $p = 3$ *(from left to right), we quote the pressure* $P(\alpha)$ *obtained with the Thiele extrapolation on the sequences of values computed with* 2^n *preimages* $(1 \leqslant n \leqslant 16)$.

n	−3	−1	1	3	−3	−1	1	3	−3	−1	1	3
1	3.73537	1.70722	−0.32093	−2.34908	0.69315	0.69315	0.69315	0.69315	4.46969	1.95200	−0.56570	−3.08340
2	4.10380	1.83003	−0.44374	−2.71751	1.76625	1.02807	0.38208	−0.17173	4.54474	1.88560	−0.38817	−2.27657
3	4.20375	1.85255	−0.45638	−2.72877	3.74582	1.69717	−0.31730	−2.57482	4.74686	1.86338	−0.41445	−2.59598
4	4.19776	1.84783	−0.44944	−2.71682	3.03953	1.46716	−0.07879	−1.62040	4.63054	1.86820	−0.40660	−2.46845
5	4.20960	1.84555	−0.44866	−2.69445	2.83257	1.41011	−0.03045	−1.48500	4.61473	1.87116	−0.40676	−2.52210
6	4.18297	1.84617	−0.44920	−2.70330	2.75040	1.31658	−0.62994	−2.00138	4.61473	1.87021	−0.40635	−2.49528
7	4.18167	1.84705	−0.44936	−2.70165	2.78933	1.37747	0.02097	−1.31022	4.61473	1.86967	−0.40685	−2.50664
8	4.18267	1.84645	−0.44928	−2.71100	2.82921	1.38667	0.01647	−1.29057	4.61789	1.86987	−0.40683	−2.50042
9	4.18319	1.84644	−0.44929	−2.70069	2.83334	1.41473	0.02393	−1.30818	4.61731	1.86994	−0.40681	−2.50307
10	4.18332	1.84646	−0.44927	−2.70124	2.83018	1.39401	0.00390	−1.33992	4.61709	1.86992	−0.40682	−2.50155
11	4.18321	1.84643	−0.44929	−2.70097	2.81978	1.39217	0.00092	−1.43138	4.61718	1.86989	−0.40682	−2.50220
12	4.18314	1.84644	−0.44929	−2.70085	2.82084	1.38651	0.00305	−1.35160	4.61729	1.86991	−0.40682	−2.50182
13	4.18310	1.84644	−0.44929	−2.70095	2.81811	1.39094	0.00453	−1.35240	4.61725	1.86991	−0.40682	−2.50198
14	4.18313	1.84644	−0.44929	−2.70099	2.82322	1.39128	0.00798	−1.35160	4.61723	1.86991	−0.40682	−2.50188
15	4.18313	1.84644	−0.44929	−2.70097	2.82275	1.39276	0.00506	−1.34752	4.61724	1.86991	−0.40682	−2.50193
16	4.18313	1.84644	−0.44929	−2.70097	2.82255	1.39155	0.00490	−1.34166	4.61725	1.86991	−0.40682	−2.50190

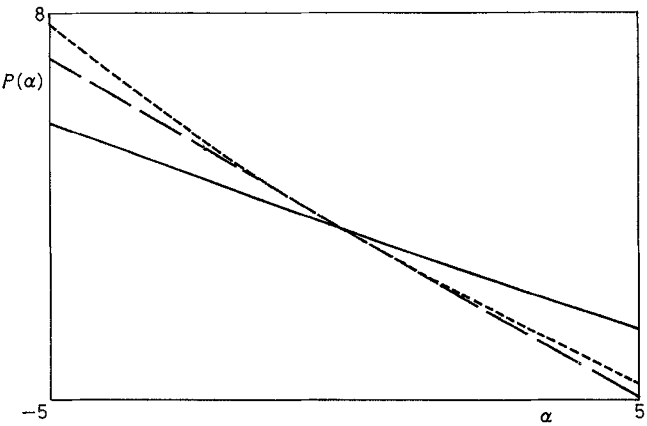

Fig. 1. – The pressure for three different maps is shown: $z' = z^2 - 3$ (continuous line), $z' = z^2 + 2$ (dashed line), $z' = z^2 - 0.15$ (dotted line).

