

CHAOTIC MOTION AND RANDOM MATRIX THEORIES

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*"Cada ciència ha mester los vocables per los quals mills sia manifestada ; e car a aquesta ciència demostrativa sien mester vocables escurs e que los hòmens lecs no han en ús, e car nòs façam aquest llibre als hòmens lecs per açò breument e ab plans vocables parlarem d'esta ciència".
Ramon Llull (Libre de Gentil, 1273 ?)*

I - INTRODUCTION

In the early usage, the word chaos referred to the darkness, the vacuum. John Milton, paraphrasing the Old Testament Genesis, wrote in the Paradise Lost

*In the beginning how the heavens and earth
Rose out of Chaos*

Since Milton, the word chaos has altered this meaning, to indicate confusion, mixing, complexity, disorder. One usually opposes chaos to order, or complexity to simplicity and one can imagine different sorts of evolutions or transitions [DH-80] : order to order, chaos to chaos, order to chaos and chaos to order. The first two are not unexpected and easy to illustrate. The third fits with everyday experience and looks consistent with the Second Principle of Thermodynamics. But the last one is surprising and intriguing. Most of the questions we will address ourselves in these lectures are of the type "How is the order diluted when going from order to chaos" and "What is the underlying order behind chaos". And the appropriate language, in most cases, will of course be the one of probability theory. In particular, it will emerge that probabilistic and deterministic points of view are not as irreconcilable as they may appear at first sight. Indeed, we will go through systems governed by causal equations of motions although their future motion does not depend on their past, i.e. we will discuss memory-losing or unpredictable systems.

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The kind of systems we shall deal with are of several types :

- i)- Points distributed on the real axis : x_1, x_2, x_3, \dots . The axis may be the energy, the points corresponding to the discrete energy levels of a quantum system (an atomic nucleus, an atom or a molecule) ; or the frequency axis, the points corresponding, for instance, to the normal frequencies of a vibrating membrane ; or the time axis, the points corresponding to successive epochs of occurrence of a given event (times of arrivals on a telephone line in queuing problems, for instance).
- Points distributed on a segment or on the circle (for instance, eigenvalues of unitary matrices).
- For illustrative purposes, we shall also briefly describe some number-theoretic functions $f(n)$ defined on the positive integers $1,2,3,\dots$
- ii)- Sequences of points $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_k, \dots$ on an N-dimensional space defined, for instance, through some transformation $\underline{x}_{n+1} = T \underline{x}_n$. For instance, snapshots of the classical trajectory of a point particle of certain dynamical systems, or sequences of values of some variables obtained through a sequential algorithm.

For the first kind of systems (discussed in this Section and in Section II), before a study of the fluctuations -one of our main goals- can be attempted, one must make a clean separation between the smooth (or average) behaviour and a fluctuating (or oscillating) part. In some cases this step will be obvious. For the second kinds of systems (treated in Section III), the attention will be focused on the structure of phase space, how it is filled by trajectories and special emphasis will be given to the question of stability of orbits.

In order to get some flavour about general ideas and concepts, let us now give a few examples coming from very different fields.

Prime Numbers (Number Theory)

The theory of numbers is at the same time one of the most elementary branches of mathematics, because it deals with the arithmetic properties of integers $1,2,3,\dots$ and one of the most difficult branches insofar as it leads to new difficult problems and techniques. Which sort of techniques ? Some titles of monographs or specialized articles reveal it : "Statistical independence in probability, analysis and number theory" [Ka-59], "On the density of certain sequences of integers", etc.. The list could be made very long. Notice the presence of words like probability, density.

Let us now consider the sequence of prime numbers $2,3,5,7,\dots$ We want to emphasize two facts [Za-77] : "The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers belong to the most arbitrary

and ornery objects studied by mathematicians... The second fact is even more astonishing, for it states just the opposite : that the prime numbers exhibit stunning regularity, that there are laws governing their behaviour, and that they obey these laws with almost military precision".

The first aspect is illustrated on Table I-1. In the interval [1-100] there are 25 primes, the largest gap between two successive primes being 97-89=8. In the interval

[1,100]	2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67, 71,73,79,83,89,97
$[10^7-100,10^7]$	9999901, 07,29,31,37,43,71,73,91
$[10^7,10^7+100]$	10 000 019, 079

Table I-1 : List of primes in three different intervals of length 100

$[10^7-100,10^7]$ there are 9 primes, the largest gap being 28. In the interval $[10^7,10^7+100]$ there are only two primes, which differ by 60. Despite this in appearance completely erratic behaviour, are there systematic patterns ? The first important feature, already known by Euclid (ca. 300 b.C.) is that the sequence of primes is infinite. But, obviously, there are much less primes than integers. For instance, the harmonic series $\sum (1/n)$ summed over integers n is known to diverge. More precisely

$$\lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{1}{n} - \ln x \right) = \gamma = 0.577 \dots \quad (\text{Euler constant}) \quad (\text{I-1})$$

What about summing over primes p only, i.e. $\sum (1/p)$? If one performs the "experiment", for instance by summing for $p \leq 10^9$ one finds 3.3, whereas summing up to $p \leq 10^{18}$ one would find $\simeq 4$. In fact, the exact asymptotic result is

$$\sum_{p \leq x} \frac{1}{p} = \ln \ln x + C + \eta(x), \quad (\text{I-2})$$

where $C \simeq 0.261497$ and $\eta(x) \rightarrow 0$ when $x \rightarrow \infty$; thus, the series diverges. This example clearly shows that in some cases, the empirical (numerical) observation is unable, not only to produce, but even to guess, the exact result. But it is not always like that. For instance, let us consider the number $\pi(x)$ of primes which are less than or equal to x . The function $\pi(x)$ is an infinite staircase, which increases by one each time

one "crosses" a prime. The first 25 steps of this irregular staircase are shown on Fig.I.1. Has this staircase function an average behaviour ? Gauss observed as early as 1792 that the density of prime numbers $d\pi(x)/dx$ appears on the average to be $1/\ln x$.

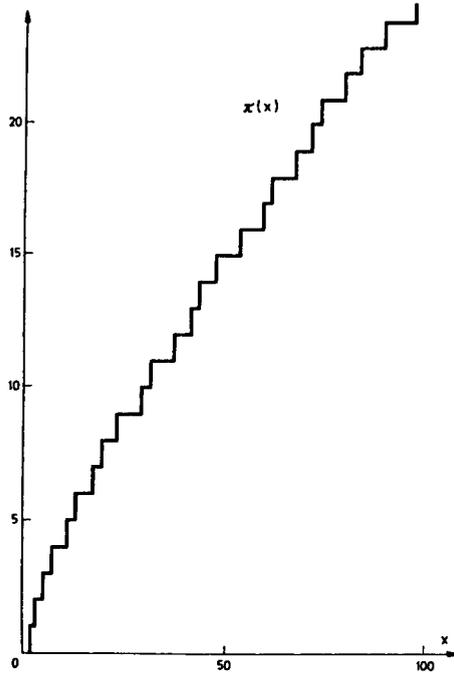


Fig.I.1 - Plot of the function $\pi(x)$ for $1 < x < 100$

He was thus led, from "empirical observation" of primes, to approximate $\pi(x)$ by the integral logarithm $Li(x)$

$$\begin{aligned} \pi(x) \simeq Li(x) &= \lim_{\epsilon \rightarrow 0} \left[\int_0^{1-\epsilon} \frac{dt}{\ln t} + \int_{1+\epsilon}^x \frac{dt}{\ln t} \right] \\ &= Li(2) + \int_2^x \frac{dt}{\ln t} = 1.04 + \int_2^x \frac{dt}{\ln t} \end{aligned} \quad (I-3)$$

$Li(x)$ admits the following expansion

$$Li(x) = \gamma + \ln \ln x + \sum_{n \geq 1} \frac{(\ln x)^n}{n \cdot n!} \quad (I-4)$$

The quality of the approximation $\pi(x) \simeq \text{Li}(x)$ is, for many purposes, very good. For instance, for $x \leq 10^7$, the relative error $|\pi(x) - \text{Li}(x)|/\pi(x)$ is smaller than 5×10^{-5} . On Fig.I.2 is plotted, for $x \leq 10^7$, the difference $\text{Li}(x) - \pi(x)$. It can be seen that although this difference is small, it is not featureless : for instance, for $x < 10^7$, it steadily increases.

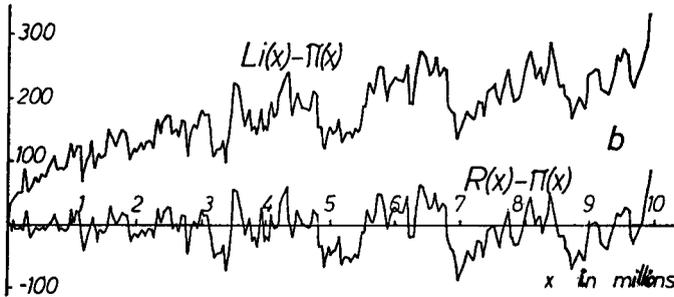


Fig.I.2 - Difference between Gauss (Riemann) approximation $\text{Li}(x)$ ($R(x)$) to $\pi(x)$ and $\pi(x)$ for $x \leq 10^7$ (taken from Ref.[Za-77])

What about rigorous, "non-empirical", results ? One of the main questions, in the middle of the 19th century, was to prove the prime number theorem (PNT) namely

$$\pi(x) \underset{x \rightarrow \infty}{\sim} \frac{x}{\ln x} \quad (I-5)$$

Notice that Gauss approximation is consistent with the PNT. The first major result in the direction of the PNT was obtained by Chebyshev in 1850, who proved that

$$0.89 \frac{x}{\ln x} < \pi(x) < 1.11 \frac{x}{\ln x} \quad (I-6)$$

for sufficiently large x . Although the PNT is true, the approximation

$$\pi(x) \simeq \frac{x}{\ln x} \quad (I-7)$$

is much poorer than the approximation introduced by Gauss (see Fig.I.3).

Riemann, in his famous memoir "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" introduced, based on empirical evidence and intuition, a better approximation $R(x)$ to $\pi(x)$

$$\pi(x) \simeq R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \text{Li}(x^{1/n}), \quad (I-8)$$

where $\mu(n)$ is the Möbius function defined as follows : $\mu(1) = 1$

$$\mu(n \neq 1) = \begin{cases} 0 & \text{if } n \text{ is divisible by a prime square} \\ 1 & \text{if } n \text{ is a product of an even number of distinct primes} \\ -1 & \text{if } n \text{ is a product of an odd number of distinct primes} \end{cases} \quad (\text{I-9})$$

For instance, $\mu(2) = \mu(3) = \mu(5) = \mu(7) = -1$; $\mu(6) = \mu(2 \times 3) = 1$; $\mu(4) = \mu(2^2) = 0$.
 $R(x)$ is an entire function of $\ln x$ with the following expansion

$$R(x) = 1 + \sum_{n \geq 1} \frac{1}{n \zeta(n+1)} \frac{(\ln x)^n}{n!}, \quad (\text{I-10})$$

where ζ is the Riemann zeta function of $s = \sigma + it$, which is defined by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \geq 2} (1 - p^{-s})^{-1} \quad (\text{I-11})$$

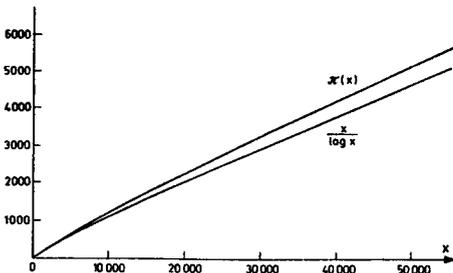


Fig.I.3 - Plot of $\pi(x)$ and $x/\ln x$ for $x \leq 5 \times 10^4$
 (taken from Ref.[Za-77])

after subtracting R to π only fluctuations are left out. In fact, Riemann, although unable to prove the PNT^(*), did something even more astonishing. He derived an exact relation for $\pi(x)$:

$$\pi(x) = R(x) - \sum_p R(x^p), \quad (\text{I-12})$$

(*)The PNT was proved simultaneously and independently by Hadamard and de la Vallée-Poussin in 1896. Hadamard was born in 1865 and de la Vallée-Poussin in 1866 and they died in 1963 and 1962 respectively !

for $\sigma > 1$ and by analytic continuation for $\sigma \leq 1, s \neq 1$. Notice the connection between $\zeta(s)$ and the prime numbers p .

For $x = 10^6$ one has $\pi(10^6) = 78498$ and the first two terms of (I-8) give $\text{Li}(10^6) - \frac{1}{2} \text{Li}(10^3) = 78628 - \frac{1}{2} \times 178 = 78539$.

On Fig.I.2 is shown the difference

$R(x) - \pi(x)$. As can be seen, no structure is anymore present. It seems now that one can consider $R(x)$ to be the smooth behaviour of $\pi(x)$;

where the sum runs over the "non trivial" ($\neq -2, -4, -6, \dots$) roots ρ of $\zeta(s)$. The first term in (I-12) gives the approximation discussed above. Putting $\rho = \frac{1}{2} + i\alpha$ (one assumes the Riemann Hypothesis (RH) to be true; see below), performing the summation over α in order of size, pairing terms of the sum corresponding to ρ and ρ^* (if ρ is a root of $\zeta(s)$, so is ρ^*), one can write

$$\pi(x) = R(x) + \sum_{k=1}^{\infty} T_k(x) \quad , \quad (\text{I-13})$$

where

$$T_k(x) = - [R(x^{\rho_k}) + R(x^{\rho_k^*})] \quad k=1,2,\dots \quad . \quad (\text{I-14})$$

The terms $T_k(x)$ are oscillating ones. The first few are shown on Fig.I.4 and on Fig.I.5 is reproduced the result obtained by adding 10 and 29 oscillating terms to $R(x)$ [Za-77, Si-79]. As can be seen, by the time when ~ 30 terms are added, one is obtaining an almost exact representation of $\pi(x)$ in the range $1 < x < 100$ (compare to Fig.I.1).

Some remarks are in order. We have mentioned that $Li(x) - \pi(x) > 0$ for $x < 10^7$ (see Fig.I.2). How incredibly weak may the position of the physicist be when looking for asymptotic results is illustrated by the following fact: it has been shown (Littlewood) that there exist numbers for which $Li(x) - \pi(x)$ is negative and Skewes proved that there is one smaller than

$$10^{10} 10^{34} \quad ,$$

a number of which Hardy said that it was surely the biggest that had ever served any definite purpose in mathematics. So, although one knows no number for which $Li(x) - \pi(x)$ is negative, one knows that this difference cannot increase steadily, as suggested by the enormous available "empirical data" coming from all the presently known primes. On the opposite, we have also seen (at least in the hands of Gauss and Riemann!) how extremely powerful and far-reaching approximations can be derived from empirical observation.

In the same vein, let us illustrate how empirism and heuristic arguments are used and differently appreciated by pure mathematicians. We have already mentioned the Riemann Hypothesis (RH), which is by universal agreement the out-standing unsolved problem in mathematics. It states that all the zeros $\rho = \sigma + i t$ of the Riemann zeta function lie on the critical line $\sigma = 1/2$, except for the "trivial zeros" lying on the real axis at the values $\sigma = -2, -4, -6, \dots$. It has been proved that there are an infinite number of zeros on the critical line (Hardy) but not that all of them are on it. It has also been proved that the RH is equivalent to

$$\lim_{x \rightarrow \infty} M(x) x^{-(1/2)-\epsilon} = 0$$

In (I-15) $M(x)$ is

$$M(x) = \sum_{n < x} \mu(n), \quad (\text{I-16})$$

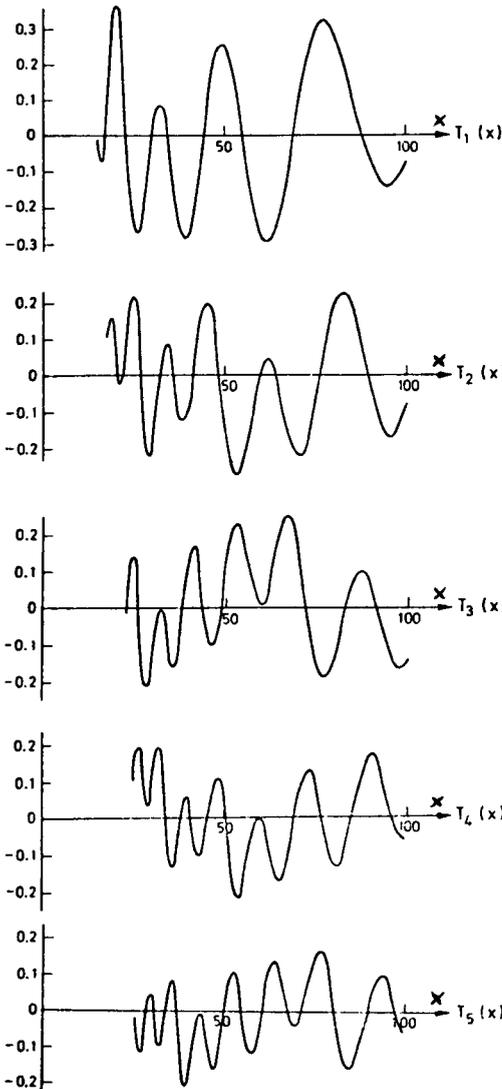


Fig. I.4 - Plot of the first five oscillating terms of Eq. I-14 (taken from Ref. [Za-77])

$$\text{for all } \epsilon > 0. \quad (\text{I-15})$$

the summation is over integers and $\mu(n)$ is the Möbius function defined above (I-9). People have tried, using heuristic arguments having a probabilistic basis, to reinforce the belief that the RH is in fact true. We reproduce one of them for its simplicity [DH-80, GC-68]. If we make a plot of the Möbius function $\mu(n)$ it looks random, in the sense that it shows no discernible regularity, except for the fact that μ is just as likely to be equal 1 or -1. Now, what is the chance for $\mu(n) \neq 0$? This will happen if n is not a multiple of 4 or a multiple of 9, or a multiple of 25 or of any other square of a prime. The probability that a number chosen at random is not a multiple of 4 is $3/4$, that it is not a multiple of 9 is $8/9$, that it is not a multiple of p^2 is $(p^2-1)/p^2$. These conditions being all independent we obtain, for the probability that $\mu(n) \neq 0$

$$\prod_{p \gg 2} \frac{p^2-1}{p^2} = \frac{6}{\pi^2}. \quad (\text{I-17})$$

Therefore, the probability that :

i) $\mu(n) = 1$ is $3/\pi^2$, ii) $\mu(n) = -1$ is $3/\pi^2$, iii) $\mu(n) = 0$ is $1 - 6/\pi^2$. Let us now examine $M(x)$

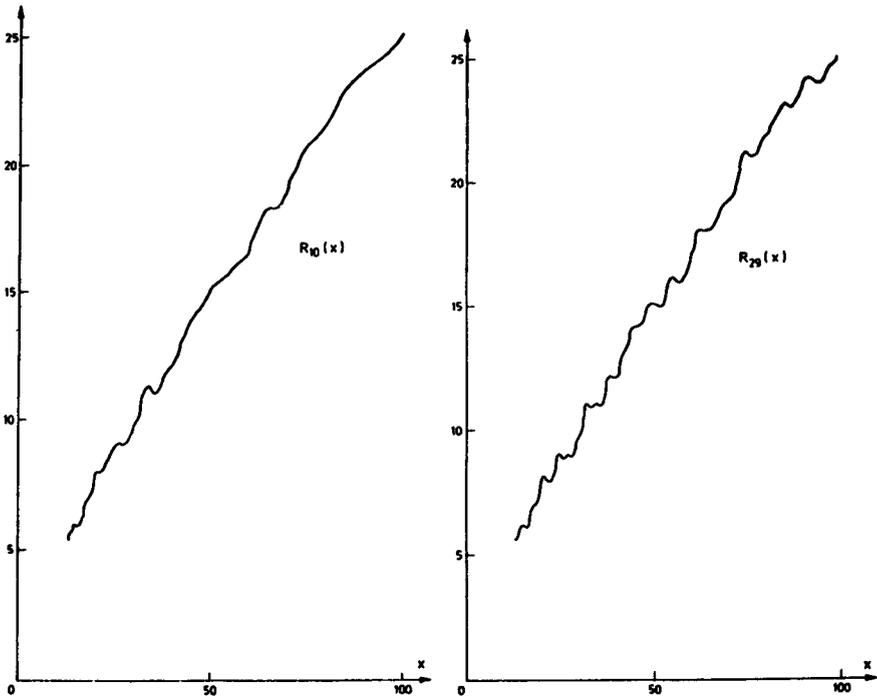


Fig.I.5 - Value of $\pi(x)$ including 10 and 29 oscillating terms in Eq.I-13 (taken from Ref. [Za-77])

and assume that each term in the summation in (I-16) can be considered as an independent random variable with the probabilities just derived. Hausdorff's inequality says then that, with probability 1, $M(x)$ grows no faster than a constant times $x^{(1/2)+\epsilon}$, exactly what is needed to prove RH ! However, we have made an unacceptable trick. Instead of adding the values of μ for the N values in the range 1 to $N = \text{Int}(x)$, we have taken N integers at random. We have done this because we feel that the table of values of μ is "random" or "unpredictable" although we know that the Möbius function is completely deterministic. The author of the authoritative work on the zeta function, H.M. Edwards [Ed-74], calls this type of heuristic reasoning "quite absurd" and Littlewood wrote : "I should also record my feeling that there is no imaginable reason why the RH should be true". Nevertheless, if one persists with the "absurd" reasoning, one predicts that the number of zeros of $\mu(n)$ between 1 and 33×10^6 is [GC-68] $33 \times 10^6 \times (1-6/\pi^2) = 12\,938\,405.6$ whereas the actual number is 12 938407, an 8 place accuracy result !

Zeros of the Riemann Zeta Function

Let $N(T)$ be the number of zeros $\rho = 1/2 + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$. Then [Mo-76]

$$N(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi} + S(T) + \frac{7}{8} + O\left(\frac{1}{T}\right). \quad (I-18)$$

The maximum order of $S(T)$ remains unknown. Probably

$$S(T) = O\left(\sqrt{\frac{\ln T}{\ln \ln T}}\right). \quad (I-19)$$

One knows the first 7×10^8 zeros of $\zeta(s)$ and stretches of 10^5 successive zeros have been computed around the 10^9 -th zero and around the 10^{11} -th zero [Od-82].

There is a conjecture that the distribution of zeros of $\zeta(s)$ resembles that of the eigenvalues of a random complex hermitian or unitary matrix [Mo-73a,74] (see Section II). There are very strong numerical indications supporting this conjecture [Od-82].

Vibrations of a membrane (the drum)

A membrane is a perfectly flexible and infinitely thin lamina of solid matter, of uniform material and thickness which is stretched in all directions by a tension so great as to remain unaltered during its vibrations. The principal subject in this field is the investigation of the transverse vibrations of membranes of different shapes, whose boundaries are fixed. One considers then (Fig.I.6) a membrane stretched over the area Ω

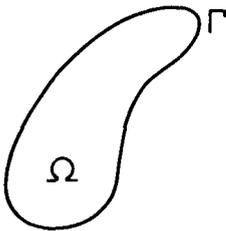


Fig.I.6

included within a fixed, closed plane boundary Γ . Taking the plane of the membrane as that of $x-y$, let ψ denote the small displacement perpendicular to its original plane. It has been known for all over a century that ψ obeys the wave function

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi, \quad (I-20)$$

where c is a constant with dimensions of a velocity, depending on the physical properties of the membrane as well as on the tension under which the membrane is held. Of special interest, both to mathematicians and to musicians, are solutions of the form

$$\psi(x, y; t) = \psi(x, y) e^{i\omega t} \quad (I-21)$$

for, being harmonic in time with frequency ω , they represent the pure tones the membrane is capable of producing. These special solutions are known as normal modes. To find the normal modes, one substitutes (I-21) in (I-20) and one finds that ψ must satisfy the eigenvalue equation

$$\nabla^2 \psi = -\frac{\omega^2}{c^2} \psi = -k^2 \psi = -E \psi \quad (\text{I-22})$$

with the condition that ψ vanishes on the boundary Γ . In (I-22) ω is a frequency, k a wave number and E an energy. Equation (I-22) possesses an infinite number of eigenvalues E_n which are real and non-negative and have no accumulation point. One has therefore

$$0 \leq E_1 \leq E_2 \leq E_3 \leq \dots, \quad \lim_{n \rightarrow \infty} E_n = \infty. \quad (\text{I-23})$$

We are now interested in properties of the sequence of eigenvalues. For instance, on what characteristics of the boundary depends the number of modes per unit energy (or unit frequency), i.e. what is the density of eigenmodes. In these lectures the emphasis will be put on the 2-dimensional case, but the problem can also be considered in 3 dimensions, or more generally in N dimensions. In three dimensions, it corresponds to the study of the vibrations of an acoustical resonator or the vibrations of an elastic body or the free electromagnetic oscillations in the interior of a cavity, with adequate changes of the boundary conditions. In one dimension, the problem corresponds to the vibrations of a string, but in this case there is no room for the influence and regularity of the shape. In quantum physics, Eq.(I-22) represents the time-independent Schrödinger equation for a free particle of mass m moving in the interior of a box with infinite walls, with $\hbar k = \sqrt{2mE}$, where E is the kinetic energy.

The asymptotic number of modes per unit frequency was first established for the case of the rectangular parallelepiped by Rayleigh (1905). Sommerfeld and H.A. Lorentz had drawn attention to the effect of the domain, or, following Kac to put it in a picturesque way, one may ask "Can one hear the shape of a drum?" [Ka-66]. Johanna Reudler, a student of Lorentz, in a Leiden dissertation, verified that the asymptotic number of modes depends only on the volume for the special cases parallelepiped, sphere and cylinder. The history of how the first important result in this field was obtained is worth to be briefly remembered [Ka-66]. Lorentz was invited to Göttingen in 1910 to deliver the Wolfskehl lectures (Wolfskehl awarded a prize for proving or disproving Fermat's last theorem^(*) and in case the prize would not be awarded, stipulated that the

(*) In 1637 Fermat stated that the diophantine equation $x^n + y^n = z^n$ with integral $n > 2$ has no solutions in positive integers x, y, z . Fermat asserted to have a "truly marvelous proof" of his statement, but today it is generally believed that his argument, which was apparently never revealed, must have been incomplete.

proceeds from the principal should be used to invite eminent scientists to lecture in Göttingen). Lorentz gave some lectures under the general title "Alte und neue Fragen der Physik". At the end of one of them he asked "In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves... The mathematical problem is to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume". If one believes an apocryphal report Hilbert predicted that the theorem would not be proved during his life. Less than two years later Hermann Weyl, who was present at the Lorentz's lecture, using the theory of integral equations which his teacher Hilbert developed only a few years before, proved the theorem, long before his death.

Let us now go back to the initial two-dimensional problem. Progress has been made since the pioneering work by Weyl. We are interested in extracting a smoothed eigenvalue distribution $N_{av}(E)$, i.e. the smoothed function giving the number of eigenvalues less than or equal to E in order to study the fluctuations or oscillations of the exact eigenvalue distribution around the averaged value $N_{av}(E)$. In the context of the previous example on prime numbers, we are searching the function $N_{av}(E)$ which has a similar relationship to $N(E)$ as $R(x)$ to $\pi(x)$. Reference [BH-76] gives a complete account of the results obtained so far in this field. The function $N_{av}(E)$ can be written

$$N_{av}(E) = \frac{\sigma}{4\pi} E - \frac{\gamma}{4\pi} \sqrt{E} + K + O\left(E^{-\eta/2} \ln \sqrt{E}\right), \quad (I-24)$$

where $0 < \eta \leq 1$. In (I-24) σ is the surface of the area Ω (Weyl's term) and γ is the perimeter of the boundary $\Gamma^{(*)}$. K is a constant term containing complex information on the geometrical and topological properties of the domain. The geometrical features contributing to the constant term are : i) Curvature contribution

$$\left(\frac{1}{12}\right) \int_{\Gamma} \kappa(l) dl, \quad (I-25)$$

where $\kappa(l)$ denotes the local curvature ; for instance, the curvature contribution for the circle is $1/6$. ii) Corners contribution ; for a square (or a rectangle), it is $4 \times (1/48)$. The topological features concern the connectivity of the surface ; for a multiply connected drum containing r holes, the contribution to the constant term is $(1-r) \times (1/6)$.

