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Average Entropy of a Quantum Subsystem

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It was recently conjectured by D. Page that if a quantum system of Hilbert space dimension nm is in a random pure state then the average entropy of a subsystem of dimension m where $m \leq n$ is $S_{m,n} = \left(\sum_{k=n+1}^{mn} 1/k\right) - (m-1)/2n$. In this Letter a simple proof of this conjecture is given. [S0031-9007(96)00569-8]

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In a recent Letter Page [1] considered a system AB with Hilbert space dimension mn. The system consisted of two subsystems A and B of dimensions m and n, respectively. Page calculated the average

$$S_{m,n} = \langle S_A \rangle$$

of the entropy S_A over all pure states $\rho = |\Psi\rangle\langle\Psi|$ of the total system where $S_A = -\mathrm{Tr}\rho_A \ln\rho_A$ and ρ_A, the density matrix of subsystem A, is obtained by taking the partial trace of the full density matrix ρ over the other subsystem, that is, $\rho_A = \mathrm{Tr}_B \rho$.

The average was defined with respect to the unitary invariant Haar measure on the space of unitary vectors $|\Psi\rangle$ in the mn dimensional Hilbert space of the total system. The quantity $\ln m - S_{mn}$ was used to define the average information of the subsystem A. It is a measure of the information that is contained in the m-subsystem A regarding the fact that the entire system AB is in a pure mn state. Using earlier work [2,3] in this area, Page was led to consider the probability distribution of the eigenvalues of ρ_A for the random pure states ρ of the entire system. He used

$$P(p_1,\ldots,p_m)\,dp_1\ldots dp_m = N\delta\left(1 - \sum_{l=1}^{m} p_l\right) \prod_{1\leq i<j\leq m} (p_i - p_j)^2 \prod_{k=1}^{m} p_k^{n-m} dp_k\,,$$

where p_i was an eigenvalue of ρ_A and the normalization constant for this probability distribution was given only implicitly by the requirement that the total probability integrated to unity. Page then showed that the average

$$S_{m,n} = -\int \left(\sum_{i=1}^{m} p_i \ln p_i\right) P(p_1,\ldots,p_m)\,dp_1\cdots dp_m = \psi(mn+1) - \frac{\int\left(\sum_{i=1}^{m} q_i \ln q_i\right) Q\,dq_1\cdots dq_m}{mn\int Q\,dq_1\cdots dq_m}\,, \tag{1}$$

where $q_i = rp_i$ for $i = 1,\ldots,m$, r is positive [1], and

$$\psi(mn+1) = -C + \sum_{k=1}^{mn} \frac{1}{k}\,,$$

C being Euler's constant, and

$$Q(q_1,\ldots,q_m)dq_1\cdots dq_m = \prod_{1\leq i<j\leq m} (q_i - q_j)^2 \prod_{k=1}^{m} e^{-q_k} q_k^{n-m} dq_k\,.$$

On the basis of evaluating $S_{m,n}$ for $m = 2, 3, 4, 5$ using MATHEMATICA 2.0, Page conjectured that the exact result for $S_{m,n}$ was

$$S_{m,n} = \left(\sum_{k=n+1}^{mn} \frac{1}{k}\right) - \frac{m-1}{2n},$$

but was not able to prove that this was the case. In this Letter, we will give a simple proof of this conjecture [4].

We first observe that the van der Monde determinant defined by

$$\Delta(q_1, \ldots, q_m) \equiv \prod_{1 \le j < i \le m} (q_i - q_j)$$

may be written

$$\Delta(q_1, \ldots, q_m) = \begin{vmatrix} 1 & \cdots & 1 \\ q_1 & \cdots & q_m \\ \vdots & \ddots & \vdots \\ q_1^{m-1} & \cdots & q_m^{m-1} \end{vmatrix}.$$

We next observe that $\Delta(q_1, \ldots, q_m)$ can be written as

$$\Delta(q_1, \ldots, q_m) = \begin{vmatrix} p_0(q_1) & \cdots & p_0(q_m) \\ p_1(q_1) & \cdots & p_1(q_m) \\ \vdots & \ddots & \vdots \\ p_{m-1}(q_1) & \cdots & p_{m-1}(q_m) \end{vmatrix} \qquad (2)$$

for any set of polynomials $p_k(q)$, $k = 0, \ldots, m-1$, which have the property, $p_0(q) = 1$, and

$$p_k(q) = q^k + C_{k-1} q^{k-1} + \cdots + C_0, \quad k = 1, \ldots, m-1.$$

This immediately follows from the fact that the value of a determinant does not change if the multiple of any one row is added to a different row.