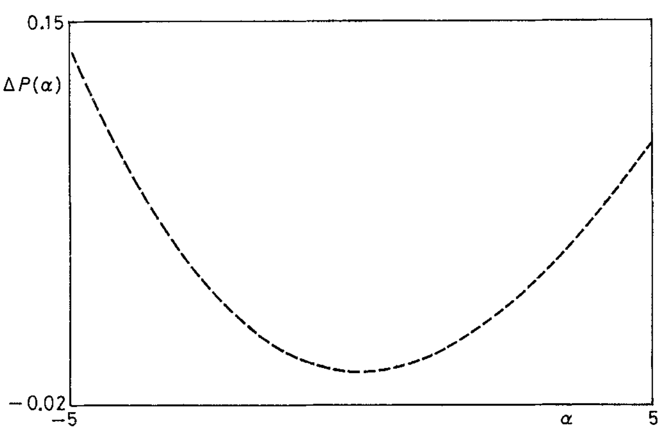

Fig. 2. – The difference $\Delta P(\alpha)$ between the pressure $P(\alpha)$ and its linear approximation (5.4) is shown for the map $z' = z^2 - 0.15$.

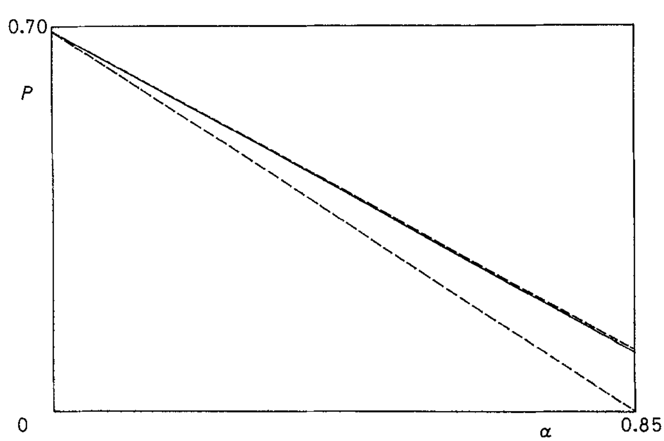

Fig. 3. – For the map $z' = z^2 - 0.15$ we compare the pressure $P(\alpha)$ (continuous line) with the bounds (5.1) (dashed lines).

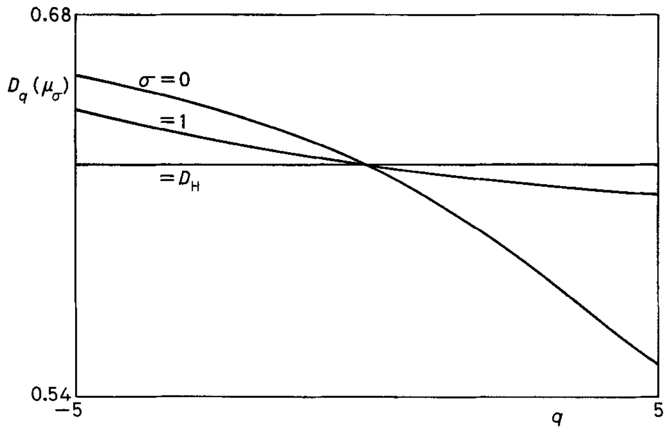

Fig. 4. – We show the generalized dimensions $D_q(\mu_\sigma)$ for *a*) $\sigma=0$, *b*) $\sigma=d$ $(d=1)$ and *c*) $\sigma=D_H$ for the map $z'=z^2-3$.