On Fig.I.7 are compared the exact function $N(E)$ and the smoothed function $N_{av}(E)$ given by Eq.(I-24) for two different shapes, namely a quarter of a circle and a stadium (see Section III). It can be seen that $N_{av}(E)$ indeed reproduces perfectly the average behaviour of $N(E)$, not only asymptotically but starting from the bottom of the spectrum.

(*) If one uses Neumann instead of Dirichlet boundary conditions, (I-24) is still valid except for the sign of the perimeter term

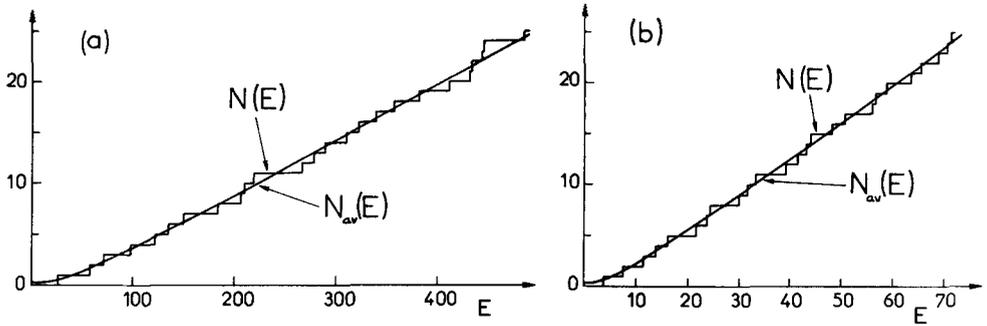


Fig.1.7 - Cumulative density of eigenvalues $N(E)$ of Eq.(1-22) and its average $N_{av}(E)$ given by Eq.(1-24) for two different shapes of the boundary : (a) a quarter of a circle of radius unity ; (b) a stadium with straight line and radius of curvature equal to one (See Section IV)(taken from [Sc-84,BGS-84b]).

How to characterize fluctuations

The examples treated so far should have convinced the reader that it is possible in many cases, for a sequence of discrete points on the real line, to separate the staircase function $N(\xi)$ in a smooth part $N_{av}(\xi)$ and a fluctuating part $N_{fl}(\xi)$

$$N(\xi) = N_{av}(\xi) + N_{fl}(\xi) . \quad (1-26)$$

Before studying fluctuations one wants to get rid of $N_{av}(\xi)$ in order to compare the fluctuation patterns of different systems whose corresponding average behaviours $N_{av}(\xi)$ are not the same. For that purpose, one "unfolds" the original spectrum ξ_i through the following mapping $\xi \mapsto x$

$$x_i = N_{av}(\xi_i) \quad i=1,2,\dots \quad (1-27)$$

Consider now the sequence $\{x_i\}$ and its corresponding smooth behaviour $\hat{N}_{av}(x)$. The effect of (1-27) is that $\hat{N}_{av}(x) = x$, i.e., the sequence $\{x_i\}$ has on the average a constant mean spacing (or a constant density) equal to unity, as can be seen from

$$N_{av}(\xi) = \int_0^\xi \rho_{av}(\xi') d\xi' = \int_0^x dx' = x = \hat{N}_{av}(x), \quad (1-28)$$

where $\rho_{av}(\xi)$ is the average density of $\{\xi_i\}$. For instance, consider the (fluctuation-free) sequence $\xi_k = k^2$ ($k = 1, 2, \dots$); then $N_{av}(\xi) = \sqrt{\xi}$ and $x_k = k$, a sequence of equally spaced points or picket fence. In summary, after unfolding, we shall study quantities related to

$$\hat{N}(x) = \hat{N}_{av}(x) + \hat{N}_{fl}(x) = x + \hat{N}_{fl}(x). \quad (1-29)$$

When considering fluctuation properties of sequences $\{x_i\}$ we shall come across different situations: i) cases in which the system is known to be, from a statistical point of view, translational invariant or stationary, i.e., the fluctuation properties are the same irrespective of which region of the spectrum (of the sequence) one is considering; ii) cases in which the system is not stationary but one is interested in asymptotic properties of the spectrum.

The question now is to discover the stochastic laws governing sequences having very different origins, as illustrated on Fig. I.8, which is inspired from a similar figure of Ref. [BFF-81]. There are displayed six spectra, each containing 50 levels.^(*) Column (a) corresponds to a Poisson system: Take a random variable s whose probability density

$p(x)$ is e^{-x} . Construct a sequence $\{x_i\}$

$$x_1 = 0, \quad x_{i+1} = x_i + s_i \quad i = 1, 2, 3, \dots, \quad (1-30)$$

where s_i are outcomes of independent trials of the variable s . The resulting spectrum is what is called a Poisson spectrum, which is obviously stationary. For instance, if one studies the counting rate of a decaying source, the successive times of decay x_i will form a Poisson spectrum, the time being measured in units of the mean life of the source. Column (b) shows an example of a segment of prime numbers in the interval [7791097-7791877] Ref. [Si-79]; column (c) the resonance energies $J^\pi = 1/2^+$ of the compound nucleus observed in the reaction $n + {}^{166}\text{Er}$ (see Section II); column (d) the eigenvalues (associated to eigenfunctions with a given symmetry) corresponding to the transverse vibrations of a membrane whose boundary is the Sinai's billiard (see Section IV); column (e) the positive imaginary part of the 1551-th to the 1600-th zero of the Riemann zeta function [HM-63]; column (f) an equally spaced sequence of levels (picket fence). Columns (a) and (f) represent two limiting cases, maximum randomness and no randomness at all respectively.

Can one deduce some features just by inspecting Fig. I.8? Arrows indicate spacings $s_i = x_{i+1} - x_i$ which are smaller than $1/4$. The Poisson spectrum shows 12 arrows out of 49 spacings, the prime number "spectrum" shows 9 arrows, the Er spectrum only 2 arrows, the frequencies of the membrane 3 arrows, the zeros of $\zeta(s)$ no arrow and, of course, the picket fence no arrow. One therefore sees a statistical

(*) The spectra have been rescaled to the same spectrum span $[0, 49]$, thereby introducing an artificial rigidity (see below).

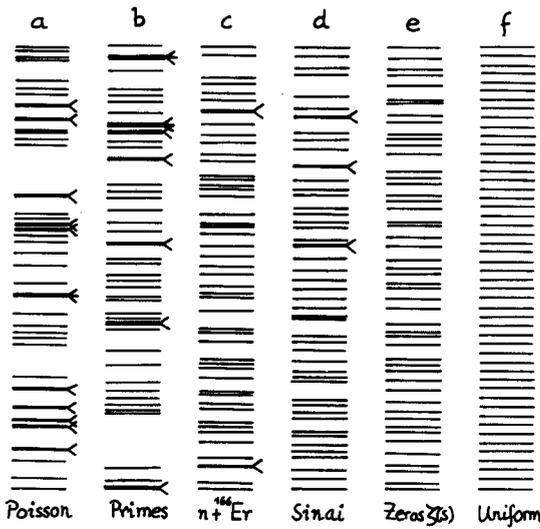


Fig.1.8 - Segments of "spectra", each containing 50 levels. The "arrowheads" mark the occurrence of pairs of levels with spacings smaller than 1/4. See text for further explanation.

similarity between (a) and (b) : large probability of small spacings and occurrence of some large spacings. On the contrary, (c) (d) and (e) show small probability of small and large spacings, the small probability of small spacings being usually referred to in the literature as the phenomenon of level repulsion. The spectrum (c), (d) and (e) deviate from (f) less strongly than (a) and (b). The picket fence (f) is a spectrum that we may qualify as absolutely rigid, in the sense that there is no departure at all from uniformity. Once the position of one level X_i is known, the position of any other level is determined, no matter how far it is from X_i . For this system the correlations between spacings are maximum and it shows perfect short and long range order. At the opposite extreme, the Poisson spectrum contains no correlations between spacings : the knowledge of a stretch of the spectrum puts no restriction on the behaviour of the spectrum beyond the interval considered (this is of course true irrespective of the form of the function $p(x)$ chosen in Eq.(1-30)). In intermediate situations between Poisson and the picket fence the degree of the spectral rigidity will depend on the nature and strength of the correlations between spacings.

Although this topic will be treated in greater detail in the next Section, let us already give some examples of characterization of fluctuation properties. We have mentioned before the spacing distribution $p(x)$ between adjacent levels. Let us reproduce a simple heuristic argument due to Wigner [Wi-56] that illustrates the presence or absence of level repulsion. Consider the probability $p(x)dx$ that, given a level at x_0 , the next level is in the small interval $dI = [x_0 + x, x_0 + x + dx]$ (see Fig. I.9). It can be represented as the product of two factors

$$p(x)dx = \text{Pr}(\text{one level in } dI / \text{no level in } I) \times \text{Pr}(\text{no level in } I) \quad , \quad (I-31)$$

where Pr means probability and $\text{Pr}(a/b)$ is the conditional probability of having a if b

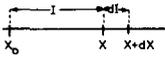


Fig.I.9

is true. One has

$$\text{Pr (no level in } l) = \int_x^{\infty} p(x') dx' \quad (\text{I-32})$$

and $\text{Pr (one level in } dl/\text{no level in } l) = \mu(x) dx$. Therefore

$$p(x) = \mu(x) \int_x^{\infty} p(x') dx', \quad (\text{I-33})$$

which can be solved to give

$$p(x) = C \mu(x) e^{-\int^x \mu(x') dx'} \quad (\text{I-34})$$

Now, for a Poisson sequence $\mu(x)$ is independent of x and one obtains $(\int p(x) dx = \int x p(x) dx = 1)$

$$p(x) = e^{-x} \quad x \geq 0 \quad (\text{I-35})$$

A linear level repulsion can be introduced by assuming $\mu(x) = \beta x$ and one obtains

$$p_w(x) = \frac{\pi}{2} x e^{-\frac{\pi}{4} x^2} \quad x \geq 0 \quad (\text{I-36})$$

The result (I-36) for the spacing distribution is known in the literature of random matrices^(*) as the Wigner surmise and it shows explicitly the level repulsion or tendency to avoid clustering of levels. Indeed, $p(x=0) = 0$ in contrast to the Poisson case, for which the corresponding spacing distribution (I-35) is maximum at the origin.

The spacing distribution $p(x)$ contains no information about spacing correlations. A simple measure of spacing correlations is the correlation coefficient C between, for instance, two adjacent spacings. Let $\{x_i\}$ be the sequence of levels and S_i the spacings $S_i = x_{i+1} - x_i$; C is given by

$$C = \frac{\sum_i (S_i - 1)(S_{i+1} - 1)}{\sum_i (S_i - 1)^2}, \quad (\text{I-37})$$

(*) (I-36) is a standard distribution in statistics, called a "Rayleigh distribution": Consider on the plane a point of cartesian coordinates (x, y) and take x and y as independent random variables normally distributed with zero mean and with the same variance $\sigma^2 = 2/\pi$. The probability density of the radius vector $r = \sqrt{x^2 + y^2}$ is (I-36).

where we assume the mean spacing equal to unity. For a Poisson spectrum, $C = 0$.

A convenient variable which is often used is $\Delta_3(L)$ [DM-63]. It measures, given an interval $[\alpha, \alpha + L]$ of length L , the least square deviation of the staircase $\hat{N}(x)$ from the best straight line fitting it (see Fig.I.10) :

$$\Delta_3(\alpha; L) = (L) \min_{A, B} \int_{\alpha}^{\alpha+L} [\hat{N}(x) - Ax - B]^2 dx \quad (I-38)$$

Notice that we are dealing with spectra for which the average part $\hat{N}_{av}(x)$ of $\hat{N}(x)$ is a straight line $y = x$. However, when considering a given interval $[\alpha, \alpha + L]$, the best straight line fitting $\hat{N}(x)$ will not just be $y = x$ but another (presumably close lying) straight line $Ax + B$ which is determined by (I-38). The value of $\Delta_3(L)$ for $L \gg 1$ for a picket fence spectrum is $1/12$ whereas the average $\langle \Delta_3(L) \rangle$ of $\Delta_3(\alpha; L)$ (take many non overlapping adjacent segments of length L of the spectrum, compute the value of Δ_3 for each segment and perform the average^(*)) for a Poisson spectrum is $L/15$. The departure of the average value of $\Delta_3(L)$ from the linear increase with L characteristic of the Poisson spectrum will give, in intermediate situations between Poisson and picket fence, information on the correlations between spacings and on the spectral rigidity or spectral stiffness.

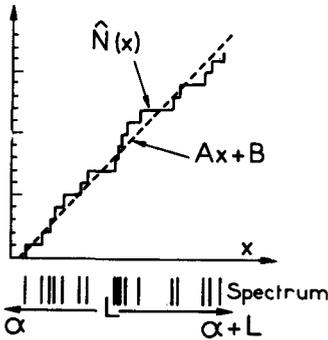


Fig.I.10- The Δ_3 -statistic of Dyson-Mehta

In Ref.[BG-75] a convenient way to compute $\Delta_3(L)$ has been given, once the ordered sequence of points x_1, x_2, \dots, x_n in the interval $[\alpha, \alpha + L]$ is known. Take as origin the center of the interval, i.e., take $\tilde{x}_i = x_i - (\alpha + \frac{L}{2})$; $\Delta_3(L)$ can then be obtained from

$$\begin{aligned} \Delta_3(\alpha; L) = & \frac{n^2}{16} - \frac{1}{L^2} \left[\sum_{i=1}^n \tilde{x}_i \right]^2 \\ & + \frac{3n}{2L^2} \left[\sum_{i=1}^n \tilde{x}_i^2 \right] - \frac{3}{L^4} \left[\sum_{i=1}^n \tilde{x}_i^2 \right]^2 \\ & + \frac{1}{L} \left[\sum_{i=1}^n (n-2i+1) \tilde{x}_i \right]. \end{aligned} \quad (I-39)$$

Another useful variable to be discussed is the "number statistic $n(L)$ "^(**): given an interval $[\alpha, \alpha + L]$ of length L , it counts the number of levels contained in the interval. It is a discrete variable which can take the values 0 (no level in the interval), 1 (one level in the interval), 2,3,... The average value of $n(L)$ is L , if the mean spacing is unity. We will consider higher moments or cumulants of $n(L)$ (variance $\Sigma^2(L)$, skewness $\gamma_1(L)$, excess $\gamma_2(L)$). Qualitatively we expect that if the

(*) when the spectrum is stationary, the average $\langle \Delta_3(L) \rangle$ does not depend on α .

(**) For the sake of simplicity and when no confusion is possible, we shall omit in the notation the α -dependence

spectrum is stiff, the variance of $n(L)$ will be small (in most cases the actual number of levels found in an interval of length L will differ only slightly from L) whereas for a non-rigid or compressible spectrum like Poisson the variance of $n(L)$ will be comparatively large. For a Poisson spectrum one has $\sum^2(L) = L$ which tells nothing but the familiar result that if one takes an interval of length L one expects to find $L \pm \sqrt{L}$ levels in the interval. At the opposite extreme, for the picket fence, one will have $L \pm 0$. Again we will be interested in what happens in intermediate situations.

BIBLIOGRAPHY

- For some general ideas and reflections on mathematical practice we recommend ref.[DH-80], a beautiful book addressed to the layman
- On prime numbers : ref.[Za-77], addressed to non mathematicians
- On the Riemann Zeta Function : ref.[Ed-74], a classic
- On properties of vibrating membranes : refs.[Ka-66,BH-76]
- On general ideas on fluctuations : Introductory Section of ref.[BFF-81].

II - RANDOM MATRIX THEORIES

The random matrix theories we shall describe have their origin in the following physical problem. In the low energy region of the excitation spectrum of a nucleus, the level density $\rho(E)$ is small and one expects to describe most of the states in a detailed way using nuclear models. However, due to the rapid increase of the level density with the excitation energy E

$$\rho(E) \approx \frac{C}{(E-\Delta)^{5/4}} \exp(a\sqrt{E-\Delta}) \quad (\text{II-1})$$

where C , Δ and a are for a given nucleus constants, by the time one reaches the region, for instance, of the neutron threshold ($E \sim 6$ MeV), the number of levels is so high that one must give up a description of microscopic detail, a description of the individual levels.

The aim of nuclear models at this and higher excitation energies is rather to describe special states, like giant resonances, analogue states, etc., which have a peculiar structure. But the detailed description of the sea of background states around the collective ones must and should be abandoned. Twenty years ago, Dyson made an eloquent resumé of the situation [Dy-62a] :

"Recent theoretical analyses have had impressive success in interpreting the detailed structure of the low-lying excited states of complex nuclei. Still, there must be a point beyond which such analyses of individual levels cannot usefully go. For example, observations of levels of heavy nuclei in the neutron-capture region give precise information concerning a stretch of levels from number N to number $(N+n)$, where N is an integer of the order of 10^6 . It is improbable that level assignments based on shell structure and collective or individual-particle quantum numbers can ever be pushed as far as the millionth level. It is therefore reasonable to inquire whether the highly excited states may be understood from the diametrically opposite point of view, assuming as working hypothesis that all shell structure is washed out and that no quantum numbers other than spin and parity remain good. The results of such an inquiry will be a statistical theory of energy levels. The statistical theory will not predict the detailed sequence of levels in any one nucleus, but it will describe the general appearance and the degree of irregularity of the level structure that is expected to occur in any nucleus which is too complicated to be understood in detail.

"In ordinary statistical mechanics a comparable renunciation of exact knowledge is made. By assuming all states of a very large ensemble to be equally probable, one obtains useful information about the over-all behaviour of a complex system when the observation of the states in all its detail is impossible. This type of statistical

mechanics is clearly inadequate for the discussion of nuclear energy levels. We wish to make statements about the fine detail of the level structure, and such statements cannot be made in terms of an ensemble of states. What is required is a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of a system but of the nature of the system itself. We picture a complex nucleus as a "black box" in which a large number of particles are interacting according to unknown laws. The problem is then to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally probable.^(*)

At the end of these lectures the reader will judge whether this programme, initiated by Wigner, has been successfully accomplished. And it will appear that partial justification of the theory may be found in concepts elaborated only recently.

The appropriate language to define an ensemble of systems is provided by random matrix theory. One considers the Hamiltonian matrix H as an $N \times N$ stochastic matrix (its matrix elements are random variables) and the question is to specify the probability density $\mathcal{P}(H)dH$. In order to proceed one must answer the following questions [Wi-67a,Dy-72]: i) what are the admissible Hamiltonians and what is the proper measure in the ensemble of these Hamiltonians ; ii) given the ensemble of Hamiltonians, are the properties in which we are interested common to the vast majority of them. Question i) is answered on the basis of general symmetry principles as well as of physical plausibility. Question ii) should be answered by the affirmative and one can then perform ensemble averages which are equivalent to averages over a generic matrix of the ensemble (ergodic property). A proper ensemble of random matrices should also fulfil the requirement that the mathematical problems involved should be tractable, in order to avoid situations in which one must exclusively rely on results obtained numerically (Monte Carlo). Of course, the final goal of the theory will be to successfully predict the empirical observations.

The systems we shall deal with are characterized by their Hamiltonians which

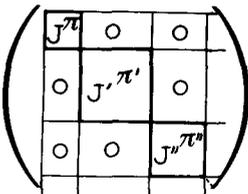


Fig.II.1

can be represented by Hermitian matrices. When there are some exact quantum numbers corresponding to exact integrals of motion, like angular momentum and parity (J^π), and if the basis states are labelled by these exact quantum numbers, the Hamiltonian matrix will split into blocks (Fig.II.1), and the matrix elements connecting different blocks will vanish. We shall assume that such a basis has already been chosen and restrict our attention to one of the diagonal blocks, an $N \times N$ Hermitian matrix in which N is a large integer, for the systems we want to describe contain many levels. The theoretical results are

(*)The underlining is ours.

in most cases derived in the limit of large N . If this asymptotic limit is reached very fast ($N \simeq 100$, for instance), the dimensionality will introduce no uncertainty when comparing to data.

II.1 GAUSSIAN ENSEMBLES [Po-65a,Me-67,Dy-62a,b]

The underlying space-time symmetries obeyed by the system put important restrictions on the admissible matrix ensembles. If the Hamiltonian is time-reversal invariant and invariant under rotations, the Hamiltonian matrices can be chosen real symmetric. If the Hamiltonian is not time-reversal invariant then, irrespective of its behaviour under rotations, the Hamiltonian matrices are complex Hermitian. Finally, if the system is time-reversal invariant but not invariant under rotations, and if it has half-odd-integer total angular momentum, the matrices are "quaternion real". In this last case all energy levels are doubly degenerate (Kramer's degeneracy).

Time Reversal Invariance	Angular Momentum	Rotation Invariance	Hamiltonian Matrix	Canonical Group	β	Number of Independent Real param.
Yes	Integer $\frac{1}{2}$ -Odd-Integer	- Yes	Real Symmetric	Orthogonal	1	$\frac{1}{2}N(N+1)$
	$\frac{1}{2}$ -Odd-Integer	No	Quaternion Real	Symplectic	4	$N(2N-1)$
No	-	-	Complex Hermitian	Unitary	2	N^2

Table II.1

Due to its importance in physical applications, we shall mainly concentrate in the case $\beta = 1$ (see Table II.1). Notice that the real symmetry property is preserved under orthogonal transformations, but not under a larger subgroup of the unitary transformations. The case $\beta = 4$ is included in Table II.1 for completeness, but no further mention will be made to it.

In order to introduce a proper measure dH in the space of matrices one can proceed as follows. A metric is defined in the matrix space to which H belongs by expressing

$$ds^2 = \text{Tr } \delta H \delta H^\dagger \quad (\text{II-2})$$

in terms of the independent variables X_μ of H , as

$$ds^2 = \sum_{\mu, \nu=1}^M g_{\mu\nu} \delta x_\mu \delta x_\nu, \quad (\text{II-3})$$

where M is the number of independent variables (matrix elements). The basic measure dH is then the measure induced by this metric, namely

$$dH = (\det g_{\mu\nu})^{1/2} \prod_{\mu=1}^M dx_\mu. \quad (\text{II-4})$$

If one considers ($\beta = 1$) real symmetric matrices $H (= H^* = H^T)$ there are $(1/2)N(N+1)$ independent variables and one has

$$ds^2 = \sum_{1 \leq i \leq N} (\delta H_{ii})^2 + 2 \sum_{1 \leq i < j \leq N} (\delta H_{ij})^2 \quad (\text{II-5})$$

so that

$$dH = 2^{N(N-1)/4} \prod_{1 \leq i \leq N} dH_{ii} \prod_{1 \leq i < j \leq N} dH_{ij}. \quad (\text{II-6})$$

For the case ($\beta = 2$) of Hermitian matrices $H (= H^+)$ one has N^2 independent variables. Writing $H = H' + iH''$ where $H' = H'^* = H'^T$, $H'' = H''^* = -H''^T$, one has

$$ds^2 = \sum_{1 \leq i \leq N} (\delta H'_{ii})^2 + 2 \sum_{1 \leq i < j \leq N} [(\delta H'_{ij})^2 + (\delta H''_{ij})^2] \quad (\text{II-7})$$

so that

$$dH = 2^{N(N-1)/2} \prod_{1 \leq i \leq N} dH'_{ii} \prod_{1 \leq i < j \leq N} dH'_{ij} dt''_{ij}. \quad (\text{II-8})$$

Any automorphism of the studied matrix space which leaves the metric (II-2) invariant will leave the associate measure dH invariant. For instance, in the case of real symmetric matrices, the invariance of (II-2) (II-5) under a real orthogonal transformation implies the invariance of (II-6).

Let us now define the Gaussian Orthogonal Ensemble (GOE) in the space of real symmetric matrices by demanding two requirements :

1. The ensemble is invariant under every orthogonal transformation

$$H' = W^T H W, \quad (\text{II-9})$$

where W is any real orthogonal matrix, i.e., the probability $\mathcal{P}(H)dH$ that a matrix H will be in the volume element dH (Eq.(II-6)) is invariant under orthogonal transformations^(*)

$$\mathcal{P}(H')dH' = \mathcal{P}(H)dH \quad (\text{II-10})$$

2. The various elements H_{ij} , ($i \leq j$) are independent random variables .

We make the first requirement because we don't want that any given state plays a particular role : all basis states, and therefore all states, should behave in the same way. The second requirement has no special physical origin. It is put for the sake of simplicity with the hope of leading to a mathematically soluble problem.

Similarly, the Gaussian Unitary Ensemble (GUE) in the space of Hermitian matrices is defined by the properties

1'. The ensemble is invariant under every unitary transformation

$$H' = U^+ H U \quad (\text{II-11})$$

where U is any unitary matrix, i.e., the probability $\mathcal{P}(H)dH$ that a matrix H will be in the volume element dH (Eq.(II-8)) is invariant under unitary transformations

$$\mathcal{P}(H')dH' = \mathcal{P}(H)dH \quad (\text{II-12})$$

2'. The various elements H_{ij}^r , ($i \leq j$), H_{ij}^i ($i < j$) are independent random variables, i.e., $\mathcal{P}(H)$ is a product of N^2 functions.

These two requirements (1. and 2. or 1'. and 2') determine uniquely the ensembles. The function $\mathcal{P}(H)$, which will also be invariant under the corresponding automorphism, can be written

$$\mathcal{P}_{NP}(H) = K_{NP} \exp \left\{ -\text{Tr}(H^2)/4\sigma^2 \right\}, \quad (\text{II-13})$$

(*)

Notice that from (II-10) and from the invariance of the measure dH , one has that $\mathcal{P}(H)$ must also be invariant under orthogonal transformations.

where $K_{N,\beta}$ is a normalization constant (N refers to the dimensionality of the matrix and $\beta = 1, 2, 4$ labels the different cases, see Table II.1). In (II-13)

$$\text{Tr } H^2 = \sum_{1 \leq i \leq N} H_{ii}^2 + 2 \sum_{1 \leq i < j \leq N} H_{ij}^2 \quad \text{for } \beta = 1 \quad (\text{II-14})$$

and

$$\text{Tr } H^2 = \sum_{1 \leq i \leq N} H_{ii}'^2 + 2 \sum_{1 \leq i < j \leq N} (H_{ij}'^2 + H_{ij}''^2) \quad \text{for } \beta = 2. \quad (\text{II-14}')$$

For the GOE one therefore has that each matrix element H_{ij} is distributed normally (or Gaussian distributed, from where the name Gaussian ensembles) with zero mean

$$\overline{H_{ij}} = 0 \quad i \leq j \quad (\text{II-15})$$

and variance

$$\overline{H_{ij}^2} = (1 + \delta_{ij}) \sigma^2 \quad i \leq j. \quad (\text{II-15}')$$

We use the notation $\overline{\cdot}$ for any quantity \mathbf{w} , $\overline{\mathbf{w}}$ is its ensemble average. For GUE one also has normal distribution of the different variables with

$$\overline{H_{ii}'} = \overline{H_{ii}''} = 0, \quad \overline{H_{ii}'^2} = 2\sigma^2 \quad i=j \quad (\text{II-16})$$

and

$$\overline{H_{ij}'} = \overline{H_{ij}''} = 0, \quad \overline{H_{ij}'^2} = \overline{H_{ij}''^2} = \sigma^2 \quad i < j. \quad (\text{II-16}')$$

We have thus defined two ensembles -GOE and GUE- depending on two parameters : a trivial scale factor σ^2 and the dimensionality N . Equation (II-13) clearly exhibits the statistical independence of the matrix elements.

