Metastable states of two-dimensional isotropic ferromagnets

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(Submitted October 4, 1975)
Pis'ma Zh. Eksp. Teor. Fiz. **22**, No. 10, 503–506 (20 November 1975)

Metastable inhomogeneous states, which can produce a finite correlation length at arbitrarily low temperatures, are found for a Heisenberg ferromagnet.

PACS numbers: 75.25.+z, 75.30.Fv, 75.10.Gj

It is well known that in a two-dimensional ferromagnet with continuous symmetry there is no spin alignment at any arbitrarily low temperature T. It is nonetheless believed[1] that the phase transition in this system can result from the fact that the ground state is degenerate as $T \to 0$. This leads to the appearance of spin waves—Goldstone particles—and to an infinite correlation radius. This reasoning, however, does not take into consideration another phenomenon that can make the correlation radius finite.[2] We consider a classical Heisenberg ferromagnet, since the long-wave fluctuations of interest to us do not depend on quantum effects.

Assume that we are calculating a certain spin correlation function $\mathbf{n}(x)$. The averaging is over all possible fields with weight

$$\exp(-H/T). \tag{1}$$

If the temperature $T\to 0$, then an important role in the averaging is played by fields that are close to those which produce the local energy minima (ground and metastable states)

$$\delta H = 0. \tag{2}$$

Usually account is taken of the trivial minimum $\mathbf{n}_0(x) = \mathrm{const}$ and of fields that differ a little from $\mathbf{n}_0(x)$. However, if there exist other solutions of (2) with finite energy $H=E$ (pseudoparticles), then they must also be taken into account for the following reason: The solutions of (2) with finite energy do not depend on the scale in the two-dimensional case. Therefore, even though the average distance between such pseudoparticles at small T is large, $r_{\mathrm{av}} \sim a\exp(E/T)$ (a is the period of the lattice), their radius, owing to scale invariance, is of the same order of magnitude. The existence of such random inhomogeneities causes the spin correlation to vanish at a distance $R > r_{\mathrm{av}}$.

In this article we demonstrate the existence of nontrivial solutions of (2) for a two-dimensional ferromagnet with three spin components $n^{\alpha}(x)$. We begin with topological arguments that prove the existence of such solutions. A similar consideration of the field of a unit vector and an expression for the degree of mapping is contained in[3,4]. However, solutions with degrees of mapping larger than unity were not investigated there. The spin field is described by a three-component unit vector $\mathbf{n}(x)$ with interaction

$$H = \int \sum_{\alpha=1}^{3} (\nabla n^{\alpha})^2. \tag{3}$$

The values of $n(x)$ can be regarded as points on the three-dimensional sphere $S^2[\mathbf{n} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)]$. The fields of interest to us satisfy a condition that follows from the fact that the energy is finite

$$\mathbf{n}(x) \to (1,0,0) \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{4}$$

The latter means that the plane $\mathbf{x}$ on which the spins are specified is topologically equivalent to another sphere S^2, and the field $\mathbf{n}(x)$ effects the mapping of the sphere $\tilde{S}^2 \to S^2$. It is clear that if two mappings $\mathbf{n}(x)$ and $\mathbf{n}_1(x)$ belong to different homotopical classes, they cannot be continuously deformed one into the other. It is well known that there exists an infinite number of classes of mappings $\tilde{S}^2 \to S^2$. Consequently, the phase space of the spin fields breaks up into an infinite number of components, each characterized by a definite integer q—the degree of mapping. We shall next obtain the mimima of the energy in each component of phase space. To do so we express the degree of mapping in terms of the field $\mathbf{n}(x)$

$$q = \frac{1}{8\pi} \int \epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu} n^{\alpha} \frac{\partial n^{\beta}}{\partial x_{\mu}} \frac{\partial n^{\gamma}}{\partial x_{\nu}} d^2x \tag{5}$$

This equation is easy to prove by changing to spherical coordinates. We then obtain for the degree of mapping

$$q = \frac{1}{4\pi} \int \sin\theta(x)\, d\theta(x)\, d\phi(x) \tag{6}$$

Consequently, q is the number of times that the sphere S^2 is covered in the course of mapping.[1)]

There exists an important inequality

$$\left(\frac{\partial n^{\alpha}}{\partial x_{\mu}} - \epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu} n^{\beta} \frac{\partial n^{\gamma}}{\partial x_{\nu}} \right)^2 \geqslant 0. \tag{7}$$

It follows from (7) and (5) that

$$H = \int \left(\frac{\partial n^{\alpha}}{\partial x_{\mu}} \right)^2 d^2x \geqslant 8\pi q. \tag{8}$$

Formula (8) yields the lower value of the energy of the metastable states in each homotopical class. The equations which these states satisfy are of the form

$$\frac{\partial n^{\alpha}}{\partial x_{\mu}} = \epsilon_{\mu\nu}\epsilon_{\alpha\beta\gamma} n^{\beta} \frac{\partial n^{\gamma}}{\partial x_{\nu}}. \tag{9}$$

To understand the meaning of (9), it is convenient to introuuce the independent variables

$$\begin{aligned} w_1 &= \mathrm{ctg}\frac{\theta}{2}\cos\phi, \\ w_2 &= \mathrm{ctg}\frac{\theta}{2}\sin\phi, \\ w &= w_1 + i w_2 = \mathrm{ctg}\frac{\theta}{2} e^{i\phi} \end{aligned} \tag{10}$$

It then follows from (9) that

$$\frac{\partial w_1}{\partial x_1} = \frac{\partial w_2}{\partial x_2}; \quad \frac{\partial w_2}{\partial x_1} = -\frac{\partial w_1}{\partial x_2} \tag{11}$$

In (11) we recognize the Cauchy-Riemann conditions. Their general solution is

$$w = w_1 + i w_2 = f(z), \qquad \text{where } z = x_1 + i x_2. \tag{12}$$

Since the spin distribution should be a continuous function of the coordinates, the only singularities of the function f are poles. Thus, the field corresponding to a metastable state with given energy $8\pi q$ and with boundary condition (4) is given by

$$w \equiv \operatorname{ctg}\frac{\theta}{2}\, e^{i\phi} = \prod_i \left(\frac{z - z_i}{\lambda}\right)^{m_i} \prod_j \left(\frac{\lambda}{z - z_j}\right)^{n_j}, \tag{13}$$

where

$$\Sigma m_i > \Sigma n_j .$$

The degree of mapping q is the number of originals of the point $w(r)$, i.e., the number of solutions of (13) that express z in terms of w, hence

$$q = \Sigma m_j . \tag{14}$$

Expression (13) can be obtained also from the known solution[3,4] with degree of mapping $q = 1$, by using the conformal invariance of the Hamiltonian (3), which exists in the two-dimensional case.

We have thus proved that a ferromagnet has inhomogeneous metastable states. This apparently means that there is a finite correlation length in the system and there is no phase transition even at very low temperatures.

We hope to consider in the future papers the quantitative influence of the metastable states on the low-temperature asymptotic form of the correlation functions.

We are grateful to C. Khokhlachev, who called our attention to the conformal invariance of (3). One of us (A. B.) thanks D. Burlankov for discussions.

[1)] Fields with a larger number of components, $k > 3$, effect the mapping $\tilde{S}^2 \to S^{k-1}$. It is known that all such mappings contract to a trivial one. There are in this case therefore no minima similar to those found below.

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