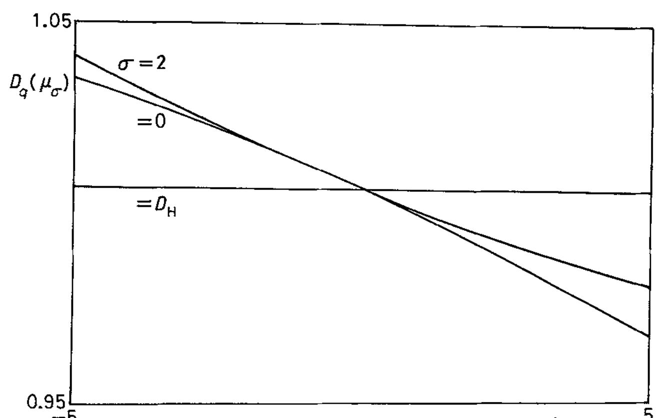

Fig. 5. – The same as fig. 4 for the map $z'=z^2-0.15$ with $d=2$.

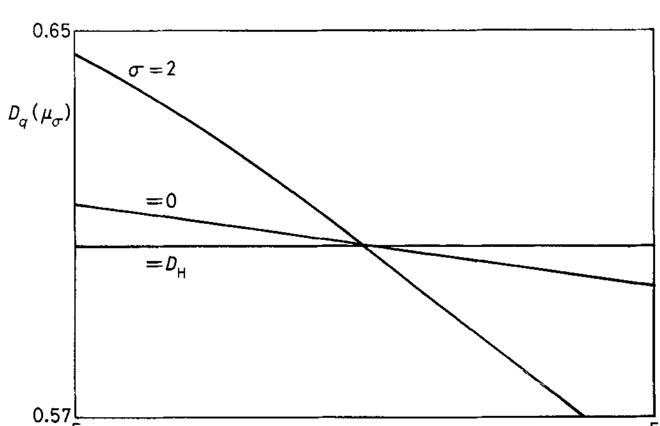

Fig. 6. – The same as fig. 4 for the map $z'=z^2+2$ with $d=2$.

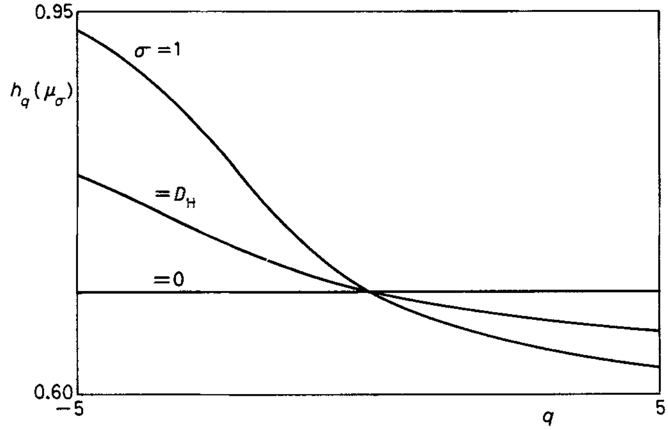

Fig. 7. – We show the entropies $h_q(\mu_\sigma)$ for $\sigma=0, \sigma=d$ $(d=1)$ and $\sigma=D_H$ for the map $z'=z^2-3$.

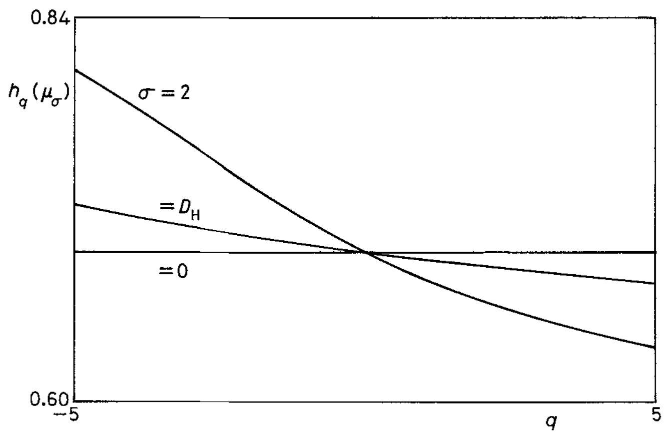

Fig. 8. – The same as fig. 7 for the map $z'=z^2-0.15$ with $d=2$.