Balian [Ba-68] has used a different and more general approach to derive (among others), the Gaussian Ensembles. He uses concepts borrowed from information theory, in particular the amount of information \mathcal{J} associated to the probability $\mathcal{P}(H)$

$$\mathcal{J} = \int \mathcal{P}(H) \ln \mathcal{P}(H) dH. \quad (\text{II-17})$$

One looks for the function $\mathcal{P}(H)$ that minimizes \mathcal{J} , which is equivalent to assuming the least possible knowledge about $\mathcal{P}(H)$. Before minimizing one has to face the following problem : in the cases we are studying, the range of variation of the matrix elements H_{ij} (or (H_{ij}', H_{ij}'')) is infinite. In order to confine the eigenvalues of H to a finite range, one has to impose a condition on its norm $(\text{Tr } H^2)^{1/2}$. That is, one asks

that $\mathcal{P}(H)$ should minimize \mathcal{J} subject to the constraints

$$\int \mathcal{P}(H) dH = C_1 \quad (\text{normalization}) \quad (\text{II-18})$$

$$\int \mathcal{P}(H) \text{Tr}(H^2) dH = C_2 \quad . \quad (\text{II-18}')$$

This leads to the result (II-13). When following this derivation of Gaussian Ensembles, the only input one needs is the invariant measure dH (plus the constraints) and the statistical independence of the matrix elements follows as a result.

Making use of the invariance properties, the joint probability density of the eigenvalues E_i can be extracted from (II-13) and one obtains :

$$\mathcal{P}_{N\beta}(E_1, E_2, \dots, E_N) = C_{N\beta} \exp\left\{-\frac{1}{4\sigma^2} \sum E_i^2\right\} \prod |E_i - E_j|^\beta, \quad (\text{II-19})$$

where $C_{N\beta}$ is a normalization constant determined by

$$\int \mathcal{P}_{N\beta}(E_1, E_2, \dots, E_N) dE_1 dE_2 \dots dE_N = 1 \quad . \quad (\text{II-19}')$$

$\mathcal{P}_{N\beta}(E_1, \dots, E_N) dE_1 \dots dE_N$ gives the probability, regardless of labelling ($\mathcal{P}_{N\beta}$ is a symmetric function under the interchange $E_i \leftrightarrow E_j$), of having one eigenvalue at E_1 , another at E_2 ..., another at E_N within each of the intervals $[E_j, E_j + dE_j]$. Equation (II-19) contains all the information concerning the eigenvalue distribution and the correlations among eigenvalues. By performing suitable integrations, one can in principle derive all the quantities related to eigenvalues which are of interest. Notice the last factor on the r.h.s. of (II-19) : when $E_i = E_j$, $\mathcal{P}_{N\beta}(E_1, \dots, E_i, \dots, E_j, \dots, E_N)$ becomes zero (there is level repulsion !). By integrating (II-19) over all the variables but one, one finds, in the limit of large N for the three cases $\beta = 1, 2, 4$, the ensemble averaged eigenvalue density $\bar{\rho}(E)$

$$\begin{aligned} \bar{\rho}(E) &= \int \dots \int \mathcal{P}_{N\beta}(E, E_2, \dots, E_N) dE_2 \dots dE_N \\ &= \begin{cases} \frac{1}{2\pi} \frac{1}{N\sigma^2} (4N\sigma^2 - E^2)^{1/2} & \text{for } |E| \leq 2\sqrt{N\sigma^2} \\ 0 & \text{for } |E| \geq 2\sqrt{N\sigma^2} \end{cases} \quad (\text{II-20}) \end{aligned}$$

which, from (II-19'), is normalized to unity. If one takes $2(N\sigma^2)^{1/2} = 2(\overline{\text{Tr}(H^2)/N})^{1/2}$ as energy unit, $(\pi/2)\bar{\rho}$ is a semi-circle and (II-20) is the so called Wigner semi-circle law

for the level density of a random matrix. Although it has been derived for large N , it is followed rather closely even for low values of N ($N \approx 20$). One should point out that it can be obtained under more general conditions than the ones fulfilled by the Gaussian Ensembles^(*). Let us finally remark that (II-20) does not reproduce the experimental level density (II-1). Whether this is a serious weakness of the theory or not will be discussed later in this Section.

II.2 FLUCTUATION PROPERTIES

Once the global behaviour of the eigenvalues has been established (the ensemble averaged eigenvalue density $\bar{p}(E)$ ^(**)) we turn now our attention to the level fluctuations or departures of the spectrum from its average behaviour. For that purpose one introduces the k-level correlation functions $\tilde{R}_k(E_1, E_2, \dots, E_k)$ defined as follows

$$\tilde{R}_k(E_1, \dots, E_k) = \frac{N!}{(N-k)!} \int \dots \int P_N(E_1, \dots, E_N) dE_{k+1} \dots dE_N \quad (1 \leq k \leq N) \quad (\text{II-21})$$

where P_N is given by (II-19). It follows that $\tilde{R}_1(E_1) = N \bar{p}(E_1)$, where $\bar{p}(E_1)$ is given by (II-20) and $\tilde{R}_N = N! P_N$ with P_N given by (II-19). $\tilde{R}_k(E_1, \dots, E_k) \times dE_1 \dots dE_k$ is the probability of finding one level, regardless of labelling, within each of the intervals $[E_j, E_j + dE_j]$. Like in the previous Section, to get rid of $\bar{p}(E)$, one introduces a new set of k-level correlation functions $R_k(x_1, \dots, x_k)$ and one considers the case of interest $N \gg k \gg 1$

$$R_k(x_1, \dots, x_k) = \lim_{N \rightarrow \infty} \frac{\tilde{R}_k(E_1, \dots, E_k)}{\tilde{R}_1(E_1) \dots \tilde{R}_1(E_k)} \quad (\text{II-22})$$

where x is obtained from the mapping $E \mapsto x$ defined by (see (I-28))

$$E \mapsto x = \int_{-\infty}^E \tilde{R}_1(E') dE' \quad (\text{II-22}')$$

The fluctuation properties of the levels are completely characterized by the set of functions $R_k(x_1, \dots, x_k)$. From the definition (II-22), one has $R_1(x_1) = 1$. This expresses the fact that the level density (or the average spacing) in the variable x

(*) For real symmetric matrices, for instance, one only needs the statistical independence of the matrix elements plus (II-15,15') for almost all the matrix elements.

(**) $N \bar{p}(E)$ plays in the context of matrix ensembles the role of $\rho_{av}(f) = \frac{d}{df} N_{av}(f)$ (see (I-26,28) in the previous Section).

is unity everywhere.

It is convenient to introduce the k-level cluster functions Y_k which are obtained from R_k by subtracting out the lower-order correlation terms

$$Y_k(x_1, \dots, x_k) = \sum_G (-)^{k-m} (m-1)! \prod_{j=1}^m R_{G_j} (x_t, \text{ with } t \text{ in } G_j). \quad (\text{II-23})$$

Here G stands for any division of the indices $[1, 2, \dots, k]$ into subgroups $[G_1, G_2, \dots, G_m]$. For instance

$$\begin{aligned} \text{--- } k=1 \quad m=1 \quad [(1)] \quad \text{and} \\ Y_1(x_1) = R_1(x_1) = 1 \end{aligned} \quad (\text{II-24})$$

$$\begin{aligned} \text{--- } k=2 \quad m=1 \quad [(1,2)] \\ m=2 \quad [(1)(2)] \quad \text{and} \\ Y_2(x_1, x_2) = -R_2(x_1, x_2) + R_1(x_1)R_1(x_2) \end{aligned} \quad (\text{II-24}')$$

$$\begin{aligned} \text{--- } k=3 \quad m=1 \quad [(1,2,3)] \\ m=2 \quad [(1)(2,3)], [(1,3)(2)], [(1,2)(3)] \\ m=3 \quad [(1)(2)(3)] \quad \text{and} \\ Y_3(x_1, x_2, x_3) = R_3(x_1, x_2, x_3) \\ - [R_1(x_1)R_2(x_2, x_3) + R_1(x_2)R_2(x_1, x_3) + R_1(x_3)R_2(x_1, x_2)] \\ + 2 R_1(x_1)R_1(x_2)R_1(x_3). \end{aligned} \quad (\text{II-24}''')$$

The inverse of (II-23) is

$$R_k(x_1, \dots, x_k) = \sum_G (-)^{k-m} \prod_{j=1}^m Y_{G_j} (x_t, \text{ with } t \text{ in } G_j). \quad (\text{II-25})$$

Thus each set of functions R_k and Y_k is easily determined in terms of the other. The advantage of the cluster functions is that they have the property of vanishing when any one (or several) of the separations $|x_i - x_j|$ becomes large. The function Y_k describes the correlation properties of a single cluster of k levels, isolated from more trivial effects of lower-order correlations.

It has been shown by Pandey [Pa-79] that the Gaussian Ensembles are stationary and ergodic in the limit of large N (this is obvious by construction for a Poisson ensemble). For instance, for one-point measures, the ensemble density $\bar{\rho}(\epsilon)$ is

equal to the spectral averaged density $\langle p(E) \rangle_S$. The mapping $E \mapsto x$ (Eq. (II-22')) (unfolding) is such that the ensemble-averaged local spacing $1/\bar{p}(x)$ is stationary (independent of x) and equal to unity. And the k -level cluster functions Y_k , functions of the variables x_j , are also stationary (they only depend on the relative coordinates $x_{ij} = x_i - x_j$). This means that the fluctuation properties of several segments of a spectrum located at different positions will be the same: from the point of view of fluctuations, the spectrum is translationally invariant. Furthermore, after unfolding, a spectral average is equal to an ensemble average.

An alternative way to characterize fluctuations consists to deal with spacing distributions^(*) and related quantities. In (II-21), instead of integrating from $-\infty$ to $+\infty$ without any restriction, one integrates some of the variables outside the interval $[\alpha, \alpha + L]$ whereas the others are integrated inside it. Assume that the unfolding (mapping $E \mapsto x$) has been performed. One defines ($N \gg k \gg 0$)

$$E(k; L) = \lim_{N \rightarrow \infty} \frac{N!}{(N-k)!} \int_{\text{in}} \dots dx_1 \dots dx_k \int_{\text{out}} \dots dx_{k+1} \dots dx_N P_N(x_1, \dots, x_N). \quad (\text{II-26})$$

If the system is stationary, $E(k; L)$ will be independent of α . $E(k; L)$ is the probability that in a sequence $\{x_i\}$ of levels with mean spacing unity an interval of length L taken at random contains exactly k levels. One useful aspect of the functions $E(k; L)$ is that they are very directly connected to the spacing distributions [MdC-72]

$$p(k; L) = (d^2/dL^2) \sum_{j=0}^k (k-j+1) E(j; L). \quad (\text{II-27})$$

In particular, for the nearest-neighbour spacing distribution $p(x)$

$$p(x) \equiv p(0; x) = (d^2/dx^2) E(0, x). \quad (\text{II-27}')$$

The probability $F(k; L)$ that in a sequence $\{x_i\}$ of levels with mean spacing unity, an interval $[x_\alpha, x_\alpha + L]$ of length L which starts at a level x_α contains exactly k levels is also given in terms of the functions $E(k; L)$:

(*) The distribution of nearest-neighbour spacings has been denoted, and will continue to be when no confusion is possible, by $p(x) \equiv p(0; x)$; $p(k; x)$ denotes the distribution of spacings $S = x_{i+k+1} - x_i$ between two levels x_i and x_{i+k+1} having k levels in-between ($k = 0, 1, 2, \dots$).

$$F(k; L) = - (d/dL) \sum_{j=0}^k E(j; L) \quad (\text{II-28})$$

and, in particular

$$F(0; L) = - (d/dL) E(0; L) . \quad (\text{II-28}')$$

From their meaning it is clear that

$$\sum_{k=0}^{\infty} E(k; L) = \sum_{k=0}^{\infty} F(k; L) = 1 \quad (\text{II-29})$$

and if there are no degeneracies

$$E(k; 0) = F(k; 0) = \delta_{k0} . \quad (\text{II-30})$$

When comparing a theoretical model with experiment, due to the limited number of high quality data available, it is impossible to make a complete comparison between the set of k-level cluster functions Y_k (II-23) and the corresponding quantities extracted from experiment. It is convenient to work with integrated quantities like

$$y_k(L) = \int_0^L \cdots \int_0^L Y_k(x_1, \dots, x_k) dx_1 \cdots dx_k . \quad (\text{II-31})$$

Consider, for instance, the random variable $n(L)$ "number of levels contained in an interval of length L " or, in short, the number statistic (see end of previous Section). Its first two moments and shape parameters^(*) are given by [Pa-78]

(*) Let ξ be a random variable whose probability density is $f(x)$. Its expectation value m_1 and central moments M_k ($k \geq 2$) are defined by $m_1 = \int f(x) dx$ and $M_k = \int f(x)(x-m_1)^k dx$. The characteristic function $\varphi(t)$ is the expectation value of the random variables $\exp(it\xi)$: $\varphi(t) = \int f(x) e^{itx} dx$. Consider the function $\psi(t) = \ln \varphi(t)$ called the cumulant generating function, because its

Mac Laurin expansion is given by $\psi(t) = \sum_{k=1}^{\infty} (K_k/k!) (it)^k$, where

K_k are the cumulants. They are related in a simple way to the centered moments M_k of ξ : $K_1 = m_1$, $K_2 = M_2$ (variance), $K_3 = M_3$, $K_4 = M_4 - 3M_2^2$, ... The shape parameters γ_k ($k \geq 1$) are defined by $\gamma_k = K_{k+2} / (K_2^{(k+2)/2})$.

γ_1 is called the excess and γ_2 the asymmetry of $f(x)$. For a Gaussian distribution $f(x) = (1/\sqrt{2\pi M_2}) \exp \left\{ - (x - m_1)^2 / 2M_2 \right\}$ one has $K_1 = m_1$, $K_2 = M_2$ and $K_\nu = 0$ for $\nu \geq 3$, i.e., $\gamma_k = 0$.

$$m_1(L) = \bar{n}(L) = \sum_{k=0}^{\infty} k E(k; L) = y_1(L) = L \quad (\text{II-32})$$

$$\Sigma^2(L) = M_2(L) = \overline{(n(L)-L)^2} = \sum_{k=0}^{\infty} (k-L)^2 E(k, L) = y_1(L) - y_2(L) \quad (\text{II-32}')$$

$$\gamma_1(L) = \left\{ \sum_{k=0}^{\infty} (k-L)^3 E(k; L) \right\} / M_2^{3/2}(L) = (y_1(L) - 3y_2(L) + y_3(L)) / M_2^{3/2}(L) \quad (\text{II-32}'')$$

$$\begin{aligned} \gamma_2(L) &= \left\{ \sum_{k=0}^{\infty} (k-L)^4 E(k; L) - 3M_2^2(L) \right\} / M_2^2(L) \\ &= (y_1(L) - 7y_2(L) + 6y_3(L) - y_4(L)) / M_2^2(L) . \end{aligned} \quad (\text{II-32}''')$$

Thus, Σ^2 , γ_1 and γ_2 are given in terms of the functions $E(k; L)$ and all values of k appear. However, they are in fact (2)-, (2+3)- and (2+3+4)- point measures respectively, as can be seen from the last equalities in (II-32). When dealing with the 2-level cluster function $Y_2(x_1, x_2)$ one uses the notation $Y_2(x)^{(*)}$. Y_2 is related to the spacing distributions by

$$1 - Y_2(x) = \sum_{k=0}^{\infty} p(k; x) \quad (\text{II-33})$$

and $(1 - Y_2(x)) dx$ gives the probability of observing a level in an infinitesimal interval dx at a distance x from a given level. An alternative form of (II-32') is

$$\Sigma^2(L) = L - \int_0^L (L-r) Y_2(r) dr . \quad (\text{II-34})$$

Finally, consider the least-square statistic $\Delta_3(L)$ introduced in the previous Section (I-38). It can be shown that its ensemble average $\bar{\Delta}_3(L)$ can be obtained as follows [Pa-79]

$$\bar{\Delta}_3(L) = (2/L^4) \int_0^L (L^3 - 2L^2 r + r^3) \Sigma^2(r) dr . \quad (\text{II-35})$$

Therefore, like $\Sigma^2(L)$, it is also a 2-point measure (some particular integral of Y_2).

(*) Take as coordinates the center of the interval $X = (x_1 + x_2) / 2$ and the relative coordinate $X = x_1 - x_2$. Then $Y_2(x_1, x_2) = Y_2(X, X)$ but as Y_2 does not depend on X , one simply writes $Y_2(x)$.

Its variance

$$\text{Var } \Delta_3(L) = \overline{(\Delta_3(L) - \bar{\Delta}_3(L))^2} \quad (\text{II-36})$$

is a (2+3+4)- point measure.

Let us now briefly describe the results obtained from different theoretical models. We shall mainly consider the GOE and also, for the sake of comparison and because it constitutes the limiting case of maximum randomness, the Poisson ensemble (an ensemble of sequences of points, not of eigenvalues of matrices).

Poisson Spectrum

- Correlation coefficient between adjacent spacings (I-37)

$$C = 0$$

- Functions E, F and spacing distributions (II-26,27,28)

$$E(k;L) = F(k;L) = p(k;L) = (L^k/k!) e^{-L} \quad (\text{II-37})$$

- k-level correlation functions (II-22)

$$R_k(x_1, \dots, x_k) = 1 \quad k \gg 1$$

- k-level cluster functions (II-23)

$$Y_1(x) = 1, \quad Y_k(x_1, \dots, x_k) = 0 \quad k \gg 2$$

and their integrals (II-31)

$$y_1(L) = L, \quad y_k(L) = 0 \quad k \gg 2$$

- Cumulants $K_\nu(L)$

$$K_\nu(L) = L$$

In particular (II-34)

$$\Sigma^2(L) = L$$

- Shape parameters (II-32)

$$\gamma_k(L) = L^{-k/2} \quad k \gg 1,$$

which means that for large L $\gamma_k \rightarrow 0$, i.e., $n(L)$ tends to be normally distributed.

- Average value of $\Delta_3(L)$ (II-35) [DM-63]

$$\bar{\Delta}_3(L) = L/15 \quad (\text{II-38})$$

and its variance (II-36) [BHP-83]

$$\text{Var } \Delta_3(L) = \left(1 + \frac{11}{30}L\right) \frac{L}{210} \tag{II-39}$$

Gaussian Orthogonal Ensemble

The derivation of the different quantities of interest, mainly due to Mehta, Gaudin and Dyson, needs beautiful classical mathematical techniques. The principal initial difficulty was encountered when trying to perform integrals of the type (II-21), difficulty that was solved by the method of integration over alternate variables invented by Mehta [Me-60]. One knows now, in closed analytical form, the different cluster functions. The functions related to the spacing distributions are only known (except for their behaviour for small and large values of the arguments) in numerical form and are connected to the spheroidal functions. For their relationship to the k-level correlation and cluster functions, see Ref.[MP-83].

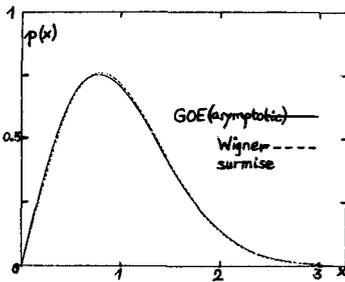


Fig.II.2 - The nearest neighbour-spacing distribution (taken from [Ga-61]).

Let us now give a brief account of the main results. On Fig.II.2 is reproduced the nearest-neighbour spacing distribution $p(x)$ (large N limit) and compared to the Wigner surmise $P_w(x)$ (I-36) derived in the previous Section. They are extremely close to each other. It is worth mentioning that $P_w(x)$ is the result corresponding to a GOE of $N \times N$ matrices of dimension $N=2$. The slope at the origin of $p(x)$ is $\pi^2/6$ to be compared to $\pi/2$ for $P_w(x)$. The two values differ by a factor of $\pi/3$, which is

close to 1.

On Fig.II.3 are reproduced the functions $E(0;L)$ and $F(0;L)$ (II-26,28').

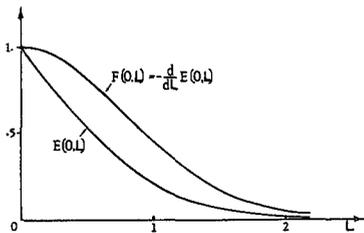


Fig.II.3 - The functions $E(0;L)$ and $F(0;L)$ for GOE

Notice that $F(0;L) - E(0;L) > 0$ for all $L > 0$. This means that the probability that in an interval of length L which starts at a level there is no level is larger than the probability that in an interval of the same length taken at random there is no level. This is actually what level repulsion means. This less conventional definition of level repulsion is sensible

not only for small values of L , but for any value of L . For a Poisson spectrum one has $F(0;L)=E(0;L)$: it is irrelevant whether the interval L starts or not at a level.

On Fig.II.4 are reproduced the functions $E(k;L)$ in the interval $0 \leq L \leq 5$. They can be used, for instance, to compute via (II-32) the values of $\gamma_1(L)$ and $\gamma_2(L)$ to be discussed later (this is a more practical method than to use the general expressions of k -level cluster functions). For adjacent spacings, the correlation coefficient (I-37) is $C = -0.271$, to be compared to 0 for a Poisson spectrum.

A closed expression has been given for the k -level cluster functions Y_k [Me-71,Dy-70]

$$Y_1(x) = 1$$

$$Y_k(x_1, \dots, x_k) = \frac{1}{2} \text{Tr} \sum_P [\sigma(x_{i_1}) \sigma(x_{i_2}) \dots \sigma(x_{i_k})] \quad k \geq 2 \quad \text{(II-40)}$$

where \sum_P denotes a sum over the $(k-1)!$ distinct cyclic permutations of the indices $(1, 2, \dots, k)$, where $x_{ij} = x_i - x_j$, and where σ is a 2-dimensional matrix given by

$$\sigma(x) = \begin{pmatrix} s(x) & Ds(x) \\ Js(x) & s(x) \end{pmatrix} \quad \text{(II-41)}$$

In (II-41) $s(x)$, $Ds(x)$ and $Js(x)$ are given by

$$s(x) = \sin \pi x / \pi x \quad \text{(II-42)}$$

$$Ds(x) = (d/dx) s(x) \quad \text{(II-42')}$$

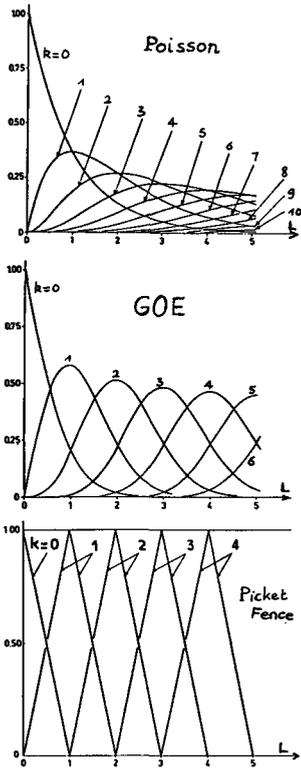


Fig.II.4 - The functions $E(k;L)$ (II-26) of a Poisson spectrum, a GOE and a picket fence (taken from [MdC-72])

$$Js(x) = Is(x) - \epsilon(x), \text{ where } \begin{cases} Is(x) = \int_0^{x'} s(x') dx' \\ \epsilon(x) = \begin{cases} 1/2 & x > 0 \\ 0 & x = 0 \\ -1/2 & x < 0 \end{cases} \end{cases} \quad \text{(II-42'')}$$

As a particular case one has

$$Y_2(r) = (S(r))^2 - Ds(r)Js(r) \quad (\text{II-43})$$

with $r = X_1 - X_2$. For small and large values of r one has

$$Y_2(r) \xrightarrow{r \rightarrow 0} 1 - (1/6)\pi^2 r + (1/60)\pi^4 r^3 + \dots \quad (\text{II-44})$$

$$Y_2(r) \xrightarrow{r \rightarrow \infty} 1/\pi^2 r^2 - (1 + \cos^2 \pi r)/\pi^4 r^4 + \dots \quad (\text{II-45})$$

The behaviour of $Y_2(r)$ for small values of r ($Y_2(0)=1$) is responsible for the level repulsion, whereas the large- r behaviour ($Y_2(r)$ tends to zero as $1/\pi^2 r^2$) determines the presence of long range order. On Fig.II.5 is displayed the function $1 - Y_2(r)$ in the range $0 \leq r \leq 2$.

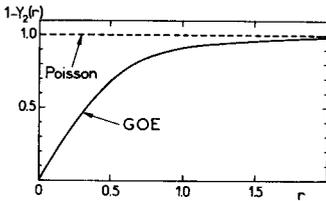


Fig.II.5 - Two-level cluster function for GOE

The value of $\sum^2(L)$ can be obtained by performing the integral in (II-34) with Y_2 given by (II-43). One obtains [DM-63,BFF-81]

$$\begin{aligned} \sum^2(L) = & (2/\pi^2) \left\{ \ln(2\pi L) + \gamma + 1 + \frac{1}{2} [Si(\pi L)]^2 \right. \\ & - \frac{\pi}{2} Si(\pi L) - \cos(2\pi L) - Ci(2\pi L) \\ & \left. + \pi^2 L [1 - (2/\pi) Si(2\pi L)] \right\} \quad (\text{II-46}) \end{aligned}$$

$$\begin{aligned} & \xrightarrow{L \gg 1} (2/\pi^2) \left\{ \ln(2\pi L) + \gamma + 1 - \pi^2/8 \right\} \\ & = \frac{2}{\pi^2} \ln L + 0.44 \quad (\text{II-46}') \end{aligned}$$

In (II-46) γ is the Euler constant and Si and Ci are the sine and cosine integrals. Note the effect of the spectral rigidity: the value of $\sum^2(L)$ is only of the order of unity for $L=100$ and even for $L \sim 10^6$ the fluctuation is not more than a couple of levels. This should be compared to $\sum^2(L) = L$ for the Poisson case.

The ensemble average $\bar{\Delta}_3(L)$ of $\Delta_3(L)$ can be obtained by integrating numerically (II-35) [HPB-82]. For $L \gtrsim 15$ one may use the asymptotic (large L) result [DM-63]

$$\bar{\Delta}_3(L) \sim \frac{1}{\pi^2} \left[\ln(2\pi L) + \gamma - \frac{\pi^2}{8} - \frac{5}{4} \right] = \frac{1}{\pi^2} \ln L - 0.007 \quad (\text{II-47})$$

with very good accuracy. Again, in comparison with the Poisson value $L/15$, the GOE spectrum is seen to exhibit long range order.

The variance of $\Delta_3(L)$ has been obtained by Monte Carlo calculations [HPB-82]. For large L , it approaches the asymptotic (large L) value [DM-63]

$$\text{Var } \Delta_3(L) \sim \frac{1}{\pi^4} \left[\frac{4\pi^2}{45} + \frac{7}{24} \right] = 0.012, \quad (\text{II-48})$$

a small constant ! By comparing to the Poisson result (II-39), once again it is seen that the fluctuations are small for GOE.

Although in what follows we shall be interested in the GOE, let us briefly mention that the fluctuation properties of the eigenvalues of Hermitian matrices of the Gaussian Unitary Ensemble (GUE, $\beta = 2$) are also known^(*). The level repulsion is stronger for GUE than for GOE : in both cases the spacing distribution $p(x)$ vanishes at the origin, but near the origin $p(x) \sim (\pi^2/6)x$ for GOE whereas for GUE it is $p(x) \sim (\pi^2/3)x^2$. The expression of the k-level cluster functions reads as in (II-40) except that the factor $(1/2)\text{Tr}$ should be dropped and $\sigma(x)$ is given by $S(x)$ (II-42). The resulting two-level cluster function $Y_2(r)$ contains only the first term of the r.h.s. of (II-43). The variance $\Sigma^2(L)$ of $n(L)$ is, for $L \gg 1$, half the variance corresponding to the GOE plus $1/8$, i.e., a GUE spectrum is more rigid than a GOE spectrum.

We have mentioned in the previous Section the conjecture that the fluctuations of the imaginary part of the zeros of the Riemann Zeta function are asymptotically (large imaginary part) identical to the fluctuations of a GUE spectrum. Indeed, one can see that the spectrum displayed in column (e) of Fig.I.8 shows a stronger level repulsion than the spectra of columns (c) and (d) which, as will be explained in the rest of this Section and in Section IV, are well described by the GOE.

II.3 COMPARISON OF GOE PREDICTIONS WITH EXPERIMENT

We shall concentrate on fluctuation properties of spectra and shall leave completely untouched the important subject of statistical properties of eigenvectors and its consequences (strength and width fluctuations) [BFF-81, MW-79, We-84]. In what follows we summarize the most significant comparisons performed so far between GOE predictions and nuclear, atomic and molecular spectra.