We now choose polynomials $p_k^{\alpha}(q)$ judiciously. We introduce orthogonal polynomials $p_k^{\alpha}(q)$ with the properties

$$p_k^{\alpha}(q) = q^k + C_{k-1}^{\alpha} q^{k-1} + \cdots + C_0^{\alpha},$$
$$p_0^{\alpha}(q) = 1,$$
$$\int_0^{\infty} dq e^{-q} q^{\alpha} p_{k_1}^{\alpha}(q) p_{k_2}^{\alpha}(q) = h_{k_1}^{\alpha} \delta_{k_1 k_2},$$
$$\alpha = n - m.$$

Polynomials with these properties are well known. They are the generalized Laguerre polynomials defined by [5]

$$p_k^{\alpha}(q) = \frac{e^q}{q^{\alpha}} (-1)^k \frac{d^k}{dq^k} (e^{-q} q^{k+\alpha}).$$

We also note, for later use, that [5]

$$p_k^{\alpha}(q) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-r+1)} q^{k-r}, \qquad (3)$$

$$\int_0^{\infty} dq\, e^{-q} q^{\alpha} p_{k_1}^{\alpha}(q) p_{k_2}^{\alpha}(q) = \Gamma(k_1+1)\Gamma(k_1+\alpha+1)\delta_{k_1,k_2}, \qquad (4)$$

$$\int_0^{\infty} dq\, q^{a-1} e^{-q} p_k^{b}(q) = (1-a+b)_k \Gamma(a) (-1)^k, \qquad (5)$$

recalling that $(1-a+b)_k = (1-a+b)(1-a+b+1)\cdots(1-a+b+k-1)$. Writing $\Delta(q_1, \ldots, q_m)$ in terms of $p_k^{\alpha}(q)$ as in Eq. (2), and using the orthogonal property of these polynomials, it immediately follows from Page's proven result, Eq. (1), that

$$S_{m,n} = \psi(mn+1) - \frac{1}{mn} \sum_{k=0}^{m-1} \int_0^{\infty} \frac{e^{-q}(q \ln q) q^{n-m} [p_k^{n-m}(q)]^2 dq}{\Gamma(k+1)\Gamma(k+1+n-m)}.$$

We thus need to evaluate the integral

$$I_{n-m}^{k} = \int_0^{\infty} (q \ln q) q^{n-m} [p_k^{n-m}(q)]^2 e^{-q}\, dq.$$

We first introduce

$$J^k(\alpha) = \int_0^{\infty} q^{\alpha+1} [p_k^{\alpha}(q)]^2 e^{-q}\, dq.$$

From the definition of the Laguerre polynomial given, it follows that $p^{\alpha}(q) = p_k^{\alpha+1}(q) - k p_{k-1}^{\alpha+1}(q)$. Using this and Eq. (4), we get

$$J^k(\alpha) = \Gamma(k+1)\Gamma(k+\alpha+2) + k^2 \Gamma(k)\Gamma(k+\alpha+1) \qquad (6)$$

and we now note that

$$I_{n-m}^{k} = \left[\frac{dJ^k(\alpha)}{d\alpha} - 2\int_0^{\infty} dq\, q^{\alpha+1} e^{-q} p_k^{\alpha} \frac{dp_k^{\alpha}}{d\alpha}\right]_{\alpha=n-m}.$$

Evaluating these two terms using Eqs. (3), (4), (5), and (6), we find

$$S_{m,n} = \psi(mn+1) - \frac{1}{mn}\sum_{k=0}^{m-1}[1+(1+2k+n-m)\psi(1+k+n-m)] + \frac{2}{mn}\sum_{k=0}^{m-1}\sum_{r=0}^{k}\binom{k}{r}(-1)^{k+r}\frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m-r+1)} \times [\psi(k+n-m+1)-\psi(k+n-m-r+1)]\frac{(r-k-1)_k\Gamma(k+n-m-r+2)}{\Gamma(k+1)\Gamma(k+n-m+1)}, \tag{7}$$

where we use the fact that $\psi(z) = \frac{1}{\Gamma(z)}\frac{d\Gamma(z)}{dz}$. We now observe that

$$\psi(mn+1) - \frac{1}{mn}\sum_{k=0}^{m-1}[1+(1+2k+n-m)\psi(1+k+n-m)] = \left(\sum_{k=n+1}^{mn}\frac{1}{k}\right) + \frac{m-1}{2n}. \tag{8}$$

This follows by examining the coefficient of $\frac{1}{r}$ in

$$\sum_{k=0}^{m-1}(1+2k+n-m)\psi(1+k+n-m)$$

after writing

$$\psi(1+k+n-m) = -C + \sum_{r=1}^{k+n-m}\frac{1}{r}.$$

The third expression in Eq. (7) above is

$$\frac{2}{mn}\sum_{k=0}^{m-1}\sum_{r=0}^{k}\binom{k}{r}(-1)^{k+r}\frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m-r+1)}[\psi(k+n-m+1)-\psi(k+n-m-r+1)] \times \frac{(r-k-1)_k\Gamma(k+n-m-r+2)}{\Gamma(k+1)\Gamma(k+n-m+1)} = \frac{2}{mn}\sum_{k=0}^{m-1}\binom{k}{1}(-1)^{2k+1} = -2\,\frac{(m-1)}{2n} \tag{9}$$

since $(r-k-1)_k = 0$, for all $r \neq 0$ and $r \neq 1$, and also $\psi(k+n-m+1) - \psi(k+n-m-r+1) = 0$ when $r = 0$. On substituting (8) and (9) back into (7), we obtain

$$S_{m,n} = \left(\sum_{k=n+1}^{mn}\frac{1}{k}\right) - \frac{m-1}{2n}, \tag{10}$$

as conjectured by Page.

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