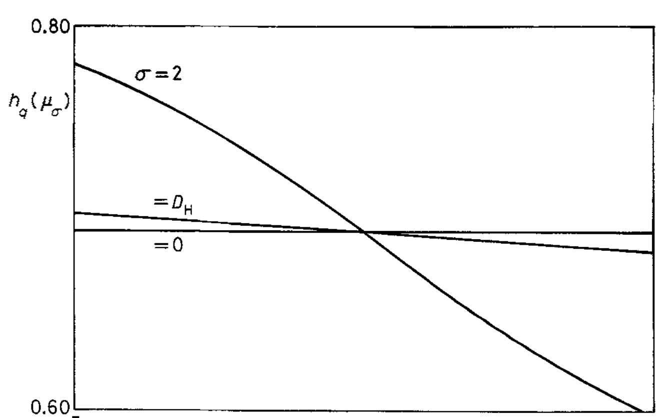

Fig. 9. – The same as fig. 7 for the map $z'=z^2+2$ with $d=2$.

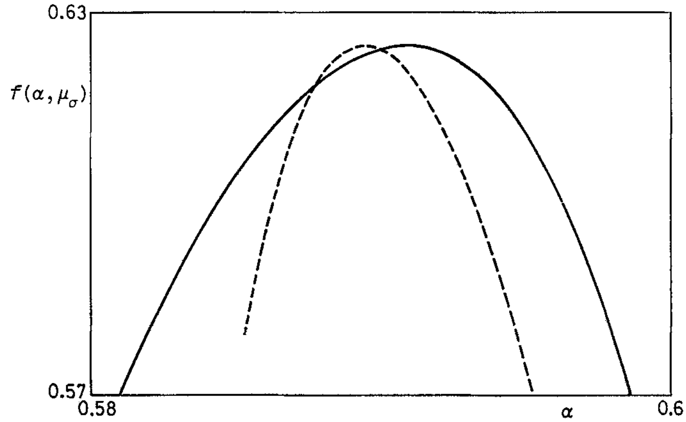

Fig. 10. – We show the scaling function $f(\alpha, \mu_\sigma)$ for $\sigma = 0$ and $\sigma = d$ ($d = 1$) for the map $z' = z^2 - 3$.

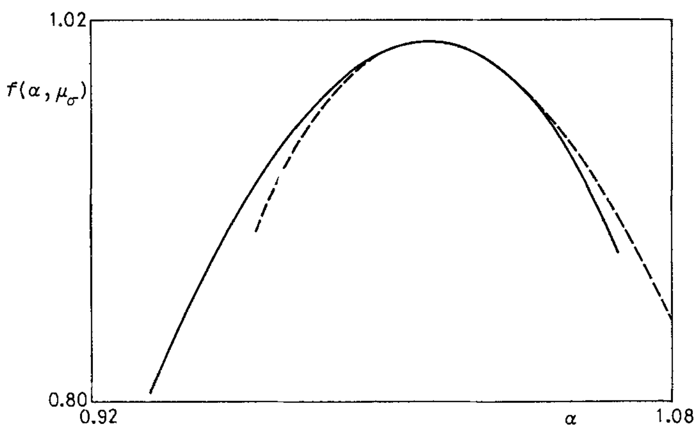

Fig. 11. – The same as fig. 10 for the map $z' = z^2 - 0.15$ with $d = 2$.