A large experimental effort has been and is still currently devoted to the measurements of positions and widths of resonances of the compound nucleus, effort motivated to a large extent by technological reasons. However, only a very limited part of the existing data, on neutron cross sections for instance, can be used for our purpose. What we want here is not a set of resonance energies but rather a

(*) And also of the eigenvalues of matrices belonging to the Symplectic Ensemble (see Table II.1, $\beta = 4$). The statistical properties of these eigenvalues are identical to those of an alternate series from the GOE.

complete and pure sequence of resonance energies corresponding to states having the same quantum numbers J^π (see Fig.II.1), in a given energy range. The very characteristic features of GOE fluctuations are rapidly lost if the sequence of levels is incomplete (missing levels) and/or polluted by spurious levels due to erroneous spin-parity assignments. Obviously, in order to test long range order effects, the pure series should contain many levels.

In the atomic and molecular cases nothing comparable to the nuclear case has been achieved, although recent and promising progress is to be noticed.

Nuclei

The main source of experimental data is provided by neutron resonance spectroscopy [Ly-68]. When transmission measurements are performed by sending neutrons on a nuclear target A, the study of the cross section σ_T as a function of the neutron energy E_n reveals the presence of sharp resonances, all over the periodic table, resonances that correspond to quasi-bound states of the compound nucleus (A+1) (see Fig.II.6 for an example). The mean spacings D and average widths Γ of the resonances near neutron threshold (excitation energy of the compound nucleus $\sim 6-7$ MeV) vary widely over the full range of nuclei. Very roughly $D \sim 100$ keV and $\Gamma \sim 10$ keV for light nuclei, $D \sim 1$ keV and $\Gamma \sim 100$ eV for medium nuclei and $D \sim 10$ eV and $\Gamma \sim 1$ eV for heavy nuclei. Most of the resonances observed in low-energy neutron cross sections are excited by s-waves, because for all but s-wave neutrons

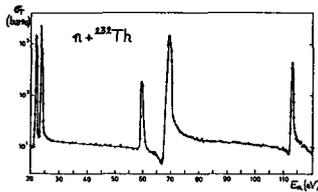


Fig.II.6 - Resonances for the reaction $n + {}^{232}\text{Th}$ (taken from the compilation Neutron cross sections (1964))

the penetration factor, which depends on the centrifugal barrier to be overcome by the entering neutron, is very small at low energies. Thus, if the target has $J^\pi = 0^+$ and E_n is very low, one populates only $(1/2)^+$ states of the compound nucleus, i.e., one produces pure sequences. The energy range that can be explored is severely limited for two reasons : i) by increasing E_n , the p-wave contribution increases and $(1/2)^-$ and $(3/2)^-$ states are also populated, the series becoming then polluted, ii) the experimental resolution being proportional to $E_n^{3/2}$, it rapidly becomes coarser than the resonance spacing. In practice this method has been extensively used, mainly by the group of Columbia University, to produce presumably pure series for a large variety of nuclei, each series containing typically of the order of 50-100 levels [HWR-78].

High resolution proton scattering experiments on medium nuclei ($A \lesssim 65$) performed in Duke University provide additional information. In this case one uses the analogue

state as an amplifying device : the fine structure states can be seen through the Coulomb force mixing of the analogue with the background sea of T_{\angle} states. The experiments are performed with proton energies of the order of a few MeV and the energy resolution attained is of the order of 10^{-4} [BLM-76].

We shall not give a complete account of comparisons between GOE predictions and nuclear data (see [BFF-81,HPB-82,BHP-83,BHP-84]) but rather present a few typical examples. The combined set of nuclear resonance-energy data of different nuclei- in short, the nuclear data ensemble (NDE)- is treated as a sampling of eigenvalues of GOE matrices. The data analyzed consist of 1762 resonance energies corresponding to 36 sequences of 32 different nuclei. In Fig.II.7 are shown the spacing distribution $p(x)$ and the average value of Δ_3 as a function of L in the range $L < 25$. In Fig.II.8 are shown the variance Σ^2 , asymmetry γ_1 and excess γ_2 of the distribution

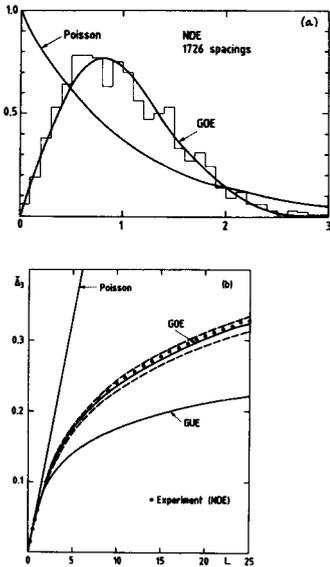


Fig.II.7 (a) Nearest-neighbour spacing histogram for NDE (experiment) ; (b) $\bar{\Delta}_3$ as a function of L ; dashed lines for GOE, to take into account finite sample size effects, correspond to one standard deviation from the average (taken from [HPB-82,BHP-83])

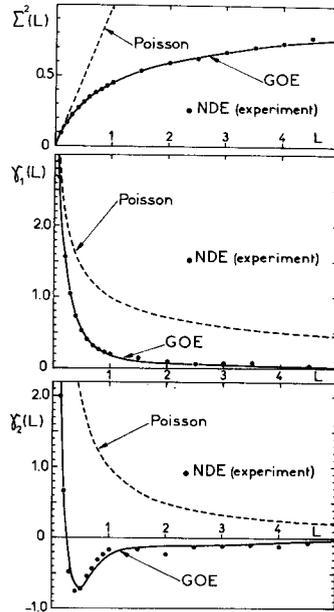


Fig.II.8 - (a) Σ^2 , (b) γ_1 , (c) γ_2 as functions of L (taken from [BHP-84])

of the number statistic $n(L)$ in the range $L < 5$. The procedure for calculations is to evaluate for each of the 36 sequences the spectral-averaged measure, say $\langle \Delta_3(L) \rangle_S$ for $\bar{\Delta}_3(L)$, and then take their average, weighted according to the size of each sequence.

Remember that Σ^2 and $\bar{\Delta}_3$ are derived from the 2-level cluster function (see (II-34,35)), whereas $\gamma_1(\gamma_2)$ derives from (2+3)-((2+3+4)-) level cluster functions respectively (see (II-32,32')). However, the restrictions imposed by $y_2(L)$ on higher-order functions $y_k(L)$ are not well understood and therefore it is not easy to know how much information not contained already in y_2 is contained in γ_1 and γ_2 .

Inspection of Figs.II.7 and II.8 illustrates the fact that all the fluctuation-measures considered so far, which include a thorough study of 2-point measures and to some extent more than 2-point measures as well, are fully consistent with the GOE predictions.

Atoms and Molecules

There exists a pioneering work by Rosenzweig and Porter [RP-60] in which atomic spectra were studied. They demonstrated that levels having the same quantum numbers do show level repulsion and that the spacing distribution $p(x)$ follows closely Wigner's prediction. It is only recently that Camarda and Georgopoulos [CG-83] have tested more systematically energy-level-fluctuation predictions of GOE with atomic spectra. These authors have analyzed energy levels of neutral and ionized atoms in the rare-earth region. In contrast with the nuclear levels studied so far, which correspond to unbound or quasi-bound states lying above particle threshold, the atomic states are bound and can decay only by photon emission. Eight different stretches of spectra containing altogether 269 atomic energy levels have been studied. The results are reproduced in Table II.2 and Fig.II.9. One can see that the agreement

	Nd	Nd	Nd ⁺	Nd ⁺	Nd ⁺	Sm ⁺	Sm ⁺	Tb	
J^π	4 ⁻	6 ⁻	7/2 ⁻	13/2 ⁻	15/2 ⁻	3/2 ⁻	9/2 ⁻	9/2 ⁻	
L	35	38	34	28	32	26	31	45	
$\bar{\Delta}_3(L)$ {	exp.	0.39	0.45	0.30	0.37	0.39	0.37	0.40	0.31
	GOE	0.35	0.36	0.35	0.33	0.34	0.32	0.34	0.38
		±0.11	±0.11	±0.11	±0.11	±0.11	±0.11	±0.11	±0.11

Table II.2 - Values of $\bar{\Delta}_3$ for atomic energy levels. Each series is identified by the angular momentum and parity (J^π). For GOE the value of $\bar{\Delta}_3$ is followed by the square root of Var Δ_3 (II-48) (taken from [CG-83]).

between GOE predictions and experiment is good.

Finally let us mention very recent studies on molecular spectra. Haller et al. [HKC-83] have studied the statistical behaviour of molecular vibronic energy

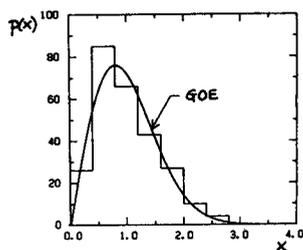


Fig.II.9- Histogram of nearest-neighbour spacing distribution $P(x)$ of eight sets of atomic energy levels containing 261 spacings (taken from [CG-83]).

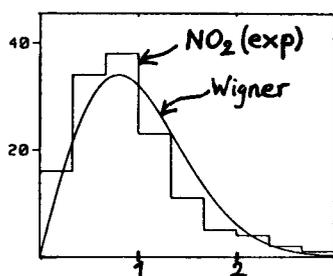


Fig.II.10 - Nearest-neighbour spacing histogram for energy levels of NO_2 (taken from [HKC-83])

levels. They have analyzed the interval $14900\text{--}19500\text{ cm}^{-1}$ above the ground state of the spectrum of the small polyatomic molecule NO_2 . On Fig.II.10 is reproduced the spacing distribution $P(x)$ corresponding to the 140 measured levels in this energy range. It is qualitatively in agreement with the Wigner prediction. However, whereas 140 levels are found, calculations predict 201 levels of a given vibronic symmetry. In ref. [HKC-83] it is suggested that there are deficiencies in the data and that the sub-interval $16600\text{--}17300\text{ cm}^{-1}$, which contains 46 levels, is most reliable. For this interval the value of

Δ_3 computed from the measured spectrum is 0.38 to be compared with the GOE prediction (II-47,48) 0.38 ± 0.11 and the value of the correlation coefficient (I-37) $C(\text{exp})$ is -0.32 to be compared to the GOE prediction -0.27 ± 0.15 .

We mention also the work of Mukamel et al. [MSP-84] who have studied level fluctuations from recent experimental data on highly vibrationally excited acetylene. The number of lines in the experiment is 65. However, due to insufficient spectral resolution, among other effects, one cannot attempt a sharp comparison between GOE fluctuations and data. The theory can, with the present quality of the data, be used to estimate the fraction

of missing levels in the experiment. But one can expect that this type of analysis will become more useful when the spectral resolution will improve.

II.4 DISCUSSION

The conclusion that can be drawn from the previous subsection is that, when systematic and accurate data are available allowing a stringent comparison with the theory, the GOE describes level fluctuations remarkably well (nuclei). For atoms the agreement is significant and for molecules the comparison is still at a primitive stage. On the other hand, we know that GOE gives for the level density a semi-circle

(II-20) in contrast with the observed increase of the level density with excitation energy (II-1). In other words, local predictions (fluctuations) are in agreement with observation whereas global ones (average or smooth behaviour, see eq.(I-26)) are not. Is this a serious drawback of the theory ? The following statement provides a guideline : "In general, events distinguished by a great disparity in size have little influence on one another, they do not communicate and so phenomena associated with each scale can be treated independently" [Wi-79]. In our context, we have a density/fluctuation separation (I-26) ([BFF-81,FK-82,Dy-72]) and GOE should be considered as a model for fluctuations^(*). Remember that all quantities (Eqs.(II-22,26)) used to describe fluctuations have been rescaled through the mapping $E \mapsto X$ (Eq.(II-22')), that we have been considering the behaviour of k points in the case $N \gg k \geq 1$, and that GOE-fluctuations have been proved to be stationary (translational invariant). We should also mention that methods have been proposed to construct ensembles of matrices with a prescribed eigenvalue density $\bar{\rho}(E)$ and the expectation is that local properties are independent of $\bar{\rho}(E)$ [Ba-68,Dy-72].

The question now is : are GOE-fluctuations specific of GOE or, on the contrary, are they really shared by other random matrix ensembles ? To answer this question, let us briefly describe other ensembles of random matrices and first the Circular Ensembles introduced by Dyson [Dy-62a]. Dyson wanted to avoid the following unsatisfactory feature of the Gaussian Ensembles (see (II-13)) : a uniform probability distribution on an infinite range being impossible to define, some arbitrary restriction of the magnitudes of the matrix elements is inevitable (II-18'). But then all interactions in GOE are not equally probable. By a small formal change, he introduced the following idea. Instead of thinking in terms of the eigenvalues E_i of the Hamiltonian matrix H , think in terms of eigenvalues $e^{i\theta_i}$ ($0 \leq \theta_i \leq 2\pi$) of a unitary matrix S connected to H , say, by

$$S = \exp(iZ H) . \quad (\text{II-49})$$

The interest of this approach is that it leads to ensembles of unitary matrices S for which the probability is uniform. For instance, using Balian's information theory approach, one proceeds as follows : Minimize (II-17) in the space of symmetric unitary matrices with the constraint (II-18). One then obtains that $\mathcal{P}(S)$ is a constant. And instead of (II-19) one has^(**)

$$P_{N\beta}(\theta_1, \dots, \theta_N) = C_{N\beta} \prod_{1 \leq i < j \leq N} |e^{i\theta_i} - e^{i\theta_j}|^\beta \quad (\text{II-50})$$

The ensemble averaged density of points θ_i on the unit circle resulting from (II-50) is constant.

(*) In a different context, is Eq.(I-13) not contradicting the density/fluctuation separation ? In fact one needs of the order of X terms in (I-13) to "see" primes which are of the order of X .

(**) For $\beta = 2$, the ensemble is defined in the space of unitary matrices ; for $\beta = 4$, in the space of selfdual unitary quaternion matrices.

The fluctuation properties of the Circular Ensembles are stationary by construction and it has been shown that, as expected, in the limit of large N they give the same fluctuation properties as the corresponding Gaussian Ensembles (orthogonal, unitary and symplectic) [Dy-70, Me-71].

In the preceding, Eq.(II-49) was suggested as one possible device to define a relation between S and H . In fact the precise connection between S and H needs not be specified. Assume only that S is a function of H so that the angles θ_j are functions of E_j . Over a small range of angles, the relation between θ_j and E_j will be approximately linear and the basic statistical hypothesis is that the fluctuation properties of k consecutive levels of H will be the same as the ones of k consecutive angles on the unit circle.

There are other matrix ensembles having GOE-fluctuations. For instance the "random-sign ensemble", which is an ensemble of real symmetric matrices with matrix elements generated by coin tossing ($H_{ij} = \pm 1$), whose eigenvalue distribution is the semi-circle level density. Or Dyson's ensembles \tilde{H} [Dy-71, BFF-81]

$$\tilde{H} = \sum_{i=1}^M A_i^T A_i, \quad (\text{II-51})$$

where the A_i are independent real asymmetric random matrices whose matrix elements are normally and independently distributed with zero mean and the same variance. A particular case of H is an ensemble proposed by Wigner [Wi-72]

$$H = \text{Re } M^* M, \quad (\text{II-52})$$

where M is an asymmetric complex matrix, the real and imaginary parts of the matrix elements being sampled normally and independently with zero mean and the same variance.

Shell-model nuclear spectra, calculated by diagonalizing a realistic (non-random) Hamiltonian in some finite subspace generated by putting n nucleons in Ω single-particle orbits, do also show GOE fluctuations [BFF-81]. In this case the average eigenvalue density is not a semi-circle but a Gaussian distribution. After adequately unfolding the spectrum, one can study the fluctuations and two examples of spacing distributions are given on Fig.II.11. In one case (Fig.II.11a) only levels having the same angular momentum and parity have been included and $p(x)$ is in agreement with the GOE-spacing law. When all levels (having different angular momenta) are included (Fig.II.11b), the spacing correlations are destroyed and the spacing distribution follows a Poisson law as expected^(*). Does this mean that the GOE correctly

(*)Consider the following example : on the interval $[0,1]$ take n points $x_i (i=1, \dots, n)$ at random uniformly distributed and construct an infinite spectrum by attaching to each point x_i a picket fence of unit spacing. The resulting spectrum, in the limit of large n , is a Poisson spectrum. So, by superposing most ordered spectra, one ends up with a Poisson spectrum.

models a shell model matrix ? This is certainly not true for in GOE, except for the symmetric nature of the matrices, all the matrix elements are independent random variables, whereas in the shell model, the Hamiltonian being a 2-body operator, the

matrix elements of H between n -particle-states ($n > 2$) are linear combinations of 2-body matrix elements.

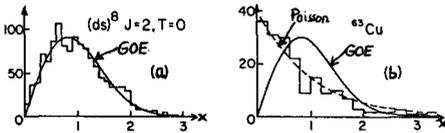


Fig.II.11 - Spacing histogram from two shell model spectra calculated with realistic interactions for ^{24}Mg [SZ-72] and ^{63}Cu [Wo-70] : (a) all states included have the same quantum numbers $J^\pi = 2^+$. (b) all states from a given spectrum span have been included ; it contains states with $J^\pi = 1/2^-, 3/2^-, \dots, 19/2^-$. (Taken from ref.[BFF-81]).

To follow closely the shell model, the microscopic model "par excellence" in nuclear physics, and to render matrix ensembles physically more plausible, a statistical extension of the shell model has been proposed : the "Two-Body Random Ensemble" (TBRE) introduced in refs.

[FW-70,BF-71] and generalized by French and collaborators ("Embedded Gaussian Orthogonal Ensemble" (EGOE) [MF-75,BFF-81]). Consider a ν -body Hamiltonian acting in an n -particle space ($n \gg \nu$), the n identical fermions being distributed over Ω distinct single-particle states ($\Omega \gg n$). The ensemble of ν -body Hamiltonians is defined by

$$H = \sum_{\alpha < \beta} w_{\alpha\beta} A_{\alpha}^{+}(\nu) A_{\beta}(\nu) , \tag{II-53}$$

where $A_{\alpha}^{+}(\nu)$ is the operator creating ν particles in single-particle states labelled by $\alpha_1, \alpha_2, \dots, \alpha_{\nu}$

$$A_{\alpha}^{+}(\nu) = C_{\alpha_1}^{+} C_{\alpha_2}^{+} \dots C_{\alpha_{\nu}}^{+} , \tag{II-54}$$

and $A_{\beta}(\nu)$ is the Hermitian conjugate of $A_{\beta}^{+}(\nu)$. The ν -body matrix elements $w_{\alpha\beta}$ are taken as statistically independent random variables normally distributed with zero mean and variances

$$\overline{w_{\alpha\beta}^2} = (1 + \delta_{\alpha\beta}) \sigma^2 . \tag{II-55}$$

When $n = \nu$ (II-53) reduces to GOE with a matrix dimension $N = \binom{\Omega}{\nu}$. For $n > \nu$, the dimensionality of the n -particle space is $\binom{\Omega}{n}$ and the matrix elements $\langle n\lambda | H | n\lambda' \rangle$ are linear combinations of the ν -body matrix elements $w_{\alpha\beta}$

$$\langle n\lambda | H | n\lambda' \rangle = \sum_{\alpha \leq \beta} C_{\alpha\beta}^{\lambda\lambda'} W_{\alpha\beta}, \quad (\text{II-56})$$

where

$$C_{\alpha\beta}^{\lambda\lambda'} = \langle n\lambda | A_{\alpha}^{+}(\nu) A_{\beta}(\nu) | n\lambda' \rangle \quad (\text{II-57})$$

and λ, λ' label the different n -particle states. The "realistic" case corresponds to $\Omega \gg n \gg \nu = 2$. It has been shown, using techniques which do not require the derivation of the joint probability density of eigenvalues $P(E_1, \dots, E_n)$, which is not known for EGOE, that, in the limit of a "dilute system" $\Omega \gg n \gg \nu \gg 1$, the eigenvalue density $\bar{\rho}(E)$ is Gaussian [MF-75, Ge-72]. For $n > \nu = 1$ the fluctuations are of Poisson-type but for $\nu \gg 2$ the methods used to derive $\bar{\rho}(E)$ fail in deriving fluctuation properties. However, there is strong numerical evidence coming from Monte-Carlo calculations that the fluctuation properties for $\nu \gg 2$ are identical to GOE-fluctuations^(*). Very recently Verbaarschot and Zirnbauer, using the method of replicated variables combined with field-theoretic tools, have obtained the first analytical evidence that EGOE spectra show long-range order [VZ-84].

One can get some insight on the generality of GOE-fluctuations by the following argument. Consider a (non-random) Hamiltonian matrix H_0 corresponding, for instance, to a realistic nuclear Hamiltonian containing all sorts of non-statistical features (large pairing and quadrupole components, etc.) and perturb it with GOE matrices

$$H = H_0 + \lambda V_{\text{GOE}} \quad (\text{II-58})$$

It has been shown [Pa-81a] that there is a rapid transition from H_0 -fluctuations to GOE-fluctuations when λ increases. Intermediate fluctuation patterns are to be expected only when the random-matrix elements are of the order of the local average spacing of the given (non-random) matrix H_0 (see also [ZVW-83]).

In summary, we have seen on the one hand that the spectra of very different systems (nuclei-light or heavy-, some atoms, maybe some molecules), when properly scaled, show identical fluctuation patterns. The scale (average spacing) covers five or six orders of magnitude, when going from a medium nucleus to a complex atom or molecule. Notice that one is considering extremely different systems, some of them governed by strong short range interactions and others by the Coulomb long range force. On the other hand, these characteristic fluctuation patterns, although

^(*) If one deals with bosons instead of with fermions, there is numerical evidence that one also has GOE-fluctuations [Ma-83]

not specific of, are well reproduced by GOE. Presumably they are shared by broad classes of models. Thus, a simple picture emerges : there exists a universality of level fluctuation laws, as well from the experimental than from the theoretical point of view. Notice also that GOE is in some respects a disappointing model : although it predicts beautifully the observed level fluctuations, it does not provide a hint on the origin of its success. It rather looks like a mathematical device that mimics perfectly the observed level fluctuations. Wigner says : "The assumption is that the Hamiltonian which governs the behaviour of a complicated system is a random symmetric matrix, with no particular properties except for its symmetric nature" [Wi-67b]. But what is, then, a complicated system ? And are the laws that we have been discussing of purely quantal nature or do they have a classical counterpart ? With the hope of finding some clues on the origin of the success of the theory as well as on its domain of validity, let us do in the next section a long excursion in the realm of classical Hamiltonian systems, the emphasis being put on the concepts of simple and complicated, or of regular and irregular systems.

BIBLIOGRAPHY

- A good general review can be found in [Wi-67a]
- In [Po-65] are reprinted all the important papers on random matrix theory published before 1965. The volume is preceded by a good and rather detailed introduction by C.E. Porter
- [Me-67] is the classical text on the subject. The emphasis is on mathematical techniques and derivation of exact results
- [BFF-81] is a recent and complete review article, containing mainly developments since [Po-65] and [Me-67], with emphasis on both theory and applications
- A review article (Bohigas, Haq and Pandey) on the comparison with experiment of random matrix predictions for level fluctuations is in preparation.

III - AN INTRODUCTION TO CLASSICAL CHAOTIC MOTION

The aim of this section is to introduce the concept of chaos in classical mechanics of conservative systems.

Giving a rigorous presentation of the material would require a highly technical language, and elaborate mathematical tools. Our purpose is rather to present an elementary intuitive approach to the subject, in order to get a physical insight into the main ideas. The material of this section is far from being exhaustive, and many fundamental questions have been omitted (such as perturbation theory, bifurcating orbits, mechanism of destruction of tori, dissipative systems, etc.).

In what follows, we limit ourselves to the study of classical Hamiltonian systems which are conservative and time-reversal invariant ; moreover, we consider only initial conditions for which the motion can take place only in a bounded region of the phase space.

III.1 FROM REGULAR TO CHAOTIC MOTION

All conservative Hamiltonian systems with N degrees of freedom have in common three essential properties :

- i) for a given set of initial conditions, the dimensionality of the accessible surface in phase space is less or equal to $(2N-1)$; since the system is conservative, the energy is constant along this "energy surface".
- ii) From Liouville's theorem, we know that the volume element in phase space is conserved. In other words, the Hamiltonian flow, which preserves the measure in phase space, is incompressible.
- iii) Trajectories in phase space cannot cross.

Apart from these features which are shared by all systems, the motion in phase space can exhibit a great variety of behaviours. For instance, one may ask how does a given volume element evolve with time : does it tend to cover the whole energy surface S_E as time goes to infinity or does it remain in a restricted part of S_E ? Does it conserve approximately its initial shape, or does it display more or less dramatic deformations with time ? According to the answers to such questions, one can define a hierarchy of regularity for dynamical systems. As we shall see now, the most regular systems, lying at the bottom of this classification, can be used as clocks, whereas, at the opposite side, the most chaotic systems^(*) can be used as random number generators.

(*)We speak here of dynamical systems in the enlarged sense of area-preserving mappings (see Sect.III.2).

Regular systems

Roughly speaking, a regular dynamical system is such that it can be integrated by quadratures. Let us be more precise, and define what are the conditions and the consequences of "extreme regularity". For a more rigorous account of the subject, see Refs. [Ar-76, AA-67].

We consider a time-independent Hamiltonian system with N degrees of freedom :

$$H = H(\underline{q}, \underline{p}) \quad , \quad \underline{q} = \{q_1, \dots, q_N\} \\ \underline{p} = \{p_1, \dots, p_N\} .$$

The equations of motion, written in Hamiltonian form, are :

$$\begin{cases} \dot{\underline{q}} = \vec{\nabla}_{\underline{p}} H(\underline{q}, \underline{p}) \\ \dot{\underline{p}} = -\vec{\nabla}_{\underline{q}} H(\underline{q}, \underline{p}) . \end{cases}$$

Definition

A time-independent Hamiltonian system with N degrees of freedom is said to be "integrable" if there exist N constants of motion $F_m(\underline{q}, \underline{p})$ (one of them being H itself) which are analytic functions of \underline{q} and \underline{p} , single-valued, functionally independent, and in involution (for the Poisson bracket) : $\{F_n, F_m\} \equiv 0 \quad \forall n, m = 1, \dots, N$.

Theorem (Liouville-Arnold)

If a system is integrable, then

i) there exist a canonical transformation to action-angle variables :

$$(q_1, \dots, q_N ; p_1, \dots, p_N) \mapsto (\theta_1, \dots, \theta_N ; I_1, \dots, I_N)$$

such that the Hamiltonian, expressed in the new variables, depends only on the actions :

$$H(q_1, \dots, q_N ; p_1, \dots, p_N) \mapsto \tilde{H}(I_1, \dots, I_N) = \tilde{H}(\underline{I}) .$$

The action variables are constants of motion :

$$\dot{\underline{I}} = 0 \quad ,$$

and the evolution of the angle variables is given by :

$$\begin{aligned} \dot{\underline{\theta}} &= \frac{\partial \tilde{H}}{\partial \underline{I}} = \underline{\omega}(\underline{I}) , \\ \underline{\theta}(t) &= \underline{\omega}(\underline{I}) t + \underline{\alpha} . \end{aligned} \quad (\text{III-1})$$

ii) For each set of initial conditions, the accessible surface in phase space is a compact manifold (we consider only bounded trajectories) having the topology of an N-dimensional torus T^N (*). These tori are called "invariant tori", because a trajectory starting on one of them remains on it for ever.

According to this theorem, the motion of an integrable system is restricted to an N-dimensional surface, instead of a (2N-1)-dimensional energy surface for a generic system having no other constant of motion than the energy. Let us notice that all conservative systems with one degree of freedom are integrable (provided they satisfy the smoothness conditions on H) : the accessible phase space is of dimension one (tori are reduced to circles). For integrable systems with two degrees of freedom, the existence of a second integral of motion reduces the dimension of accessible phase space from three to two.