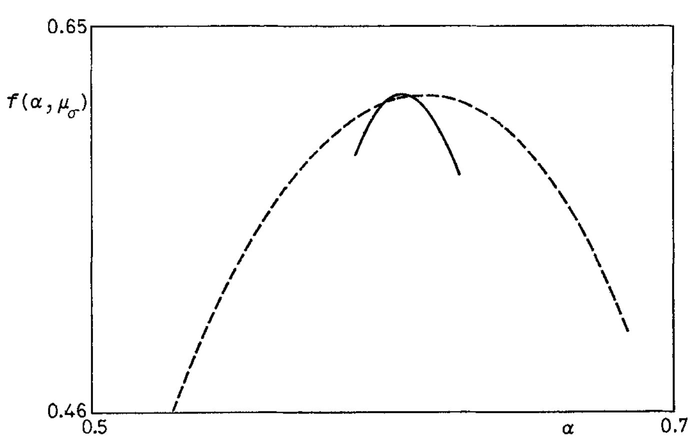

Fig. 12. – The same as fig. 10 for the map $z' = z^2 + 2$ with $d = 2$.

constant for $\sigma=0$ and exhibit variations, for the other two measures, which can be as high as 25%. The scaling functions $f(\alpha;\mu_\sigma)$ have all a maximum which is D_H for all measures; the variation from one measure to another is remarkable. The results for $\sigma=0$ are in agreement with those obtained by Kadanoff and co-workers (22).

Appendix A

We want to show that the Gibbs measures μ_σ and $\mu_\sigma^{(n)}$ of an atom $A_{l_1\dots l_n}$ of the Markov partition $\mathscr{A}^{(n)}$ of a nonlinear Cantor set are boundedly equivalent for large n, so that the two measures become the same in the limit $n\to\infty$, since they are ergodic.

By the Walter's theory of the Ruelle-Perron-Frobenius operator (19), we can bound the μ_σ-measure of an atom $A_{l_1\dots l_n}$ (17):

$$C_2|A_{l_1\dots l_n}|^\sigma \exp[n[-P(\sigma)-\varepsilon]] \leqslant \mu_\sigma(A_{l_1\dots l_n}) \leqslant C_1|A_{l_1\dots l_n}|^\sigma \exp[n[-P(\sigma)+\varepsilon]], \tag{A.1}$$

where C_1 and C_2 are finite constants which do not depend on n and l, ε is an arbitrarily chosen positive number and σ is taken positive without loss of generality.

By the uniformity of the limit (4.2), if we choose n sufficiently large (depending on ε), we can replace $P(\sigma)$ with $(1/n)P_n(\sigma)$ in (A.1). Putting, for the sake of simplicity, $\mathrm{diam}(\Omega)=1$ in (4.1) we obtain

$$C_2\mu_\sigma^{(n)}(A_{l_1\dots l_n})\exp[-n\varepsilon] \leqslant \mu_\sigma(A_{l_1\dots l_n}) \leqslant C_1\mu_\sigma^{(n)}(A_{l_1\dots l_n})\exp[n\varepsilon]. \tag{A.2}$$

Taking the two limits $\varepsilon\to 0$ and hence $n\to\infty$, we obtain the desired result.

Appendix B

We prove that

$$\lim_{n\to\infty} D_H^{(n)} = D_H$$

for one-dimensional nonlinear $\mathscr{C}^2$ maps T defined on the unit interval. We call $I_k^{(n)}$ an element of $T^{-n}([0,1])$ (see sect. 1 and (1.23)). We begin observing that, for a well-known distorsion argument, there exists a constant $G\geqslant 1$ such that for every pair of points x, y in the same $I_k^{(n)}$ and for every $n>0$ we have