Each value of \underline{I} defines the torus T^N , whereas the vector $\underline{\theta}(t)$ gives the position of the trajectory on the torus at each time t. If the frequencies ω_i are mutually commensurable i.e. if there exist (N-1) independent relations

$$\underline{\omega} \cdot \underline{m} = 0 , \quad (\text{III-2})$$

where the vector \underline{m} has integer (positive or negative) components and $\underline{m} \neq 0$, then the trajectory is closed on the torus ; in this case, the motion in phase space takes place on a one-dimensional region, and the motion is periodic. Systems having this property exhibit therefore the strongest degree of regularity, as an "ideal clock" would do.

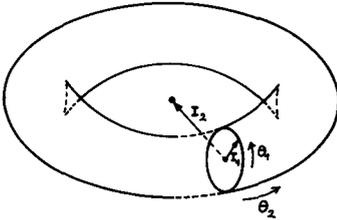


Fig.III.1 - An invariant torus of an integrable system with two degrees of freedom ; the actions I_1 and I_2 are the radii of the two circles defining the torus T^2 , and the angles θ_1 and θ_2 define a point on T^2 .

(*) An N-dimensional torus T^N is a direct product of n circles. A point on T^N can be defined by N angular coordinates $(\theta_1, \dots, \theta_N)$ [see Fig.III.1]. The torus is often represented as an N-dimensional hypercube defined by $\{(\theta_1, \dots, \theta_N) : 0 \leq \theta_i \leq 2\pi\}$.

At the opposite, if no relation of the type (III-2) holds, the orbit never closes, but densely covers the torus after infinite time : such a spiralling orbit is "ergodic" on the torus (see footnote (**)) -called "irrational", or "non-resonant" torus-, and exhibits a strong regularity. Intermediate cases where p independent relations ($0 < p < N-1$) like (III-2) hold can also occur ; orbits then lie on $(N-p)$ -dimensional manifolds of T^N .

It should be noticed that in the generic integrable case (see below the harmonic oscillator and the Kepler motion as exceptions), the frequencies depend on the actions (which define the invariant tori), i.e. on the set of initial conditions fixing the values of the N first integrals of the motion. Consequently for a generic integrable system, there exist simultaneously -corresponding to different sets of initial conditions- non-resonant tori, covered by ergodic trajectories and resonant tori ; measure theoretic arguments show that for such systems, the irrational tori form a set which is almost everywhere dense [Ar-76]. In other words, almost all the tori of a generic integrable system are irrational, in the same way as almost all real numbers are irrational.

Examples of regular systems

i) for $N=1$, we already saw that all conservative systems are regular (therefore periodic)

ii) for $N=3$, all systems submitted to a central field force $V(r)$ are regular : \vec{L} and L_z (orbital moment) are conserved, together with H .

Two particular systems of this kind are of special interest :

- for the Kepler motion ($V(r) = -\frac{k}{r}$) :

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2 \sin^2 \theta} - \frac{k}{r} ,$$

The Hamiltonian is well known to write, in terms of the action variables $I_i = \oint p_i \cdot dq_i$ (each integral is over the period corresponding to the q_i) as :

$$\tilde{H} = \frac{-2\pi^2 m k^2}{(I_r + I_\theta + I_\varphi)^2}$$

- for the isotropic harmonic oscillator ($V(r) = \omega^2 \frac{r^2}{2}$), the Hamiltonian takes the form

$$\tilde{H} = \omega (I_1 + I_2 + I_3) .$$

In both cases, for all values of the actions, i.e. for all initial conditions, the three frequencies coincide; the orbits are closed, with period $\tau = \frac{2\pi}{\omega}$. This means that for these two particular systems, all tori are rational (no orbit covering densely a torus). To understand why such a situation is an exception, one has to remember Bertrand's theorem [Be-1873] : Consider the motion of a point-mass under the action of a spherically

symmetric potential $V(r)$, supposed to be a C^3 function of r . Then a necessary and sufficient condition for all the bounded trajectories to be closed is that $V(r)$ behaves like r^2 or $(-\frac{1}{r})$. As is well known, the origin of this particular behaviour of spherical harmonic oscillator and Kepler motion is the existence of a further first integral ("dynamical invariance"), due to the particular shape of the potential, whereas the other three independent first integrals are merely consequences of rotational invariance. For this reason, these systems are called "overintegrable" systems. For Kepler motion, the extra integral is the Runge-Lenz vector, which is carried by the focal axis. Another peculiarity of Kepler motion and harmonic oscillator lies in the fact that they are "separable" in orthogonal coordinate systems. Most of the integrable systems we are used to consider possess this property ; however, separability is a very special case of integrability. In fact, no general method is available to know whether a system is integrable or not. Moreover, even if one system is known to be integrable, finding action-angle variable is not guaranteed !

- The system of a particle in a parallelepipedic box of sides a, b, c is also a regular system ; the energy writes

$$E = \frac{\pi^2}{2m} \left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2} + \frac{I_3^2}{c^2} \right)$$

iii) The N -dimensional (anisotropic or isotropic) oscillator of frequencies ω_i ($i=1, \dots, N$) is also a separable regular system :

$$\tilde{H} = \underline{\omega} \cdot \underline{I}$$

iv) There are some other systems which are known to be regular (see below regular billiards) : these are all the systems which are solved by quadratures in textbooks or articles... but they are not so numerous !

Regular systems had, up to one or two years ago, a so strong monopoly in textbooks, that one might believe their properties as being generic of any classical systems [Fo -83]. The fact is that regular systems are an exception among dynamical systems, and that, for other kinds of systems, many difficult questions remain open.

Ergodic systems

The most popular definition of ergodicity is the equality of time averages and phase space averages. In more abstract language, one often defines an ergodic system as a dynamical system whose phase space is metrically indecomposable under the Hamiltonian flow. Let us get in some more detail into these two equivalent definitions of ergodicity,

and quote some important consequences.

Let us consider a conservative dynamical system, of phase space X . Let $x(t) = [q_1(t), \dots, q_N(t), p_1(t), \dots, p_N(t)]$ be any point of X . The evolution of x with time can be described by an operator T :

$$\begin{aligned} X &\xrightarrow{T_t} X : \\ x &\longmapsto T_t x = x_t . \end{aligned}$$

The transformations T_t have trivially the property of forming a continuous group T depending on one parameter t . This group is often called the "dynamical group" or the "Hamiltonian flow".

The Liouville measure of a subset V of X (which is the volume of V) :

$$\mu(V) = \int_V dq_1 \dots dq_N dp_1 \dots dp_N$$

is, according to Liouville's theorem, invariant under the Hamiltonian flow :

$$\mu(T_t V) = \mu(V) . \quad (\text{III-3})$$

For a conservative system, one can define a surface element of the energy surface S_E as

$$d\sigma = d\mu \|\vec{g} \cdot \text{grad } H\|_{H=E}$$

where

$$\|\vec{g} \cdot \text{grad } H\|^2 = \sum_{i=1}^N \left[\left(\frac{\partial H}{\partial p_i} \right)^2 + \left(\frac{\partial H}{\partial q_i} \right)^2 \right] ,$$

and get from (III-3) the invariance of the induced measure σ under the group T . In the following, the measure σ will be normalized to one (i.e. $\sigma(S_E) = 1$).

The system is said to be ergodic, or metrically transitive (or indecomposable) if it is indecomposable into non trivial subsets which are invariant under T , i.e. :

$$\nexists A \subset S_E \quad (\sigma(A) \neq 0, \sigma(A) \neq 1) \quad \text{such that} \quad T_t A = A . \quad (\text{III-4})$$

The fact that the only subset of S_E of non-zero measure which is invariant under T is the whole energy surface S_E (or any subset of S_E having the same area as S_E itself) intuitively suggests that a typical trajectory cannot be confined in a restricted region of

phase space, but, on the contrary, explores the whole energy surface (except for a set of zero measure)^(*). In fact, an even stronger property holds for ergodic systems : one can show that almost every orbit explores almost every point of the energy surface S_E (densely covers S_E), spending in any part A a time proportional to the area of A . Indeed, let $\alpha \in S_E$ and $t(\alpha, A, \tau)$ be the time that α spends in $A \subset S_E$ between instants 0 and τ . According to the ergodic theorem (Birkhoff-von Neumann), the limit

$$\lim_{\tau \rightarrow \infty} \frac{t(\alpha, A, \tau)}{\tau}$$

does exist for almost all α 's, and is equal to the area of A :

$$\lim_{\tau \rightarrow \infty} \frac{t(\alpha, A, \tau)}{\tau} = \sigma(A)$$

if the system is ergodic.

The equality of phase averages and time averages constitutes an alternative definition of ergodicity (i.e. is equivalent to metrical transitivity) ; it can be formulated as follows : a system is ergodic iff, for any integrable function $f(x)$ ($x \in X$, $\int f(x) d\sigma < \infty$),

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t dt f(T_t \alpha) = \int_{S_E} f(x) d\sigma \quad (\text{III-5})$$

for almost all $\alpha \in S_E$. Eq.(III-5) is often referred to as a version of the law of large numbers, the left hand side representing the infinitely many trials approaching the probability in the right hand side.

Equilibrium Statistical Mechanics is built on the ergodic hypothesis. But ergodicity is not sufficient for a system to reach an equilibrium state : one additional property -called mixing- is needed, which concerns the way any volume element evolves with time.

(*) It is clear that no trajectory can explore the whole energy surface (i.e. ergodicity in the sense of Boltzmann can never hold). Indeed, a trajectory which would pass through any point of S_E should intersect with itself, which is impossible. What we call ergodicity here is often referred to as the quasi-ergodic hypothesis in Statistical Mechanics.

Mixing systems

The concept of mixing was first introduced by Gibbs. Qualitatively, the underlying idea of mixing is the property, for any part A of the energy surface ($\sigma(A) \neq 0$), of getting spread uniformly over the energy surface as $t \rightarrow \infty$. To get an "experimental feeling" of this feature, let us imagine a shaker in which one puts 80 % of orange juice, and 20 % of vodka ; then, one continuously stirs the shaker. If the system is mixing, one will find, as time tends to infinity, 20 % of vodka in any volume of the liquid, i.e. the vodka will be uniformly mixed with the orange juice in any part of the cocktail. Of course, such a property implies for the initial volume of vodka (or of any part of the energy surface) being strongly deformed as time runs, i.e. transforming with time in a thinner and thinner filament which ultimately will be present in any volume of the liquid (or will densely cover the energy surface).

This intuitive description of mixing makes now the mathematical definition very easy to understand :

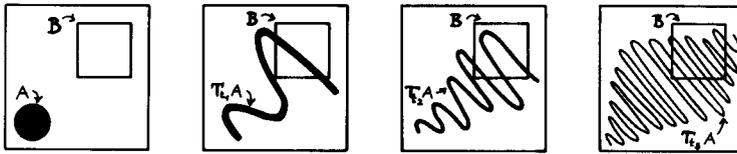


Fig.III.2 - Schematic representation of the mixing property : B is any fixed area of the surface energy, and A is a surface element considered at initial time t_0 , whose evolution is drawn for successive times $t_1 < t_2 < t_3$. As time is running, the Hamiltonian flow T_t tends to dilute the initial surface A uniformly in the whole energy surface. As time tends to infinity, the fraction of the test area B occupied by $T_t A$ is equal to the fraction of the energy surface S_E initially occupied by A .

A system is said to be mixing if

$$\forall A, B \subset S_E, \quad \lim_{t \rightarrow \infty} \frac{\sigma(B \cap T_t A)}{\sigma(B)} = \sigma(A). \quad (\text{III-6})$$

$(\sigma(A) \neq 0 \neq \sigma(B))$

Let us look at the right hand side of Eq.(III-6). One can write it as :

$$\sigma(A) = \frac{\sigma(T_t A)}{\sigma(S_E)},$$

since $\sigma(T_t A) = \sigma(A)$ from Liouville's theorem, and $\sigma(S_E) = 1$. So the r.h.s. of Eq.(III-6)

represents the fraction of the energy surface occupied by $T_t A$, while the l.h.s. represents the fraction of B occupied by $T_t A$. According to the uniform spreading of A over S_E , these two terms should be equal.

Some important consequences of mixing should be noticed. First, one can show that mixing implies ergodicity. Indeed, suppose that the mixing property holds; take for A an invariant set of S_E , and choose $B \equiv A$. Then $(B \cap T_t A)$ is simply A , and Eq.(III-6) implies that $\sigma(A)$ is zero or one, i.e. there does not exist any non-trivial subset A of S_E which is invariant by T_t . This is exactly the indecomposability property characterizing ergodic systems (see (III-4)).

Notice that the converse is not true, i.e. ergodicity does not imply mixing; indeed, one can imagine systems for which any surface element explores the whole energy surface without being deformed with time (see Fig.III.3 and example III-3-b-1);

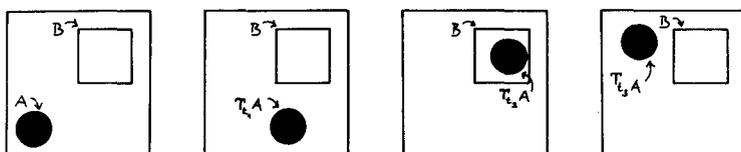


Fig.III.3 - Schematic representation of an ergodic system which does not have the mixing property. The surface element A tends to explore the whole energy surface as time is running, spending equal times in equal areas (ergodic property), but its shape remains unchanged, so that Eq.(III-5) (mixing property) cannot hold.

in such systems, the distance between two points of S_E remains of the same order of magnitude as time is running. On the contrary, in a mixing system, the spreading property allows two points which are initially close to each other to get as far from each other, in the limit $t \rightarrow \infty$, as is permitted by the constrain of staying on the energy surface. Thus, mixing implies instability with respect to initial conditions, or, in other words, loss of information with time, since a small error in measurement of $x(t=0)$ can propagate with time and eventually induce so large errors on $x(t)$ that long time predictions for the system are prohibited.

Finally, let us mention an important characteristic (i.e. which could be used as a definition) property of mixing, which is the decay of correlations between two functions: in mixing systems, any two square integrable function (for the measure σ) asymptotically become statistically independent, i.e.

$$\forall f, g \in L^2_\sigma(S_E), \quad \int f(x) g(T_t x) d\sigma \xrightarrow{t \rightarrow \infty} [\int f(x) d\sigma] [\int g(x) d\sigma].$$

As a particular case, let us take for g a non-equilibrium normalized distribution $\rho(\alpha)$:

$$\int f(\alpha) \rho(T_t \alpha) d\sigma \xrightarrow{t \rightarrow \infty} \int f(\alpha) d\sigma,$$

which means that, in average, the distribution $\rho(\alpha)$ tends to a uniform equilibrium probability density.

We shall not discuss the subtle relations of mixing with irreversibility. Let us only emphasize in this respect that the equations of motion allow to recover any point $\alpha(t=0)$ knowing $\alpha(t>0)$; however, all memory of the initial state is lost as time tends to infinity, and it is only in this limit that one can speak of irreversibility.

K-systems

Without no further assumption than mixing, one cannot give any quantitative information about the separation of orbits with time.

The so-called K-systems are mixing systems which possess so strong instability that most orbits starting from close points separate, in the average, exponentially with time. For such systems, the knowledge of $\alpha(t_i)$ for all discrete times $\{t_i\} = \{-\infty, \dots, t_0-1, t_0\}$ does not provide any useful information on the behaviour of the system for times $t > t_0$. Such a strange feature means that, though the system is completely deterministic -i.e. is governed by causal equations of motion-, the evolution of a generic point in phase space is very irregular, and exhibits some kind of random behaviour; this is why one associates with K-systems, for which the motion does not depend on their distant past history, the idea of unpredictability.

The mathematical definition of a K-system would go beyond the scope of our "qualitative" introduction to chaotic phenomena (see for instance Refs.[AA-67,Sh-73,Or-74] for a rigorous approach). Let us only give the main ideas of characterization of K-systems, and try to understand roughly in which sense one can say that the past does not determine the future. All the concepts used are borrowed from information theory [SW-49,Bi-78,ME-81]. First, one introduces any finite ordered partition P of the energy surface S_E into cells (atoms), in order to define a measurement : the result of the measurement of the system associated with P at time t_0 is the n° of the cell of P which is crossed by the orbit at time t ; then, an experiment associated with P is a sequence of measurements for equally spaced times going from t_0 to ∞ . Now, one can define the entropy $h(T,P)$ of the hamiltonian flow T relative to the partition P , which represents the mean rate (averaged over the whole sequence of measurements

of the experiment) of information^(*) generated by the time running for the experiment associated with P. Consider, for instance, a very regular system, whose motion is periodic in phase space : for such a system, the knowledge of a finite set of sequential measurements is sufficient to determine the outcomes of all further measurements, i.e. the information produced by all but a finite set of measurements is zero, and therefore the average of information generated for a given experiment is zero. For strongly irregular systems, even the knowledge of an infinite set of sequential measurements is not sufficient to know, with probability one, the outcomes of later measurements, which means that the average of information -or entropy- is positive.

K -systems are now defined as systems for which the entropy relative to any finite partition P is positive :

$$h(T, P) > 0 \quad \forall P .$$

These are therefore systems whose motion looks irregular and unpredictable whatever may be the experiments one can imagine. A quantity of interest to measure the degree of irregularity is the so-called metric entropy, or Kolmogorov-Sinai (K-S) entropy : it is defined as the maximum value of $h(T, P)$, taken over all the possible finite partitions P :

$$h_{K-S}(T) = \max_{P_i} h(T, P_i) \quad (\text{III-7})$$

It is clear from their definition that K-systems have positive K-S entropy. Let us emphasize that this entropy does not depend on any dynamical state under consideration, contrarily to the entropy commonly used in Statistical Mechanics, but is an intrinsic quantity associated to the dynamical system considered as a whole at a given energy E. This K-S entropy provides a measure of the strength of mixing of the system, and is related to the mean rate of exponential separation of trajectories.

Bernouilli systems

These are, among the K-systems, the most unpredictable ones. Indeed, they have the property that there exist a partition P_0 such that the sequence of measurements of the associated experiment are completely uncorrelated, as would be the outcomes of games with a roulette wheel ! This partition P_0 must also satisfy another condition

(*) Notice that the amount of information gained by making a measurement is equal to the lack of information (i.e. to the uncertainty) before making the measurement, which is also called the entropy associated with the experiment.

(whose statement would require entering in abstract mathematical considerations : see for instance [Sh-73] or [Pa-81]) in order for the system to be Bernouilli.

For a Bernouilli system, the K-S entropy is simply equal to the entropy of the particular partition P_0 , i.e.

$$h_{K-S}(T) = - \sum_i \mu_i \ln \mu_i, \quad (\text{III-8})$$

where μ_i is the measure of the i^{th} atom of P_0 (i.e. the area of the i^{th} cell of $P_0 \subset S_E$), and the sum runs over all the atoms of P_0 .

The different types of systems we have just introduced illustrate the transition from regularity to chaos, i.e. randomness produced by causal equation of motion. Several remarks should be made at this stage :

i) Part of the vocabulary introduced above ("ergodicity", "mixing") sounds familiar, and might evoke Statistical Mechanics. We emphasize, however, that all the considerations made in the preceding are not restricted to systems with a great number of degrees of freedom, but really can apply to very simple systems, eventually having only two degrees of freedom (see example in the following).

ii) For physical systems which are suspected to belong to a given member of the hierarchy just presented (except for certain integrable systems), it is in general extremely difficult to prove that indeed they are ergodic, or mixing, etc.. The main rigorous results available so far concern billiards, and we shall devote a particular attention to this kind of systems. Also are known interesting results for "abstract" dynamical systems, many of them being very "educative". Several examples will be presented in Sect.III.3, as illustrations of the rather abstract ideas introduced up to now.

iii) One question of interest is the following : can "most" of the dynamical systems be classified according to the preceding scheme, i.e. are physical systems, in their majority, either integrable, or ergodic ? There is at least a very peculiar^(*) kind of systems for which the answer to this question is trivially "no" : these are conservative systems having a number n of independent first integrals, with $n > 1$, but $n < N$; for such systems, the motion in phase space is restricted to a $(2N - n)$ -dimensional surface. Thus they are non-integrable systems which are not ergodic on the whole energy surface S_E . We can therefore reformulate the question as : "are conservative systems having no other constant of motion than the energy E ergodic ? Much attention has been paid to this problem during the first half of the century, and we recall in a few words the revolution produced in the 1960's by the so-called Kolmogorov-Arnold-Moser theorem.

(*) Here, the word peculiar refers to the rarity of such systems among the family of all dynamical systems ; however, these are generic systems for the physicist ! (isolated physical system possess other constants of motion than the energy).

Quasi-integrable systems

Conservative quasi-integrable systems are, roughly speaking, obtained by inducing a slight perturbation preserving the energy conservation on an integrable system. In 1892, Poincaré showed that, for such systems, apart from exceptional kinds of perturbations, all the constants of motions other than the energy are destroyed. It was probably this theorem which encouraged people to try to demonstrate -or at least to convince themselves- that most of the integrable systems can be made ergodic by small perturbations. (For instance Fermi [1923] and Birkhoff [1927] attempted to develop arguments in favour of this assumption : see Ref.[Mo- 73b]). In view of this, the result of Kolmogorov, Arnold, and Moser, now known as the KAM theorem, was a major turning point in the history of dynamical systems. Indeed, the main information of this theorem is that, under rather general conditions, "most" of the irrational tori persist (though slightly deformed) under a small perturbation.

From this fundamental result, one can easily imagine that, for a "generic" dynamical system, the structure of phase space is very complicated, showing both regular and chaotic regimes (there is an enormous gap between integrability and ergodicity !). In this respect, the above classification, according to increasing degree of chaos, is irrelevant for most systems, and may appear somewhat "academic". However, to study open problems such as the characterization of quantum chaos, it seems natural to begin with systems whose classical analogues are reasonably well understood -for instance K-systems-, even though they are known to be rare among the whole family of chaotic dynamical systems.

III.2 - MEASURE-PRESERVING MAPPINGS

III.2.a) Surfaces of section

For the sake of simplicity, let us restrict for a moment to physical conservative systems with two degrees of freedom. As already mentioned, it is far from being easy, even for these "simplest" systems, to study analytically their degree of irregularity. However, one can get for them quite useful informations from very simple numerical experiments, as we shall see now.

For the systems considered here, the accessible portion of phase space at a given energy E -the surface energy S_E - is a three-dimensional bounded surface. The method of Poincaré surfaces of section consists in choosing an "appropriate" two-dimensional surface Σ in the phase space, and look at the figure generated by successive crossings of one given orbit with Σ ; by "appropriate", we mean that

- i) the trajectory should cross the surface Σ an infinite number of times, as time

goes from zero to infinity

ii) the area bounded by any closed curve on Σ should be conserved with time (see Eq. (III-9)).

Let us illustrate the interest of the method by considering a particular surface Σ_y , defined in the following way. Let us suppose that the surface $\{x = 0\}$ intersects with the energy surface $H(x, y, p_x, p_y) = E$; we call this intersection Σ_y , which is a two-dimensional surface. A point on Σ_y can be defined by the coordinates (y, p_y) . The integral of motion $H = E$ defines p_x^2 as a function of (p_y, x, y) ; therefore, one can attach to a point on Σ_y a set of coordinates $(x=0, y, \epsilon|p_x|, p_y)$, with the sign $\epsilon = \pm 1$ of p_x depending on the sense with which a given trajectory crosses Σ_y . To get this sign unambiguous, we choose a given side of Σ_y , and look at the successive intersections Q_n of a given trajectory with Σ_y which, for instance, go out of the chosen side of Σ_y :

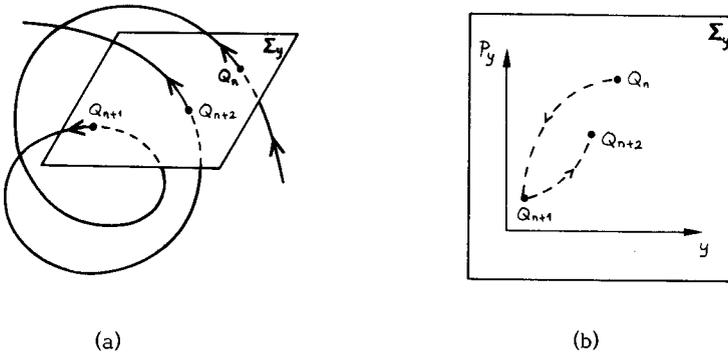


Fig.III.4 - (a) A trajectory crossing the surface of section Σ_y ; (b) Points of the Poincaré map on Σ_y

Now, the knowledge of the coordinates of Q_n on Σ_y completely specifies the whole set of coordinates of Q_n in phase space, i.e. the knowledge of one pair of conjugate variables on Σ_y completely determines the dynamical state. What has been lost by reducing the initial three-dimensional problem to a two-dimensional one is the "story" of the orbit between two successive crossings with Σ_y , but this is not essential for our purpose.

The repeated crossings with Σ_y define a mapping M :

$$\Sigma_y \xrightarrow{M} \Sigma_y$$

Given an initial condition $\underline{x}_0 = (x_0=0, y_0, p_{x0}, p_{y0})$ for a point on Σ_y , the study of the dynamical system is now reduced to the study of the set of points

$$\{ \underline{\xi}_n = M^n \underline{\xi}_0, \quad n=0,1,2,\dots,\infty \}.$$

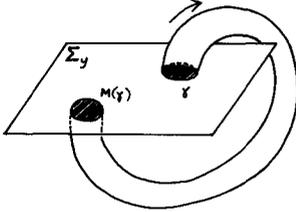


Fig.III.5 - Illustration of the property (III-9): Σ_y is a surface of section, γ a closed curve on Σ_y , and $M(\gamma)$ the image of γ on Σ_y under the Hamiltonian flow. The areas bounded by γ and $M(\gamma)$ are equal.

The property ii) stated above means that the mapping M is area (or measure)-preserving, i.e. that, given any closed curve γ on Σ_y , the area bounded by the image curve $M(\gamma)$ of γ by the mapping M should be equal to the area bounded by the curve γ :

$$\oint_{\gamma} p_y dy = \oint_{M(\gamma)} p_y dy. \quad (\text{III-9})$$

This can be easily shown from the Hamiltonian character of the motion (see e.g. Ref.[AA-67, Appendix 3]).

Now, let us look at the pattern generated on Σ_y by the points $M^n \underline{\xi}_0$, ($n = 0, 1, 2, \dots, \infty$), and see which kind of information we can learn from it. First, let us consider a system which has, besides the energy, another constant of motion F in involution with H (i.e. an integrable system)

$$F(x, y, p_x, p_y) = C, \quad H(x, y, p_x, p_y) = E. \quad (\text{III-10})$$

From (III-10) one can eliminate two of the canonical variables, say x and p_x , which yields

$$p_y = p_y(x, y). \quad (\text{III-11})$$

Since $x = 0$ on Σ_y , Eq.(III-11), "projected" on to Σ_y is the equation of a curve Γ , which is invariant under the mapping M , and which is simply the intersection of the 2-dimensional invariant torus T^2 with Σ_y . Therefore, the successive crossings of the orbit with Σ_y all lie on a unique smooth curve ($M(\Gamma) \equiv \Gamma$), for any set of initial conditions, which is a signature of integrability. Two possibilities can occur in such a case:

i) the invariant torus T^2 is resonant (or rational). Then, the motion is periodic on the torus, i.e. the orbit is closed, and has only a finite number k of crossings with Σ_y , which are all fixed points for M , since $M^k \underline{\xi}_0 = \underline{\xi}_0$ (see Fig.III.6).

ii) if, on the contrary, the invariant torus T^2 is irrational, the orbit densely covers the torus (is ergodic on the torus), and the points generated by successive iterations of the mapping will ultimately densely cover the invariant curve Γ (see Fig.III.7).

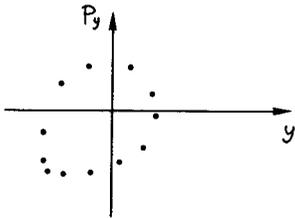


Fig.III.6 - Typical pattern of a Poincaré map of two-dimensional integrable system for initial conditions defining a rational torus

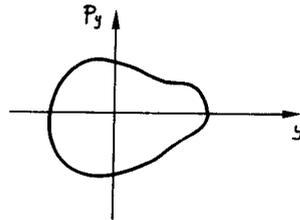


Fig.III.7 - Same as Fig.III.6, but for an irrational invariant torus

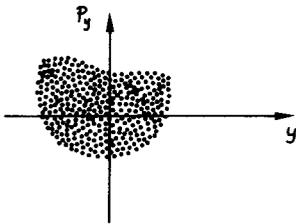


Fig.III.8 - Idealized pattern of a Poincaré map of an ergodic system

Let us next consider a dynamical system which is ergodic (on the energy surface). Then, almost all trajectories densely cover a three-dimensional region of phase space, and their intersections with Σ_y fill densely some area (see Fig.III.8).

We should notice that they may exist "non-typical" trajectories for ergodic systems such as, for instance, periodic trajectories (see examples for chaotic billiards in Sect.III.4); but these form a set of zero measure.

Between integrable and ergodic systems, there is a great variety of systems for which chaotic and regular regions coexist in phase space ; for such systems, the pattern of the Poincaré map shows both invariant curves and sparse points, as can be seen on the example of Fig.III.12. (It is usual to represent on the same Poincaré map impacts of several trajectories corresponding to different initial conditions).

Comments

i) The method of surfaces of section is highly efficient, and provides valuable informations on the structure of phase space. (The example of the Toda lattice presented below

nicely illustrates the success of the method. Many other examples can be found in Ref.[He-83]). However, it should, in any case, remain no more than a guide for further theoretical studies, and can never be used as a proof, neither of integrability, nor of ergodicity.

ii) Of course, the pattern gets more and more informative as the number of iterations increases. But the number of iterations necessary to get a reasonable image of the structure of phase space highly depends of the nature of the system, and of the choice of initial conditions. This is particularly true for strongly chaotic systems, for which a trajectory may happen to be trapped in a given region of phase space, and to escape only after a long time ; therefore, the true pattern appears only after a great number of iterations. In this respect, an "abusive" use of Poincaré maps for such systems might suggest that the system is not ergodic, since a typical trajectory for an ergodic system must have spent equal times in equal areas after an infinite time ; however, if a trajectory happens to be confined for a long time in a restricted region of the energy surface S_E , it will visit again this region much less frequently than other regions of S_E , in order to preserve the ergodic property in the limit $t \rightarrow \infty$.

iii) Among the limitations of the method, one should quote its inefficiency to detect any kind of instability of the system ; for instance, the method is completely inadequate to distinguish simple ergodicity from mixing or K-property. These properties are indeed very delicate to study, even by numerical means.

iv) More "exotic" possibilities than those presented before for the pattern of Poincaré maps are open for dissipative systems and will not be discussed here. For instance, the points may be restricted to lie on "fractal" sets of the surface of section (as can happen in strange attractors).

v) We presented the method only for systems with $N = 2$. degrees of freedom. It can, in fact, be generalized to any number N of degrees of freedom ; in this case, the surface of section is $(2N-2)$ -dimensional, which makes the study more difficult.

III.2.b) Abstract dynamical systems

An abstract dynamical system (X, μ, T) is usually defined given a space X , a measure μ , and a mapping T , in the following way. X is a measurable space with respect to the measure μ ; moreover, μ is a probability measure, i.e. $\mu(X) = 1$. T is a mapping $X \xrightarrow{T} X$, which is surjective, and which preserves the measure μ , i.e.

$$\mu(T^{-1}A) = \mu(A) \quad \forall A \subset X : \mu(A) \neq 0.$$

Moreover, the mapping T is often supposed to define a one parameter group of transformations T_t (the parameter may take continuous or discrete values). In this case,

T is one-to-one, and the measure-preserving property can also be expressed as

$$\mu(TA) = \mu(A) .$$

The most natural abstract dynamical system associated with a physical Hamiltonian system is such that X refers to the energy shell S_E , μ to the measure induced on S_E by the Liouville measure, and T is what we called the dynamical group, or Hamiltonian flow (see Sect.III.1) ; the property that μ is an invariant measure for T is simply the Liouville's conservation of surface on S_E . In this case the group T depends on a continuous parameter t , and the transformation T_t is a diffeomorphism on X (i.e. is a C^∞ automorphism on X). Another way of defining an abstract dynamical system given a physical Hamiltonian system is to take for X a surface of section Σ , for μ the normalized measure on it, and for T the mapping generating the successive crossings of an orbit with Σ . This time, the mapping is discrete, and its measure-preserving property is associated with the conservation of the area $\oint_{\Gamma \subset \Sigma} p dq$.

The discrete mapping associated with a Poincaré surface of section is in fact induced by the continuous dynamical flow, i.e. can be obtained by solving the differential equations of motion. But really abstract discrete mappings are defined by a "discrete time" transformation, which is simply a deterministic algorithm ; among them, the discrete area-preserving mappings of the plane are often considered. These are defined as follows : consider a compact surface X in the plane (x, y) with the measure $d\mu = dx dy$, and an application T :

$$X \xrightarrow{T} X$$

$$(x_n, y_n) \longmapsto T(x_n, y_n) = (x_{n+1}, y_{n+1}) .$$

The mapping T is area-preserving if

$$\det \begin{bmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{bmatrix} = 1 \quad (\text{III-12})$$

Examples of such mappings will be presented in the next section.

These notions of dynamical measurable space and measure-preserving transformation have been used as key points in Sect.III.1 to introduce the hierarchy of irregularity for physical Hamiltonian systems. It should be clear now that the definitions given above of ergodicity, mixing, K-property and B-property (Bernouilli) trivially extend to abstract dynamical systems.

III.3 EXAMPLES

Let us now illustrate the concepts just introduced by concrete examples.

III.3.a) Physical Hamiltonian systems

There are now numerous Poincaré maps of physical systems available. Among them, the Toda lattice and the Hénon-Heiles systems are probably the most striking illustrations of the interest of surfaces of section.

The Toda lattice [To-67,To-70] is the system of three particles moving on a ring with exponential repulsive forces between them :

$$H = \frac{1}{2} (P_1^2 + P_2^2 + P_3^2) + e^{-(\phi_1 - \phi_2)} + e^{-(\phi_2 - \phi_1)} + e^{-(\phi_3 - \phi_2)} - 3.$$

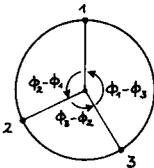


Fig.III.9 - The three-particle Toda lattice

This system possesses, besides the energy, another constant of motion :

$$P = p_1 + p_2 + p_3 = ct$$

associated with the invariance of H under a rigid rotation.

Its true number of degrees of freedom is thus reduced from three to two^(*). After some manipulation, it

is possible to eliminate one irrelevant dynamical variable,

and express the Hamiltonian in terms of two coordinates and their canonical momenta :

$$\bar{H} = \frac{1}{2} (P_x^2 + P_y^2) + \frac{1}{24} \left[e^{2y + 2\sqrt{3}x} + e^{2y - 2\sqrt{3}x} + e^{-4y} \right] - \frac{1}{8}. \quad (\text{III-13})$$

Now, the accessible phase space at a given energy $\bar{H} = E$ is located on the energy shell S_E , which is three-dimensional. To investigate the structure of the motion on

(*) In the given example, the existence of the extra-constant P reduces the dimension of the accessible phase space from five to three (i.e. the Toda lattice is truly a conservative system with two degrees of freedom). This is because the total momentum P can be considered -by means of an appropriate canonical transformation- as the conjugate momentum of an ignorable (or cyclic, or kinosthenic) variable, i.e. a coordinate which does not appear in the Hamiltonian function (see for instance Refs.[La-70,Go-51] for the general procedure of elimination of cyclic variables, and Ref.[LL-83] for the reduction of numbers of degrees of freedom of the Toda lattice).

S_E , Ford et al. [FST-73] studied the pattern of the motion on the Poincaré surface of section $\Sigma_y = (y, p_y)$:

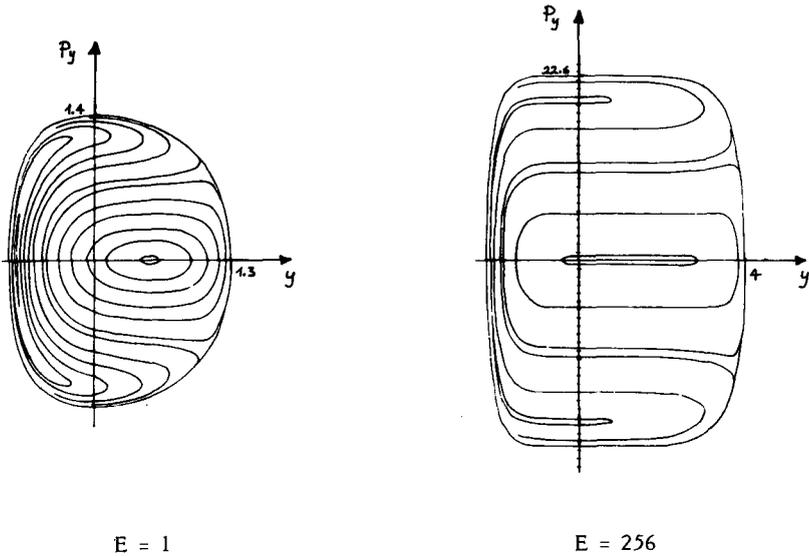


Fig.III.10 - Poincaré maps of the Toda lattice, at energies $E = 1$ and $E = 256$ (taken from Ref.[FST-73])

At any energy they studied, they got points of the Poincaré map lying on smooth curves, as if the system had, besides the energy, a further first integral. And indeed, another constant of motion was discovered by Hénon (He-74) and independently by Mikanov [Ma-74], whose explicit form is :

$$F = 8 p_x (p_x^2 - 3 p_y^2) + (p_x + \sqrt{3} p_y) e^{2y - 2\sqrt{3}x} - 2 p_x e^{-4y} + (p_x - \sqrt{3} p_y) e^{2y + 2\sqrt{3}x} = ct.$$

Thus the Toda lattice is an integrable system for any set of initial conditions. One can notice that in the limit $(x, y) \rightarrow (0, 0)$, the Hamiltonian (III-13) reduces to a two-dimensional isotropic harmonic oscillator Hamiltonian :

$$\bar{H}_0 = \frac{1}{2} (p_x^2 + p_y^2 + x^2 + y^2), \quad (\text{III-14})$$

and the second integral of motion becomes simply proportional to the orbital momentum :

$$F \longrightarrow 12 (y p_x - x p_y) .$$

$$\begin{array}{l} (x, y) \rightarrow (0, 0) \\ (p_x, p_y) \rightarrow (0, 0) \end{array}$$

Now, let us look at the Hénon-Heiles Hamiltonian [HH-64], which can be viewed as deduced from (III-13) by truncation after third order terms :

$$H = \frac{1}{2} (p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{y^3}{3} . \quad (\text{III-15})$$

This system is often considered as a model in several physical problems : vibrating three-atomic molecule, effective mean field (produced by other stars) description of the motion of a star in the galaxy. Contrarily to the Toda lattice, the Hénon-Heiles system has not a bounded energy surface S_E for all values of the energy E , but only for $E \leq \frac{1}{6}$. Fig.(III.1) shows the Poincaré map for the surface of section $\Sigma_y = (y, p_y)$ at $E = \frac{1}{12}$ (several trajectories are reproduced on the same map).

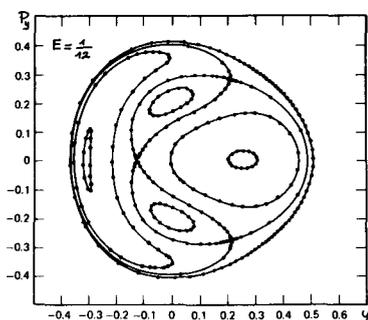


Fig.III.11 - Poincaré map of the Hénon-Heiles system at energy $E = \frac{1}{12}$ (taken from Ref.[HH-64])

For some initial conditions, the successive points again lie on a smooth curve ; but there are also points distributed erratically (which all correspond to the same set of initial conditions). Thus, a chaotic region of the energy surface has appeared by increasing the energy from $\frac{1}{12}$ to $\frac{1}{8}$: the truncation of the Toda lattice Hamiltonian

For each trajectory, the emerging pattern seems to be an invariant curve, and one may be tempted to conclude that the system is integrable. Therefore, Fig.(III.12) [obtained at $E = \frac{1}{8}$] is a surprise :

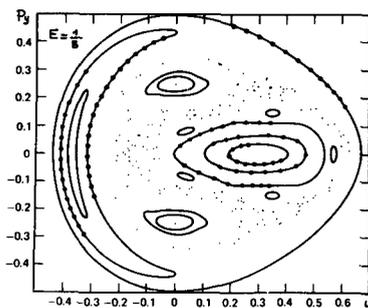


Fig.III.12 - Same as Fig.III.11, for $E = \frac{1}{8}$ (taken from Ref.[HH-64])

has destroyed integrability ! Looking next at $E = \frac{1}{6}$ (Fig.III.13), one sees that the chaotic region occupies now most of the area of the energy surface, and invariant curves are reduced to small islands.

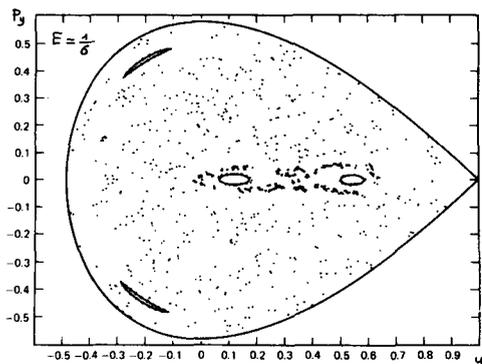


Fig.III.13 - Same as Fig.III.11, for $E = \frac{1}{6}$ (taken from Ref.[HH-64])

comment about Fig.(III.12). One may interpret the invariant curves as traces of distorted invariant tori which remain present after perturbing the integrable limit (III.14) of (III.15) ; however, the persistence of such tori cannot be predicted by KAM theorem, which requires (only as a sufficient condition) the frequency ratio of the unperturbed oscillator to be "sufficiently irrational", which of course is not true for the Hénon-Heiles system ($\omega_1 = \omega_2 = 1$).

III.3.b) Number-theoretic abstract dynamical systems

1) Rotations of the Circle

Consider a unit circle C (circumference length = 1), take some point on C as origin O , and choose a sense to define the arc joining any point A of C with O (i.e. to measure the abscissa of A). Define now the mapping of C on itself by the rotation

$$\left\{ \begin{array}{l} C \xrightarrow{T} C \\ x \in C \mapsto Tx = x + \omega, \end{array} \right.$$

where x is the abscissa of any given point of C , and where ω is any real number.

The Hénon-Heiles system is a very simple example showing nicely the complicated structure of the phase space, and the sudden changes of this structure by increasing the energy (More comments can be found in Ref.[He-83]). Many other examples have been considered, in particular those illustrating the mechanism of destruction of the invariant tori of a perturbed integrable system ; but these notions require more technical background than proposed here. We would just like to make the following comment

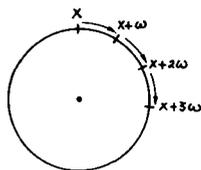


Fig.III.14 - Rotations of the circle

This transformation can also be viewed as the mapping

$$\begin{cases} X = [0,1[& \xrightarrow{T} [0,1[\\ x \in X & \longmapsto \text{Frac}(x + \omega), \end{cases} \quad (\text{III-16})$$

where "Frac" denotes the fractional part of a number. The transformation T clearly preserves the Lebesgue measure $d\mu = dx$ on X . Without loss of generality, one can assume that $0 < \omega \leq 1$.

Suppose first that ω is a rational number, which can be written as the irreducible fraction $\omega = \frac{p}{q}$ (p and q are prime to each other). One can easily see that the mapping generates exactly q different points -which therefore are fixed points- $Tx, T^2x, \dots, T^q x = x$, i.e. the initial point x is recovered for the first time at the q^{th} iteration: every orbit is periodic, and the system is not ergodic.

Take next ω irrational; now, $T^n x$ ($n = 1, 2, \dots$) takes infinitely many different values. It can be shown that, this time, the mapping is ergodic for the Lebesgue measure (see for instance [AA-67]). It is clear, however, that T is not mixing, since two points initially close to each other remain so for ever: the shape of any volume in X -or length in this case- remains strictly unchanged by the successive iterations. This mapping is therefore an example of a dynamical system which is ergodic, but not chaotic in the sense that it is perfectly stable with respect to initial conditions (it is intuitively evident that the entropy of this system is zero).

The preceding considerations can be extended without difficulty to the translations^(*) of the N -dimensional torus T^N . The problem is reformulated by replacing x and ω by vectors \underline{x} and $\underline{\omega}$, the space X_N is now the N -direct product $[0,1[\times [0,1[\times \dots \times [0,1[$, and the measure is the Lebesgue measure $dx_1 \dots dx_N$ on X_N . One gets readily the generalization of the preceding result: if the torus T^N is irrational (see Sect.III.1), the mapping is ergodic on T^N . Notice that this mapping can be viewed as the discrete time evolution of a integrable Hamiltonian system on an irrational torus according to Eq.(III-1). Indeed, $T^n \underline{x} = n\underline{\omega} + \underline{x}$ ($n = 1, 2, \dots$), which is exactly equivalent to Eq.(III-1) taken at $t = 1, 2, \dots$. In some sense, the system is an "ergodic clock"^(**).

(*) It seems that the term "translations" is preferred in the literature for the torus T^N , whereas "rotations" is more frequently used for the circle, i.e. for the torus T^1 .

(**) As already emphasized, the concept of ergodicity is relative to a given space X . Here, we mention the ergodicity of a class of integrable physical systems on its N -dimensional invariant torus. But it is clear that the physical system itself is not ergodic on the energy shell, except for $N = 1$.

Let us finally mention a nice application of the preceding results, proposed in [AA-67]. Consider the numbers $\alpha_n = 2^{-n}$ ($n = 1, 2, \dots$) and ask what is the probability distribution P_k of k ($k = 1, 2, \dots, 9$) for the first decimal of α_n ; the answer is $P_k = \log_{10}(1 + \frac{1}{k})$. What is remarkable in this example is that the result can be derived in a very simple way, using only the ergodicity of (III-16) (see the proof in [AA-67]).

2) Continued fraction expansion

Let us define the one-dimensional automorphism T :

$$\left\{ \begin{array}{l} X = [0, 1[\xrightarrow{T} [0, 1[\\ x \in X \longmapsto Tx = \begin{cases} \text{Frac}(1/x) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{array} \right.$$

This transformation is related to the continued fraction expansion of the starting point :

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \dots}}$$

($a_i(x)$ ($i = 1, 2, \dots$) are positive integers) by :

$$a_n(x) = \text{Int}\left(\frac{1}{T^{n-1}x}\right)$$

where "Int" denotes the integer part of a number.

The mapping T does not preserve the Lebesgue measure, but preserves the Gauss measure $d\mu = \frac{1}{\ln 2} \frac{dx}{1+x}$. The dynamical system (X, μ, T) is a K-system, and even a Bernouilli system, whose Kolmogorov-Sinai entropy has been calculated by Rohlin [Ro-61]:

$$h_{K-S}(T) = \frac{\pi^2}{6 \ln 2}.$$

Various measure-theoretic results are known for continued fraction expansions. Most of them are due to Khinchin (see [Kh-64] for a very clear account), and also to Lévy [Le-54], who were very successful in proving highly non-trivial results, using old-fashioned methods. Many of their results can now be obtained as straightforward consequences of ergodicity. For instance, let us quote the following properties, whose simple derivation can be found in Ref.[Bi-78].

- The asymptotic relative frequency P_k of k among the partial quotients $a_1(x), a_2(x), \dots, a_n(x), \dots$, is known to be

$$P_k = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)} \quad (\text{III-17})$$

for almost all x .

- $\lim_{n \rightarrow \infty} \sqrt[n]{a_1(x) \cdots a_n(x)} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2 + 2k}\right)^{\ln k / \ln 2} \quad \text{a.e.}$

("a.e." stands for "almost everywhere")

- $\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| x - \frac{P_n(x)}{q_n(x)} \right| = -\frac{\pi^2}{6 \ln 2} = -h_{x-s}(T) \quad \text{a.e.},$

where $\frac{P_n(x)}{q_n(x)}$ is the n^{th} -order convergent of x (or rational approximation to x):

$$\frac{P_n(x)}{q_n(x)} = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \dots + \frac{1}{a_n(x)}}}$$

($P_n(x)$ and $q_n(x)$ are prime to each other)

- $\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(x) = \frac{\pi^2}{12 \ln 2} \quad \text{a.e.}$

Notice that all the properties derived from ergodicity hold on $X = [0,1]$, except for a set of numbers of zero measure. In particular, rational numbers are trivially excluded from X for the applicability of these results, since they have only a finite number of non-zero partial quotients a_k . Also excluded are quadratic irrational numbers (i.e. irrational roots of a quadratic equation with integer coefficients), whose sequence of partial quotients $\{a_k(x)\}$ becomes periodic for sufficiently large values of k : for any quadratic irrational, there exist two integers k_0 and p such that, for every $k \gg k_0$, $a_k = a_{k+p}$. (Evidently, Eq.(III-17) cannot hold for such numbers).

3) Pseudo-random number generator

The following mapping is often quoted as a one of the nicest examples of complete randomness generated by a deterministic algorithm. Define T as

$$\left\{ \begin{array}{l} X = [0, 1] \\ x \in X \end{array} \right. \begin{array}{l} \xrightarrow{T} [0, 1[\\ \longmapsto Tx = \text{Frac}(rx), \end{array} \quad (\text{III-18})$$

where r is any positive integer.

One should first note that this application is not one-to-one. Indeed, if we choose $r = 2$, we see that the inverse image of a point x_{n+1} is not uniquely defined, since $x_n = \frac{x_{n+1}}{2}$ and $x'_n = \frac{x_{n+1} + 1}{2}$ are both possible solutions of $Tx_n = x_{n+1}$. Anyway, this "time reversal invariance breaking" is not essential.

The transformation T -called the r -adic transformation- is known to leave the Lebesgue measure invariant^(*). Let us first give a very simple interpretation of T ; for illustration, let us fix $r = 10$. We write x as :

$$x = 0. a_0 a_1 \dots a_n \dots, \quad (\text{III-19})$$

where $\{a_i/a_i = 0, 1, 2, \dots, 9\}$ is the sequence of digits of the representation of x in decimal basis. Clearly, the effect of T on x is simply to shift the decimals one position to the left :

$$T^n x = 0. a_n a_{n+1} \dots \quad (\text{III-20})$$

The same result (III-20) holds in the general case where Eq.(III-19) represents the number x , written in base- r (now, $a_i = 0, 1, \dots, r-1$), as can be seen from the general expression)

$$x = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}.$$

Now, let us ask for the degree of irregularity of this dynamical system ; simple intuition suggests that it should be strongly chaotic. Indeed, let us consider two initial points x and x' which are close to each other in the sense that their first k digits are the same, and look at the distance between $x_n = T^n x$ and $x'_n = T^n x'$ as n increases from one to infinity. For n less than k , the first $(k-n)$ digits

(*) To "visualize" this property, take for instance $A = [0, \frac{1}{2}]$.

Then, for $r = 2$, $T^{-1}A = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]$, so $\mu(A) = \mu(T^{-1}A) = \frac{1}{2}$

of x_n and x'_n remain the same, i.e. the distance $d(x_n, x'_n)$ does not increase in an "uncontrollable" way. But as soon as $n > k$, the two points x_n and x'_n have completely forgotten their original relationship, so that the distance $d(x_n, x'_n)$ may now evolve in a completely erratic manner as n increases (The knowledge of the first p digits of a number x tells nothing about the following digits of x). Therefore, this dynamical system seems to exhibit the strongest possible sensitivity to a perturbation of initial conditions.

This intuition is indeed corroborated by the mathematical property of the dynamical system to be a Bernouilli flow, i.e. to be at the top of the irregularity scale defined in III.1. Moreover, one can show in this case that the successive numbers of the sequence of r -based digits $\{a_0, a_1, \dots, a_n, \dots\}$ for a typical initial condition x (by "typical" we mean for almost all x of the interval X) are completely uncorrelated; this can be explained in the following way. As mentioned in Sect.III.1, a necessary condition for a system to be Bernouilli is that there exist a partition of the dynamical space having the property that the outcomes of successive measurements of the system are completely uncorrelated. In this case, this means that one can find a partition P_0 of the interval $[0,1]$ in such a way that the successive numbers $x_n = T^n x$ fall into the different atoms of the partition P_0 completely at random; and this partition P_0 happens to be precisely the set of r segments $[0, \frac{1}{r}, \frac{2}{r}, \dots, \frac{r-1}{r}, 1]$, so that the property of the successive measurements to be uncorrelated simply, means that the sequence $\{a_0, a_1, \dots, a_n, \dots\}$ is completely random, i.e. the successive applications of the mapping T can be viewed as a sequence of games with a roulette wheel having r equiprobable outcomes.

Applying the formula (III-8), we get for the Kolmogorov-Sinai entropy of $T^{(*)}$:

$$h_{K-S}(T) = - \sum_{i=1}^r \frac{1}{r} \ln \frac{1}{r} = \ln r. \quad (\text{III-21})$$

Notice that this value of h_{K-S} is exactly the classical result in information theory for the entropy of a system of Bernouilli trials with r equiprobable possibilities. This entropy, or a priori lack of information on the system, is clearly an increasing function of r , since the degree of uncertainty increases with the number of possibilities (a player who would have the choice of $r > 1$ for a given fixed, r -independent gain, would evidently choose $r = 2$, the coin toss!). This property is reflected in (III-21).

Let us now investigate how the mapping T can be used as a pseudo-random number generator^(**) and, for this purpose, first look at the consequences of ergodicity. Consider

(*) Here, we admit that the partition P_0 satisfies another necessary condition (that we did not mention) for the system to be Bernouilli.

(**) The process T , applied to a given X having an infinite number of digits, is a truly random number generator. The term "pseudo" refers to the finiteness of the number of digits with which the computer works in practice.

the characteristic property of ergodicity (III-5), translated for our discrete time process as :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{p=0}^{n-1} f(T^p x) = \int_0^1 f(x) dx \quad \text{a.e.}, \quad (\text{III-22})$$

where $f(x)$ is integrable on $[0,1]$ for the Lebesgue measure dx . Choose now f to be the indicator I_Ω (or characteristic function) of any subset Ω of $[0,1]$ (of non-zero measure) :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{p=0}^{n-1} I_\Omega(T^p x) = \int_\Omega f(x) dx = L(\Omega) \quad \text{a.e.}, \quad (\text{III-23})$$

where $L(\Omega)$ is the length (the Lebesgue measure) of Ω . This equation simply tells us that the proportion of numbers generated by successive applications of T which belong to any interval $\Omega \subset [0,1[$ asymptotically tends to the length $L(\Omega)$ of Ω . Notice that this property is shared by all dynamical systems which are ergodic on $[0,1]$ for the Lebesgue measure, and in particular holds for the example 1) (rotations of the circle) : all these systems are such that the numbers generated by $\{T^m : m = 1, 2, \dots\}$ asymptotically have a uniform distribution on $[0,1]$. However, all these systems cannot be used as random number generators. Consider for example the rotations of the circle, which, as we have seen, produce points in an inexorable regular pattern : clearly, in this case, the successive points obtained by the mechanism T have the strongest correlations that can be imagined. On the contrary, the algorithm (III-18) produces completely uncorrelated points, or, in other words, the successive trials associated with successive applications of T are, in this case, completely independent.