$$G^{-1}|(T^n)'(y)| \leqslant |(T^n)'(x)| \leqslant G|(T^n)'(y)|. \tag{B.1}$$

Then, adapting Theorem 8.8 in ref. (8) to our case, if we have for each $n>1$

$$q_i|x-y| \leqslant |T_i^{-n}(x)-T_i^{-n}(y)| \leqslant r_i|x-y|$$

for all x, $y \in [0, 1]$ and $i = 1, \ldots, s^n$, then $w \leq D_{\rm H} \leq t$, where w and t are defined by

$$\sum_{i=1}^{s^n} q_i^w = 1 = \sum_{i=1}^{s^n} r_i^t .$$

Actually

$$\max_{\zeta \in T_i^{-n}[0,1]} |(T^n)'(\zeta)|^{-1}|x-y| \leq |T_i^{-n}(x) - T_i^{-n}(y)| \leq \min_{\zeta \in T_i^{-n}[0,1]} |(T^n)'(\zeta)|^{-1}|x-y| .$$

Since every inverse branch of T^n is $\mathcal{C}^2$ on $[0, 1]$, there exists a point $\zeta_i' \in T_i^{-n}[0, 1]$, $i = 1, \ldots, s^n$ such that

$$\lambda_{i,n} = |T_i^{-n}(1) - T_i^{-n}(0)| = |(T^n)'(\zeta_i')|^{-1} .$$

Using this fact and condition (B.1) we get

$$G^{-1}\lambda_{i,n}|x-y| \leq |T_i^{-n}(x) - T_i^{-n}(y)| \leq G\lambda_{i,n}|x-y|$$

for all x, $y \in [0, 1]$. Hence

$$\sum_{i=1}^{s^n} \lambda_{i,n}^t = G^{-t}, \qquad t > 0 ,$$

$$\sum_{i=1}^{s^n} \lambda_{i,n}^w = G^w, \qquad w > 0 .$$

We consider the first; it can be rewritten as

$$\sum_{i=1}^{s^n} \lambda_{i,n}^{D_{\rm H}^{(n)}} \lambda_{i,n}^{(t-D_{\rm H}^{(n)})} = G^{-t} .$$

We replace each $\lambda_{i,n}$ in the second factor with the maximum $\lambda_{\rm M}$; since $\sum_{i=1}^{s^n} \lambda_{i,n}^t$ is a decreasing function of $t \in \boldsymbol{R}$ and recalling that, by (1.21) and (1.23), $\sum_{i=1}^{s^n} \lambda_{i,n}^{D_{\rm H}^{(n)}} = 1$, we have, for n sufficiently large so that $\lambda_{\rm M}^{-1} > G$,

$$t \leq \frac{D_{\rm H}^{(n)} \log \lambda_{\rm M}}{\log G + \log \lambda_{\rm M}} .$$

A similar argument applies to the equation in w, thus we get

$$\frac{D_{\rm H}^{(n)} \log \lambda_{\rm M}}{\log \lambda_{\rm M} - \log G} \leq D_{\rm H} \leq \frac{D_{\rm H}^{(n)} \log \lambda_{\rm M}}{\log \lambda_{\rm M} + \log G} .$$

When $n \to \infty$, $\lambda_{\rm M} \to 0$ and $D_{\rm H}^{(n)}$ converges to $D_{\rm H}$.

* * *

One of us (SV) was supported, during the present work, by a CNR-NATO fellowship and thanks G. Paladin for useful discussions.

● RIASSUNTO

Tramite il calcolo diretto della pressione topologica sono fornite stime rigorose su dimensioni generalizzate, entropie, indici di Lyapunov e funzioni di scala di Julia sets iperbolici per una misura di Gibbs qualsiasi.

Обобщенные динамические переменные и величины для систем Джулиа.

Резюме (*). — Используя непосредственные вычисления топологического давления, мы приводим строгие оценки размеров, энтропий, характерных показателей экспонент и функции подобия для гиперболических систем Джулиа, для произвольной меры Гиббса. Затем предложенный метод обобщается на случай негиперболических систем Джулиа.

(*) *Переведено редакцией.*