Several consequences of the property just discussed are worth to be mentioned.

i) consider the case where the subset Ω is any of the segments $\Omega_k = \left[\frac{k}{r}, \frac{k+1}{r} \right]$ ($k = 0, 1, 2, \dots, r-1$). Then, $I_\Omega(T^p x)$ is equal to one if the p^{th} digit of x is k , and zero otherwise. Thus, the repeated experiments $\{T^p x \ (x = 01, \dots, n-1)\}$ in or out of Ω_k can be viewed as independent trials for a random variable which takes the value k ($k = 0, \dots, r-1$) with probability $\frac{1}{r}$. One can therefore interpret Eq.(III-23) as follows. The l.h.s. represents the relative frequency of the value k , and the r.h.s. the probability $\frac{1}{r}$ for getting k . According to Eq.(III-23), the probability that the l.h.s. is equal to the r.h.s. is one, i.e. the relative frequency is almost surely equal to the probability, which is the strong law of large numbers (i.e. except for a set of starting points of zero measure).

ii) Another illustration of the strong law of large numbers is due to Borel, and can be recovered very nicely directly from the ergodic theorem. A number $x \in [0,1[$ is said to be normal to base r if each digit of its expansion in this basis has the relative frequency $\frac{1}{r}$. A number is said to be normal if it is normal to every basis r . As a consequence

of ergodicity of T , one can show that almost all numbers are normal. To prove this statement, it is sufficient to show that almost all numbers are normal to base r , for each fixed r . Indeed, this ensures that all the numbers are normal to base r , except for a set A_r of zero measure ; therefore, the union of all numbers being normal to any base remains of measure one (the measure of $\bigcup_r A_r$ is zero). Let us illustrate this property for $r=2$. We define

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases} .$$

Then, the ergodic property (III-22) for $f(x)$ gives

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{p=0}^{n-1} f(T^p x) = \frac{1}{2} \quad \text{a.e.} \quad (\text{III-24})$$

To interpret the l.h.s. of Eq.(III-24), let us write the explicit expression of x in binary basis :

$$x = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}} , \quad (a_n = 0, 1)$$

then

$$T^p x = \sum_{n=0}^{\infty} \frac{a_{n+p}}{2^{n+1}} .$$

We see that $\begin{matrix} \text{if } T^p x < \frac{1}{2} & \text{then } a_{n+p} = 0 \\ \text{if } T^p x \geq \frac{1}{2} & \text{then } a_{n+p} = 1 \end{matrix}$, whereas

Therefore,

$$f(T^p x) = \begin{cases} 0 & \text{if } a_{n+p} = 0 \\ 1 & \text{if } a_{n+p} = 1 \end{cases} ,$$

and the l.h.s. of Eq.(III-24) is simply the relative frequency of the digit one in the binary expansion of x , which does converge, for almost all x , to its probability $\frac{1}{2}$.

III.3.c) Two-dimensional abstract dynamical systems

1) Translations on the torus T^2

This mapping has been studied in III.3.c). We noticed that it is ergodic, but not mixing, if the two parameters of the translation are rationally independent.

2) Arnold's cat

The Arnold's cat mapping T is the linear automorphism of the two-dimensional torus $T^2 : \{ (x, y) \bmod 1 \}$ for the measure $dx dy$, defined as

$$T(x, y) = (x + y, x + 2y) \pmod{1}. \quad (\text{III-25})$$

One can also consider T as a mapping of the unit square or to itself :

$$X = [0, 1[\times [0, 1[\xrightarrow{T} [0, 1[\times [0, 1[.$$

To construct a geometrical realization of this automorphism, one first maps the unit square X into the rhombus MX , where $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, and then brings back any point ξ lying outside the square into the square by subtracting unity as many times as necessary (one or two times) to each coordinate of ξ .

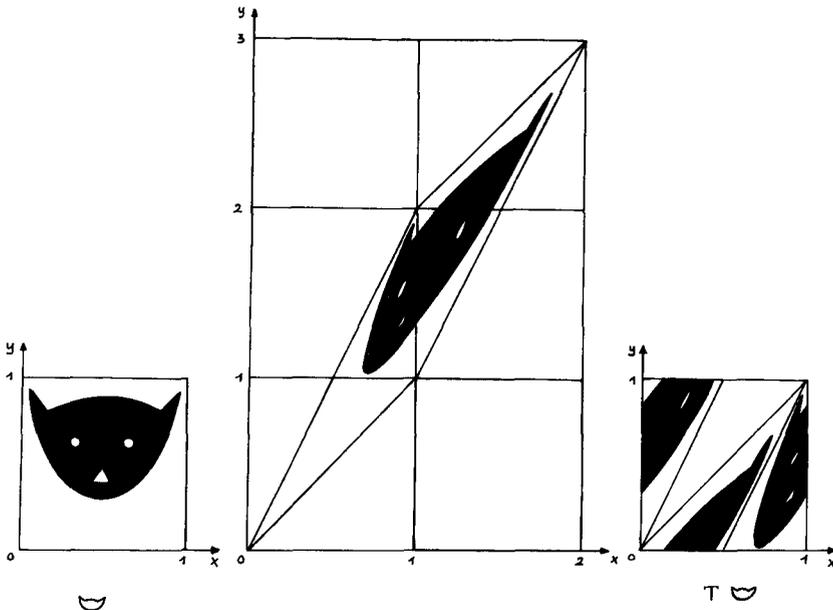


Fig.III.15 - The Arnold's cat mapping : \smile is the starting figure, $T\smile$ represents the first iteration (taken from Ref.[AA-67])

The automorphism T is known to be a K-system ([AA-67]), and the example is nice because this property can nearly be seen (if one does not care about mathematics)

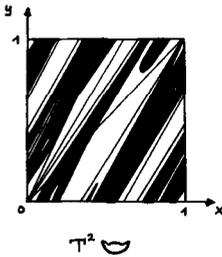


Fig.III.16 - The Arnold's cat mapping : second iteration

"with the eyes". Fig.(III.16) represents the second iteration of T . The cat is already completely unrecognizable, which confirms the first impression of Fig.(III.15) that T undergoes drastic deformations of shapes. It is easy to imagine that the cat will asymptotically be "diluted" into the whole square, which is the mixing property. Now, consider the eigenvalues λ_1 and λ_2 :

$$\lambda = \lambda_1 = \frac{1}{2} (3 + \sqrt{5}) > 1 > \lambda_2 = \lambda_1^{-1} > 0 .$$

One sees that the direction D_1 defined by the eigenvector associated with λ_1 is expanding under the action of T , whereas the direction D_2 associated to the other eigenvector is contracting : at each step of the process, a rectangle whose sides are parallel to D_1 and D_2 is stretched along D_1 by a factor λ , and compressed along D_2 by the same factor λ^{-1} . This mechanism is responsible for exponential separation of trajectories, indicating the K-property. After p steps, the expanding factor is simply $\lambda^p = \exp(p \ln \lambda)$, which suggests that the K-S-entropy of the system is simply (this result can be derived rigorously)

$$h_{K-S}(T) = \ln \lambda .$$

Finally, let us notice that the origin O is an invariant point of T , and that the orbits of all points $(x, y) = (\frac{p}{q}, \frac{p'}{q})$ are periodic ; one can also show the existence of two invariant curves which are both dense in the square (see again [AA-67]).

3) Baker's transformation

Consider again the unit square, and define the transformation T as

$$\left\{ \begin{array}{l} X = [0,1[\times [0,1[\xrightarrow{T} X \\ (x,y) \in X \longmapsto T(x,y) = \begin{cases} (2x, \frac{y}{2}) \text{ if } 0 \leq x < \frac{1}{2} \\ (2x-1, \frac{y+1}{2}) \text{ if } \frac{1}{2} \leq x < 1 \end{cases} \end{array} \right.$$

The name of the transformation is illustrated in Fig.(III.17), in which the unit square transformed under T looks like a piece of dough worked by a baker.

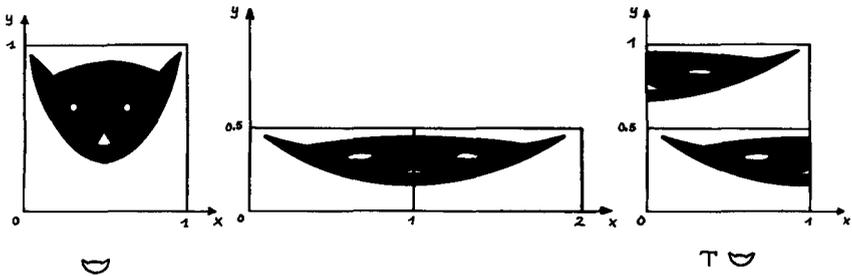


Fig.III.17 - The Baker's transformation : Cat is the starting figure, and $T \text{ Cat}$ represents the first iteration (taken from Ref.[AA-67]).

The area-preservation of T can be seen on the figure. Also is clear the fact that each iteration of T doubles the number of horizontal "slices" in which the cat is more and more "compressed" and this property suggests the mapping to be mixing. In fact, the dynamical system is much more chaotic than simple mixing, since it is a Bernoulli system. We can easily imagine this strong irregularity by noting that the image of the abscissa x_n of any point of X is, in any case

$$x_{n+1} = \text{Frac}(2x_n),$$

which is the dyadic transformation of Eq.(III-18). The results obtained for the one-dimensional mapping Eq.(III-18) ensure that if we consider the partition P_0 of the square into the two atoms $0 \leq x < \frac{1}{2}$ and $\frac{1}{2} \leq x < 1$, then the sequence of measurements of $\{T^n x : n = 0, 1, \dots\}$ with respect to the partition P_0 will be completely random, i.e. completely equivalent to a sequence of coin tosses.

Up to now, we have seen some examples of Bernoulli flows, all being abstract dynamical system. Another famous example of Bernoulli mapping is generated by geodesics on surfaces of negative curvature. We shall now investigate a class of physical systems, some of them having also the Bernoulli property.

III.4 EXAMPLES AGAIN : BILLIARDS

A billiard is a dynamical system produced by the free motion of a point particle in a domain D of the plane \mathbb{R}^2 bounded by a closed curve Γ -or, more generally, by a set of closed curves Γ_i -, with elastic reflections at the boundary Γ of D (i.e. the tangential component of any trajectory remains constant at Γ , whereas its normal component changes sign)^(*). The boundary Γ may contain singular points (vertices) where the tangent is not defined ; we disregard all trajectories coming at some time to a vertex of Γ , which form a set of zero measure. A billiard is therefore a two degrees of freedom system, with at least one constant of motion, the energy $E = \frac{1}{2} m v^2$. Notice that the behaviour of any orbit in phase space depends neither on the mass m of the point particle nor on its velocity v (this peculiar feature of billiards is due to the fact the motion is simply governed by the laws of geometrical optics) ; therefore the properties of this kind of dynamical systems do not depend on the energy E , all the energy surfaces having exactly the same structure.

An orbit in configuration space is an infinite sequence of segments, each one corresponding to a constant velocity vector \vec{v} . Therefore, it is natural to search for a Poincaré surface of section Σ such that the successive crossings with Σ correspond to the successive bounces on the "wall" Γ . To do this, let us take as coordinates of any bounce Q_n its curvilinear abscissa s_n along Γ (once an origin O and a sense have been defined on Γ), and $p_n = \sin \alpha_n$, where α_n is the angle of the trajectory leaving Q_n with the normal to Γ at Q_n , pointing towards the interior of D . It is easy to show that the mapping M defined by

$$\begin{pmatrix} s_{n+1} \\ p_{n+1} \end{pmatrix} = M \begin{pmatrix} s_n \\ p_n \end{pmatrix}$$

is area-preserving, i.e. that (see Ref. [Be-81 a] for details)

$$\det \begin{bmatrix} \frac{\partial s_{n+1}}{\partial s_n} & \frac{\partial s_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial s_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{bmatrix} = 1 .$$

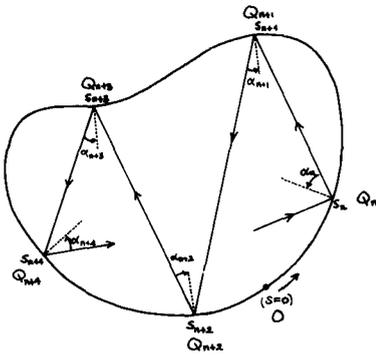


Fig.III.18 - Surface of section coordinates for a billiard

(*) The domain D may also be a non-compact part of \mathbb{R}^2 : see below the example of the Sinai's billiard.

The study of billiards is of interest in many respects :

- i) they are, among the physical Hamiltonian systems, the most studied from a mathematical point of view, and some important analytical properties are known
- ii) billiards are very rich in their behaviours : there exist systems lying at the top, as well as at the bottom of the hierarchy of regularity described in Sect.III.1, and also systems which cannot be classified according to this hierarchy, i.e. for which the phase space contains both regular and chaotic regions
- iii) the energy surface of billiards is of dimension two, which is the smallest one needed to observe chaotic motion
- iv) for our purpose -trying to find signatures of chaos in quantum systems whose classical analogues are chaotic by statistical studies of the spectrum-, drums (quantum billiards) are the most convenient systems : they have an infinite discrete spectrum, and efficient numerical methods are available to compute long series of adjacent levels.

Regular billiards

The rectangle, the circle, and the ellipse are regular billiards, i.e. dynamical systems whose energy surface is a torus. The two independent constants of motion are

- i) for the rectangle : the two projections $|v_x|$ and $|v_y|$ of the moduli of the velocity on two axes parallel to the sides of the rectangle
- ii) for the circle : the energy, and the angular momentum $L_2 = |\vec{L}|$ with respect to the center of the billiard
- iii) for the ellipse : the energy, and the product $L_1 L_2$ of the angular momenta with respect to the two foci

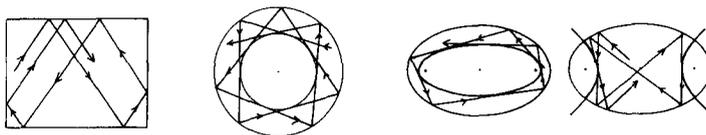
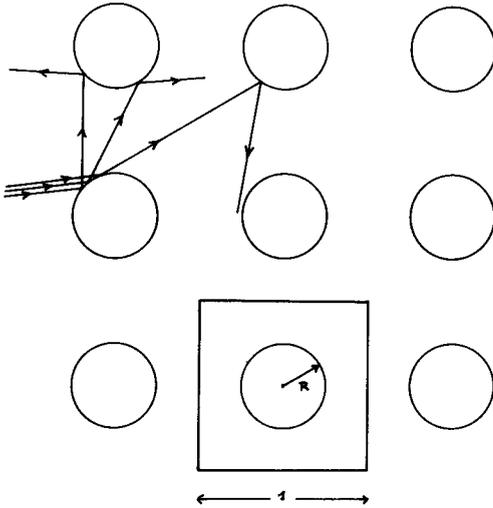


Fig.III.19 - Trajectories in regular billiards. Caustics can be seen for the circular billiard (the caustics are circles of same center as the billiard) and for the elliptic billiard (the caustics are ellipses and hyperbolae confocal with the ellipse defining the boundary of the billiard)

For the circle and the ellipse, the integrability manifests itself by the existence of families of caustics, which reveal the existence of tori (see again Ref. [Be-81 a], and also Ref.[Be-83]).

Chaotic billiards

As a first example, let us consider the Sinai's billiard : the point mass moves in the plane \mathbb{R}^2 and bounces on reflecting disks, all of the same radius, forming a periodic



lattice as shown in Fig.III.20. By trivial symmetry arguments, one can reduce the study of this system to the study of the motion of a particle in a square, with a circular obstacle around the center. Simple intuition suggests that the dynamical system associated with the Sinai's billiard is strongly chaotic. Indeed, as illustrated in Fig.III.20, a bundle of nearby trajectories reflecting on convex obstacles is drastically defocused, and spreads in a large part of the configuration space after a small number of bounces. The effect of this instability with respect to initial conditions can be estimated :

Fig.III.20- The Sinai's billiard

suppose, for instance, that one wants to calculate on a computer an orbit for $R = 0.1$; then, one can show that at each bounce, about one decimal digit is lost for the angle determining the direction of the orbit. This means that, typically, one cannot compute the trajectories in Sinai's billiard for a number of bounces greater than 10-20, which illustrates the dramatic instability of the system ! (see Ref.[Be-78] for more details, and also for other illustrations of the degree of instability, in terms of external perturbations of the physical system).

In fact, it has been proved by Sinai [Si-70] that this billiard is a K-system ; it has even been shown more recently to be a B-system. The proof by Sinai -which is very long and very elaborate- has initiated further works on other billiards, conjectured to be also chaotic, and the K-property (and also the B-property) is now known to hold for several billiards, such as, for example the stadium (two half-circles of radius R , joined by two segments parallel to the line passing through their centers), and also the diamond (see Fig.III.21). For the diamond, the chaotic behaviour has the same origin as for the Sinai's billiard, which is the negative curvature of the obstacles. The origin of the instability of the stadium is somewhat less intuitive. A small bundle of trajectories contracts after reflecting on a circular -focusing- part of the boundary, but subsequently expands

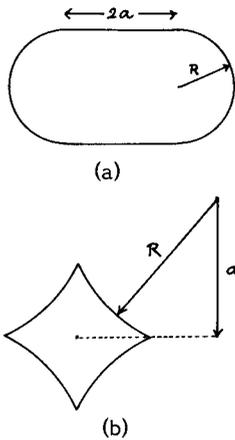


Fig.III.21- (a) the stadium
(b) the diamond

in such a way that the synchronization of successive contractions is made impossible : expansions prevail over contractions, producing the stochastic properties of the billiard.

Notice that in the limits $R = 0$ (Sinai's billiard : see Fig.III.20) or $a = 0$ (Stadium : see Fig.III.21-a), the Sinai's billiard and the stadium become integrable systems (a square and a circle respectively). However, it should be emphasized that the K-property (and the B-property) holds for any value of $R > 0$ (Sinai's billiard) or of $a > 0$ (stadium). This means that a regular billiard may be transformed into a strongly chaotic billiard by an infinitely slight perturbation of its boundary ; evidently, the KAM theorem -which would predict the persistence of tori under a small perturbation of the integrable system- does not apply here.

This is because the smoothness conditions required for the KAM theorem to be applicable are not satisfied by billiards. So far there exists no equivalent to KAM theorem for billiards. Some results in this direction are known but their field of applicability is still very peculiar ; for instance, Lazutkin [La-73] proved the existence of tori for convex billiards having a very smooth boundary ("very smooth" means that the radius of curvature as a function of the arc length is at least a $C^{5/3}$ function... which is only a sufficient condition for the tori to exist !).

Finally, let us recall that the ergodicity of these chaotic systems does not exclude the existence of periodic trajectories (these form a set of zero measure of trajectories which do not visit the neighbourhood of all the points of the energy surface). For instance, the trajectory in the stadium passing through the two centers of the circle is periodic; also are periodic the trajectories hitting the horizontal segments of the stadium at right angle, etc... These periodic orbits can form a family or be isolated. We shall not discuss the questions related to the periodic orbits, though they play a crucial role in the quantization of the classical systems (see [Be-81 b] and [Be-83]).

Pseudo-integrable billiards - Polygons

The chaotic billiards presented above have in common the property that at least one part of the boundary is not a straight line ; this is, in fact, an essential condition to produce the K-property. Indeed, it has been shown that the K-S entropy of billiards in polygons is zero, i.e. trajectories in polygons never show exponential instability ; but can one say more about polygons ?

As a particular kind of polygon, we have already mentioned the rectangle, which is integrable. Also integrable are three kinds of triangles : the equilateral one, and the two right triangles ($90^\circ, 45^\circ, 45^\circ$) and ($90^\circ, 30^\circ, 60^\circ$). Putting apart the integrable systems, polygons can be divided into two families :

i) irrational polygons (i.e. polygons whose angles are irrational multiples of π). These billiards are conjectured to be ergodic, and perhaps mixing systems (see Ref.[Ho-75]).

ii) rational polygons (all angles are rational multiples of π). For such systems, it is clear that the angles of the segments defining any trajectory with a fixed direction can take only a finite number of different values. Consequently, these systems are certainly not ergodic. In fact, one can show the existence of another first integral F which is independent of H and in involution with H only almost everywhere, i.e. except for a discrete set of singular points (see e.g. Ref.[ZK-75] ; such systems are called pseudo-integrable. Due to the existence of F , the allowed region in phase space is of dimension two, like for an integrable billiard, but the singular points just mentioned prohibit the invariant surfaces from being tori ; these surfaces can be viewed as tori with more than one hole, i.e. two-dimensional manifolds of genus $g > 1$.

Integrable, chaotic and pseudo-integrable billiards will be discussed in Section IV from a quantal point of view.

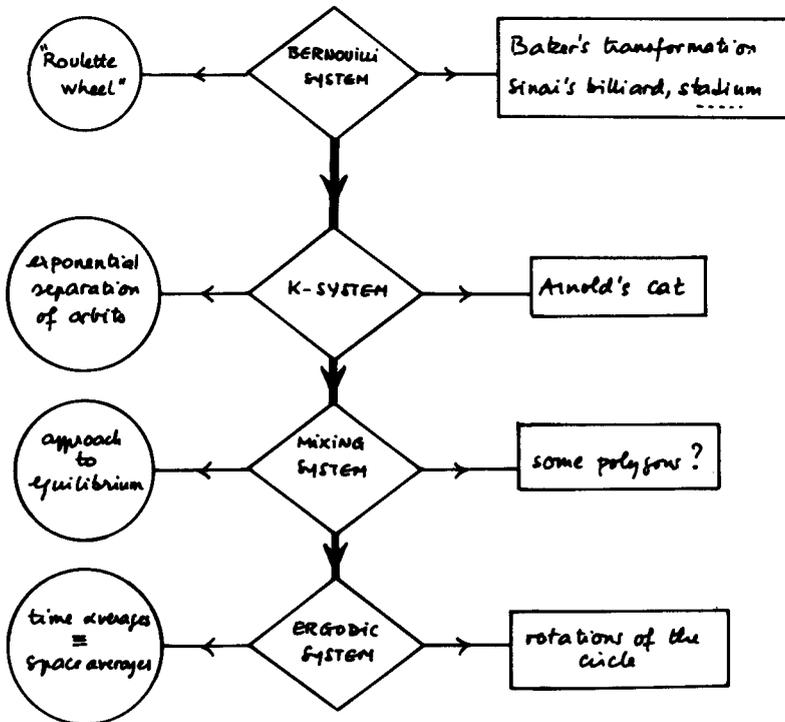


Fig.III.22 - Ergodic theory : the hierarchy of chaos

BIBLIOGRAPHY**Ergodic theory**

For an introduction to the concepts of ergodic theory we suggest Refs.[LP-73, Fo-83], in which are stated the main ideas, and [Ba-75, TKS- 83] for a first detailed approach. For a more advanced mathematical study, see [AA-67,Pa-81,CFS- 82] (general ergodic theory), and also [Sh-73,Bi-78,ME-81] (more specialized topics).

Chaotic motion

Among the enormous literature on classical chaotic motion of conservative systems the following references are both elementary enough, and rich in material : [LL-83,Be-78, He-80] (general reports) and [Fo -74,Wh-77] (emphasis on KAM theorem). An extensive bibliography, classified by topics, can be found in [He-80].

IV - QUANTUM CHAOS

As shown in the preceding section, classical chaotic motion is now reasonably understood, at least for the "pure" systems which enter in the classification of the ergodic theory as summarized in Fig.III.22. It is then natural to ask whether there are quantum-mechanical manifestations of classical chaotic motion. We shall use the term "quantum chaotic system" in the precise, and restricted, sense of "quantum system whose classical analogue is chaotic". General discussions on quantum chaotic systems can be found in ref.[Ca-84] ; it appears that there is a large variety, and even disparity, of opinions.

It should be clear from the beginning that the notions which are of most relevance in the classical case may be obvious and useless in the quantum case. Consider for instance the quantum analogue of a classical chaotic billiard, namely a free particle in a box. Remember that its spectrum, determined by the eigenvalue equation (I-22), contains an infinite number of discrete eigenvalues E_n . The time evolution of a quantum state $\Psi(\underline{r}, t)$ is given by

$$\Psi(\underline{r}, t) = \sum_n a_n \psi_n(\underline{r}) \exp\left(-\frac{i}{\hbar} E_n t\right), \quad (\text{IV-1})$$

which is quasi-periodic. Therefore there will never be exponential separation of the difference of wave-functions of two close initial states, in contrast to the classical orbits which do show the exponential divergence characteristic of classical chaotic motion (K-systems). We thus expect that the "dictionary" translating classical chaos \leftrightarrow quantum chaos will not be simple.

In what follows we shall restrict ourselves, in the search for manifestations of chaos, to properties of the spectrum and nothing will be said on properties related to wave functions, like structure of nodal lines, properties of the Wigner function, etc.. The tools to characterize fluctuation properties of spectra, described in Section II in connection with random matrix theories, will be thoroughly used.

The first property which has been considered as a possible tool to discriminate between integrable and chaotic systems is the behaviour of the nearest-neighbour spacing distribution for small spacings, i.e., $p(x)$ as $x \rightarrow 0$. Berry and Tabor [BT-77], using torus quantization, predict level clustering for integrable systems $p(x) \underset{x \rightarrow 0}{\sim} e^{-x}$ in the asymptotic high energy regime^(*). This feature can be understood by examining

^(*)This prediction excludes harmonic oscillators, for which the arguments of the proof are not applicable. It can be shown, in fact, that the distribution $p(x)$ does not follow the Poisson law for the two-dimensional harmonic oscillator [BGP-84].

the occurrence of degeneracies in a parameter space (see [Be-84]). It can also be viewed as the result of mixing independent spectra^(*). Let us give a heuristic argument illustrating this idea. Consider the circular membrane, which is classically integrable (circular billiard), or better, to avoid degeneracies, the semi-circular drum. The eigenfrequencies of

$$(\Delta + k^2) \Psi = (\Delta + E) \Psi = 0 \quad (\text{IV-2})$$

with Dirichlet boundary conditions on the semi-circle are given by the zeros of the Bessel functions $J_\nu(x)$ ($\nu = 1, 2, 3, \dots$). Let $j_{\nu, s}$ ($s = 1, 2, 3, \dots$) denote the s -th zero of $J_\nu(x)$. The $j_{\nu, s}$'s ($s = 1, 2, \dots$) extend from $\approx \nu$ to infinity with a density

$$\frac{1}{\pi} \sqrt{1 - \frac{\nu^2}{k^2}} \quad \text{for } k > \nu \quad (\text{IV-3})$$

and are practically fluctuation-free. Consider now an interval at high energy containing N levels k_n ($n = 1, 2, \dots, N$) ordered with increasing value of k_n . The successive values of k_n , which are the zeros $j_{\nu, s}$, correspond to the order of N different (and unordered) values of ν . The point now is that zeros of J_ν and $J_{\nu'}$, with ν not too close to ν' , are likely to be uncorrelated. Consider, for instance, a stretch of ten eigenfrequencies near the 1400th eigenfrequency. The labelling (ν, s) of successive eigenfrequencies is as follows: (82,4), (70,7), (37,18), (45,15), (5,32), (32,20), (3,33), (1,34), (18,26), (60,10). We are therefore in a similar situation as when randomly superposing picket fences, which leads to a Poisson spectrum^(**).

It has been predicted that a drastic change in the spacing distribution $p(x)$ takes place when going from an integrable case (Poisson) to a non-integrable case. Berry [Be-83, Be-84] predicts $p(x) \sim x^{\sigma}$, like in GOE, whereas Zaslavsky [Za-81] predicts $p(x) \sim x^{\sigma}$ where σ is related to the Kolmogorov entropy h_{KS} of the corresponding classical system. On the other hand, Richens and Berry [RB-81] predict that level repulsion will also be present for systems which classically are pseudo-integrable. In what follows we shall discuss some recent results concerning fluctuation-

(*) In section II we saw that the result of superposing randomly highly correlated spectra -the argument works even for picket fences- is to produce a Poisson spectrum. We also saw that the effect of superposing shell model spectra corresponding to different values of J^π , each one having level repulsion, is to destroy all kinds of correlations, and in particular level repulsion (see Fig.II.11).

(**) The keypoint of this heuristic argument is the independence of the positions of successive levels in a given interval for large k (or E). Of course, this does not hold for the spectrum $m + \alpha n$ ($m, n = 1, 2, 3, \dots; \alpha$ irrational) of the two-dimensional harmonic oscillator, which leads to strongly correlated levels in any interval of the spectrum.

properties of spectra of systems whose classical analogues are either integrable or chaotic. We shall not confine the discussion to the presence or absence of level repulsion but, for reasons that should be clear to the reader from the discussion of Section II, take as reference patterns the entire Poisson- and GOE- fluctuations.

IV.1 VIBRATIONS OF THE MEMBRANE (QUANTUM BILLIARDS)

Towards the end of Section III, reasons were given for putting special emphasis in the study of billiards. And examples were shown of integrable, chaotic and pseudo-integrable billiards. Let us now study some of their properties in the quantum case, specifically their spectral properties. The cases to be discussed are shown on

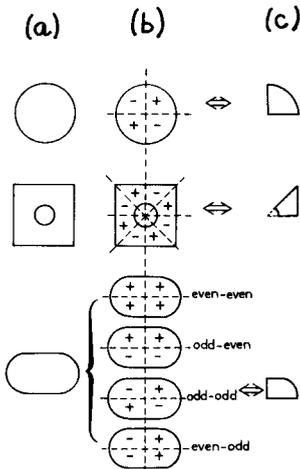


Fig.IV.1 - Column (a) : shapes of different membranes, whose spectral fluctuations are discussed. Column (b) : examples of the symmetry character of the eigenfunctions considered. For the circle the symmetry is given by Eq.IV-4 and the eigenvalues coincide with those of the quarter of a circle. For Sinai's billiard shape, the symmetry is given by Eq.IV-5 and the eigenvalues are those of the desymmetrized Sinai'billiard, as shown on column (c). For the stadium, the four possible symmetry classes are indicated (see Eqs. IV-6). For the odd-odd symmetry, the eigenvalues are those of the quarter of stadium, as indicated on column (c).

Fig.IV.1 : circle (integrable), Sinai's billiard and stadium (both strongly chaotic, in fact Bernoulli systems). Mention should be made of the pioneering work in the direction we follow : Refs.[MK-79,CVG-80] for the stadium and ref.[Be-81b] for Sinai's billiard. To determine the eigenvalues of Eq.IV-2 with Dirichlet boundary conditions, an efficient method has been proposed by Berry [Be-81b]. It is inspired on the work in solid state physics, by Korringa, Kohn and Rostoker, to determine the high-energy bands at the centre of the Brillouin zone. Once a sequence of eigenvalues is obtained, there is no ambiguity in separating the average part and the fluctuating part of the spectrum. Indeed, we know that for the systems we are considering the smooth part is given by Eq.(I-24) (see Fig.I.7). The procedure is therefore to first compute a sequence of eigenvalues $\{E_i\}$ and then to unfold the spectrum via Eq.(I-27), where N_{ν} is given by Eq.(I-24). One finally has a sequence of points $\{x_i\}$ with mean spacing equal to unity, all over the spectrum.

Let us now present some results. To illustrate how the analysis is performed, we consider first a "trivial" case, the case of the circular membrane. To avoid two-fold degeneracies, we take a semi-circle or a

quarter of a circle. Equivalently, the quarter of a circular membrane corresponds to eigenvalues of the full circle whose associated eigenfunctions have the symmetry property illustrated on column (b) of Fig.IV.1, namely

$$\Psi(x,y) = -\Psi(-x,y) = -\Psi(x,-y) \quad (IV-4)$$

The eigenvalues are given by the squares of the roots $j_{\nu,s}$ of the Bessel functions $J_{\nu}(x)$, with $\nu = 2,4,6,\dots$ and $s = 1,2,3,\dots$ On Fig.I.7a is shown the cumulative density $N(E)$ for the first eigenvalues. On Fig.IV.2 are shown the results for the spacing distribution $p(x)$ and for the average value of Δ_3 as a function of the length L . On

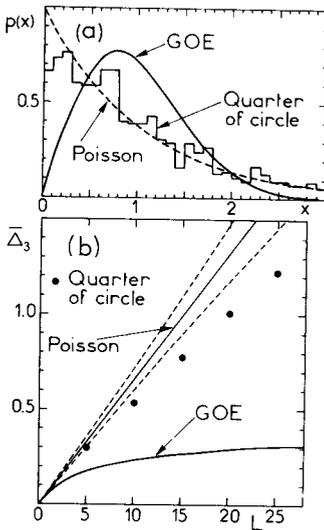


Fig.IV.2 - Results of level fluctuations for the first 675 eigenvalues of a circular membrane corresponding to eigenfunctions with the symmetry indicated on column (b) of Fig.IV.1 : (a) Nearest-neighbour spacing histogram. (b) $\bar{\Delta}_3$ as a function of L ; dashed lines for Poisson and small bars for GOE indicate the effect of the finiteness of the sample as predicted by the theory (one standard deviation). Curves corresponding to the Poisson case (stretch of uncorrelated levels) and to the random matrix theory predictions (GOE) are drawn for comparison.

Table IV.1 is given the value of the correlation coefficient C between adjacent spacings. As can be seen, the computed values are far from GOE-values and rather close to, although not consistent with, the Poisson-values (except for C , which is consistent with the Poisson-value $C=0$). We interpret these results as follows. The spectra under study are not translational invariant or stationary. We expect that the characteristic fluctuation patterns (GOE or Poisson) are valid in the asymptotic regime (high energy). The departures of the results presented for the circular membrane from Poisson indicate that the asymptotic regime has not yet been reached. Work is in progress to determine the rate of convergence to the asymptotic regime (Poisson in the present case).

Let us now consider chaotic systems. We treat the desymmetrized Sinai's billiard as shown on column (c) of Fig.IV.1, or, equivalently, the solutions of the Schrödinger equation IV-2 with Sinai's billiard as the boundary and with the symmetry

$$\Psi(x,y) = -\Psi(-x,y) = -\Psi(x,-y) = -\Psi(y,x) \quad (IV-5)$$

	1/4 Circle	Sinai	Stadium (one symme- try)	Stadium (mixed sym- metries)	GOE	Poisson
Number of levels	675	740	810	3200		
C	-0.04 ± 0.04	-0.30 ± 0.04	-0.31 ± 0.04	-0.18 ± 0.02	-0.27	0

Table IV.1

Values of the correlation factor C between adjacent spacings (Eq.I-37) for different systems investigated. GOE- and Poisson-values are given for comparison.

To improve the statistical significance of the results, we consider four different cases by changing the value of the parameter R and perform a joint analysis of the results as explained in [BGS-83,BGS-84a]. The results are presented on Fig.IV.3 and Table IV.1. In contrast to the case of the circular membrane, for which the results are close to the Poisson spectrum, we see now that the results are fully consistent with GOE-predictions for $p(x)$ (not only for small values of x), for $\overline{\Delta}_3$ and for C.

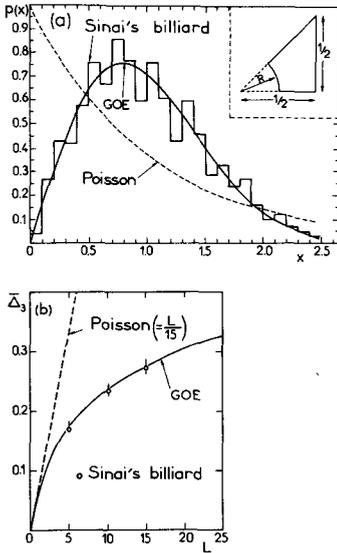


Fig.IV.3 - Results of energy level fluctuations for desymmetrized Sinai's billiards as specified on the upper right corner of the figure. 740 levels have been included in the analysis, corresponding to the 51-th to 268-th level for R=0.1, 21-th to 241-th level for R=0.2, 16-th to 194-th level for R=0.3, 11-th to 132-th level for R=0.4. See caption of Fig.IV.2 for further explanation (taken from [BGS-83,BGS-84a]).

The results are close to the Poisson spectrum, we see now that the results are fully consistent with GOE-predictions for $p(x)$ (not only for small values of x), for $\overline{\Delta}_3$ and for C.

Let us investigate another system whose classical analogue is chaotic, namely one particle in a two-dimensional box whose boundary is the stadium. The solutions of eq.(IV-2) can be classified according to four different symmetry classes :

even-even

$$\Psi(x,y) = \Psi(-x,y) = \Psi(x,-y) \quad (IV-6)$$

odd-eren

$$\Psi(x,y) = \Psi(-x,y) = -\Psi(x,-y) \quad (IV-6')$$

odd-odd

$$\Psi(x,y) = -\Psi(-x,y) = -\Psi(x,-y) \quad (IV-6'')$$

even-odd

$$\Psi(x,y) = -\Psi(-x,y) = \Psi(x,-y) \quad (IV-6''')$$

and the spectrum, despite the symmetry of the problem, contains no degeneracies. Take now the eigenvalues corresponding to a definite symmetry-class, for instance

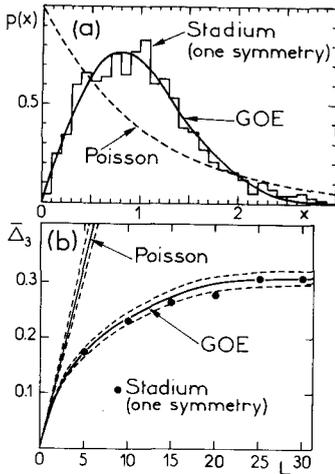


Fig.IV.4 - Results of level fluctuations for the first 810 eigenvalues of a membrane whose boundary is a stadium. They correspond to eigenfunctions with odd-odd symmetry (see Fig.IV.1). The ratio $2a/R$ of the straight line segment to the radius is 1 (see Fig.III.21a). See caption of Fig.IV.2 for further explanations (taken from [Sc-84,BGS-84b]).

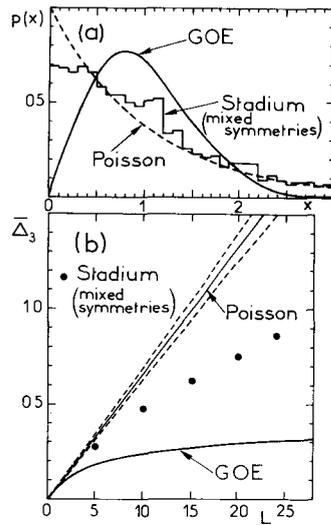


Fig.IV.5 - Same as in Fig.IV.4 but with the four different symmetry classes, as specified in Fig.IV.1. The spectrum analyzed contains the first 3200 eigenvalues. See caption of Fig.IV.2 for further explanations (taken from [Sc-84, BGS-84b]).

the odd-odd case. Results are presented on Fig.IV.4 and Table IV.1. Again we have a remarkable agreement with GOE-predictions. A similar agreement is obtained when analyzing the eigenvalues belonging to the other three symmetry classes. Consider finally the spectrum which contains all levels corresponding to the four symmetries of the stadium (IV-6,6',6'',6'''). The results change drastically. They are shown on Figs.I.7b, IV.5 and Table IV.1. The spectrum fluctuations are intermediate between GOE and Poisson. The results would be closer to Poisson-fluctuations if more than four different families characterized by different quantum numbers would be present. This is in exact analogy with what happens when superposing several different GOE spectra. Or with compound nucleus resonances, when no attention is paid to quantum numbers and the spectrum results from mixing several pure series.

IV.2 OTHER EXAMPLES

The systems treated in the preceding subsection were classically either integrable or strongly chaotic (highly-unstable ergodic systems) and correspondingly we saw that the spectra show Poisson- or GOE- fluctuation patterns. What happens then for systems displaying a more complicated structure in phase-space ? We have seen in Section III that there exist systems whose Poincaré sections look as belonging to an integrable system at some particular energies but that, by changing the energy, islands of chaoticity appear and increase until they cover the full Poincaré sections, showing that in fact the system is non-integrable and chaotic. Does this behaviour have some manifestation in the corresponding quantum systems ? And is there a transition parameter characterizing the relative importance of chaotic/non-chaotic regions which manifests in the spectral fluctuations ? The problem is very interesting but difficult and, although several efforts in this direction are worth mentioning -the study of a two-dimensional harmonic oscillator plus quartic terms in ref.[HYK-84], of a system of kinetically coupled Morse oscillators in ref.[MT-84], of the Hénon-Heiles Hamiltonian (Eq.III-15) in ref.[PS-83]- it is still premature to draw general conclusions from them. Mention should also be made of the study of a one-parameter family of billiards which classically goes continuously from the integrable to the chaotic regime ; in the quantum case the spacing distribution $P(x)$ seems to continuously go from a Poisson to a Wigner distribution [Ro-84].

In this direction, a very recent work by Seligman, Verbaarschot and Zirnbauer deserves special mention [SVZ-84]. These authors consider a two-dimensional system consisting of two interacting particles moving in one-dimensional potential wells. The Hamiltonian is

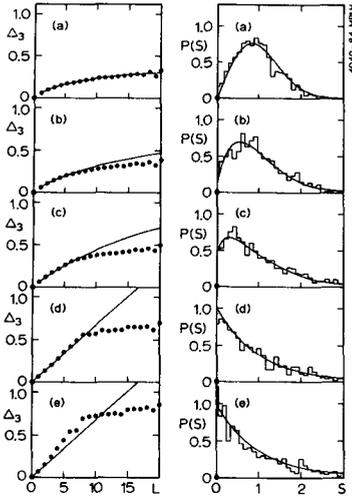
$$H = \frac{1}{2} (p_1^2 + p_2^2) + V_1(x_1) + V_2(x_2) + V_{int}(x_1 - x_2), \quad (IV-7)$$

where V_1 , V_2 and V_{int} have the same functional form

$$V_j(x) = \lambda_j (x^2 + \mu_j x^4 + \nu_j x^6) \quad j=1,2,int \quad (IV-8)$$

In Eq.IV-8 λ_j , μ_j and ν_j are parameters. With an adequate choice of the numerical values of μ_j , ν_j , λ_1 and λ_2 , the properties of the system are studied as a function of a single parameter λ_{int} . For $\lambda_{int} = 0$ the system is separable and therefore integrable. The study of classical trajectories and Poincaré sections indicate that the system is probably, for large values of λ_{int} , classically chaotic. By varying the strength of the interaction λ_{int} , the fraction f of phase space filled by chaotic trajectories, in the energy region occupied by the first 400 levels of each parity, can be varied from 0 to ~ 1 . Some results for level fluctuations are reproduced on Fig.IV.6. Figs.IV.6(a) to (e) correspond to the fraction of phase space

taking the estimated values $\approx 1.0, 0.7, 0.4, 0.1,$ and $0.0,$ respectively. (a) corresponds



to the chaotic regime and (e) to the integrable case. As can be seen, (a) agrees beautifully with GOE and (e) with Poisson. In situations characterized by $1 > f > 0,$ the fluctuation patterns are intermediate between GOE and Poisson. For the integrable or quasi-integrable case ((e) and (d)), there is a flattening of $\overline{\Delta_3}(L)$ for $L \gg 9$ which is not fully understood.

Fig.IV.6 - Results of level fluctuations ($\Delta_3(L)$ and $P(x)$) for the Hamiltonian IV.7. Dots and histograms represent the results obtained from the 40-th to the 400-th level of each parity. The continuous lines correspond : in (a) and (e) to GOE and Poisson respectively ; in (b), (c) and (d) to a continuous interpolation between GOE and Poisson derived from a one-parameter random matrix model (taken from [SVZ-84]).

IV.3 DISCUSSION

We close this Section with some remarks and a short discussion. A more general one is postponed until the next Section.

Two main conclusions can be drawn from the results discussed : Integrable systems show Poisson fluctuation patterns whereas strongly chaotic systems show GOE fluctuations. An example has been given of a strongly chaotic system whose states belong to four different symmetry classes and the fluctuations are, separately, of GOE type, in perfect analogy with what happens, for instance, in real physical systems like nuclei, with pure sequences. We have also seen a nice example of a continuous

transition from integrable to chaotic regime with a corresponding transition in the spectral fluctuations from Poisson to GOE. However, many questions remain open. We have already mentioned that the spectral fluctuations of the systems discussed are not translational invariant and that the characteristic fluctuation patterns are expected to be asymptotic. It is important to determine the rate of convergence to the asymptotic regime and, at a given energy E , the range on which Poisson or GOE fluctuations are expected to be valid. Clearly, the range does not extend from zero to E and in this connection an important role is played by the different sorts of classical closed orbits (see [Be-81b,84]). Another question of current interest is to determine the parameter that governs the spectral fluctuations when the regime is not fully chaotic. The authors of ref.[SVZ-84] tentatively propose an average of the fraction of phase space filled by chaotic trajectories in the energy range considered. This seems very plausible but more experience is needed before a final conclusion can be drawn.

Let us also mention some problems created when heavily relying on numerical results (see the introductory Section for extremistic views and warnings). Obviously, the longer the spectrum span studied, the more difficult and time consuming is the work to be furnished. But in order to have a chance to guess asymptotic properties, it is of crucial importance to test the approximate translational invariance of the results and one thus needs many levels. It may therefore be misleading to rely exclusively on results obtained from the first few dozens of lowest eigenvalues as is often done. Furthermore, it is also clear that if one is comparing, for instance, two billiards differing only by small irregularities, in order to see differences one needs small wavelengths, i.e., high energies. Also worth mentioning are some other types of numerical difficulties. At the end of Section IV some properties of rational and irrational polygon billiards were mentioned. It is our understanding that, without a solid theoretical tool, it is hardly possible to investigate on a computer questions where the rationality or irrationality plays an essential role. (In this respect, the study of the harmonic two-dimensional oscillator is illuminating [BGP-84]).

We have, for instance, studied a billiard in a rational polygon (i.e. a pseudo-integrable system) discussed in ref.[RB-81]. Richens and Berry predict level repulsion. We have computed several spectra containing each of the order of 300 levels but no sign of attaining an asymptotic regime has been found, for instance for the average value of Δ_3 . Presently we cannot draw conclusions from these numerical experiments.

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- Ref.[Be-83], entitled "Semiclassical Mechanics of Regular and Irregular Motion", is an excellent review of the subject.
- In ref.[Ca-84] the point has been made on the present state of the art ; it includes the more recent developments.

V- SUMMARY, REMARKS, CONCLUSIONS

Our main concern in these lectures has been the study of fluctuation properties of spectra. We have seen that as a prerequisite, one needs a clean identification of the smooth behaviour of the spectrum. What is left out from the average behaviour are precisely the level fluctuations. Ensembles of random matrices have been introduced to describe the Hamiltonian and special emphasis has been put in the description of the Gaussian Orthogonal Ensemble (GOE), the prototype model valid for time-reversal and rotational invariant systems. The derivation of the GOE using information theory is most enlightening; it appears as the model obtained when the only ingredient of the theory is the real symmetric nature of the Hamiltonian. It is important to remember that GOE is a parameter-free theory. Reasons have been given to consider the failure of GOE in describing global properties (experimental level densities) as unimportant. Indeed, one expects that global and local properties are disconnected. On the other hand, it has been shown that GOE-fluctuations are not specific of GOE. They are shared presumably by a large class of ensembles of random matrices. We have examined some of them.

We have discussed how to characterize level fluctuations. In full generality, one needs the set of k -level cluster functions Y_k . Some of the most relevant qualitative features of GOE fluctuations have been emphasized : level repulsion (small probability of occurrence of small spacings) and spectral rigidity (for instance, logarithmic increase with L of the variance of the number of levels to be found in an interval of length L). This is in contrast with what happens for a spectrum obtained by adding spacings coming from random independent trials distributed like e^{-x} , viz. a Poisson spectrum. In this case there is by construction no level repulsion but level clustering (the variance of the number of levels increases linearly with L). The effect of level repulsion is that levels appear rather evenly distributed, and when spectral rigidity is present the spectrum looks incompressible. It is important to notice that the spacing distribution $p(x)$ contains no information about spacing correlations, one of the main characteristics of GOE-fluctuation patterns. The role of exact symmetries is prominent and GOE-predictions apply to levels having the same set of exact quantum number (J^π).

The comparison between GOE-fluctuation predictions and experimental data has been reviewed. Due mainly to a thorough effort in high resolution measurements of compound nucleus resonances, a very stringent comparison between theory and nuclear data can be performed. GOE-predictions are fully consistent with experiment, not only for 2-point measures, where the comparison may become significant at as low as a few percent level, but even for measures containing up to 4-point cluster functions.

The present quality of atomic data allow a comparison which, although quite significant, is yet far from what can be achieved for the nuclear case. There are a few very recent and promising attempts to compare GOE-fluctuations with molecular spectra. An extensive experimental effort in the atomic and molecular case is called for. It is worth mentioning that the close agreement between GOE-predictions and data in the nuclear case can be used to impose restrictions on mechanisms that would change the fluctuations. In particular if time reversal invariance is not an exact symmetry, the appropriate model would be the Gaussian Unitary Ensemble (GUE, see Fig.II.7b). But even a small perturbation of a GOE-matrix by a GUE-matrix induces major changes in the fluctuation properties. This notion is being pursued to derive an upper bound on the time-reversal symmetry breaking part of the nuclear Hamiltonian [FK-82]. It is not yet clear whether with the amount of presently available nuclear data such an upper bound may be competitive with the ones derived by other means (sensitivities down to 10^{-3}). In passing, we mention a conjecture supported by extensive numerical data [Od-82] : the fluctuation properties of the imaginary part of the zeros of the Riemann zeta function are those of the eigenvalues of GUE matrices (or of unitary matrices belonging to the equivalent circular unitary ensemble).

In summary, we are facing a remarkable generality of fluctuations which is two-fold, experimental (nuclei, atoms, probably molecules, covering five or six orders of magnitude in scale, which is fixed by the average level spacing) and theoretical (it is shared by a large variety of matrix ensembles). It applies to very different systems, governed as well by strong short range interactions than by electromagnetic long range interactions. What is then the origin of this universality of level fluctuation laws ? They apply to complex systems, but complex in which sense ? What is the origin of the randomness ? To get insight in these questions we turn our attention to classical conservative Hamiltonian systems, where the notions of simple and complicated have been thoroughly investigated. In the classical case, we follow the path that leads from extreme regularity (integrable systems) to strongly chaotic motion. The most important tools are shortly reviewed -structure of phase space, Poincaré sections, how the phase space is filled when the system evolves, stability of orbits, etc.- and the hierarchy of irregular systems is defined : ergodic, mixing, K- and Bernoulli systems. It is discussed in which sense a system governed by causal equations of motion may be unpredictable. Examples of very different nature are given, illustrating the different categories. In particular, the properties of several two- dimensional systems are described using the tools introduced. It is seen that in general the structure of the Poincaré sections depends on the energy and that for some systems, by varying the energy, one undergoes a transition from almost regular motion to chaotic motion. Special emphasis is put on billiards, which have the simplifying feature that all their energy surfaces have the same structure. It appears that there is no need of many degrees of freedom (in fact, two are sufficient) in order to attain strongly chaotic motion and several two-dimensional strongly defocusing

billiards are discussed (Bernouilli systems).

We then proceed to search for quantum-mechanical manifestations of classical chaotic motion and restrict ourselves to spectral properties. To simplify the discussion and with the aim of finding clean signatures of chaoticity, we treat as pure as possible limiting systems, namely the Schrödinger equation for one particle moving in a box in two dimensions (equivalently the transverse vibrations of a membrane) corresponding to integrable and to strongly chaotic billiards (Bernouilli systems). It appears that the spectrum-fluctuations are of the Poisson type for the integrable case and of GOE-type for the chaotic billiards. We are thus led to make the following Conjecture 1 : The spectrum of the Laplacian with Dirichlet (or Neumann) boundary conditions on an irregular boundary has asymptotically (high energy) GOE-fluctuations^(*). By irregular we mean such that the corresponding classical billiard is a Bernouilli system (possibly that ergodic is sufficient). These results, of course, apply also to transverse vibrations of a membrane or, in three dimensions, to the electromagnetic oscillations of a cavity. Porter has foreseen this possibility very clearly. He wrote : "That matrix ensembles will most likely be relevant to the fields of acoustics and elasticity is rather evident" [Po-65a]. But experiments should be performed, for instance by observing the resonances of a microwave cavity of irregular shape [Dy-83]. Therefore, the following picture emerges : At a "macroscopic scale", we have universality properties for these systems. The number of eigenvalues up to a given energy depends only on macroscopic features of the boundary, such as surface, perimeter. At the other extreme, at a "microscopic scale", fluctuations also show universality patterns : Poisson-pattern for integrable systems, GOE-pattern for strongly chaotic systems.

It is unfortunate that in the literature one uses the terms "regular" and "irregular" spectra to indicate spectra originating from a regular and irregular system respectively. With this denomination, a spectrum like the one shown in Fig.I.8a would be "regular" whereas the ones shown in Fig.I.8c,d would be "irregular". But, as we have seen, levels from an "irregular" spectrum are much more regularly distributed than from a "regular" spectrum. The interplay between the symmetry or regularity of the shape of a box or a potential and the regularity of the associated eigenfrequencies or eigenvalues is an important and familiar topic in acoustics and nuclear physics. In acoustics, for instance, one major practical problem consists in designing auditoriums such that the response is as uniform as possible and it is well known from acousticians that for that purpose irregular walls are desirable, whereas when the room is more symmetrical the response of the room is more irregular [Mo-81]. In nuclear physics

(*) One can imagine variations to this Conjecture 1. To attack the problem mathematically, it may be convenient, instead of putting the complication in the shape of the boundary, to put it in the metric of the space and consider free motion without walls. We remind, for instance, that the geodesic flow on a surface of negative curvature is a Bernouilli system [Or-74].

it is known that there are many facts well explained by the properties of the mean field. In particular, shell effects belong to everyday practice of nuclear physicists and are responsible for dramatic effects in a multitude of phenomena like the abundances of elements, the occurrence of stable deformations, the heights of fission barriers, the existence of shape isomerism, etc.. And large shell effects are nothing but extreme cases of the clustering or bunching of eigenvalues that we have been discussing. Would the mean field be completely irregular, shell effects would not exist.

However, the origin of the success of GOE in describing the fluctuations of the compound nucleus resonances has not to be searched in properties of the mean field because we very well know that a single-particle theory is absolutely unable to correctly predict the positions and widths of these resonances. The origin of this success is to be found in more general properties which are not specific of shapes of boxes or potentials. Remember that spectra of other systems (like atoms) show GOE-fluctuations. Remember also that some particular systems, not necessarily billiards, which have been studied, when they undergo a transition from the integrable to the ergodic strongly chaotic regime in the classical case, the corresponding spectrum fluctuations in the quantum case undergo a transition from Poisson-patterns to GOE-patterns. We thus make Conjecture 2, which is more general than Conjecture 1 : Spectrum fluctuations of quantal time-reversal invariant systems whose classical analogues are strongly chaotic have GOE fluctuation patterns [BGS-84a]. If the conjecture happens to be true, it will then have been established the universality of the laws of level fluctuations in spectra already found in nuclei, to a lesser extent in atoms and to a much lesser extent in molecules. They should be tested systematically and should also be found in other systems, such as hadrons, etc..

One of the main themes in these lectures has been the translation of a scheme like the one shown in Fig.III.22 into the quantum case. We think that although some significant steps have been performed, most remains to be done. For instance, one should look for formal proofs of the equivalence of GOE- and spectral fluctuations of chaotic billiards (we remember that most of the arguments rely only on a numerical basis, although some attempts to give formal proofs should be noticed [Pe-83,Be-84]). One should discover "simple" systems showing spectrum fluctuations corresponding to the other canonical ensembles of random matrices, namely GUE and GSE. For instance, we expect that chaotic systems which are not time-reversal invariant will show GUE fluctuations ; work in this direction is in progress.

To close these lectures let us finally remark that we have been dealing with objects which are, at first sight, disconnected : quantal objects like compound nucleus resonances, classical objects like eigenmodes of vibrating membranes, frequencies in a cavity, mathematical objects like eigenvalues of random matrices, the structure in phase space of irregular motions, the zeros of the Riemann zeta function, etc.. The unifying ability is one of the great privileges of theoretical physics, which moves

between two distinct worlds, the world of physical phenomena and the world of mathematics. Wigner, the "physicist", who has been at the origin of the theory of level fluctuations in terms of random matrices, says, in his article suggestively entitled "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" : "The first point is that mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections. Secondly, just because of this circumstance, and because we do not understand the reasons of their usefulness, we cannot know whether a theory formulated in terms of mathematical concepts is uniquely appropriate" [Wi-67b]. Whereas Poincaré, the "mathematician", who has been at the origin of the study of the stability of dynamical systems, says : "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes..., elle nous fait pressentir la solution"... Almost with perfect reflexion anti-symmetry !

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